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On the Initial-Boundary-Value Problem for $u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqq 2$, with Lipschitz boundary $\partial\Omega$, and let $T \in (0, \infty)$. In this paper we consider the problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \qquad \text{in } Q_T \equiv \Omega \times (0, T), \tag{1.1a}$$

$$u = 0 \qquad \text{on } S_T \equiv \partial\Omega \times (0, T), \tag{1.1b}$$

$$u(x, 0) = u_0(x) \quad \text{on } \Omega, \tag{1.1c}$$

in the case where u_0 lies in $L^1(\Omega)$ and $1 < p < 2$. We establish the following theorem.

Theorem A. *Let Ω, T, u_0, and p be given as above. Then there exists a unique renormalized solution to* (1.1).

In the generality considered in Theorem A, an estimate of the type

$$|\nabla u| \in L^q(Q_T), \quad q \geqq 1,$$

is no longer possible. This suggests that solutions of (1.1) display new phenomena that cannot be incorporated into the classical weak formulation. To define a suitable notion of a weak solution, we employ the idea of renormalization, first introduced by DIPERNA & LIONS [DL]. We see from Theorem A that our notion of a renormalized solution does encompass the new phenomena here. Also, the uniqueness in Theorem A implies that this notion of a solution is the "physically" correct one for (1.1).

If $u_0 \geqq 0$, then Theorem A can be inferred from [DH]. However, the argument presented in [DH] relies on the nonnegativeness of u_0 in an essential way. In fact, our problem here was proposed as an open problem in [DH]; also see [D] where a comprehensive account of problems of the type (1.1) is presented. Thus Theorem A gives a positive answer to this open problem. We remark that the nonlinear semigroup theory can also be employed to establish an existence assertion for (1.1). However, in this existence assertion the sense in which the partial differential equation (1.1a) is satisfied is unclear.

The main gap between the case $u_0 \geqq 0$ and the case where u_0 may change sign is that in the latter case an estimate of the type

$$\int_s^T \int_\Omega \frac{u_t^2}{(1+|u|)^{1+\varepsilon}}\,dx\,dt < \infty, \quad \varepsilon > 0, \quad s \in (0, T),$$

is no longer available. To overcome this difficulty, we develop an analysis that is based upon suitable versions of the chain rule and the comparison principle.

This work is organized as follows. In Section 2, we introduce the notion of a renormalized solution for (1.1). Then we proceed to prove the uniqueness of such a solution. The existence of a renormalized solution is established in Section 3.

We conclude this section by making some remarks on notation. We use $\|\cdot\|_p$ to denote the norm in L^p. For $s \in \mathbb{R}$, $s^+ = \max\{s, 0\}$ and $s^- = \max\{-s, 0\}$. If f is a measurable function on Q_T, for each $M \in \mathbb{R}$, we denote by $\{f \geqq M\}$ the set $\{(x, t) \in Q_T : f(x, t) \geqq M\}$.

2. Uniqueness

In this section, the notion of a renormalized solution is introduced for (1.1). Then we establish the uniqueness of such a solution.

Before we give our definition of a renormalized solution, we need to introduce some notations and function spaces. For each $M > 0$ define

$$P_M(s) = \min\{|s|, M\}\,\mathrm{sign}(s). \tag{2.1}$$

Assume that u is a measurable function in Q_T with

$$P_M(u) \in L^p(0, T; W^{1,p}(\Omega)) \quad \text{for each } M > 0.$$

Then we can construct a function $g: Q_T \to \mathbb{R}^N$ so that

$$g = \nabla P_M(u) \quad \text{almost everywhere on } \{|u| < M\} \quad \text{for all } M > 0. \tag{2.2}$$

Clearly, g, as is determined by (2.2), is measurable and is unique up to a set of measure 0. In this case, we call g the spatial gradient of u in the almost everywhere sense. If, in addition, u lies in $L^1(0, T; W^{1,1}(\Omega))$, then $g = \nabla u$ almost everywhere on Q_T. In general, we also denote g by ∇u.

Recall that a function $\eta(s)$ on $\mathbb{R}$ is said to be piecewise continuous if there exist $-\infty < a_0 < a_1 < \cdots < a_j < \infty$ such that η is continuous on $(-\infty, a_0) \cup (a_0, a_1) \cup \cdots \cup (a_j, \infty)$ and $\eta(a_i \pm)$ exist for $i = 0, 1, \ldots, j$. Define $\mathscr{A} = \{\theta \in C(\mathbb{R})$: θ is a Lipschitz function whose derivative θ' is piecewise continuous and the set $\{s \in \mathbb{R}$:

$\theta'(s) \neq 0$ or $\theta'(s)$ does not exist$\}$ is bounded$\}$. It is easy to see that

$$\mathscr{A} \subset W^{1,\infty}(\mathbb{R}). \tag{2.3}$$

Define

$$\mathscr{A}_c = \{\theta \in \mathscr{A}\text{: the support of } \theta \text{ is compact}\}.$$

Now we are ready to present the definition of a renormalized solution.

Definition. A measurable function u on Q_T is said to be a *renormalized solution* of (1.1) if

(i) $u \in C([0, T]; L^1(\Omega))$.
(ii) For each $\theta \in \mathscr{A}$, $\theta(u) \in L^p(0, T; W^{1,p}(\Omega))$ and $\nabla\theta(u) = \theta'(u)\nabla u$ almost everywhere on Q_T, where $\theta'(u)$ is understood to be 0 if $u \in B_\theta \equiv \{s \in \mathbb{R}\text{: } \theta'(s) \text{ does not exist}\}$ and ∇u is taken in the almost everywhere sense. If, in addition, $\theta(0) = 0$, then $\theta(u) \in L^p(0, T; W_0^{1,p}(\Omega))$.
(iii) $|\nabla u|^{p-1} \in L^1(Q_T)$ and

$$- \int_{Q_T} \int_0^u \theta(s)\, ds \varphi_t \, dx\, dt + \int_{Q_T} |\nabla u|^{p-2}\nabla u(\nabla\theta(u)\varphi + \theta(u)\nabla\varphi)\, dx\, dt$$
$$= \int_\Omega \varphi(x, 0) \int_0^{u_0(x)} \theta(s)\, ds\, dx \quad \text{for all } \theta \in \mathscr{A} \text{ and all } \varphi \in W^{1,\infty}(Q_T) \tag{2.4}$$

such that $\varphi(x, T) \equiv 0$ and $\varphi\theta(u) \in L^p(0, T; W_0^{1,p}(\Omega))$.

