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LESSONS FROM PSEUDO SCOTUS

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1. INTRODUCTION

Medieval logicians have a great deal to teach their modern descendants.[1] They discussed issues that are of contemporary importance with an ingenuity and sophistication lost till this century. We will illustrate this by considering an argument produced by a medieval logician, fated to become known as 'Pseudo-Scotus'. It was rescued from oblivion by Bocheński in 1938 and has been commented on more recently, particularly by Benson Mates and Stephen Read.[2] However, a good deal more can be learnt from Scotus' argument as we will show. Specifically we will formulate separate lessons in Sections 2, 4 and 6.

2. AGAINST THE CLASSICIAL ACCOUNT OF VALIDITY AND RELATEDNESS LOGIC

According to a standard account, an argument is valid iff it is impossible for the premises to be true and the conclusion false (not true). The account gives rise to different logics, depending on how 'impossible' is cashed out. (For example classicial truth conditions on possible worlds give rise to classical (or modal) logical consequence, whereas 'forcing' truth conditions give rise to intuitionistic logical consequence.) But it is in fact enshrined in most modern formal logics (with the exception of relevant logics, as will become clear). Nonetheless the account is wrong.[3] One of the things that Scotus' argument teaches us is just that.[4]

According to the account of validity in question (which we will call the 'classical account') if an argument has a necessarily true conclusion it must be valid, since *a fortiori* it is impossible for the premises to be true and the conclusion false. Scotus gave an argument whose conclusion was necessarily true but which was invalid, thus refuting the classical account. The argument is:

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'God exists. Therefore this argument is invalid' (where Scotus took the premise to be necessarily true). For the purpose of analysis we will formalize Scotus' argument. An argument from α to β is valid iff (that) α entails (that) β, i.e. iff '$\alpha \Rightarrow \beta$' is true.[5] Hence we may formulate Scotus' argument as

$$\text{(A)} \qquad \frac{\mu}{\neg(\mu \Rightarrow \lambda)}$$

where $\lambda = \neg(\mu \Rightarrow \lambda)$ and μ is necessarily true. If it can be shown that λ is necessarily true, then since λ says the argument is invalid, we are home. Now consider the following deduction.[6]

(B)

1.	μ_1		Hyp
2.		$\mu \Rightarrow \lambda_2$	Hyp
3.		μ_1	Reit
4.		$\lambda_{1,2}$	$\Rightarrow E$
5.		$\neg(\mu \Rightarrow \lambda)_{1,2}$	Since $\lambda = \neg(\mu \Rightarrow \lambda)$
6.	$\neg(\mu \Rightarrow \lambda)_1$		$\neg I$

(Alternatively one might arrive at 5 from lines 2, 3 and the principle that an argument with a true premise and a false conclusion is invalid.) $\neg(\mu \Rightarrow \lambda)$ is seen to follow from μ. Moreover by a natural and acceptable principle of modality, whatever is deducible from a necessary truth is necessarily true. And since μ is, by hypothesis, necessarily true, so is $\neg(\mu \Rightarrow \lambda)$ as required. (A) is a counterexample to the classical account of validity, which is therefore incorrect. As a bonus it is also a counterexample to the account of validity associated with 'relatedness logics'.[7] According to this, an inference is valid iff it is classically valid and the premise is related to the conclusion. What exactly relatedness *is*, need not concern us. For sharing a common propositional variable is taken to be sufficient for relatedness. Thus inference(A) comes out as valid according to this account. We have seen that it is not.

3. AGAINST EVASION

Is it possible to avoid this conclusion? According to both Mates and Read, it is. Thus Mates says (*op. cit.*, p. 138)

> [If] we accept [the classical account of validity] then [Scotus'] argument becomes an antinomy... (A) is both sound and not sound since if it were sound it would have a true antecedent and a false consequent, which is impossible in a sound *consequentia*. But

on the other hand (A) is sound. For since [μ] is counted as a necessary truth, we have just shown the consequent of (A) is a necessary truth and by [the classicial account] this suffices for the soundness of (A).

