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Asymptotic Behavior of One-Dimensional Discrete-Velocity Models in a Slab

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Abstract

We prove results on the asymptotic behavior of solutions to discrete-velocity models of the Boltzmann equation in the one-dimensional slab $0 < x < 1$ with general stochastic boundary conditions at $x = 0$ and $x = 1$. Assuming that there is a constant "wall" Maxwellian $M = (M_i)$ compatible with the boundary conditions, and under a technical assumption meaning "strong thermalization" at the boundaries, we prove three types of results:
I. If no velocity has x-component 0, there are real-valued functions $\beta_1(t)$ and $\beta_2(t)$ such that in a measure-theoretic sense

$$f_i(0, t) \to \beta_1(t)M_i, \quad f_i(1, t) \to \beta_2(t)M_i$$

as $t \to \infty$. β_1 and β_2 are closely related and satisfy functional equations which suggest that $\beta_1(t) \to 1$ and $\beta_2(t) \to 1$ as $t \to \infty$.
II. Under the additional assumption that there is at least one non-trivial collision term containing a product $f_k f_l$ with $v_k = v_l$, where v_k denotes the x-component of the velocity associated with f_k, we show that in a measure-theoretic sense $\beta_1(t)$ and $\beta_2(t)$ converge to 1 as $t \to \infty$. This entails L^1-convergence of the solution to the unique wall Maxwellian. For this result, $v_k = v_l = 0$ is admissible.
III. In the absence of any collision terms, but under the assumption that there is an irrational quotient $(v_i + |v_j|)/(v_l + |v_k|)$ (here $v_i,\ v_l > 0$ and $v_j, v_k < 0$), renewal theory entails that the solution converges to the unique wall Maxwellian in L^∞.

1. Introduction

We are concerned with the long-time behavior of global solutions to initial-boundary-value problems for discrete-velocity models of the Boltzmann equation in the one-dimensional "slab" $0 < x < 1$, with stochastic boundary conditions compatible with a steady Maxwellian. As in [8], we consider a discrete-velocity gas of particles moving with a finite number of velocities $\boldsymbol{v}_i \in \mathfrak{R}^3$, $i \in \Lambda = \{1, \ldots, m\}$.

By $f_i(\cdot, t)$ we denote the density distribution function of the particles moving with the i-th velocity. We assume that there is homogeneity in the y- and z-spatial directions, so that all f_i depend only on x and t and satisfy the equations

$$\partial_t f_i + v_i \partial_x f_i = Q_i(f, f), \tag{1.1}$$

$i \in \Lambda$. Here, v_i is the x-component of $\boldsymbol{v}_i$, and

$$Q_i(f, f) = \sum_{jkl} (A_{kl}^{ij} f_k f_l - A_{ij}^{kl} f_j f_i),$$

$i \in \Lambda$. The transition rates $A_{ij}^{kl}(A_{kl}^{ij})$ are nonnegative constants, which we assume to satisfy

$$A_{kl}^{ij} = A_{kl}^{ji} = A_{lk}^{ji} \tag{1.2}$$

(indistinguishability of the particles),

$$A_{kl}^{ij}(v_j + v_i - v_l - v_k) = 0 \tag{1.3}$$

(momentum conservation) and

$$A_{kl}^{ij} = A_{ij}^{kl} \tag{1.4}$$

(microreversibility or detailed balance).

The equations (1.1) are complemented by general stochastic boundary conditions at $x = 0$ and $x = 1$. We use KAWASHIMA's notation [8].

Let $\Lambda_+ = \{i \in \Lambda;\, v_i > 0\}$, $\Lambda_- = \{i \in \Lambda; v_i < 0\}$. At $x = 0$, the boundary conditions are

$$f_i(0, t) = \sum_{j \in \Lambda_-} B_{ij}^0 f_j(0, t), \quad i \in \Lambda_+, \tag{1.5}$$

and at $x = 1$,

$$f_i(1, t) = \sum_{j \in \Lambda_+} B_{ij}^1 f_j(1, t), \quad i \in \Lambda_-. \tag{1.6}$$

The transition coefficients B_{ij}^ν, $\nu = 0, 1$, are nonnegative constants. We use the abbreviations $\sum_j^-$ and $\sum_j^+$ for $\sum_{j \in \Lambda_-}$ and $\sum_{j \in \Lambda_+}$, respectively. In order to guarantee mass conservation, we impose the following conditions on the B_{ij}^ν:

$$\begin{aligned} &\sum_i^+ B_{ij}^0 v_i + v_j = 0, \quad j \in \Lambda_-, \\ &\sum_i^- B_{ij}^1 v_i + v_j = 0, \quad j \in \Lambda_+. \end{aligned} \tag{1.7}$$

The conditions (1.7) not only imply mass conservation, but also allow the proof of an entropy theorem (see Section 2), as demonstrated for the discrete case in [8] and, for the full Boltzmann equation, in [2].

Our next restriction on the boundary conditions is that there exist constant Maxwellian equilibria $M = (M_i)_{i\in\Lambda}$ such that

$$M_i = \sum_j^{-} B^0_{ij} M_j, \quad i \in \Lambda_+,$$
$$M_i = \sum_j^{+} B^1_{ij} M_j, \quad i \in \Lambda_-. \tag{1.8}$$

A vector M is called Maxwellian in this context if all the M_i are positive and if

$$A^{ij}_{kl}(M_i M_j - M_k M_l) = 0 \tag{1.9}$$

for any $i, j, k, l \in \Lambda$.

We mention that (1.5)–(1.9) are modelled after the corresponding boundary conditions for the full Boltzmann equation (see [2]). The discrete analogue considered here is discussed in detail in [8]. A general introduction to discrete-velocity models of the Boltzmann equation is given in [7].

Clearly, the Maxwellians satisfying (1.8) form a cone in a subspace of $\Re^m$.

In addition to these boundary conditions, we supplement equations (1.1) by initial conditions

$$f_i(x, 0) = f_{i,0}(x), \quad i \in \Lambda, \tag{1.10}$$

where $f_{i,0} \in C^1_+[0, 1]$. To guarantee classical solvability of the initial-boundary-value problem, we require that the initial data satisfy the compatibility conditions

$$f_{i,0}(0) = \sum_j^{-} B^0_{ij} f_{j,0}(0), \quad f_{i,0}(1) = \sum_j^{+} B^1_{ij} f_{j,0}(1). \tag{1.11}$$

We further assume the normalization

$$\sum_{i\in\Lambda} \int_0^1 f_{i,0}(x)\,dx = 1. \tag{1.12}$$

The conditions and assumptions made so far are physically natural. In addition, we make the following more technical assumptions, which are used in our present proofs, but can certainly be relaxed.

A1. No v_i is zero.
A2. All the B^0_{ij}, B^1_{ij} are positive.
A3. There are indices i, j, k, l such that $v_i > 0$, $v_j < 0$, $v_k = v_l$ and $A^{kl}_{ij} > 0$.

Assumption A1 ensures that every particle eventually meets the boundary and assumption A2 implies that there is "good mixing" at the boundary (as is true for real wall Maxwellians). A3 is a more technical assumption which we need to apply a specific method. We point out that A2 implies, by the Perron-Frobenius Theorem, uniqueness of the wall Maxwellian given by (1.8) modulo a factor. We henceforth always assume that this wall Maxwellian is normalized such that

$$\sum_{i\in\Lambda} M_i = 1. \tag{1.13}$$

If some of the v_i are zero, our methods still apply if the collision terms are such that the corresponding M_i are uniquely determined from the M_k with $v_k \neq 0$ and the conditions (1.9) and (1.13). The Broadwell model (see (1.14), (1.15) below) is the standard example for this situation. In Section 3, where we treat the full problem, we can relax condition A1 if the Maxwellian remains unique. In Section 4, where we treat the collisionless case, we have to insist on condition A1.

Our objective in this paper is to prove that under conditions A1–A3, every global solution of (1.1), (1.5), (1.6) and (1.10) must, eventually get arbitrarily close in L^1, to the Maxwellian defined by (1.8). A slightly weaker result is obtained without A3.

This result generalizes a recent convergence result for the Broadwell model in a box (see [1]), and our research was indeed motivated by the result and the methods from [1]. For the Broadwell model

$$\begin{aligned}(\partial_t + \partial_x)v &= z^2 - vw,\\ (\partial_t - \partial_x)w &= z^2 - vw,\\ \partial_t z &= \tfrac{1}{2}(vw - z^2),\end{aligned} \tag{1.14}$$

with reflecting boundary conditions

$$v(0,t) = w(0,t), \quad v(1,t) = w(1,t) \tag{1.15}$$

the Maxwellian cone consists of the constant vectors (a, a, a) $(a > 0)$, and convergence of the global solution to (1.14), (1.15) to the unique Maxwellian follows from the observation that as a consequence of the H-Theorem, v, w and z eventually vary very slowly along their characteristics, while $z^2 \approx vw$ with the exception of sets of small measure. Note that A1 is not satisfied for the Broadwell model.