Let us make a few remarks about the definition. For each $M > 0$, $P_M \in \mathscr{A}$, and thus ∇u can be calculated in the almost everywhere sense. Fix $\theta \in \mathscr{A}$. Then $\int_0^u \theta(s)\, ds \in C([0, T]; L^1(\Omega))$, because $\int_0^s \theta(\tau)\, d\tau$ is Lipschitz continuous on $\mathbb{R}$. Now, let $M > 0$ be so large that $\operatorname{supp}\theta' \subset [-M, M]$. Then

$$|\nabla u|^{p-2}\nabla u \nabla\theta(u) = |\nabla P_M(u)|^p \theta'(u) \quad \text{almost everywhere on } Q_T.$$

Thus, the second integral in (2.4) makes sense because $P_M(u) \in L^p(0, T; W_0^{1,p}(\Omega))$. Similarly, if $\theta \in \mathscr{A}_c$, then $\operatorname{div}(\theta(u)|\nabla u|^{p-2}\nabla u) \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ and $\theta(u)|\nabla u|^p \in L^1(Q_T)$. We can conclude that our renormalized solution is well-defined. It is easy to see that if, in addition, $u \in L^p(0, T; W_0^{1,p}(\Omega))$, then u is a classical weak solution.

In the following lemma we state a version of the chain rule [ET, p. 27] that is suitable for our purpose.

Lemma 2.1. *Let $\xi(s)$ be a Lipschitz function on $\mathbb{R}$ with $\xi(0) = 0$. Assume that*

$$u \in V(0, T) \equiv \{u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))\text{:}$$
$$u_t \in Y \equiv L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(0, T; L^1(\Omega))\}.$$

Then the function $\eta(t) \equiv \int_\Omega \int_0^{u(x,t)} \xi(s)\,ds\,dx$ *is absolutely continuous on* $[0, T]$ *and*

$$\frac{d}{dt}\eta = (u_t, \xi(u)) \quad \textit{almost everywhere on } (0, T), \tag{2.5}$$

where $(\cdot,\cdot)$ *denotes the duality pairing between* $W^{-1,p'}(\Omega) + L^1(\Omega)$ *and* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. A result in [B] shows that

$$L^\infty(Q_T) = L^\infty(0, T; L^\infty(\Omega)), \quad L^1(Q_T) = L^1(0, T; L^1(\Omega)).$$

We easily see that $(u_t, \xi(u)) \in L^1(0, T)$. To obtain (2.5), it is enough to show that

$$-\int_{Q_T} \psi'(t) \int_0^{u(x,t)} \theta(s)\,ds\,dx\,dt = \int_0^T (u_t, \xi(u))\psi(t)\,dt \quad \text{for all } \psi \in C_0^\infty(0, T).$$

For this purpose, let ρ be the C_0^∞ function defined by

$$\rho(s) = \begin{cases} c\exp\dfrac{1}{s^2 - 1} & \text{if } |s| < 1, \\ 0 & \text{if } |s| \geqq 1. \end{cases}$$

Here, $c > 0$ is such that $\int_{-\infty}^{\infty} \rho(s)\,ds = 1$. Next, set

$$\rho_k(s) = k\rho(ks) \quad \text{for each } k \in \{1, 2, \ldots\}.$$

Now fix $\psi \in C_0^\infty(0, T)$. Then we can select $0 < a < b < T$ so that

$$\operatorname{Supp}\psi \subset (a, b).$$

For each $t \in [a, b]$, and each k such that $\frac{1}{k} < \min\{a, T - b\}$, define $u_k \in C^\infty([a, b]; W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$ by

$$u_k(x, t) = \int_0^T \rho_k(t - \tau)u(x, \tau)\,d\tau.$$

An elementary calculation indicates that

$$u_k \to u \quad \text{strongly in } L^p(a, b; W_0^{1,p}(\Omega)),$$

$$u_k \to u \quad \text{almost everywhere on } Q_T \text{ and weak* in } L^\infty(a, b; L^\infty(\Omega)),$$

$$\frac{\partial u_k}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{strongly in } L^{p'}(a, b; W^{-1,p'}(\Omega)) + L^1(a, b; L^1(\Omega)).$$

Keeping these limits in mind, we compute that

$$-\int_{Q_T} \psi'(t) \int_0^u \xi(s)\,ds\,dx\,dt = -\lim_{k\to\infty} \int_{Q_T} \psi'(t) \int_0^{u_k} \xi(s)\,ds\,dx\,dt$$

$$= \lim_{k\to\infty} \int_a^b \int_\Omega \psi \xi(u_k) \frac{\partial}{\partial t} u_k \, dx\, dt$$

$$= \lim_{k\to\infty} \int_0^T \left(\frac{\partial}{\partial t} u_k, \xi(u_k) \right) \psi \, dt = \int_0^T \left(\frac{\partial}{\partial t} u, \xi(u)\psi \right) dt.$$

This completes the proof.

Remark. Note that the dual space of $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ is much larger than Y. Our proof relies upon the assumption that $u_t \in Y$ in an essential way.