Clearly Mates does not take Scotus' argument seriously. By calling it an antinomy, he wishes to downgrade it from the falsification it is, to a mere anomaly. This tactic has been well discussed by Kuhn.[8] During periods of normal science scientists *presuppose* that the governing paradigm is correct. Falsifying evidence is classed as anomalous and put in the 'given sufficient time we'll get it sorted out' bin. Now whatever the legitimacy of this kind of move in general, Mates manoeuver will not work now. For we are no longer in a period of normal science governed by the classical paradigm.[9] Time has run out for the classical paradigm: its cup of anomalies overfloweth. During revolutionary periods, the like of which we are in now, theories must face the full glare of their refutations. Scotus' argument must be taken seriously.

In fact the line Mates would take if he were pushed further is clear. He points out the similarities between Scotus' argument and the semantic paradoxes. Since there are a number of proposed 'solutions' to these, one might suppose that one or all of these 'solutions' would serve to remove Scotus 'anomaly'. We do not think that any of the proposed 'solutions' to the paradoxes work.[10] However, even if they did, they would not save the classical account of validity. This can be seen as follows.

Essentially the semantic paradoxes, such as the liar, are generated by the assumption that the paradoxical sentence of the natural language in question is either true (or states a truth) or false (or states a falsity), but not both. All the common solutions to the paradoxes reject this assumption. They distinguish a third category of sentences. Let us call them 'defective'. On exactly what is wrong with defective sentences, the various approaches differ. According to some, defective sentences are neither true nor false. According to others, they make no statements. According to yet others, they are not grammatical (*really* grammatical that is). Yet others combine these variants and add even more. (The chaos of the situation is in fact indicative of a deep malaise here.) All this matters not. What matters is that sentence are divided up into the good (which are true or make true statements), the bad (which are false or make false statements), and the ugly (which are defective). The paradox is then 'solved' by insisting (usually *ad hocly*) that the offending sentence belongs to the defective category.

But how is all this meant to apply to Scotus' argument? Presumably in some way such as this. If an argument contains as premise or conclusion a sentence which is defective, then the argument will itself be defective in a corresponding sense, and therefore not up for appraisal as either valid or invalid. Thus the standard valid/invalid dichotomy will have to be scrapped. In its place we will have to suppose that forms of argument expressed in natural language will fall into the trichotomy: valid (or expressing a valid argument), invalid (or expressing an invalid argument), or defective. Given this trichotomy we can now avoid the conclusion that (A) is a counterexample to the classical account, by putting it in the defective box. Formally this works as follows. The sub-proof of argument (B) shows that if (A) is valid, it is invalid. Thus whether (A) is valid or invalid, it is invalid. If the categories valid/invalid were exhaustive, we could conclude that (A) is invalid *simpliciter*. However, since they are not, we are no longer forced to this conclusion and line 6 of proof (B) breaks down.

Unfortunately, despite all this, the classicial account is not saved. (A) may no longer be a counterexample to the classical account but (A′) is:[11]

$$(\text{A}') \qquad \frac{\mu}{\neg(\mu \Rightarrow \lambda) \text{ or '}\mu\text{. Therefore } \lambda\text{' is defective}}$$

where λ is the conclusion and μ is logically necessary. Now, exactly as before, the assumption that $\mu \Rightarrow \lambda$ leads to the conclusion that: $\neg(\mu \Rightarrow \lambda)$ or 'μ. Therefore λ' is defective. So by a quite legitimate *reductio* we can conclude that λ is true, and thus, necessarily true. There is no analogous way out of the problem posed by (A′). To insist that (A′) is a defective argument form doesn't help at all. It just produces a contradiction. We might just as well stop fighting against the odds and accept that (A) is an invalid argument, and that the classical account is incorrect.