The generalization of the convergence theorem for (1.14) which we present here needs improvements of the methods developed in [1], which we present in Section 3.

In the course of this research, we naturally encountered the question of to what extent the boundary conditions enforce convergence to equilibrium. To this end, we consider in Section 4 the collisionless (or free-flow) problem, i.e., the case where $Q_i(f,f)$ is replaced by 0 for all $i \in \Lambda$, and A3 is replaced with

A4. If $\Gamma = \{\gamma_{ij};$ there is an $i \in \Lambda_+$ and a $j \in \Lambda_-$ such that $\gamma_{ij} = |v_i| + |v_j|\}$, then there is at least one irrational quotient γ_{ij}/γ_{kl}.

A4 implies that the mixing guaranteed by conditions A1 and A2 is eventually "spread out" in time.

We demonstrate in Section 4 that under assumption A4, the free-flow problem can be recast as a Markov renewal process (see ÇINLAR [3]) with non-arithmetic probability distribution, and a generalization of the rather profound renewal theorem (see [4]) to this situation implies that the solutions of the free-flow equations with the boundary conditions (1.5), (1.6) converge in L^∞ to a constant vector.

In Section 2 we review the basic properties of the system, mainly the mass conservation law and the entropy theorem, and we formulate the necessary global existence and uniqueness result as proved in [8]. In Section 3 we generalize the estimates from [1], based on the entropy theorem, to the present case, and prove our main theorem.

We remark that the main obstacle towards a generalization of our main result to two or more dimensions is the lack of a satisfactory global existence and uniqueness theory in this situation.

2. Global Solutions, Mass Conservation and the Entropy Theorem

The following theorem is proved in [8]. The proof is based on an adaptation of known techniques for the pure initial-value problem to the present case.

Theorem 2.1. *The initial-boundary-value problem* (1.1), (1.5), (1.6), (1.10) *subject to all the constraints* (1.2)–(1.4), (1.7), (1.8) *has a global nonnegative classical solution.*

Remark. The results in [8] contain no information about uniform bounds on the solution, although such bounds are certainly to be expected.

Theorem 2.2. *The global solution given by Theorem 2.1 satisfies*

$$\frac{d}{dt}\sum_{i\in\Lambda}\int_0^1 f_i(x,t)\,dx = 0 \tag{2.1}$$

(*mass conservation*) *and for each constant Maxwellian M satisfying* (1.8),

$$\begin{aligned}\sum_{i\in\Lambda}\int_0^1 & f_i \ln\frac{f_i}{M_i}(x,t)\,dx \\ & +\frac{1}{4}\int_0^t\int_0^1\sum_{ijkl}A_{ij}^{kl}\,(f_i f_j - f_k f_l)\ln\frac{f_i f_j}{f_k f_l}\,dx d\tau \\ & +\sum_{i\in\Lambda}\int_0^t v_i f_i(1,\tau)\ln\frac{f_i(1,\tau)}{M_i}\,d\tau - \sum_{i\in\Lambda}\int_0^t v_i f_i(0,\tau)\ln\frac{f_i(0,\tau)}{M_i}\,d\tau \\ & = \sum_{i\in\Lambda}\int_0^1 f_{i,0}\ln\frac{f_{i,0}}{M_i}(x)\,dx.\end{aligned} \tag{2.2}$$

Remark. The boundary terms in (2.2) are the difference with respect to the case of the pure initial-value problem.

Proof. The proofs of (2.1), (2.2) are also given in [8]. Equation (2.1) is an easy exercise; we include the proof of (2.2) here because of the central importance of (2.2) for our result.

Multiply (1.1) by $1+\ln f_i$ and sum over $i \in \Lambda$. Standard manipulations based on (1.2), (1.4) yield

$$\partial_t \sum_i f_i \ln f_i + \partial_x \sum_i v_i f_i \ln f_i = -\frac{1}{4} \sum_{ijkl} A_{kl}^{ij} (f_i f_j - f_k f_l) \ln \frac{f_i f_j}{f_k f_l}. \tag{2.3}$$

Similarly, let $M = (M_i)_{i \in \Lambda}$ be the constant Maxwellian satisfying (1.8) and (1.13). Multiplying (1.1) by $\ln M_i$, summing over i and using (1.9), we find

$$\begin{aligned} &\partial_t \sum_i (f_i \ln M_i) + \partial_x \sum_i v_i f_i \ln M_i \\ &\qquad = -\tfrac{1}{4} \sum_{ijkl} A_{kl}^{ij} (f_i f_j - f_k f_l)(\ln M_i + \ln M_j - \ln M_k - \ln M_l) \\ &\qquad = 0 \end{aligned} \tag{2.4}$$

by (1.9). Substracting (2.4) from (2.3), we get

$$\partial_t \sum f_i \ln \frac{f_i}{M_i} + \partial_x \sum v_i f_i \ln \frac{f_i}{M_i} = -\frac{1}{4} \sum_{ijkl} A_{kl}^{ij} (f_i f_j - f_k f_l) \ln \frac{f_i f_j}{f_k f_l}.$$

Integrating this from 0 to 1 with respect to x, and from 0 to t with respect to time, we arrive at (2.2). q.e.d.

Theorem 2.3. *The boundary terms in* (2.2) *satisfy the inequalities*

$$\sum v_i f_i(0,t) \ln \frac{f_i(0,t)}{M_i} \leqq 0, \tag{2.5}$$

$$\sum v_i f_i(1,t) \ln \frac{f_i(1,t)}{M_i} \geqq 0. \tag{2.6}$$

Under assumptions A.1 *and* A.2, *equality in* (2.5) *holds exactly if there is a factor* $\beta_1(t)$ *such that* $f_j(0,t) = \beta_1(t) M_j$ *for all* $j \in \Lambda_-$, *and equality in* (2.6) *holds exactly if there is a* $\beta_2(t)$ *such that* $f_j(1,t) = \beta_2(t) M_j$ *for all* $j \in \Lambda_+$.

Proof. Let $h(\eta) = \eta \ln \eta$ (for $\eta > 0$), $h(0) = 0$. h is a convex continuous function, and we can write the entropy flux $\sum_i v_i f_i \ln \frac{f_i}{M_i}$ as $\sum_i v_i M_i h\left(\frac{f_i}{M_i}\right)$. For $i \in \Lambda_+$ and $x = 0$, it follows from Jensen's inequality and (1.8) that (we suppress the arguments 0 and t)

$$h\left(\frac{f_i}{M_i}\right) = h\left(\sum_j^{-} \left(B_{ij}^0 \frac{M_j}{M_i}\right) \cdot \frac{f_j}{M_j}\right) \leqq \sum_j^{-} B_{ij}^0 \frac{M_j}{M_i} h\left(\frac{f_j}{M_j}\right),$$

and therefore, by (1.7), that

$$\begin{aligned} \sum v_i f_i \ln \frac{f_i}{M_i} &= \sum_i^{+} v_i M_i h\left(\frac{f_i}{M_i}\right) + \sum_i^{-} v_i M_i h\left(\frac{f_i}{M_i}\right) \\ &\leqq \sum_i^{+} v_i M_i \sum_j^{-} B_{ij}^0 \frac{M_j}{M_i} h\left(\frac{f_j}{M_j}\right) + \sum_i^{-} v_i M_i h\left(\frac{f_i}{M_i}\right) \end{aligned}$$

$$= \sum_j^{-} \left(\sum_i^{+} v_i B_{ij}^0 + v_j \right) M_j h\left(\frac{f_j}{M_j} \right)$$
$$= 0.$$

This inequality was first obtained by GATIGNOL [6]. It follows that

$$- \sum_{i \in \Lambda} v_i f_i(0,t) \ln \frac{f_i(0,t)}{M_i} \geqq 0, \tag{2.7}$$

and equality holds exactly if the values $f_j(0,t)/M_j$ are all equal for each $j \in \Lambda_-$.

At the other end ($x = 1$), we use the same estimates. As above, for $i \in \Lambda_-$,

$$h\left(\frac{f_i(1,t)}{M_i} \right) \leqq \sum_j^{+} B_{ij}^1 \frac{M_i}{M_j} h\left(\frac{f_j(1,t)}{M_j} \right),$$

and as $v_i < 0$ for $i \in \Lambda_-$,

$$\sum^{-} v_i M_i h\left(\frac{f_i(1,t)}{M_i} \right) \geqq \sum_i^{-} v_i M_i \sum_j^{+} B_{ij}^1 \frac{M_j}{M_i} h\left(\frac{f_j(1,t)}{M_j} \right).$$

By repeating the estimates preceding (2.7) we obtain

$$\sum v_i f_i(1,t) \ln \frac{f_i(1,t)}{M_i} \geqq 0.$$

q.e.d.

