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Some Modifications of Scott's Theorem on Injective Spaces

Abstract. D. Scott in his paper [5] on the mathematical models for the Church-Curry λ-calculus proved the following theorem.

A topological space X is an absolute extensor for the category of all topological spaces iff a contraction of X is a topological space of "Scott's open sets" in a continuous lattice.

In this paper we prove a generalization of this theorem for the category of $\langle \alpha, \delta \rangle$-closure spaces. The main theorem says that, for some cardinal numbers α, δ, absolute extensors for the category of $\langle \alpha, \delta \rangle$-closure spaces are exactly $\langle \alpha, \delta \rangle$-closure spaces of $\langle \alpha, \delta \rangle$-filters in $\langle \alpha, \delta \rangle$-semidistributive lattices (Theorem 3.5).

If $\alpha = \omega$ and $\delta = \infty$ we obtain Scott's Theorem (Corollary 2.1). If $\alpha = 0$ and $\delta = \omega$ we obtain a characterization of closure spaces of filters in a complete Heyting lattice (Corollary 3.4). If $\alpha = 0$ and $\delta = \infty$ we obtain a characterization of closure space of all principial filters in a completely distributive complete lattice (Corollary 3.3).

1. Introduction

Recall that roughly speaking a Heyting lattice L is a lattice of classes of provably equivalent formulas in a constructive (in intuitionistic sense) theory T. Moreover, there is a natural correspondence between filters in the Heyting lattice L and intuitionistic theories which are extensions of T. From the intuitive standpoint a closure space which satisfies the compactness theorem is a consequence operator for a certain logic in which we think using only finite deduction rules.

In [2] we can find the following theorem.

A contraction of a closure space X is a Heyting lattice iff X is an absolute extensor for the category of all closure spaces which satisfies the compactness theorem.

This paper is a continuation of [5], [2], and [1]. We give an idea of uniform approach to the problem of the characterization of absolute extensors for categories of topological spaces (Scott's Theorem) and closure spaces using distributive laws for lattices. The paper uses the notation of [1].

2. $\langle \alpha, \delta \rangle$-pseudolattices

We shall say that a partially ordered set $\langle P, \leqslant \rangle$ is an $\langle \alpha, \delta \rangle$-*pseudolattice* provided that for every subset $D \subseteq P$, if $\overline{\overline{D}} < \alpha$ or D is a downward

δ-directed set then there exists a greatest lower bound of D in P. Note that, if P is an $\langle\alpha, \delta\rangle$-pseudolattice then P has a largest element.

Let L be an $\langle\alpha, \delta\rangle$-pseudolattice and let W be a subset of L.

We shall say that W is an $\langle\alpha, \delta\rangle$-*pseudofilter* in L, provided that:

(i) $W = \uparrow W$,

(ii) if $D \subseteq W$ and $\overline{\overline{D}} < \alpha$ then $inf_P D \in W$,

(iii) if $D \subseteq W$ is a downward δ-directed set then $inf_P D \in W$.

Let $V_{\alpha,\delta}(L)$ be a family of all $\langle\alpha, \delta\rangle$-pseudofilters in L. Moreover, let $\hat{V}_{\alpha,\delta}(L) = \langle V_{\alpha,\delta}(L), \leqslant\rangle$, where:

$$W_1 \leqslant W_2 \quad \text{iff} \quad W_2 \subseteq W_1.$$

Notice that then $\langle L, V_{\alpha,\delta}(L)\rangle$ is a closure space and $\hat{V}_{\alpha,\delta}(L)$ is a complete lattice.

Let $\hat{\mathscr{P}}(L) = \langle \mathscr{P}(L), \leqslant\rangle$ be a lattice of all subsets of L equipped with a dual order of $\subseteq$. It means that for $Z_1, Z_2 \subseteq L$ we have

$$Z_1 \leqslant Z_2 \quad \text{iff} \quad Z_2 \subseteq Z_1.$$

PROPOSITION 2.1. *The inclusion* $in: \hat{V}_{\alpha,\delta}(L) \hookrightarrow \hat{\mathscr{P}}(L)$ *is an* $\langle\alpha, \delta\rangle$-*monotone map.*

PROOF. Let $\{W_i\}_{i\in I}$ be an $\langle\alpha, \delta\rangle$-set in $\hat{V}_{\alpha,\delta}(L)$. We must show that $W = \bigcup_{i\in I} W_i$ is an $\langle\alpha, \delta\rangle$-pseudofilter.