Lemma 2.2. *Let u be a renormalized solution of* (1.1). *Then for each* $\varepsilon > 0$,

$$\lim_{M\to\infty} \int_{\{M \leqq |u| \leqq M+\varepsilon\}} |\nabla u|^p \, dx\, dt = 0.$$

Proof. For each $M > 0$, and each $\varepsilon > 0$, define

$$J_{M,\varepsilon}(s) = \begin{cases} 1 & \text{if } 0 \leqq s \leqq M, \\ -\dfrac{1}{\varepsilon}(s - M - \varepsilon) & \text{if } M < s < M + \varepsilon, \\ 0 & \text{if } s \geqq M + \varepsilon, \\ J_{M,\varepsilon}(-s) & \text{if } s < 0. \end{cases} \tag{2.6}$$

For simplicity, we write J for $J_{M,\varepsilon}$ in this proof. Now, select a sequence of functions $\{\xi_k\}$ from $W^{1,\infty}(0, T)$ so that

$$\xi_k(0) = 1, \quad \xi_k(T) = 0,$$

$$\xi_k'(t) \leqq 0 \quad \text{almost everywhere on } [0, T], \tag{2.7}$$

$$\lim_{k\to\infty} \xi_k(t) = 1 \quad \text{for each } t \text{ in } [0, T). \tag{2.8}$$

Set $\theta = J(s^+) - 1$, and $\varphi = \xi_k$ in (2.4) to obtain

$$- \int_{Q_T} \int_0^u (J(s^+) - 1)\, ds\, \xi_k' \, dx\, dt + \int_{Q_T} |\nabla u|^{p-2} \nabla u \nabla J(u^+) \xi_k \, dx\, dt$$
$$= \int_\Omega \int_0^{u_0} (J(s^+) - 1)\, ds\, dx. \tag{2.9}$$

Note that

$$J(s^+) - 1 = 0 \quad \text{on } (-\infty, M].$$

This, together with (2.7), shows that the first term in (2.9) is nonpositive. By virtue of (ii),

$$\nabla J(u^+) = J'(u^+) \operatorname{sign}^+(u) \nabla u = 0 \quad \text{almost everywhere on } \{u \leqq M\} \cup \{u \geqq M + \varepsilon\}.$$

We conclude from (2.9) that

$$\frac{1}{\varepsilon}\int_{\{M<u<M+\varepsilon\}} |\nabla u|^p \xi_k \, dx\, dt \leqq \int_\Omega \int_0^{u_0} (1-J(s^+))\, ds\, dx \leqq \int_\Omega (u_0-M)^+\, dx.$$

This, in conjunction with (2.8), implies that

$$\lim_{M\to\infty} \frac{1}{\varepsilon}\int_{\{M<u<M+\varepsilon\}} |\nabla u|^p\, dx\, dt = 0.$$

Similarly, we can establish that

$$\lim_{M\to\infty} \int_{\{-M-\varepsilon<u<-M\}} |\nabla u|^p\, dx\, dt = 0.$$

This completes the proof.

Theorem 2.3. *Let f, g be two functions in $L^1(\Omega)$. Assume that u is a renormalized solution of (1.1) corresponding to f, and v a renormalized solution of (1.1) corresponding to g. If $f \leqq g$, then $u \leqq v$.*

Proof. For each $M>0$, let $J_M = J_{M,1}$. Then set

$$A_M = \int_0^u J_M(s)\, ds - \int_0^v J_M(s)\, ds.$$

For each $\theta \in \mathscr{A}_c$ and each renormalized solution w, we infer from (iii) that

$$\frac{\partial}{\partial t}\int_0^w \theta(s)\, ds - \operatorname{div}(\theta(w)|\nabla w|^{p-2}\nabla w) + \theta'(w)|\nabla w|^p = 0 \quad \text{in } \mathscr{D}'(Q_T).$$

This implies that

$$\left(\frac{\partial}{\partial t}A_M, \varphi\right) + \int_\Omega (J_M(u)|\nabla u|^{p-2}\nabla u - J_M(v)|\nabla v|^{p-2}\nabla v)\nabla\varphi\, dx$$
$$+ \int_\Omega (J_M'(u)|\nabla u|^p - J_M'(v)|\nabla v|^p)\varphi\, dx = 0 \tag{2.10}$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and almost every $t\in(0,T)$.

Fix $\varepsilon>0$, and let θ_ε be the function defined by

$$\theta_\varepsilon(s) = \begin{cases} 1 & \text{if } s \geqq \varepsilon, \\ \frac{1}{\varepsilon}s & \text{if } |s|<\varepsilon, \\ -1 & \text{if } s\leqq -\varepsilon. \end{cases}$$

Note that $A_M \in V(0, T)$. Set $\varphi = \theta_\varepsilon^+(A_M)$ in (2.10), apply Lemma 2.1, and integrate the resulting equation with respect to t over $(0, \tau)$, $\tau > 0$, to get

$$\int_\Omega \int_0^{A_M(x,\tau)} \theta_\varepsilon^+(s)\,ds\,dx + \int_{Q_\tau} (J_M(u)|\nabla u|^{p-2}\nabla u - J_M(v)|\nabla v|^{p-2}\nabla v)\nabla\,\theta_\varepsilon^+(A_M)\,dx\,dt$$
$$+ \int_{Q_\tau} (J'_M(u)|\nabla u|^p - J'_M(v)|\nabla v|^p)\theta_\varepsilon^+(A_M)\,dx\,dt = \int_\Omega \int_0^{A_M(x,0)} \theta_\varepsilon^+(s)\,ds\,dx, \quad (2.11)$$

where $Q_\tau = \Omega \times (0, \tau)$. A result in [O, p. 145] shows that

$$(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v) \geqq (p-1)\frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}}. \quad (2.12a)$$