We now draw information from the entropy equality (2.2). Recall that the Maxwellian M in (2.2) has been normalized such that $\sum_i \int_0^1 f_{i,0}(x)\,dx = \sum_i M_i = 1$. We rewrite (2.2) as

$$H_M[f](t) + \tfrac{1}{4} \int_0^t e(\tau)\,d\tau = H_M[f](0) + \int_0^t \mathscr{E}_M(0,\tau)\,d\tau - \int_0^t \mathscr{E}_M(1,\tau)\,d\tau, \tag{2.8}$$

where $H_M[f](t) = \sum_{i \in \Lambda} \int_0^1 f_i \ln \frac{f_i}{M_i}(x,t)\,dx$ is the H-functional relative to M,

$$e(\tau) = \int_0^1 \sum_{ijkl} A_{ij}^{kl} (f_i f_j - f_k f_l) \ln \frac{f_i f_j}{f_k f_l}(x,\tau)\,dx$$

is the (nonnegative) entropy production due to particle interactions in the interval at time τ, and

$$\mathscr{E}_M(0,\tau) = \sum_{i \in \Lambda} v_i f_i \ln \frac{f_i}{M_i}(0,\tau), \quad \mathscr{E}_M(1,\tau) = \sum_{i \in \Lambda} v_i f_i \ln \frac{f_i}{M_i}(1,\tau)$$

are the (negatives of the) entropy production terms at the boundaries at time τ.

From Theorems 2.2 and 2.3 we read off the following facts:

Theorem 2.4. ○ *H_M is decreasing (and by Jensen's inequality, $\geqq 0$).*
○ *$\int_0^t e(\tau)\,d\tau$ is uniformly bounded.*
○ *$\int_0^t \mathscr{E}_M(0,\tau)\,d\tau$ and $\int_0^t \mathscr{E}_M(1,\tau)\,d\tau$ are uniformly bounded.*

Proof. It is enough to show the first statement; the rest then follows from identity (2.8) and Theorem 2.3. Let $h(x) = x \ln x$; then the strict convexity of h and Jensen's inequality imply

$$\int_0^1 \sum_j h\left(\frac{f_j}{M_j}\right)(M_j\,dx) \geqq h\left(\sum_j \int_0^1 f_j\,dx\right) = 0.$$

Equality applies exactly if $f_j = M_j$ a.e. q.e.d.

The powerful information which the last theorem gives us is that the entropy relative to M and the integrals

$$\int_0^t \int_0^1 \sum_{ijkl} A_{ij}^{kl}(f_i f_j - f_k f_l) \ln \frac{f_i f_j}{f_k f_l}\,dx\,d\tau \tag{2.9}$$

as well as the integrals over the boundary terms in (2.2) are uniformly bounded.

3. The Main Theorem

For later reference, we formulate a lemma about the compactness properties of the solution family $\{f_i(\cdot, t)\}_{i \in \Lambda}$ in L^1. The proof is a straightforward generalization of that given in [1] for the Broadwell model. We denote the Lebesgue measure on [0, 1] by λ.

Lemma 3.1. *For every* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that for all* $t > 0$ *and all* $\Sigma \subset [0, 1]$ *with* $\lambda(\Sigma) < \delta$,

$$\sum_{i \in \Lambda} \int_\Sigma f_i(t, x)\,dx < \varepsilon.$$

Remark. This is a consequence of the entropy theorem (Theorem 2.2) and the facts stated in Theorem 2.4.

From Lemma 1, together with the mass conservation law, it follows that every family $\{f_i(\cdot, t)\}_{t \geqq 0}$, $i \in \Lambda$, forms a weakly relatively compact set in L^1.

Next we recall the concept of a "renormalized solution", which for our problem is equivalent to the concept of a classical solution. The advantage which we gain is that, as a consequence of Theorem 2.2, the effect of the renormalized collision terms can be shown to become weaker and weaker for large times. This is the assertion of Lemma 3.2 below.

Definition. $\{f_i(t, x)\}_{i \in \Lambda}$ is called a *renormalized solution* of (1.1), (1.5), (1.6) and (1.10) if

$$(\partial_t + v_i \partial_x)[\ln(1 + f_i)] = \frac{Q_i(f, f)}{1 + f_i} \tag{3.1}$$

and if the initial and boundary conditions are satisfied.

Now let $t_N \to \infty$ and let $C_N > 0$ be a given sequence. Define rectangles $B_N = [t_N, t_N + C_N] \times [0, 1]$, and let

$$a_N = \int_{B_N} \sum_{ijkl} A_{ij}^{kl} (f_i f_j - f_k f_l) \ln \frac{f_i f_j}{f_k f_l} \, dx \, dt.$$

By Theorem 2.4, $a_N \to 0$ as $N \to \infty$ (formally, we can even set $C_N = \infty$).

Our next objective is to use the entropy bounds given by Theorem 2.4 to estimate the absolute values of the various terms whose sum is $Q_i(f, f)$. To simplify our notation, let

$$|Q_i|(f, f) = \sum_{jkl} A_{ij}^{kl} | f_k f_l - f_i f_j |.$$

Lemma 3.2. *If $C_N a_N \to 0$, then there exists a sequence $\varepsilon_N \searrow 0$ such that for all i*

$$\int_{B_N} \frac{|Q_i|(f, f)}{1 + f_i} \, dx \, dt \leqq \varepsilon_N.$$

Proof. For every $\delta > 0$, there are constants M_δ, $N_\delta < \infty$, such that if $s > -1$, *then*

$$\begin{aligned} &1) \ |s| \leqq M_\delta s \ln(1 + s) \quad \text{for } |s| \geqq \delta, \\ &2) \ |s|^2 \leqq N_\delta s \ln(1 + s) \quad \text{for } |s| < \delta \end{aligned} \tag{3.2}$$

($s \ln(1 + s)$ is superlinear away from the origin, and quadratic in small neighborhoods). Now fix a δ. Then we can write

$$\int_{B_N} \frac{|Q_i|(f, f)}{1 + f_i} \, dx \, dt \leqq \int_{B_N} \sum_{jkl} A_{ij}^{kl} \frac{f_i f_j}{1 + f_i} \left| \frac{f_k f_l}{f_i f_j} - 1 \right| dx \, dt.$$

$$= \sum_{jkl} \int_{B_N^+} \cdots + \sum_{ijk} \int_{B_N^-} \cdots,$$

where $B_N^+ = \left\{ (x, t); \left| \frac{f_k f_l}{f_i f_j} - 1 \right| \geqq \delta \right\}$, and $B_N^- = B_N - B_N^+$. We suppress the dependence of B_N^+ on the indices i, j, k, l.

Now use the estimates 1) and 2) in (3.2), with $s = \frac{f_k f_l}{f_i f_j} - 1$, to obtain

$$\sum_{jkl} \int_{B_N^+} \cdots \leqq \sum_{jkl} A_{ij}^{kl} \int_{B_N^+} \frac{f_i f_j}{1 + f_i} M_\delta \left(\frac{f_k f_l}{f_i f_j} - 1 \right) \ln \frac{f_k f_l}{f_i f_j} \, dx \, dt.$$

$$\leqq M_\delta a_N.$$

For the second integral, we use the Cauchy-Schwarz inequality and estimate

$$\sum_{jkl} \int_{B_N^-} \cdots = \sum_{jkl} A_{ij}^{kl} \int_{B_N^-} \sqrt{\frac{f_i f_j}{1+f_i}} \sqrt{\frac{f_i f_j}{1+f_i}} \left| \frac{f_k f_l}{f_i f_j} - 1 \right| dx\,dt$$

$$\leqq \sum_{jkl} A_{ij}^{kl} \left(\int_{B_N^-} f_j \right)^{\frac{1}{2}} \left(\int_{B_N^-} \frac{f_i f_j}{1+f_i} \left| \frac{f_k f_l}{f_i f_j} - 1 \right|^2 dx\,dt \right)^{\frac{1}{2}}$$

$$\leqq \sum_{jkl} A_{ij}^{kl} \sqrt{C_N} \left(\int_B N_\delta (f_k f_l - f_i f_j) \ln \frac{f_k f_l}{f_i f_j}\, dx\,dt \right)^{\frac{1}{2}}$$

$$\leqq \sqrt{C_N} \sqrt{N_\delta a_N}\,.$$

We have also used the mass conservation law. The assertion of the lemma now follows by taking $\hat{\varepsilon}_N = M_\delta a_N + \sqrt{N_\delta C_N a_N}$ and $\varepsilon_N = \sup_{n \geqq N} \hat{\varepsilon}_N$. q.e.d.