Of course $W = \uparrow W$. If $D \subseteq W$ and $\overline{\overline{D}} < \alpha$ then there exists an $i_0 \in I$ such that $D \subseteq W_{i_0}$. It means that $inf_P D \in W_{i_0} \subseteq W$.

Now let $D \subseteq W$ be a downward δ-directed set. If there is an i_0 such that $D \subseteq W_{i_0}$, then $inf_P D \in W$. Assume that for every $i \in I$ there exists a $d_i \in D - W_i$. Note that

$$\overline{\overline{\{d_i\}_{i\in I}}} \leqslant \overline{\overline{I}} < \delta$$

and D is a downward δ-directed set.

Thus there is a lower bound $d \in D$ of $\{d_i\}_{i\in I}$. Let $d \in W_{i_0}$. Hence $d_{i_0} \in \in W_{i_0}$ and

$$d_{i_0} \in D - W_{i_0}. \quad \square$$

Moreover, notice that the inclusion $\hat{V}_{\alpha,\delta}(L) \hookrightarrow \hat{\mathscr{P}}(L)$ preserves all joins from $\hat{V}_{\alpha,\delta}$. By Freyd's Adjoint Functor Theorem (see [4] pp. 116) there exists a right adjoint functor

$$[\ ]: \hat{\mathscr{P}}(L) \to \hat{V}_{\alpha,\delta}(L)$$

preserving all meets in $\hat{\mathscr{P}}(L)$. In particular, $[\ ]: \hat{\mathscr{P}}(L) \to \hat{V}_{a,\delta}(L)$ is an $\langle a, \delta\rangle$-monotone map. Note that for $Z \subseteq L$

$$[Z]$$

is an $\langle a, \delta\rangle$-pseudofilter in L generated by Z. Of course

$$\langle \mathscr{P}(L), \subseteq\rangle \simeq \hat{\mathscr{P}}(L).$$

It means that $\hat{V}_{a,\delta}(L)$ is a retraction of the lattice of all subsets in the category of all $\langle a, \delta\rangle$-semilattices and $\langle a, \delta\rangle$-monotone maps.

Thus by Theorem 4.2 in [1] we have:

LEMMA 2.1. *Assume that the monomorphism*

$$K \hookrightarrow \hat{V}_{a,\delta}(L)$$

is an $\langle a, \delta\rangle$-monotone map and there exists an $\langle a, \delta\rangle$-monotone retraction

$$r\colon \hat{V}_{a,\delta}(L) \to K.$$

Then K is an absolute extensor for the category of all $\langle a, \delta\rangle$-semilattices and $\langle a, \delta\rangle$-monotone maps. Moreover, if $P(a, \delta, \overline{\overline{L}}) = 1$ then $F_{a,\delta}(K)$ is an absolute extensor for the category of all $\langle a, \delta\rangle$-closure spaces. □

We shall say that a subset $G \subseteq L$ is an $\langle a, \delta\rangle$-*directed set in* L, provided that:

(i) if $K \subseteq G$ and $\overline{\overline{K}} < a$ then there exists a lower bound of K in G,
(ii) if K is a downward δ-directed set in L and $K \subseteq \uparrow G$ then there exists a lower bound of K in G.

Note that:

REMARK 2.1.

(i) G is an $\langle a, \infty\rangle$-directed set in L iff G is a downward a-directed set,
(ii) if $\overline{\overline{G}} < a$ then G is an $\langle a, \delta\rangle$-directed set iff $inf_L G \in G$,
(iii) if $a \geqslant \delta$ then a subset G of L is an $\langle a, \delta\rangle$-directed set in L iff $inf_L G \in G$.