The second integral in (2.11), denoted by I, can be estimated as follows:

$$I = \int_{\{|u|<M\}\cap\{|v|<M\}\cap Q_\tau} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\theta_\varepsilon^+)'(A_M)(\nabla u - \nabla v)\,dx\,dt$$
$$+ \int_{Q_\tau\setminus\{|u|<M\}\cap\{|v|<M\}} (\theta_\varepsilon^+)'(A_M)((J_M(u))^2|\nabla u|^p + (J_M(v))^2|\nabla v|^p$$
$$- J_M(u)J_M(v)|\nabla v|^{p-2}\nabla v\,\nabla u - J_M(u)J_M(v)|\nabla u|^{p-2}\nabla u\,\nabla v)\,dx\,dt$$
$$\geqq \int_{\{|u|<M\}\cap\{|v|\geqq M\}\cap Q_\tau} (\theta_\varepsilon^+)'(A_M)J_M(v)(-|\nabla v|^{p-2}\nabla v\,\nabla u$$
$$- |\nabla u|^{p-2}\nabla u\,\nabla v)\,dx\,dt$$
$$+ \int_{\{|u|\geqq M\}\cap\{|v|<M\}\cap Q_\tau} (\theta_\varepsilon^+)'(A_M)J_M(u)(-|\nabla v|^{p-2}\nabla u\,\nabla v$$
$$- |\nabla v|^{p-2}\nabla v\,\nabla u)\,dx\,dt$$
$$+ \int_{\{|u|\geqq M\}\cap\{|v|\geqq M\}\cap Q_\tau} (\theta_\varepsilon^+)'(A_M)J_M(u)J_M(v)(-|\nabla v|^{p-2}\nabla v\,\nabla u$$
$$- |\nabla u|^{p-2}\nabla u\,\nabla v)\,dx\,dt$$
$$\equiv I_1 + I_2 + I_3. \quad (2.12b)$$

We wish to show that

$$\lim_{M\to\infty} I_i = 0 \quad \text{for } i = 1, 2, 3. \quad (2.12c)$$

To this end, observe from the definition of ∇v that $\nabla v = 0$ almost everywhere on any set where v is a constant. Hence, the integrand in I_1 may not be zero only when the following all hold:

$$M < |v| < M + 1, \quad (2.13)$$

$$0 < A_M = \int_0^u J_M(s)\,ds - \int_0^v J_M(s)\,ds < \varepsilon, \tag{2.14}$$

$$|u| < M. \tag{2.15}$$

Combining (2.14) and (2.15) yields

$$\int_0^v J_M(s)\,ds < u < \varepsilon + \int_0^v J_M(s)\,ds. \tag{2.16}$$

Note that $\int_0^M J_M(s)\,ds = M$, $\int_0^{-M} J_M(s)\,ds = -M$, and $\int_0^s J_M(\tau)\,d\tau$ is nondecreasing on $\mathbb{R}$. We conclude from (2.13) and (2.16) that

$$\{M < v < M+1\} \cap \left\{\int_0^v J_M(s)\,ds < u < \varepsilon + \int_0^v J_M(s)\,ds\right\} \cap \{|u| < M\} = \emptyset,$$

$$\{-M-1 < v < -M\} \cap \left\{\int_0^v J_M(s)\,ds < u < \varepsilon + \int_0^v J_M(s)\,ds\right\} \cap \{|u| < M\}$$
$$\subset \{-M-1 < v < -M\} \cap \{-M < u < -M+\varepsilon\}.$$

Keeping these in mind, we calculate that

$$|I_1| \leqq \frac{1}{\varepsilon} \int_{\{-M<u<-M+\varepsilon\}\cap\{-M-1<v<-M\}\cap Q_\tau} (|\nabla v|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla v|)\,dx\,dt$$
$$\leqq \frac{1}{\varepsilon}\left[\left(\int_{\{-M-1<v<-M\}} |\nabla v|^p\,dx\,dt\right)^{1/p'}\left(\int_{\{-M<u<-M+\varepsilon\}} |\nabla u|^p\,dx\,dt\right)^{1/p}\right.$$
$$\left. + \left(\int_{\{-M<u<-M+\varepsilon\}} |\nabla u|^p\,dx\,dt\right)^{1/p'}\left(\int_{\{-M-1<v<-M\}} |\nabla v|^p\,dx\,dt\right)^{1/p}\right].$$

By Lemma 2.2,

$$\lim_{M\to\infty} I_1 = 0.$$

Similarly, we can show that

$$\lim_{M\to\infty} I_j = 0, \quad j = 2, 3.$$

It follows from (2.12b) and (2.12c) that

$$\liminf_{M\to\infty} I \geqq 0. \tag{2.17}$$

Taking $M \to \infty$ in (2.11), and keeping in mind (2.17) and Lemma 2.2, we get

$$\int_\Omega \int_0^{u-v} \theta_\varepsilon^+(s)\,ds\,dx \leqq \int_\Omega \int_0^{f-g} \theta_\varepsilon^+(s)\,ds\,dx.$$

Letting $\varepsilon \to 0$ gives

$$\int_\Omega (u - v)^+ \, dx \leqq \int_\Omega (f - g)^+ \, dx. \tag{2.18}$$

This implies the desired result.

An easy consequence of this theorem is that there exists at most one renormalized solution to (1.1). If we use θ_ε instead of θ_ε^+ in our proof, we can obtain

$$\max_{0 \leqq t \leqq T} \int_\Omega |u(x, t) - v(x, t)| \, dx \leqq \int_\Omega |f - g| \, dx. \tag{2.19}$$

3. Existence

The main result of this section is

Theorem 3.1. *Assume that* $u_0 \in L^1(\Omega)$, *and* $1 < p < 2$. *Then* (1.1) *has a renormalized solution.*

Proof. For $k \in \{1, 2, \ldots\}$, set $f_k = \min\{u_0^+, k\}$, $g_k = \min\{u_0^-, k\}$. Clearly, we have

$$0 \leqq f_k \leqq f_{k+1} \leqq u_0^+ \quad \text{for all } k, \tag{3.1}$$

$$0 \leqq g_k \leqq g_{k+1} \leqq u_0^- \quad \text{for all } k, \tag{3.2}$$

$$f_k \to u_0^+ \quad \text{strongly in } L^1(\Omega), \tag{3.3}$$

$$g_k \to u_0^- \quad \text{strongly in } L^1(\Omega). \tag{3.4}$$

Consider the regularized problems

$$\frac{\partial}{\partial t} u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \qquad \text{in } Q_T, \tag{3.5a}$$

$$u = 0 \qquad \text{on } S_T, \tag{3.5b}$$

$$u(x, 0) = u_{0k} \equiv f_k - g_k \quad \text{on } \Omega, \quad (k = 1, 2, \ldots). \tag{3.5c}$$