Lemma 3.2 is a generalization, and the proof is a simplification, of Lemma 3 from [1] and its proof.

We discuss what happens if $C_N = 1$ for all N and if $t_N = N$. In this case, the entropy theorems from the previous section imply that $\sum_N a_N < \infty$. Lemma 3.2 then says that

$$\int_{B_N} \frac{|Q_i|(f,f)}{1+f_i}\, dx\,dt \leqq M_\delta a_N + \sqrt{N_\delta a_N}\,. \tag{3.3}$$

Unfortunately, because we do not know whether $\sum \sqrt{a_N} < \infty$, we cannot conclude directly from Lemma 3.2 that $\int_0^\infty \int_0^1 \frac{|Q_i|(f,f)}{1+f_i}\, dx\,dt < \infty$.

In the sequel we always assume that $t_N = N$. This is not essential, but it simplifies the discussion. From the Čebyshev inequality we get the following useful consequence of Lemma 3.2.

Corollary 3.3. $\lambda^2 \left\{ (x,t) \in B_N; \frac{|Q_i|(f,f)}{1+f_i} > \sqrt{\varepsilon_N} \right\} \leqq \sqrt{\varepsilon_N}\,.$

Proof. $\sqrt{\varepsilon_N}\, \lambda^2 \left\{ (x,t) \in B_N; \frac{|Q_i|(f,f)}{1+f_i} > \sqrt{\varepsilon_N} \right\} \leqq \int_B \int \frac{|Q_i|(f,f)}{1+f_i}\, dx\,dt \leqq \varepsilon_N.$ q.e.d.

Let $P = (x,t)$ be a point in B_N. By $L_i(P)$ we denote the characteristic associated with the velocity v_i passing through P, extended forward and backward until it reaches the boundaries. It follows from (3.3) that there is a sequence ε_N converging to zero (we use again the symbol ε_N to denote this sequence) such that

$$\int_0^1 \int_{L_i(x,t_N)} \frac{|Q_i|(f,f)}{1+f_i}\, ds\,dx \leqq \varepsilon_N,$$

and because there are only finitely many velocities, the sequence ε_N can be chosen independently of i. By using the Čebyshev inequality again, we then have

Corollary 3.4. $\lambda\left\{x;\ \max_{i\in A}\int_{L_i(x,t_N)}\frac{|Q_i|(f,f)}{1+f_i}ds > \sqrt{\varepsilon_N}\right\} \leqq \sqrt{\varepsilon_N}.$

Lemma 3.5. *There is a constant $C_1 > 0$ such that for all $t \geqq 0$, all $i \in A$ and all $m \geqq 2$,*

$$\lambda\{x; f_i(x,t) \geqq mM_i\} \leqq C_1/(m\ln m).$$

Proof. This follows from the boundedness of the functional H_M and the estimates

$$M_i m\cdot\lambda\{f_i(x,t) \geqq M_i m\} \leqq \frac{1}{\ln m}\int_{\{x;\ f_i(x,t)\geqq M_i m\}} f_i \ln\frac{f_i}{M_i}$$

$$\leqq \frac{1}{\ln m}(H_M(0)+C),$$

where in the last step we have used that $x\ln\frac{x}{M_i}$ is bounded below. q.e.d.

Corollary 3.6. *The sets of x where $f_i \geqq \varepsilon_N^{-p}$ $(p > 0)$ are of measure $o(1)$ as $N \to \infty$.* (See Lemma 6 in [1].)

Corollary 3.7. *Except on a set of points x of measure $\leqq \sqrt{\varepsilon_N}$,*

$$\mathrm{var}_{L_i(x,t_N)}\ln(1+f_i) \leqq \sqrt{\varepsilon_N}.$$

There is a constant $C > 0$ such that if P_1 and P_2 are two points on $L_i(x,t_N)$, then except for $x \in [0,1]$ in a set of measure $o(1)$

$$|f_i(P_1) - f_i(P_2)| \leqq C\varepsilon_N^{1/4}.$$

Proof. The first assertion follows from Corollary 3.4 by noting that

$$|(\partial_t + v_i\partial_x)\ln(1+f_i)| \leqq \frac{|Q_i|(f,f)}{1+f_i}.$$

From this inequality,

$$\ln\frac{(1+f_i)(P_1)}{(1+f_i)(P_2)} \leqq \sqrt{\varepsilon_N},$$

with the exception of points (x,t_N) of measure $o(1)$. By applying elementary manipulations to this inequality and using the fact that by Corollary 3.6 the set where $f_i(x,t_N) \geqq \varepsilon_N^{-1/4}$ is of measure $o(1)$, the second assertion follows. q.e.d.

Remark. The above discussion shows that we can actually control $\int_{L_i(x,t_N)}|Q_i|(f,f)\,ds$ except for points x of small measure. It follows that we have control of $\mathrm{var} f_i$ along most characteristics.

We now abbreviate $I_N = [N, N+1]$. Then, by the proof of Lemma 3.2, we obtain

Corollary 3.8. *Let* $0 < p < 1$. *Then, for* $q < 1 - p$, $\sum A_{ij}^{kl} |f_k f_l - f_j f_i| \leqq \varepsilon_N^q$ *on* $I_N \times [0,1]$, *with the exception of a set of two-dimensional Lebesgue measure* $o(1)$.

Proof. First note that if $\lambda\{x; f_i(x,t) \geqq \varepsilon_N^{-p}\} = o(1)$ for $t \in I_N$, then $\lambda^2\{(x,t) \in I_N \times [0,1]; f_i(x,t) \geqq \varepsilon_N^{-p}\} = o(1)$. Therefore,

$$\begin{aligned}
&\lambda^2\{(x,t) \in B_N; |Q_i|(f,f) \geqq \varepsilon_N^q\} \\
&= \lambda^2\left\{(x,t) \in B_N; |Q_i|(f,f)\frac{1+f_i}{1+f_i} \geqq \varepsilon_N^q\right\} \\
&= \lambda^2\left\{(x,t) \in B_N; |Q_i|(f,f)\frac{1+f_i}{1+f_i} \geqq \varepsilon_N^q \text{ and } f_i \geqq \varepsilon_N^{-p}\right\} \\
&\quad + \lambda^2\left\{(x,t) \in B_N; |Q_i|(f,f)\frac{1+f_i}{1+f_i} \geqq \varepsilon_N^q \text{ and } f_i < \varepsilon_N^{-p}\right\} \\
&\leqq o(1) + \lambda^2\left\{(x,t) \in B_N; |Q_i|(f,f)\frac{1+\varepsilon_N^{-p}}{1+f_i} \geqq \varepsilon_N^q\right\} \\
&\leqq o(1) + \lambda^2\left\{(x,t) \in B_N; \frac{|Q_i|(f,f)}{1+f_i} \geqq \frac{1}{2}\varepsilon_N^{q+p}\right\} \\
&\leqq o(1) + \left(\int_{B_N} \frac{|Q_i|(f,f)}{1+f_i}\, dx\, dt\right) 2\varepsilon_N^{-p-q} \\
&\leqq o(1) + 2\varepsilon_N^{1-p-q}.
\end{aligned}$$

This estimate completes the proof.

The previous lemmas and corollaries put us in a position to draw further conclusions from the Entropy Theorem 2.4. First, let $C > 0$ be an arbitrary but fixed constant. From Theorem 2.4 we know that

$$\int_N^{N+1} \mathscr{E}_M(0,\tau)\, d\tau \to 0$$

as $N \to \infty$. Let $I_N(C) = \{t \in I_N; \forall i\ f_i(0,t) \leqq C\}$. Because we know from Theorem 2.4 that $\mathscr{E}_M(0,\tau)$ does not change sign, it follows that

$$\int_{I_N(C)} \mathscr{E}_M(0,\tau)\, d\tau \to 0.$$

Now note that $h(y) = y \ln y$ is strictly convex on $(0, C]$ ($h''(y) = 1/y$), and use the definition of $\mathscr{E}_M$ and the boundary conditions to get

$$\mathscr{E}_M(0,\tau) = \sum_i^{+} v_i M_i h\left(\sum_j^{-} B_{ij}^0 \frac{M_j}{M_i}\frac{f_j(0,\tau)}{M_j}\right) - \sum_i^{+} v_i M_i \sum_j^{-} B_{ij}^0 \frac{M_i}{M_j} h\left(\frac{f_j(0,\tau)}{M_j}\right).$$

These facts together imply

Lemma 3.9. *For all* $\varepsilon > 0$,

$$\lambda\left\{t \in I_N(C);\ \exists i,j \in \Lambda \text{ such that } \left|\frac{f_i(0,\tau)}{M_i} - \frac{f_j(0,\tau)}{M_j}\right| > \varepsilon\right\} = o(1)$$

as $N \to \infty$.