Let us assume that for $a, b \in L$

$$a \ll b,$$

provided that for every $\langle a, \delta\rangle$-directed set G in L, if $inf_L G \leqslant a$ then there exists an element $g \in G$ such that

$$g \leqslant b.$$

An $\langle a, \delta\rangle$-pseudolattice L is an *$\langle a, \delta\rangle$-continuous lattice*, provided that L is a complete lattice and for every $l \in L$

$$l = inf\{x \in L \mid l \ll x\}.$$

For example the lattice of all subsets $\langle \mathscr{P}(X), \subseteq\rangle$ is an $\langle a, \delta\rangle$-continuous lattice. Moreover, by Remark 2.1 we have:

REMARK 2.2. Let L be a complete lattice,

(i) if for every $l \in L$ we have $l \ll l$ then L is an $\langle a, \delta\rangle$-continuous lattice,

(ii) if for every $l \in L$ $\overline{\overline{\uparrow\{l\}}} < a$, then L is an $\langle a, \delta\rangle$-continuous lattice and in particular, if $\overline{\overline{L}} < a$ then L is an $\langle a, \delta\rangle$-continuous lattice,

(iii) if $a \geqslant \delta$, then L is an $\langle a, \delta\rangle$-continuous lattice,

(iv) a lattice L is an $\langle \omega, \infty\rangle$-continuous lattice iff the dual lattice L^{op} is a continuous lattice (see [5], [3]). □

If L is a complete lattice, then for $W \in \hat{V}_{a,\delta}(L)$ let $\Pi W = inf_L W$. Notice that $\Pi\colon \hat{V}_{a,\delta}(L) \to L$ is a right adjoint to the functor $\uparrow\colon L \to \hat{V}_{a,\delta}(L)$ (where $\uparrow(l) = \uparrow\{l\}$).

For $l \in L$ let:

$$\twoheaduparrow(l) = \{x \in L \mid l \ll x\}.$$

PROPOSITION 2.2. *For every $\langle a, \delta\rangle$-pseudolattice L the function $\twoheaduparrow$ is a functor from L to $\hat{V}_{a,\delta}(L)$. Moreover, if L is a complete lattice then $\twoheaduparrow$ has a left adjoint iff L is an $\langle a, \delta\rangle$-continuous lattice. If L is an $\langle a, \delta\rangle$-continuous lattice then a left adjoint to $\twoheaduparrow$ is the functor $\Pi\colon \hat{V}_{a,\delta}(L) \to L$.*

PROOF. First we prove that $\twoheaduparrow(l)$ is an $\langle a, \delta\rangle$-pseudofilter in L. It is sufficient to show that for every $D \subseteq \twoheaduparrow(l)$, if $\overline{\overline{D}} < a$ or D is a downward δ-directed set in L then for every $\langle a, \delta\rangle$-directed set G in L such that $inf_L G \leqslant l$ there exists an element g of G such that $g \leqslant inf_L D$.

If $\overline{\overline{D}} < a$ then for every $d \in D$ there exists a $g_d \in G$ such that

$$g_d \leqslant d$$

and $\overline{\overline{\{g_d\}_{d\in D}}} < a$. Thus there exists an element g of G which is a lower bound of D. It means that $g \leqslant inf_L D$. Hence, $inf_L D \in \twoheaduparrow(l)$.

If D is a downward δ-directed set in L then $D \subseteq \uparrow G$ and there exists a lower bound of D in G. It means that $inf_L D \in \twoheaduparrow(l)$.

This proves that $\twoheaduparrow(l)$ is an $\langle a, \delta\rangle$-pseudofilter in L.

Now let us assume that $l \in L$ and W is an $\langle a, \delta\rangle$-pseudofilter in L. Notice that, if $\Pi W \leqslant l$ and $x \in \twoheaduparrow(l)$ then there exists an element w of W such that

$$w \leqslant x.$$

Hence, if $\Pi W \leqslant l$ then $\Uparrow(l) \subseteq W$ (i.e. $W \leqslant \Uparrow(l)$).

If $\Uparrow(l) \subseteq W$ and L is an $\langle a, \delta\rangle$-continuous lattice then

$$\Pi W \leqslant \Pi \Uparrow(l) = l.$$

Thus, if L is an $\langle a, \delta\rangle$-continuous lattice then $\Uparrow$ has a left adjoint.