According to a result in [DH], for each k there exists a unique u_k such that

$$u_k \in C([0, T]; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)), \quad \frac{\partial}{\partial t} u_k \in L^{p'}(0, T; W^{-1,p'}(\Omega)),$$

$$\frac{\partial}{\partial t} u_k - \operatorname{div}(|\nabla u_k|^{p-2}\nabla u_k) = 0 \quad \text{in } W^{-1,p'}(\Omega) \quad \text{for almost every } t \in (0, T), \tag{3.6}$$

$$u_k(x, 0) = u_{0k} \quad \text{in } L^2(\Omega). \tag{3.7}$$

The remaining proof is divided into several assertions.

Assertion 1. *For each* $M > 0$, $\varepsilon > 0$, $k \in \{1, 2, \dots\}$,

$$\max_{0 \leqq t \leqq T} \tfrac{1}{2} \int_\Omega |P_M(u_k(x,t))|^2\, dx + \int_{Q_T} |\nabla P_M(u_k)|^p\, dx\, dt \leqq 2\|u_0\|_1 M, \tag{3.8}$$

$$\int_{Q_T} \frac{1}{(|u_k|+1)^{1+\varepsilon}} |\nabla u_k|^p\, dx\, dt \leqq \|u_0\|_1 \frac{1}{\varepsilon}. \tag{3.9}$$

Proof. By the proof of Lemma 2.1, we see that for each locally Lipschitz function ξ on $\mathbb{R}$ with $\xi(0) = 0$, the function $t \to \int_\Omega \int_0^{u_k(x,t)} \xi(s)\, ds\, dx$ is absolutely continuous on $[0, T]$, and

$$\frac{d}{dt} \int_\Omega \int_0^{u_k(x,t)} \xi(s)\, ds\, dx = \left(\frac{\partial}{\partial t} u_k, \xi(u_k) \right) \quad \text{almost everywhere on } (0, T), \tag{3.10a}$$

where $(\cdot, \cdot)$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. We use $P_M(u_k)$ as a test function in (3.6), and keep (3.10a) in mind to obtain

$$\frac{d}{dt} \int_\Omega \int_0^{u_k(x,t)} P_M(s)\, ds\, dx + \int_\Omega P_M'(u_k) |\nabla u_k|^p\, dx = 0 \quad \text{almost everywhere on } (0, T). \tag{3.10b}$$

Note that $\int_0^s P_M(\tau)\, d\tau \geqq \frac{1}{2}(P_M(s))^2$ for all $s \in \mathbb{R}$. Integration of (3.10b) with respect to t gives (3.8). To obtain (3.9), define

$$\phi_\varepsilon(s) = \begin{cases} 1 - \dfrac{1}{(1+s)^\varepsilon} & \text{if } s \geqq 0, \\ -\phi_\varepsilon(-s) & \text{if } s < 0. \end{cases}$$

Use $\phi_\varepsilon(u_k)$ as a test function in (3.6) to get

$$\frac{d}{dt} \int_\Omega \int_0^{u_k(x,t)} \phi_\varepsilon(s)\, ds\, dx + \int_\Omega \frac{\varepsilon}{(1+|u_k|)^{\varepsilon+1}} |\nabla u_k|^p\, dx = 0 \quad \text{almost everywhere on } (0, T).$$

This implies (3.9). The proof is complete.

In light of (3.10a), we have that

$$\int_0^T \left(\frac{\partial}{\partial t} u_k, \theta(u_k)\varphi \right) dt = -\int_{Q_T} \int_0^{u_k} \theta(s)\, ds\, \varphi_t\, dx\, dt \tag{3.10c}$$

for all $\theta \in \mathscr{A}$ and all $\varphi \in L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(Q_T)$ such that $\varphi_t \in L^2(Q_T)$, $\varphi(x, 0) = \varphi(x, T) \equiv 0$, and $\theta(u_k)\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$. Thus for each k, u_k is a renormalized solution of (3.5). We infer from (2.19) that

$$\max_{0 \leqq t \leqq T} \int_\Omega |u_m(x, t) - u_k(x, t)|\, dx \leqq \int_\Omega |u_{0m} - u_{0k}|\, dx.$$

By (3.3) and (3.4), there exists a function $u \in C([0, T]; L^1(\Omega))$ such that

$$u_k \to u \quad \text{strongly in } C([0, T]; L^1(\Omega)). \tag{3.10d}$$

This, in conjunction with (3.8), shows that

$$\theta(u_k) \to \theta(u) \quad \text{weakly in } L^p(0, T; W^{1,p}(\Omega)) \quad \text{for each } \theta \in \mathscr{A}. \tag{3.11}$$

If $\theta \in \mathscr{A}$ is such that $\theta(0) = 0$, then $\theta(u_k) \in L^p(0, T; W_0^{1,p}(\Omega))$, and thus $\theta(u) \in L^p(0, T; W_0^{1,p}(\Omega))$. Now we can calculate ∇u in the almost everywhere sense.

Assertion 2. *Let $E \subset Q_T$ be measurable. Assume that there exists an $M > 0$ such that*

$$|u_k| \leqq M \quad \textit{almost everywhere on } E \textit{ for each } k.$$

Then

$$\nabla u_k \to \nabla u \quad \textit{strongly in } (L^p(E))^N.$$

Proof. We infer from (3.6) that

$$\frac{d}{dt} \int_\Omega \int_0^{u_m(x,t) - u_k(x,t)} P_{2M}(s)\, ds\, dx$$
$$+ \int_\Omega P'_{2M}(u_m - u_k)(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_m - \nabla u_k)\, dx = 0 \tag{3.12}$$

almost everywhere on $(0, T)$.