Proof. By Čebyshev's inequality $\mathscr{E}_M(0,\tau) = o(1)$ on I_N, except on sets of measure $o(1)$ as $N \to \infty$. We are only concerned with times τ for which all $f_i(0,\tau) \leqq C$. Since, by assumption, there is an $i \in \Lambda_+$ with $M_i > 0$ such that $B^0_{ij}\dfrac{M_i}{M_j} > 0$ for all $j \in \Lambda_-$, it follows from the strict convexity of h on $(0, C]$ and the lower bound $1/C$ on h'' that $\mathscr{E}_M(0,\tau)$ can only be small if all the $f_j(0,\tau)/M_j, j \in \Lambda_-$, are close to each other (if there were one pair i, j such that $\left|\dfrac{f_i}{M_i} - \dfrac{f_j}{M_j}\right|$ were large, then $\mathscr{E}_M$ would be large). By applying the boundary condition we get the assertion of the Lemma for all $j \in \Lambda$. q.e.d.

Corollary 3.10. *For all* $\varepsilon > 0$ *there is a sequence* $C_N \to \infty$ *such that*

$$\lambda\left\{t \in I_N(C_N);\ \exists i, j \in \Lambda \text{ such that } \left|\frac{f_i(0,t)}{M_i} - \frac{f_j(0,t)}{M_j}\right| > \varepsilon\right\} = o(1).$$

Proof. Choose ε arbitrary but fixed and let

$$\alpha_N(C) = \lambda\left\{t \in I_N(C);\ \exists i, j \in \Lambda \text{ such that } \left|\frac{f_i(0,\tau)}{M_i} - \frac{f_j(0,\tau)}{M_j}\right| > \varepsilon\right\}.$$

By Lemma 3.9, $\alpha_N(C) \to 0$ for each C. Next choose a sequence $C_N \to \infty$ and consider the family of sequences $\alpha_N(C_M)$, from which we choose an appropriate diagonal sequence. The assertion then follows. q.e.d.

Corollary 3.11. *For all* $\varepsilon > 0$,

$$\lambda\left\{t \in I_N;\ \exists i, j \in \Lambda \text{ such that } \left|\frac{f_i(0,t)}{M_i} - \frac{f_j(0,t)}{M_j}\right| > \varepsilon\right\} = o(1)$$

as $N \to \infty$.

Proof. By Corollary 3.10, we only need to show that $\lambda(I_N \backslash I_N(C_N)) = o(1)$ as $N \to \infty$, and this follows from Corollaries 3.5 and 3.7. q.e.d.

In the sequel we use the symbol "$\approx$" to denote equality up to order $o(1)$ as $N \to \infty$, with the exception of sets of measure $o(1)$ (in one or two dimensions, depending on the situation). So we have just proved that there is a function $\beta_1(t)$ such that in this sense on I_N

$$f(0,t) \approx \beta_1(t) M$$

(if, e.g., $v_1 > 0$, we can take $\beta_1(t) = f_1(0,t)/M_1$; this shows that β_1 can be chosen to be continuous). Similarly, there is a function $\beta_2(t)$ (which can be chosen as $f_1(1,t)/M_1$) such that

$$f(1,t) \approx \beta_2(t)M.$$

Remark. Note that so far we have not used any information supplied by the control of the collision terms. In fact, everything said so far applies to the free-flow problem, i.e., the case where the collision terms are replaced by zero in the equations, with the same boundary conditions.

For the full problem we now use Corollary 3.8 to collect further information about the functions β_1 and β_2. Recall that $\sum A_{ij}^{kl}|f_k f_l - f_i f_j| = o(1)$ on B_N, with the exception of sets of two-dimensional measure $o(1)$. This implies that $|f_k f_l - f_i f_j| = o(1)$ whenever $A_{ij}^{kl} > 0$, except on sets of measure $o(1)$ on B_N. Since by Corollary 3.5, every f_i (f_j, f_k, f_l, respectively) varies slowly along most of its characteristics, and since $f_i \approx \beta_1(t)M_i$ on the left boundary with the exception of small sets, we get that

$$o(1) = |(f_k f_l - f_i f_j)(x,t)| \approx |\beta_1(P_i)\beta_1(P_j)M_iM_j - \beta_1(P_k)\beta_1(P_l)M_kM_l|$$

(see Fig. 1), where $P_i = P_i(x,t)$, etc.

Recalling that M is a Maxwellian, we have that $M_iM_j = M_kM_l$, and hence

$$\beta_1(P_i)\beta_1(P_j) \approx \beta_1(P_k)\beta_1(P_l). \tag{3.4}$$

Notice that the location of the points P_i, P_j, etc. depends on the particular part of the collision term under consideration. We assume for the rest of the discussion that $v_j < v_l < 0 < v_k < v_i$, as indicated in Fig. 1, but this assumption is just for

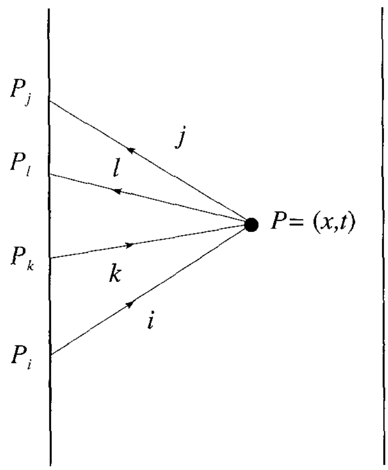

Figure 1.

convenience. If the point $P = (x, t)$ is moved along the characteristic associated with v_i, then P_i remains fixed, but the other points move in such a way that

$$\frac{P_k - P_i}{P_j - P_i} \quad \text{and} \quad \frac{P_l - P_i}{P_j - P_i}$$

remain fixed and depend only on the velocities. In fact, a short calculation shows that the first quotient is $(v_i - v_k)/(v_i + |v_j|)$, and the second is $(v_i + |v_l|)/(v_i + |v_j|)$. We denote these quotients by a and b respectively; then our assumption on the velocities implies that $0 < a < b < 1$.

We can also move the point P vertically, i.e., we can keep x fixed and vary t; the points P_i, etc., then also move vertically, with the same speed. Let $P_i(x, t) = (0, \tau)$ and $P_j(x, t) = (0, \tau + s)$. Then we realize that (3.4) can be rewritten as

$$\beta_1(\tau)\beta_1(\tau + s) = \beta_1(\tau + as)\beta_1(\tau + bs) + o(1) \tag{3.5}$$

where $s \leqq v_i + |v_j|$, except on sets of measure $o(1)$ in $I_N \times [0, v_i + |v_j|]$ with respect to (t, s). We have therefore proved our first main result.

Theorem 3.12. *There is a continuous function $\beta_1(t)$, $t \geqq 0$, such that*

$$\lambda\{t \in I_N; |f_i(0, t) - \beta_1(t) M_i| > \varepsilon\} \to 0 \tag{3.6}$$

as $N \to \infty$, and $\beta_1(t)$ satisfies the functional equation (3.5) *in $I_N \times \{0, v_i + |v_j|]$, with the exception of sets of measure $o(1)$. Except for sets of measure $o(1)$ as $N \to \infty$ the function $\beta_2(t)$ is just a shift of $\beta_1(t)$.*

Proof. Only the last statement has not yet been proved, but is an immediate consequence of Corollary 3.5. q.e.d.

It is completely trivial that $\beta_1(t) \equiv 1$ satisfies (3.5). If we could show that this is the only solution of (3.5) as $t \to \infty$, it would easily follow, from the convergence in measure spelled out in (3.6) and from the slow variation of f_i along most of its characteristics, that $f_i \to M_i$ in L^1 as $t \to \infty$. Unfortunately, we failed to find a rigorous proof that $\beta_1(\tau) \to 1$ as $\tau \to \infty$ follows from (3.5). It is true that β_1 can be considered continuous and "almost periodic" (because of the slow variation of the f_i along their characteristics, the values of β_1 repeat, with small errors, after times $\frac{1}{v_i} + \frac{1}{|v_k|}$, where $i \in \Lambda_+$, $k \in \Lambda_-$), but problems arise from the error term in (3.5) and the fact that (3.5) applies only up to small sets.