Now let us assume that $\Uparrow$ has a left adjoint then for every $l \in L$

$$\Uparrow(l) \leqslant \Uparrow(l)$$

and by the adjunction we have

$$\Pi \Uparrow(l) \leqslant l.$$

Hence L is an $\langle a, \delta\rangle$-continuous lattice. □

It is interesting that if L is an $\langle a, \delta\rangle$-continuous lattice and $a \ll b$ in L then there exists a $c \in L$ such that $a \ll c \ll b$ (if $a = \omega$, $\delta = \infty$ the proof can be found in [5] and [3] p. 288).

We now know that if L is an $\langle a, \delta\rangle$-continuous lattice then the functor

$$\Pi: \hat{V}_{a,\delta}(L) \to L$$

has right adjoint $\uparrow$ and left adjoint $\Uparrow$. Recall that if a functor is a right (resp. left) adjoint then it preserves all the limits (resp. colimits) which exist in its domain. Hence and by Lemma 2.1 we obtain.

THEOREM 2.1. *If K is an $\langle a, \delta\rangle$-continuous lattice then:*

(i) *K is an absolute extensor for the category of all $\langle a, \delta\rangle$-semilattices and $\langle a, \delta\rangle$-monotone maps,*

(ii) *if $P(a, \delta, \overline{\overline{K}}) = 1$ then the closure space $F_{a,\delta}(K)$ is an absolute extensor for the category of all $\langle a, \delta\rangle$-closure spaces.* □

Notice that if $\delta = \infty$ or $a \geqslant \delta$ then every $\langle a, \delta\rangle$-monotone retraction of the lattice of all subsets is an $\langle a, \delta\rangle$-continuous lattice.

Thus by Theorem 4.1 in [1], Lemma 2.1 in [1] and by Theorem 2.1 we have:

THEOREM 2.2. *If $\delta = \infty$ or $a \geqslant \delta$ then a closure space X is an absolute extensor for the category of all $\langle a, \delta\rangle$-closure spaces iff a contraction of X is the closure space of all $\langle a, \delta\rangle$-filters in an $\langle a, \delta\rangle$-continuous lattice.* □

By Theorem 2.2 and by Remark 2.2 we have:

COROLLARY 2.1 (D. Scott [5], $a = \omega$, $\delta = \infty$). *A topological space X is an absolute extensor for the category of all topological spaces iff a contraction of X is the topological space of all "Scott's open sets" in a continuous lattice.* □

COROLLARY 2.2. *If $a \geqslant \delta$ then a closure space X is an absolute extensor for the category $S_{a,\delta}$ iff a contraction of X is a closure space of all upper sets in a complete lattice.* □

COROLLARY 2.3. *A partially ordered set P is an absolute extensor for the category of all partially ordered sets and monotone maps iff P is a complete lattice.* □

3. Distributive laws for absolute extensors

Note that if L is a complete lattice then for every family $\{a_{t,s}\}_{t\in T, s\in S}$ of elements of L we have:

$$\sup_{f\in S^T} \inf_{t\in T} a_{t,f(t)} \leqslant \inf_{t\in T} \sup_{s\in S} a_{t,s},$$

$$\sup_{t\in T} \inf_{s\in S} a_{t,s} \leqslant \inf_{f\in S^T} \sup_{t\in T} a_{t,f(t)}.$$

THEOREM 3.1. *An $\langle\alpha, \delta\rangle$-pseudolattice L is an $\langle\alpha, \delta\rangle$-continuous lattice iff the inequality*

$$\text{(i)} \qquad \inf_{f\in P^K} \sup_{k\in K} b_{k,f(k)} \leqslant \sup_{k\in K} \inf_{p\in P} b_{k,p}$$

holds for every family $\{b_{k,p}\}_{k\in K, p\in P}$ of elements of L such that for every $k \in K$ the set

$$\{b_{k,p}\}_{p\in P}$$

is an $\langle\alpha, \delta\rangle$-directed set in L.