We deduce from (3.12) and (2.12a) that

$$(p-1) \int_{Q_T} P'_{2M}(u_m - u_k) \frac{|\nabla u_m - \nabla u_k|^2}{(|\nabla u_m| + |\nabla u_k|)^{2-p}}\, dx\, dt$$
$$\leqq \int_\Omega \int_0^{u_{0m} - u_{0k}} P_{2M}(s)\, ds\, dx \leqq 2M \int_\Omega |u_{0m} - u_{0k}|\, dx. \tag{3.13}$$

With the aid of (3.8) we estimate that

$$\int_E |\nabla u_m - \nabla u_k|^p\, dx\, dt = \int_E |\nabla P_{2M}(u_m - u_k)|^p\, dx\, dt \tag{3.14}$$
$$= \int_E P'_{2M}(u_m - u_k)|\nabla u_m - \nabla u_k|^p\, dx\, dt$$
$$= \int_E P'_{2M}(u_m - u_k) \frac{|\nabla u_m - \nabla u_k|^p}{(|\nabla u_m| + |\nabla u_k|)^{((2-p)p)/2}} (|\nabla u_m| + |\nabla u_k|)^{(2-p)p/2}\, dx\, dt \leqq$$

$$\leqq \left(\int_E P'_{2M}(u_m - u_k) \frac{|\nabla u_m - \nabla u_k|^2}{(|\nabla u_m| + |\nabla u_k|)^{2-p}} dx\, dt \right)^{p/2}$$

$$\cdot \left(\int_E P'_{2M}(u_m - u_k)(|\nabla u_m| + |\nabla u_k|)^p \, dx\, dt \right)^{(2-p)/2}$$

$$\leqq \left(\frac{2M}{p-1} \int_\Omega |u_{0m} - u_{0k}| \, dx \right)^{p/2} (2^{2+p} \| u_0 \|_1 M)^{(2-p)/2}.$$

We conclude that $\{\nabla P_M(u_k)\}$ is a Cauchy sequence in $(L^p(E))^N$. This, together with (3.11), implies the desired result.

Assertion 3. *The sequence $\{|\nabla u_k|\}$ is bounded in $L^q(Q_T)$ for each $0 < q < p/2$.*

Proof. Fix $q \in (0, p/2)$. Then we can find $\varepsilon_0 > 0$ so that $q = 1/(2 + \varepsilon_0)p$. We deduce from (3.9) and (3.10d) that

$$\int_{Q_T} |\nabla u_k|^q \, dx\, dt = \int_{Q_T} \frac{1}{(|u_k| + 1)^{(1+\varepsilon_0)q/p}} |\nabla u_k|^q (|u_k| + 1)^{(1+\varepsilon_0)q/p} \, dx\, dt$$

$$\leqq \left(\int_{Q_T} \frac{1}{(|u_k| + 1)^{1+\varepsilon_0}} |\nabla u_k|^p \, dx\, dt \right)^{q/p}$$

$$\cdot \left(\int_{Q_T} (|u_k| + 1)^{(1+\varepsilon_0)q/(p-q)} \, dx\, dt \right)^{(p-q)/p}$$

$$\leqq \frac{c}{\varepsilon_0^{q/p}} \left(\int_{Q_T} (|u_k| + 1) \, dx\, dt \right)^{(p-q)/p} \leqq c(\varepsilon_0). \tag{3.15}$$

Assertion 4. *For each $\theta \in \mathscr{A}$, $\theta(u) \in L^p(0, T; W^{1,p}(\Omega))$ and $\nabla \theta(u) = \theta'(u) \nabla u$ almost everywhere on Q_T.*

The proof is a slight modification of that of Claim 4 in [X]. We omit it here.

Assertion 5. *Let $f \in L^1(\Omega)$ be such that f does not change sign. Then there exists a unique renormalized solution u to the problem*

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \textit{in } Q_T, \tag{3.16a}$$

$$u = 0 \quad \textit{on } S_T, \tag{3.16b}$$

$$u(x, 0) = f \quad \textit{on } \Omega. \tag{3.16c}$$

Furthermore, the following conditions hold:

(a) $\int_{Q_T} |\nabla P_M(u)|^p \, dx\, dt \leqq 2\|f\|_1 M$ *for each* $M > 0$.
(b) *For each* $s \in (0, T)$ *and each* $\theta \in \mathscr{A}$, $(\partial/\partial t)\,\theta(u) \in L^2(\Omega \times (s, T))$, *and thus* $\partial u/\partial t$ *can be taken in the almost everywhere sense on* Q_T. *If, in addition,* $\theta \in C^1(\mathbb{R})$, *then* $\partial/\partial t\, \theta(u) = \theta'(u)\, \partial u/\partial t$ *almost everywhere on* Q_T.
(c) *For each* $s \in (0, T)$, *and each* $\varepsilon > 0$, *there exists a positive number* $c(s, \varepsilon)$ *with*

$$\int_s^T \int_\Omega \frac{1}{(1 + |u|)^{1+\varepsilon}} u_t^2 \, dx\, dt \leqq c(s, \varepsilon). \tag{3.17}$$

The proof of this assertion is essentially contained in [DH]. Later, we shall indicate a different proof based upon our development here.