4. AGAINST E

We have learnt the first lesson from Scotus' argument. However, there are others to come. In his (1979) Read discusses Scotus' counterexample and concludes that "the classical account of validity emerges unscathed from Pseudo-Scotus' attack" (p. 273) on the grounds that "whatever solution one takes to the paradoxes of self reference will undercut [Scotus' argument]" (p. 267). As we have seen, such solutions may undecut (A), but they don't undercut its simple extension (A′). Hence Read's main conclusion is wrong.

Part of Read's case that solutions to the semantical paradoxes must demolish (A) is that "A leads to a contradiction independently of the classical of validity" (p. 267). To learn the next lesson, let us analyse this claim.

According to Read the following slight extension of argument (B) is valid in E. (See his fn. 2.)

(C)

	Main proof	Sub-proof	
1.	μ_1		Hyp
2.		$\mu \Rightarrow \lambda_2$	Hyp
3.		μ_1	Reit
4.		$\lambda_{1,2}$	$\Rightarrow E$
5.		$\neg(\mu \Rightarrow \lambda)_{1,2}$	Since $\lambda = \neg(\mu \Rightarrow \lambda)$
6.	$\neg(\mu \Rightarrow \lambda)_1$		$\neg I$
7.	λ_1		Since $\lambda = \neg(\mu \Rightarrow \lambda)$
8.	$\mu \Rightarrow \lambda$		$\Rightarrow I$

Actually this is not strictly true. Line 3 does not follow in in *FE* since only entailments are reiterable in sub-proofs. However, since all we have supposed about μ so far is that it is a necessary truth, we can choose μ to be of the form $\alpha \Rightarrow \alpha$. The proof is then E-valid. Now this argument is an unconditional proof of $\mu \Rightarrow \lambda$, i.e. a proof that (A) is valid. As we have seen (and as one would expect from looking at it anyway), (A) is invalid. Hence the proof (C) must be wrong.[12] Thus E gives an incorrect account of entailment. This is the next lesson to be learnt from Scotus' argument. Actually something slightly stronger can be learnt. Since E is incorrect then *a fortiori* anything stronger, such as strict implication, the relevant implication of R, etc. must be incorrect.[13]

5. AGAINST THE SUPPRESSION OF INNOCENT PREMISES

So far all we have argued is that the E-proof (C) is unacceptable. However we can be a little more precise and explain one of the reasons why. One of the main things wrong with E is that it allows the suppression of premises.[14] Given an *FE* sub-proof with hypothesis A_k and conclusion B_α, $\Rightarrow I$ allows us to infer $A \Rightarrow B_{\alpha - \{k\}}$; even though premises other than A were used in the proof, namely those reiterated into the sub-proof from outside. True, not anything may be reiterated (as it can be in R); only things of the form $C \Rightarrow D$. Nonetheless, the fact remains that B has not been deduced from A on its own, but from A and all the assumptions obtained by reiteration. Thus it is not true that A entails B. A in conjunction with the other assumptions entails B. It is clear then that the rule $\Rightarrow I$, which allows the other formulas to be suppressed (into the relevance subscripting) is wrong.

On p. 15 of (1975) Anderson and Belnap themselves say

> in our usual mathematical or logical proofs, we demand that all the conditions required for the conclusion be stated in the hypothesis of the theorem. After the word 'PROOF': in a mathematical treatise, mathematical writers seem to feel that no more hypotheses may be introduced; and it is regarded as a criticism of the proof if not all the required hypotheses are stated explicitly at the outset.

Quite so. The suppression of premises in the form of reiteration followed by an application of $\Rightarrow I$ ought not to be allowed.

Anderson and Belnap have to motivate the reiteration of entailments somehow. The above quotation continues:

> Of course, additional machinery *may* be invoked in the proof but this must be of a logical character, i.e. in addition to the hypotheses, we may use in the argument only propositions tantamount to statements of logical necessity.