Under the additional assumption A3 from the Introduction, we can prove a stronger result. The method we employ is a generalization of the one used in [1]. Again, we use the symbol " $\approx$ " to denote "approximate equality except on sets of measure $o(1)$".

Suppose now that A3 applies and that $v_k = v_l > 0$, $A_{ij}^{kl} > 0$. By Corollary 3.8, $f_k f_l \approx f_i f_j$. Consider next a point $P \in I_N \times [0, 1]$. f_i varies slowly along $L_i(P)$, so $f_i(P) \approx \beta_1(\tau) M_i$ (see Fig. 2).

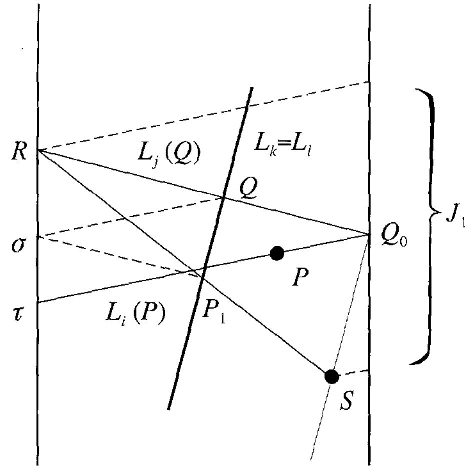

Figure 2.

Let Q be a point on the characteristic L_j leaving the point Q_0 where $L_i(P)$ reaches the right boundary. As f_j varies slowly along L_j, and as $f_j(Q_0) \approx \beta_1(\tau)\, M_j$, by the boundary condition and Theorem 3.12, it follows that

$$f_j(Q) \approx \beta_1(\tau) M_j, \quad f_i(Q) \approx \beta_1(\sigma) M_i, \quad f_j(P_1) \approx \beta_1(\sigma) M_j \tag{3.7}$$

and, as f_k and f_l vary slowly along $L_k = L_l$,

$$f_k f_l(P_1) \approx f_k f_l(Q). \tag{3.8}$$

Finally, $f_i f_j(Q) \approx f_k f_l(Q)$ and $f_i f_j(P_1) \approx f_k f_l(P_1)$. It follows from the last three identities that

$$f_i f_j(Q) \approx f_i f_j(P_1). \tag{3.9}$$

Notice that we need, for this step, no information about f_k and f_l except (3.8). It is for this reason that we can also allow $v_k = v_l = 0$.

We now need a lemma which says that for large enough N, β_1 cannot be close to zero except on sets with asymptotically vanishing measure. Specifically, let $C(N, \delta) = \lambda\{t \in [N, N+1];\ \beta_1(t) \leqq \delta\}$. Then we have

Lemma 3.13. *For every $\varepsilon > 0$ there are a $\delta > 0$ and an N_0 such that $C(N, \delta) < \varepsilon$ for all $N \geqq N_0$.*

We defer the proof of Lemma 3.13 until the end of this section.

The identity (3.9) can be rewritten as

$$\beta_1(\sigma) M_i \beta_1(\tau) M_j \approx f_i(P_1) \beta_1(\sigma) M_j,$$

and because by Lemma 3.13 $\beta_1(\sigma) \not\approx 0$ except on sets of arbitrarily small measure (explicitly $\beta_1(\sigma) > \delta$ except for σ in a set of measure ε),

$$f_i(P_1) \approx \beta_1(\tau) M_i.$$

Now note that as Q varies along $\overline{Q_0R}$ (see Fig. 2), P_1 varies along a line $\overline{RS}$ transversal to the v_i-characteristics. Since f_i varies slowly along its characteristics, it follows that

$$f_i(1,t) \approx f_i(P) \approx \beta_1(\tau)\,M_i$$

for $t \in J_1$, where J_1 is the interval indicated in Fig. 2. But this implies that β_2 is approximately constant on J_1.

By using simple overlap arguments and mass conservation, we readily see that in the sense of measure

$$\beta_1(\tau) \to 1, \quad \beta_2(\tau) \to 1$$

as $\tau \to \infty$. This means that every f_i approaches M_i on I_N in the sense of measure. But convergence in the sense of measure together with the weak compactness given by Lemma 3.1 imply L^1-convergence, which is our main result.

Theorem 3.14. *Under conditions* A1–A3, *or under conditions* A2, A3 *if the Maxwellian M is unique,*

$$\lim_{t\to\infty} \sum_{i\in\Lambda} \int_0^1 |f_i(x,t) - M_i|\,dx = 0.$$

Proof of Lemma 3.13. The proof is by contradiction. If the assertion of the lemma is false, there must be an $\varepsilon > 0$ such that for all $\delta > 0$ and all N_0 there is an $N \geqq N_0$ with $C(N,\delta) \geqq \varepsilon$. In addition, mass conservation, the compactness property from Lemma 3.1 and Corollary 3.7 imply that there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$\lambda\{t \in [N, N+1],\ \beta_1(t) > C_1\} \geqq C_2 \tag{3.10}$$

(in other words, β_1 cannot be close to zero almost everywhere, and the mass cannot concentrate on small sets by the entropy theorem; we omit a detailed verification of (3.10)).

Let G be the set in $[N, N+1]$ where $\beta_1 > C_1$. A simple geometric argument shows that there is a constant $K > 0$ (depending only on the angle between the i-th and j-th characteristics) such that the two-dimensional measure of the set of points $(x,t) \in [N, N+1] \times [0,1]$ for which both $L_i(x,t)$ and $L_j(x,t)$ meet G is at least $K(\lambda(G))^2$. In Fig. 3, for convenience and without any loss of generality, we have indicated G as a pair of intervals and (some of) the intersection set as the resultant parallelogram. Since f_i and f_j vary slowly along most of their characteristics, we have that $f_i f_j > C_1^2 M_i M_j$ on most of the intersection set. Using again that $f_k f_l \approx f_i f_j$, it follows that $f_k f_l > C_1^2 M_k M_l$ on most of this set. The characteristics L_k and L_l are identical and form a strip S indicated in Fig. 3. Let $\bar{f}_k$ and $\bar{f}_l$ denote the values of f_k and f_l at a point R in the intersection set D of the strip S and the strip E formed by the characteristics L_i emerging from the set where $\beta_1(t) < \delta$. By assumption, this set has macroscopic measure. Since $\bar{f}_k \bar{f}_l - f_k f_l = (\bar{f}_k - f_k)\bar{f}_l + f_k(\bar{f}_l - f_l)$ and since $|\bar{f}_k - f_k| < \varepsilon_N^{1/2}$ (except on a small set) and $\bar{f}_l < \varepsilon_N^{-1/4}$ except on a small set, it follows that $\bar{f}_k \bar{f}_l > \frac{1}{2} C_1^2 M_k M_l$ on most of D. However, by the same

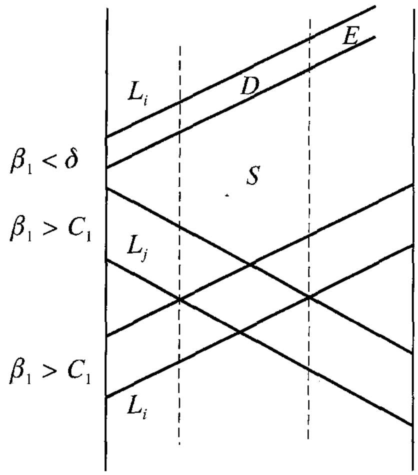

Figure 3.

reasoning applied to the strip $E, f_i f_j <$ const. $\sqrt{\delta}$ in D. As δ can be arbitrarily small and D has macroscopic measure, we have a contradiction to Corollary 3.8, and the proof of Lemma 3.13 is complete.

4. The Collisionless Case

The case where $Q_i(f, f)$ is replaced by zero appears to be simpler at first glance, because the explicit solution of the initial-boundary-value problem is immediate: The f_i's are constant along their characteristics, and the boundary conditions redistribute incoming to outgoing densities at the boundaries. The entropy theorem (Theorems 2.2 and 2.3) applies, but the term $e[f](t)$ is replaced by zero. Entropy increase is entirely due to mixing at the boundary.

In addition to the conditions which we have assumed so far, we now make the additional assumption:

A4. Let $\Gamma = \{\gamma_{ij} = |v_i| + |v_j|; i \in \Lambda_+, j \in \Lambda_-\}$. There are velocities such that at least one quotient γ_{ij}/γ_{kl} is irrational.