To continue our proof of Theorem 3.1, denote by u_+ the renormalized solution of (3.16) with $f = u_0^+$, and u_- the renormalized solution of (3.16) with $f = -u_0^-$. Then we can conclude from (3.1), (3.2), and Theorem 2.3 that

$$u_- \leqq u_k \leqq u_+ \quad \text{almost everywhere on } Q_T \text{ for all } k. \tag{3.18}$$

Now, we are ready to prove (iii), in the case $\theta \in \mathscr{A}_c$. For this purpose, let $\eta(s) \in \mathscr{A} \cap C^1(\mathbb{R})$ be such that

$$\eta(s) = 1 \quad \text{on } [1, \infty), \tag{3.19}$$

$$\eta(s) = 0 \quad \text{on } (-\infty, 0], \tag{3.20}$$

$$\eta' \geqq 0 \quad \text{on } (0, 1)\,. \tag{3.21}$$

For each $i \in \{1, 2, \ldots\}$ define

$$\xi_1(s) = \begin{cases} 1 & \text{if } s \in \left[\dfrac{2}{i}, T\right], \\[2ex] i\left(s - \dfrac{1}{i}\right) & \text{if } s \in \left(\dfrac{1}{i}, \dfrac{2}{i}\right), \\[2ex] 0 & \text{if } s \in \left[0, \dfrac{1}{i}\right]. \end{cases} \tag{3.22}$$

Let $\theta \in \mathscr{A}_c$, $\varphi \in C_0^\infty(\mathbb{R}^N \times (-\infty, T))$ be such that $\theta(u_k)\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$. Now for $M > 0$, set

$$\psi = \varphi\left(1 - \eta\left(\frac{u_+}{M}\right)\right)\left(1 - \eta\left(\frac{-u_-}{M}\right)\right)\xi_i(t). \tag{3.23}$$

It is easy to verify that

$$\psi \in L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(Q_T), \quad \psi_t \in L^2(Q_T). \tag{3.24}$$

Keeping (3.10c) in mind, we use $\theta(u_k)\psi$ as a test function in (3.6) to obtain

$$-\int_{Q_T}\int_0^{u_k}\theta(s)\,ds\,\psi_t\,dx\,dt+\int_{Q_T}|\nabla u_k|^{p-2}\nabla u_k\nabla\theta(u_k)\psi\,dx\,dt$$
$$+\int_{Q_T}|\nabla u_k|^{p-2}\nabla u_k\theta(u_k)\nabla\psi\,dx\,dt=0. \tag{3.25}$$

Denote by I_2 the second integral in (3.25). Set $E_M=\{u_+\leqq M\}\cap\{u_-\geqq -M\}$. Observe from (3.18) that

$$|u_k|\leqq M \quad \text{almost everywhere on } E_M \text{ for all } k.$$

We may apply Assertion 2 to obtain

$$|\nabla u_k|^{p-2}\nabla u_k\to|\nabla u|^{p-2}\nabla u \quad \text{strongly in } (L^{p'}(E_M))^N.$$

Note that

$$\psi=0 \quad \text{on } Q_T\setminus E_M.$$

We conclude from (3.11) that

$$\lim_{k\to\infty}I_2=\lim_{k\to\infty}\int_{E_M}|\nabla u_k|^{p-2}\nabla u_k\nabla\theta(u_k)\psi\,dx\,dt=\int_{Q_T}|\nabla u|^{p-2}\nabla u\,\nabla\theta(u)\psi\,dx\,dt.$$

We can take $k\to\infty$ in (3.25) to get

$$-\int_{Q_T}\int_0^{u}\theta(s)\,ds\,\psi_t\,dx\,dt+\int_{Q_T}|\nabla u|^{p-2}\nabla u\,\nabla\theta(u)\psi\,dx\,dt$$
$$+\int_{Q_T}|\nabla u|^{p-2}\nabla u\theta(u)\nabla\psi\,dx\,dt=0. \tag{3.26}$$

Let $v\in\{u_-,-u_-\}$. With the aid of (a) we estimate that

$$\int_{Q_T}\left|\nabla\left(1-\eta\left(\frac{v}{M}\right)\right)\right|^p dx\,dt=\frac{1}{M^p}\|\eta'\|_\infty^p\int_{\{|v|\leqq M\}}|\nabla v|^p\,dx\,dt$$
$$=\frac{c}{M^p}\int_{Q_T}|\nabla P_M(v)|^p\,dx\,dt$$
$$\leqq\frac{c}{M^{p-1}}\to 0 \quad \text{as } M\to\infty. \tag{3.27}$$

Consequently, we have that

$$\nabla\psi=\nabla\varphi\xi_i\left(1-\eta\left(\frac{u_+}{M}\right)\right)\left(1-\eta\left(\frac{-u_-}{M}\right)\right)$$
$$+\varphi\xi_i\nabla\left(1-\eta\left(\frac{u_+}{M}\right)\right)\left(1-\eta\left(\frac{-u_-}{M}\right)\right)$$
$$+\varphi\xi_i\left(1-\eta\left(\frac{u_+}{M}\right)\right)\nabla\left(1-\eta\left(\frac{-u_-}{M}\right)\right)\to\nabla\varphi\xi_i$$

strongly in $(L^p(Q_T))^N$ as $M \to \infty$. Now set $Q_i = \Omega \times \left(\frac{1}{i}, T\right)$, and fix $\varepsilon \in (0, 1)$. We derive from (3.17) that

$$\int_{Q_i} \left| \frac{\partial}{\partial t}\left(1 - \eta\left(\frac{v}{M}\right)\right)\right|^2 dx\,dt \leqq \frac{\|\eta'\|_\infty^2}{M^2} \int_{\theta_i \cap \{|v| \leqq M\}} v_t^2\,dx\,dt$$

$$\leqq \frac{c\left(\frac{1}{i}, \varepsilon\right)}{M^2}(1 + M)^{1+\varepsilon} \to 0 \quad \text{as } M \to \infty. \tag{3.28}$$

This implies that $\partial/\partial t\,\psi \to \partial/\partial t(\varphi\xi_i)$ strongly in $L^2(Q_T)$ as $M \to \infty$. Take $M \to \infty$ in (3.26) to obtain

$$-\int_{Q_T}\int_0^u \theta(s)\,ds\,\varphi_t\xi_i\,dx\,dt - \int_{Q_T}\int_0^u \theta(s)\,ds\,\varphi\xi_i'(t)\,dx\,dt$$
$$+ \int_{Q_T} |\nabla u|^{p-2}\nabla u\,\nabla\theta(u)\varphi\xi_i\,dx\,dt + \int_{Q_T} |\nabla u|^{p-2}\nabla u\theta(u)\nabla\varphi\xi_i\,dx\,dt = 0. \tag{3.29}$$