This seems to us somewhat disingenuous. Of course additional machinery is involved in the proof: rules of logic are applied to draw inferences. However, these are not additional hypotheses (premises of the argument) as Lewis Carroll taught us.[15] We do not draw conclusions *from* them but *in accordance with them*. And if we ever do draw conclusions from them, as Anderson and Belnap say, they need to be stated explicitly as premises.

Of course, it is not only applications of $\Rightarrow I$ which suppress premises. Any rule which discharges an hypothesis can suppress a premise used but obtained by reiteration. Thus take the rule $\neg I$ of FE:

$$\begin{array}{l} A_k \\ \vdots \\ \neg A_\alpha \\ \hline \neg A_{\alpha - \{k\}} \end{array}$$

This is a form of *reductio ad absurdum*. Given A and all the other premises reiterated into the sub-proof, $B_1, \ldots, B_n$, we can deduce an absurdity. What follows from this is that you can't have A and all the other premises, i.e. $\neg(A \wedge B_1 \wedge \ldots \wedge B_n)$. Hence $\neg I$ works only by suppressing all the premises $B_1, \ldots, B_n$ into the relevance subscripting. Hence $\neg I$ too is guilty of premise suppression. This becomes even clearer when we note that $\neg I$ could, in the presence of other FE rules, be replaced by the rule

$$\frac{A \Rightarrow \neg A_\alpha}{\neg A_\alpha} \qquad (\Pi)$$

Then an application of $\neg I$ is the same as juxtaposed applications of $\Rightarrow I$ and (Π), the former committing suppression. One of the things wrong with (C) is now quite clear. The application of $\neg I$ at line 6 suppresses the premise μ. Hence it is incorrect.

Before we leave the topic of suppression, there is one more point to be made. An application of $\neg I$ suppresses not only all the premises obtained by reiteration but also the law of the excluded middle. As we noted in Section 3, the rationale for $\neg I$ is that either A or $\neg A$ holds. Then if A implies $\neg A$, we have $\neg A$ in either case. But if some *tertium* is *datur*, this no longer follows. We are still faced with a choice between $\neg A$ and the *tertium*. Let us make this explicit by writing all but the last step of (C) in the more perspicuous natural deduction form:

$$\dfrac{(\mu \Rightarrow \lambda) \vee \neg(\mu \Rightarrow \lambda) \qquad \dfrac{\dfrac{\mu \quad \dfrac{}{\mu \Rightarrow \lambda}\,(1)}{\lambda}}{\neg(\mu \Rightarrow \lambda)} \qquad \dfrac{}{\neg(\mu \Rightarrow \lambda)}\,(1)}{\neg(\mu \Rightarrow \lambda)}\,(1)$$

This proof no longer uses $\neg I$. In its place, the more fundamental $\vee E$ and law of excluded middle are used. This proof contains two subproofs. One is a deduction of $\neg(\mu \Rightarrow \lambda)$ from $\mu \Rightarrow \lambda$ and the other is a subproof of $\neg(\mu \Rightarrow \lambda)$ from $\neg(\mu \Rightarrow \lambda)$. The final line of the proof is an application of $\vee E$ which discharges the assumptions $\mu \Rightarrow \lambda$ and $\neg(\mu \Rightarrow \lambda)$, in these subproofs. (This is indicated by the (1)'s.) It now becomes obvious that we are faced, not with a proof of $\neg(\mu \Rightarrow \lambda)$ from μ, but from μ and $(\mu \Rightarrow \lambda) \vee \neg(\mu \Rightarrow \lambda)$. The final application of $\Rightarrow E$ in (C) to obtain $\mu \Rightarrow \neg(\mu \Rightarrow \lambda)$, also therefore commits a suppression, viz. the suppression of $(\mu \Rightarrow \lambda) \vee \neg(\mu \Rightarrow \lambda)$.