Under assumptions A1, A2 made in Section 1 and the additional assumption A4, we shall prove

Theorem 4.1. *Let* $f_{i,0} \in C_+[0, 1]$, $i \in \Lambda$, *be fixed initial data satisfying* (1.11), (1.12). *Then, under the conditions* A1–A3, *the solution* $f_i(x, t)$ *to the collisionless initial-boundary-value problem*

$$(\partial_t + v_i \partial_x) f_i = 0$$

with initial condition (1.10) *and boundary conditions* (1.5), (1.6) *satisfies*

$$\lim_{t\to\infty} f_i(x,t) = M_i \tag{4.1}$$

where the $M_j > 0$ *form the (unique) Maxwellian equilibrium given by* (1.8), (1.9) *and where the convergence is uniform in* $x \in [0,1]$.

We remark that it is sufficient to establish (4.1) when $x = 0$ (or $x = 1$), for then the uniformity follows immediately by the constancy of the f_i along characteristic trajectories.

The method we are going to use to prove Theorem 4.1 employs probabilistic techniques. In order to apply these, we first have to renormalize the B^{γ}_{ij} to obtain row stochastic matrices. The key for this is the condition (1.8).

Let

$$M^+ = (M_i)_{i\in\Lambda_+}, \quad M^- = (M_i)_{i\in\Lambda_-},$$
$$B^0 = (B^0_{ij})_{i\in\Lambda_+, j\in\Lambda_-}, \quad B^1 = (B^1_{ij})_{i\in\Lambda_-, j\in\Lambda_+}.$$

Equation (1.8) becomes

$$M^+ = B^0 M^-, \quad M^- = B^1 M^+. \tag{4.2}$$

Splitting $\mathfrak{R}^{\Lambda} = \mathfrak{R}^{\Lambda_+} \times \mathfrak{R}^{\Lambda_-}$, we form the vector $M = (M^+, M^-)$ and the matrices

$$B = \begin{pmatrix} 0 & B^0 \\ B^1 & 0 \end{pmatrix}, \quad V = \operatorname{diag}(M_j).$$

If $\hat{B} = V^{-1}BV$, then

$$\hat{B}_{ij} = \frac{1}{M_i} B_{ij} M_j \tag{4.3}$$

and in particular, $\hat{B}$ has exactly the same block structure as B. Since $M = BM$, we have

$$\hat{B}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{4.4}$$

i.e., $\hat{B}$ is row-stochastic. As $\hat{B}$ is irreducible (this follows easily from the block structure and assumptions A1, A2) and nonnegative, it follows from the Frobenius Theorem (see [5]) that $(1, \ldots, 1)$ is the unique eigenvector associated with the eigenvalue 1. The left stationary vector for $\hat{B}$ is also easily found.

Set $v^+ = (v_i)_{i\in\Lambda_+}$, $v^- = (v_i)_{i\in\Lambda_-}$. Then $v^+ B^0 = -v^-$ and $v^- B^1 = -v^+$, so if $v = (v^+, -v^-) \in \mathfrak{R}^{\Lambda}$, it follows that $vB = v$. We see that $\nu = (|v_i| M_i)_{i\in\Lambda}$ satisfies

$$\nu\hat{B} = \nu. \tag{4.5}$$

For each x, t set $f^+(x,t) = ((f_i(x,t))_{i\in\Lambda_+}, 0^{\Lambda_-})$, $f^-(x,t) = (0^{\Lambda_+}, (f_i(x,t))_{i\in\Lambda_-}) \in \mathfrak{R}^{\Lambda}$, where 0^n denotes the zero element in $\mathfrak{R}^n$. The equations (1.5), (1.6) become

$$f^+(0,t) = Bf^-(0,t), \quad f^-(1,t) = Bf^+(1,t), \tag{4.6}$$

which, in the new coordinate system

$$q^+(x,t) = V^{-1}f^+(x,t), \quad q^-(x,t) = V^{-1}f^-(x,t), \tag{4.7}$$

may be written as

$$q^+(0,t) = \hat{B}q^-(0,t), \quad q^-(1,t) = \hat{B}q^+(1,t). \tag{4.8}$$

In component form, the constancy of the solution along characteristic trajectories and (4.8) together yield

$$\begin{aligned} q_i(0,t) &= \sum_{j\in\Lambda_-} \hat{B}_{ij} q_j\left(1, t - \frac{1}{|v_j|}\right) \quad \forall i \in \Lambda_+, \\ q_i(1,t) &= \sum_{j\in\Lambda_+} \hat{B}_{ij} q_j\left(0, t - \frac{1}{|v_j|}\right) \quad \forall i \in \Lambda_-. \end{aligned} \tag{4.9}$$

We now turn to the Markov renewal version of our problem. The notation and motivation follow that of ÇINLAR [3].

For $i, j \in \Lambda$ and $t \geqq 0$, define the functions

$$Q(i,j,t) = \begin{cases} \hat{B}_{ij} & \text{if } t \geqq \dfrac{1}{|v_j|}, \\ 0 & \text{if } 0 \leqq t < \dfrac{1}{|v_j|}. \end{cases} \tag{4.10}$$

We observe the properties

$$Q(i,j,t) \to 0, \tag{4.11}$$

$$Q(i,j,s) \leqq Q(i,j,t) \quad \text{for } s \leqq t, \tag{4.12}$$

$$\forall i \in \Lambda, \quad \sum_j \lim_{t\to\infty} Q(i,j,t) = \sum_j \hat{B}_{ij} = 1. \tag{4.13}$$

Let X_n and T_n, $n = 0, 1, 2, \ldots$, be random variables on a probability space (Ω, P) satisfying $X_n \in \Lambda$ and $T_n \in \mathfrak{R}_+ = [0, \infty]$ for all n, and

$$0 = T_0 \leqq T_1 \leqq \ldots \leqq T_n \leqq \ldots.$$

Assume that the process (X_n, T_n) evolves according to a rule

$$\begin{aligned} P\{X_{n+1} = j,\ & T_{n+1} - T_n \leqq t \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n, T_0 = t_0, \ldots, T_n = t_n\} \\ &= P\{X_{n+1} = j, T_{n+1} - T_n \leqq t \mid X_n = i_n\} \\ &= Q(i_n, j, t). \end{aligned} \tag{4.14}$$

In the terminology of ÇINLAR, (X_n, T_n) is a Markov renewal process (henceforth abbreviated as MRP) with semi-Markov kernel $Q(i,j,t)$. Given the kernel $Q(i,j,t)$ satisfying (4.11)–(4.13), it is easy to see that there exists a MRP evolving according to (4.14).

Since $\hat{B}$ is irreducible, the Markov process X_n is irreducible. As $\hat{B}_{jj} = 0$ and $(\hat{B}^2)_{jj} > 0$ for all $j \in \Lambda$, each state recurs after exactly two steps in the process. The sojourn time between two occurrences of state $j_0 \in \Lambda_+$ passing through state $i \in \Lambda_-$ is $\frac{1}{|v_{j_0}|} + \frac{1}{|v_i|}$. Thus the (cumulative) distribution function $F(j_0, j_0, t)$ of these sojourn times between recurrence to state j_0 may be computed explicitly.

Set $\Gamma_t(j_0) = \left\{ i \in \Lambda_-; \frac{1}{|v_{j_0}|} + \frac{1}{|v_i|} \leqq t \right\}$; then $F(j_0, j_0, t) = \sum_{i \in \Gamma_t(j_0)} \hat{B}_{j_0 i}$, so condition A4 ensures that F is *non-arithmetic* (see ÇINLAR [3]). A similar expression may be derived for $j_0 \in \Lambda_-$. Accordingly, the MRP (X_n, T_n) is said to be *irreducible and aperiodic.*

Before we can state our main result, we need to establish some notation: For $\tau > 0$, define the hitting times to the interval (τ, ∞) by

$$N_\tau = \sup\{n;\ T_n \leqq \tau\}.$$

So $T_{N_\tau + 1}$ is the first time that $T_n > \tau$. Given $i, j \in \Lambda, \tau, \xi > 0$, we also define hits to the interval $(\tau, \tau + \xi]$ starting at zero:

$$H(i, j, \tau, \xi) = P\{X_{N_\tau + 1} = j,\ T_{N_\tau + 1} \in (\tau, \tau + \xi] \,|\, X_0 = i\}.$$

For $i, j \in \Lambda$, $t \geqq 0$, and $n = 0, 1, \ldots$ set

$$r_n(i, j, t) = P\{X_n = j,\ T_n \leqq t \,|\, X_0 = i\}.$$

Define

$$R(i, j, t) = \sum_{n \geqq 0} r_n(i, j, t).$$

R is the so-called *Markov Renewal Function.*

$v = (|v_j| M_j)_{j \in \Lambda}$ is the left stationary vector for $\hat{B}$ (see (4.5)). Finally, we define the *expected sojourn time* during visits to j by

$$m_j = \int_0^\infty \{1 - \sum_k Q(j, k, t)\}\, dt.$$

Using (4.10), we easily see that

$$m_j = \sum_k \hat{B}_{jk} \frac{1}{|v_k|}.$$

Let $m = (m_j)_{j \in \Lambda}$.