In view of the definition of ξ_i, we have

$$\int_{Q_T}\int_0^u \theta(s)\,ds\,\varphi\xi_i'(t)\,dx\,dt = i\int_{1/i}^{2/i}\int_\Omega\int_0^u \theta(s)\,ds\,\varphi(x, t)\,dx\,dt$$

$$\to \int_\Omega \varphi(x, 0)\int_0^{u_0(x)} \theta(s)\,ds\,dx \quad \text{as } i \to \infty,$$

because $\int_0^u \theta(s)\,ds \in C([0, T]; L^1(\Omega))$. Taking $i \to \infty$ in (3.29) yields

$$-\int_{Q_T}\int_0^u \theta(s)\,ds\,\varphi_t\,dx\,dt + \int_{Q_T} |\nabla u|^{p-2}\nabla u(\nabla\theta(u)\varphi + \nabla\varphi\theta(u))\,dx\,dt$$

$$= \int_\Omega\int_0^{u_0} \theta(s)\,ds\,\varphi(x, 0)\,dx \tag{3.30}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N \times (-\infty, T))$ and all $\theta \in \mathscr{A}_c$ such that $\varphi\theta(u) \in L^p(0, T; W_0^{1,p}(\Omega))$.

In light of the proof of Lemma 2.2, we have that

$$\frac{1}{M}\int_{\{M < u_k < 2M\}} |\nabla u_k|^p\,dx\,dt \leqq \int_\Omega (f_k - M)^+\,dx. \tag{3.31}$$

By Assertion 2 and (3.18), we may assume that $\nabla u_k \to \nabla u$ almost everywhere on Q_T. Set

$$B_M = \left\{(x, t) \in Q_T\colon \lim_{k\to\infty} u_k(x, t) = M \text{ or } 2M\right\}.$$

It is not difficult to see from a device in [X] that

$$\nabla u_k \to 0 \quad \text{almost everywhere on } B_M.$$

Hence,

$$\chi_{\{M < u_k < 2M\}} |\nabla u_k|^p \to \chi_{\{M < u < 2M\}} |\nabla u|^p \quad \text{almost everywhere on } Q_T.$$

Invoking Fatou's lemma, we derive from (3.31) that

$$\frac{1}{M} \int_{\{M < u < 2M\}} |\nabla u|^p \, dx \, dt \leqq \int_\Omega (u_0 - M)^+ \, dx.$$

Consequently,

$$\lim_{M \to \infty} \frac{1}{M} \int_{\{M < u < 2M\}} |\nabla u|^p \, dx \, dt = 0. \tag{3.32}$$

Similarly, we can show

$$\lim_{M \to \infty} \frac{1}{M} \int_{\{-2M < u < -M\}} |\nabla u|^p \, dx \, dt = 0. \tag{3.33a}$$

Next, we prove that (iii) follows (3.30), (3.32), and (3.33a). To see this, fix $\theta \in \mathscr{A}$. Then, by (3.30), we have

$$- \int_{Q_T} \int_0^u \theta(s) J_{M,M}(s) \, ds \, \varphi_t \, dx \, dt + \int_{Q_T} |\nabla u|^{p-2} \nabla u \nabla(\theta(u) J_{M,M}(u)) \varphi \, dx \, dt$$

$$+ \int_{Q_T} |\nabla u|^{p-2} \nabla u \theta(u) J_{M,M}(u) \nabla \varphi \, dx \, dt = \int_\Omega \varphi(x, 0) \int_0^{u_0} \theta(s) J_{M,M}(s) \, ds \, dx, \tag{3.33b}$$

where $\varphi \in C_0^\infty(\mathbb{R}^N \times (-\infty, T)$ is such that $\theta(u)\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$. By Assertion 3, $|\nabla u|^{p-1} \in L^1(Q_T)$, and so

$$\lim_{m \to \infty} \int_{Q_T} |\nabla u|^{p-2} \nabla u \theta(u) J_{M,M}(u) \nabla \varphi \, dx \, dt = \int_{Q_T} |\nabla u|^{p-2} \nabla u \theta(u) \nabla \varphi \, dx \, dt.$$

We estimate

$$\left| \int_{Q_T} |\nabla u|^{p-2} \nabla u \theta(u) \nabla J_{M,M}(u) \varphi \, dx \, dt \right|$$

$$\leqq \|\varphi\|_\infty \|\theta\|_\infty \frac{1}{M} \int_{\{M < |u| < 2M\}} |\nabla u|^p \, dx \to 0 \quad \text{as } M \to \infty.$$

Taking $M \to \infty$ in (3.33b) yields (iii). This completes the proof.

Remark. We indicate here a proof of Assertion 5 which seems to be much simpler than that in [DH]. Without loss of generality, assume that $f = u_0^+$. We assert that the limit u of the sequence $\{u_k\}$, obtained by setting $g_k \equiv 0$ in (3.5), is a renormalized solution of (3.16). First, according to Theorem 2.3,

$$0 \leqq u_k \leqq u_{k+1} \leqq u \quad \text{almost everywhere on } Q_T \text{ for all } k.$$

Then, due to the nonnegativity of f_k, the estimate

$$\int_s^T \int_\Omega \frac{1}{(u_k+1)^{\varepsilon+1}} \left(\frac{\partial}{\partial t} u_k\right)^2 dx\, dt \leqq c(s, \varepsilon), \quad T > s > 0,\ \varepsilon > 0,$$

holds, where $c(s, \varepsilon)$ is a positive constant depending on s, ε (see [DH]). This implies that u satisfies items (b) and (c). Item (a) is an easy consequence of (3.8). We can set $u_- = 0$ and $u_+ = u$ in our earlier argument to conclude the proof.

An interesting remaining question is: Is it possible to prove an existence and uniqueness theorem in the case where $\Omega = \mathbb{R}^N$ and $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$? I shall try to answer this question in a forthcoming paper.

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