Moreover, an extra defect of the original argument hidden by the blanket suppression of $\neg I$, now becomes clear, viz. μ is actually used in only one of the possible cases. Thus the application of $\vee E$ at (1) is illicit even in E-terms! We see that the application of $\neg I$ in (C) is objectionable not only because of suppression but also because it cloaks niceties of relevance.

6. AGAINST ASSERTION

There is yet one more lesson to be learnt from Scotus' argument. There is often more than one route to any desired conclusion. Argument (C) is no exception to this. For the final lesson, just consider the following proof of $\mu \Rightarrow \lambda$, couched again in natural deduction terms:

$$\text{i.e.}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\lambda\Rightarrow\neg\lambda}{\neg\lambda\Rightarrow\neg\neg(\mu\Rightarrow\lambda)}\qquad \neg\neg(\mu\Rightarrow\lambda)\Rightarrow(\mu\Rightarrow\lambda)}{\neg\lambda\Rightarrow(\mu\Rightarrow\lambda)}\qquad \neg\lambda\Rightarrow\neg\lambda}{\neg\lambda\Rightarrow(\neg\lambda\wedge(\mu\Rightarrow\lambda))}\qquad (\neg\lambda\wedge(\mu\Rightarrow\lambda))\Rightarrow\neg\mu\ \ (1)}{\neg\lambda\Rightarrow\neg\mu}}{\mu\Rightarrow\lambda}$$

Again, since the argument establishes that the argument (A) is valid, when it is not, there must be something wrong with it. In fact there is only one part of it which is not entirely unproblematic. All the premises used, save (1), are substitution instances of first degree entailments, and all the rules used (transitivity, contraposition and consequent conjunction) are acceptable principles of first degree entailment. Now first degree entailment is the core of all relevant systems and, as such, above suspicion.[16] Hence it must be the premise (1) which is incorrect:

$$(\neg\beta \wedge (\alpha \Rightarrow \beta)) \Rightarrow \neg\alpha$$

This is just the *modus tollens* form of the principle assertion (sometimes called *modus ponens*).

$$(\alpha \wedge (\alpha \Rightarrow \beta)) \Rightarrow \beta$$

which obviously stand or fall together. In fact they should fall. For assertion is well-known to be problematic: it leads to Curry paradoxes.[17] There are a number of arguments against it.[18] We have just found another. Scotus' argument shows assertion to be incorrect.[19]

This allows us to draw a couple of more general conclusions. First, assertion is part of all the strong relevant logics *E, R* and *T*. Hence all the strong relevant logics must be rejected. Secondly the arguments *B* and *C* of Read's paper (1979), closely related to (A), are both invalid. Proofs that they are valid all require illicit suppression by $\neg I$, or assertion in one form or another.

7. FOR DEPTH RELEVANT LOGICS

We have learnt a number of lessons from Pseudo-Scotus' argument. They have all been desctructive. It is now time to be a little more constructive. The classical account of validity is wrong. What is to replace it? The answer has already been provided by Chrysippus, Lewis, McColl and others: an argument is valid iff its premise is inconsistent with the negation of its conclusion. However, this doesn't get us far until we have a satisfactory analysis of consistency. (If we take 'α is consistent with β' to be $\Diamond(\alpha \wedge \beta)$, we are back with the classical account.) A relevant account of consistency is very satisfactory. We take 'α is consistent with β' to be $\neg(\alpha \Rightarrow \neg\beta)$. Thus an argument from α to β is valid iff α is inconsistent with $\neg\beta$ iff $\neg\neg(\alpha \Rightarrow \neg\neg\beta)$ iff $\alpha \Rightarrow \beta$, as one would expect.[20] However, which relevant logic are we to accept? We have learnt that we must reject all strong relevant logics, which contain assertion and allow suppression. This means that we must look to depth relevant logics such as *DK* of Routley (1977) and the system of Priest (1980).[21]