Lemma 4.2. *For all $i, j \in \Lambda$ and all $\xi \geqq 0$,*

$$\lim_{\tau \to \infty} H(i, j, \tau, \xi) = \frac{1}{v \cdot m} \sum_{k \in \Lambda} v_k \int_0^\infty \varphi_\xi(k, j, s)\, ds,$$

where $\varphi_\xi(k, j, s) = Q(k, j, s + \xi) - Q(k, j, s)$.

Proof. Let $n \geqq 0$ be fixed. Summing over $k \in \Lambda$ and integrating over $x \in [0, \tau]$, one obtains

$$P\{X_{n+1} = j,\ T_{n+1} \in (\tau, \tau + \xi] \mid X_0 = i\}$$
$$= \sum_{k \in \Lambda} \int_0^\tau P\{X_{n+1} = j,\ T_{n+1} - T_n \in (\tau - x, \tau + \xi - x] \mid X_0 = i,$$
$$X_n = k,\ T_n = x\}\, dr_n(i, k, x)$$
$$= \sum_{k \in \Lambda} \int_0^\tau P\{X_{n+1} = j,\ T_{n+1} - T_n \in (\tau - x, \tau + \xi - x] \mid X_n = k\}\, dr_n(i, k, x),$$

where we have twice used (4.14). This may be written as

$$\sum_{k \in \Lambda} \int_0^\tau \varphi_\xi(k, j, \tau - x)\, dr_n(i, k, x).$$

Summing now over $N_\tau = n$ (disjoint events) and using the Monotone Convergence Theorem yield

$$H(i, j, \tau, \xi) = P\{X_{N_\tau + 1} = j,\ T_{N_\tau + 1} \in (\tau, \tau + \xi] \mid X_0 = i\}$$
$$= \sum_{k \in \Lambda} \int_0^\tau \varphi_\xi(k, j, \tau - x)\, dR(i, k, x).$$

By the Markov Renewal Theorem (Prop. 4.9, p. 331 in ÇINLAR [3]) this latter expression converges, as $\tau \to \infty$, to

$$\frac{1}{v \cdot m} \sum_{k \in \Lambda} v_k \int_0^\infty \varphi_\xi(k, j, s)\, ds.$$

q.e.d.

As we know the form of Q, v and m explicitly, we obtain

Corollary 4.3.

$$\lim_{\tau \to \infty} H(i, j, \tau, \xi) = \begin{cases} \dfrac{1}{\sum_k M_k} |v_j| M_j \xi & \text{for } 0 \leqq \xi \leqq \dfrac{1}{|v_j|}, \\ \dfrac{M_j}{\sum_k M_k} & \text{for } \xi > \dfrac{1}{|v_j|}. \end{cases}$$

It is now easy to see how this result determines the values $q_j(0, t)$, $j \in \Lambda_-$, and $q_j(1, t)$, $j \in \Lambda_+$, for large values of t. The equations (4.9) can be written in terms of integration with respect to point measures. For $\tau > 0$,

$$\forall i \in \Lambda_+, \quad \int_{\Lambda \times \Re} q_p(0, s)\, \delta_{i, \tau}(dp, ds)$$
$$= \int_{\Lambda \times \Re} q_p(1, s) \sum_{j \in \Lambda_-} \hat{B}_{ij} \delta_{j, \tau - \frac{1}{|v_j|}}(dp, ds), \tag{4.15}$$

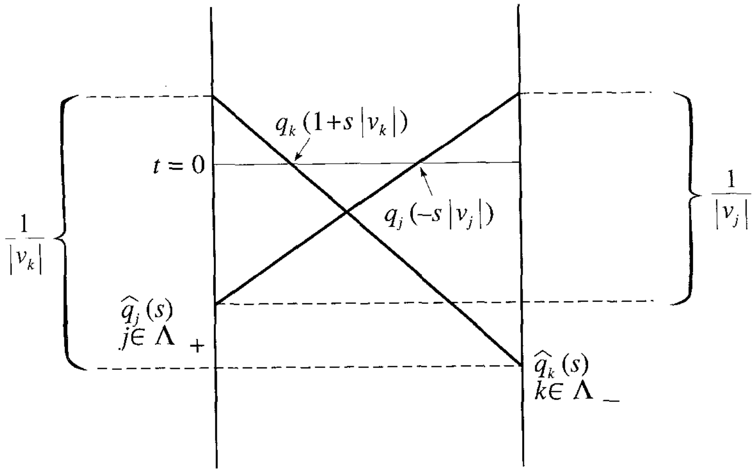

Figure 4.

$$\forall i \in \Lambda_-, \int_{\Lambda \times \mathfrak{R}} q_p(1,s)\,\delta_{i,\tau}(dp,ds)$$

$$= \int_{\Lambda \times \mathfrak{R}} q_p(0,s) \sum_{j \in \Lambda_+} \hat{B}_{ij}\delta_{j,\tau-\frac{1}{|v_j|}}(dp,ds), \tag{4.16}$$

where $\delta_{i,t}$ denotes the point mass at $(i,t) \in \Lambda \times \mathfrak{R}$.

The distribution of the measures on the right-hand sides of (4.15), (4.16) is

$$\sum_{j \in \Lambda} Q(i,j,\tau - \xi),$$

so if we iterate these expressions and stop the point masses when they first reach the set $\Lambda \times (-\infty, 0)$, the distribution of these discrete measures is the same as the hitting distribution of (X_n, T_n) to (τ, ∞), namely $H(i,j,\tau,\xi)$. Since the $H(i,j,\tau,\xi)$ converge pointwise as $\tau \to \infty$, the stopped discrete measures coverge weakly to

$$\mu_{i,j}(d\xi) = \frac{1}{\sum_k M_k} |v_j|\, M_j \lambda(d\xi) = \mu_j(d\xi)$$

where $\lambda(d\xi)$ denotes the one-dimensional Lebesgue measure on $(-\infty, 0)$. Note the independence of the right-hand side from i. We conclude that for all $i \in \Lambda$,

$$\lim_{\tau \to \infty} q_i(0,\tau) = \sum_{j \in \Lambda_+} \int_{-\infty}^{0} \hat{q}_j(s) m_j(ds) + \sum_{j \in \Lambda_-} \int_{-\infty}^{0} \hat{q}_j(s)\, m_j(ds) \tag{4.17}$$

where the $\hat{q}_j$ are obtained by projecting the values of q_j to the real axis along characteristic lines (see Fig. 4):

$$\begin{aligned} \hat{q}_j(s) &= q_j(-s|v_j|) \quad \text{for } j \in \Lambda_+, \\ \hat{q}_j(s) &= q_j(1 + s|v_j|) \quad \text{for } j \in \Lambda_-. \end{aligned} \tag{4.18}$$

Finally, by a change of variables on each integral in (4.17) we may integrate over [0, 1] to obtain

$$\lim_{\tau \to \infty} q_i(0, \tau) = \left(\sum_{k \in \Lambda} M_k \right)^{-1} \sum_{k \in \Lambda} \int_0^1 q_j(x) M_j \, dx. \tag{4.19}$$

But from (4.7) $q_j M_j = f_j$, and we combine this with the normalization (1.12) to obtain (4.1). This completes the proof of Theorem 4.1.

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References

1. L. ARKERYD & R. ILLNER, *The Broadwell Model in a Box: Strong Convergence to Equilibrium.* SIAM J. Appl. Math., to appear.
2. C. CERCIGNANI, *Theory and Application of the Boltzmann Equation.* Springer-Verlag (1988).
3. ÇINLAR, *Introduction to Stochastic Processes.* Prentice-Hall (1975).
4. W. FELLER, *An Introduction to Probability Theory and its Applications, II.* Wiley (1966).
5. F. R. GANTMACHER, *Application of the Theory of Matrices.* Interscience (1959).
6. R. GATIGNOL, *Kinetic Theory Boundary Conditions for Discrete Velocity Gas.* Phys. Fluids **20**, 2022-2030 (1977).
7. R. ILLNER & T. PLATKOWSKI, *Discrete Velocity Models of the Boltzmann Equation: A Survey on the Mathematical Aspects of the Theory.* SIAM Review **30**, 213–255 (1988).
8. S. KAWASHIMA, *Global Solutions to the Initial Boundary Value Problems for the Discrete Boltzmann Equation.* Nonlin. Anal. Th. Meths. Appls. **17**, 577–597 (1991).

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