One final point: in Section 4 we criticized and rejected argument (B) on which Scotus' counterexample depends. Have we cut the grounds from under our own feet? The answer is 'No'. For we can formulate the argument in a perfectly acceptable way. The following is a proof of the sequent $\mu, (\mu \Rightarrow \lambda) \vee \neg(\mu \Rightarrow \lambda) : \neg(\mu \Rightarrow \lambda)$ in the system of Priest (1980).

i.e.

$$\dfrac{\dfrac{\dfrac{\mu:\mu \quad \mu\Rightarrow\lambda:\mu\Rightarrow\lambda}{\mu,\mu\Rightarrow\lambda:\lambda}}{\mu,\mu\Rightarrow\lambda:\neg(\mu\Rightarrow\lambda)} \qquad \neg(\mu\Rightarrow\lambda):\neg(\mu\Rightarrow\lambda)}{\mu,(\mu\Rightarrow\lambda)\vee\neg(\mu\Rightarrow\lambda):\ \neg(\mu\Rightarrow\lambda)}$$

Since both antecedents of the sequent are necessarily true and sequents are truth preserving, the conclusion of the sequent is necessarily true, as required.

Thus Scotus' argument supports the case of depth relevant logics, depth relevant logics support the case of Scotus' argument and all's well with at least this logical segment of the world.

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NOTES

[1] This claim was made at the end of Priest and Read (forthcoming). The present paper goes some way towards bearing it out.
[2] See Mates (1965), which gives details of Bocheński's work and Read (1979) which gives details of other modern commentators.
[3] It is criticized, for example, in Priest (1979), Section 5, (1980), Routley and Meyer (forthcoming) Ch. 1 and a number of other places.
[4] Unfortunately Scotus' account of validity (see Read (1979), Section 5) is less inspiring than his counterexample to the classical account.
[5] This is analytic and defended as such in Routley and Meyer (forthcoming) Ch. 10.
[6] For reasons that will become obvious later, we will put it in the proof and sub-proof style of *FE* of Anderson and Belnap (1975).
[7] See, e.g. Copeland (1980) and Epstein (1979).
[8] See T. Kuhn (1962).
[9] As explained in Priest (forthcoming).
[10] We prefer the paraconsistent approach. See Priest (1979a) and Routley (1977).
[11] Inspired by strengthened paradoxes, which also suffice to sink most paradox 'solutions'.
[12] It is open to a supporter of E who is also a paraconsistentist to claim that the proof is in order and that (A) is both valid and invalid. We think that this move, being little more than an *ad hoc* evasion, has little to recommend it. However, as far as we know, the class of E-paraconsistentists is empty anyway. So we will leave the point unless and until it is a real issue.
[13] Few would, of course, claim that the arrow of R is a correct account of entailment. However, a number would claim that its necessitation is. Note then that the proof (C) goes through *as it stands* in R. Hence its conclusion can be necessitated in NR. Thus this account of entailment is incorrect too. The proof (C) does not go through in T. However, T gets its just deserts in Section 6.
[14] What is wrong with suppression has been discussed in some detail in Routley and Routley (1972). What follows relates this specifically to E.
[15] See Lewis Carroll (1895).
[16] It is defended in detail in Routley and Meyer (forthcoming) Chs 2, 3.
[17] See Meyer *et al.* (1979).
[18] See Priest (1980), Section 8.
[19] Through possibly it may be acceptable in certain restricted domains.
[20] The account of validity offered in Priest (1980) appears somewhat different: an argument from α to β is valid iff the sense of α contains that of β. However, as the rest of the paper shows, this is equivalent to $\alpha \Rightarrow \beta$, where $\Rightarrow$ is relevant. Thus this account is, in fact, the same.
[21] Depth relevant logics can be defended independently, as they are in the papers cited. We take the term 'depth relevant' from Brady (forthcoming).

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