PROOF. Assume that L is an $\langle\alpha, \delta\rangle$-continuous lattice and $\{b_{k,p}\}_{k\in K, p\in P}$ is a family of elements of L such that for every $k \in K$ the set $\{b_{k,p}\}_{p\in P}$ is an $\langle\alpha, \delta\rangle$-directed set in L. Let

$$l = \inf_{f\in P^K} \sup_{k\in K} b_{k,f(k)},$$

$$r = \sup_{k\in K} \inf_{p\in P} b_{k,p}.$$

By the assumption we have

$$r = \inf\{y \in L \mid r \ll y\}.$$

It is sufficient to show that for every

$$x \in \{y \in L \mid r \ll y\}$$

we have $l \leqslant x$. So assume that

$$r \ll x.$$

Note that if $k \in K$ then

$$\inf_{p\in P} b_{k,p} \leqslant r,$$

and since $\{b_{k,p}\}_{p\in P}$ is an $\langle a, \delta\rangle$-directed set then there exists $\bar{f}(k) \in P$ such that $b_{k,\bar{f}(k)} \leqslant x$. Hence

$$l \leqslant \sup_{k\in K} b_{k,\bar{f}(k)} \leqslant x.$$

Thus we get (i).

Now let us assume (i). We must show that for every $z \in L$

$$z = \inf\{x \in L \mid z \ll x\}.$$

Let us put

$$\mathscr{D} = \{G \subseteq L \mid G \text{ is an } \langle a, \delta\rangle\text{-directed set in } L \text{ such that } \inf G \leqslant z\}.$$

We can assume that

$$\mathscr{D} = \big\{\{b_{k,p}\}_{p\in P} \mid k \in K\big\}.$$

By (i)

$$\inf_{f\in P^K} \sup_{k\in K} b_{k,f(k)} \leqslant \sup_{k\in K} \inf_{p\in P} b_{k,p},$$

and since $\{z\} \in \mathscr{D}$ we obtain that

$$\sup_{k\in K} \inf_{p\in P} b_{k,p} = z.$$

Now we will prove that

$$\{\sup_{k\in K} b_{k,f(k)} \mid f \in P^K\} \subseteq \{x \mid z \ll x\}.$$

For this purpose let $f\colon K \to P$ and let $G \in \mathscr{D}$. Hence

$$G \cap \{b_{k,f(k)}\}_{k\in K} \neq \emptyset.$$

It means that there exists a $g \in G$ such that

$$g \leqslant \sup_{k\in K} b_{k,f(k)}.$$

Thus we have proved that

$$\inf\{x \mid z \ll x\} \leqslant \inf_{f\in P^K} \sup_{k\in K} b_{k,f(k)} \leqslant z,$$

which combined with inequality

$$z \leqslant \inf\{x \mid z \ll x\}$$

completes the proof of the theorem. □

We shall say that a complete lattice L is an $\langle a, \delta\rangle$-*semidistributive lattice* provided that the inequality

$$\inf_{t\in T} \sup_{s\in S} a_{t,s} \leqslant \sup_{f\in S^T} \inf_{t\in T} a_{t,f(t)}$$

holds for every family $\{a_{t,s}\}_{t\in T, s\in S}$ of elements of L fulfilling the following three conditions:

(a) $\overline{\overline{T}} < \delta$,
(b) the family $\{\downarrow\{a_{t,s}\}_{s\in S}\}_{t\in T}$ is a downward α-directed set in $\langle \mathscr{P}(L), \subseteq\rangle$,
(c) for every $f\colon T\to S$ there exists a $g\colon T\to S$ such that:
 (c_1) $\{a_{t,g(t)}\}_{t\in T}$ is a downward α-directed set in L,
 (c_2) $\{a_{t,g(t)}\}_{t\in T} \subseteq \uparrow\{a_{t,f(t)}\}_{t\in T}$.

REMARK 3.1.

(i) A complete lattice L is a $\langle 0, \delta\rangle$-semidistributive lattice iff the inequality

$$\inf_{t\in T} \sup_{s\in S} a_{t,s} \leqslant \sup_{f\in S^T} \inf_{t\in T} a_{t,f(t)}$$

holds for every family $\{a_{t,s}\}_{t\in T, s\in S}$ of elements of L such that $\overline{\overline{T}} < \delta$,

(ii) a lattice L is a $\langle 0, \infty\rangle$-semidistributive lattice iff L is a completely distributive complete lattice,

(iii) a lattice L is a $\langle 0, \omega\rangle$-semidistributive lattice iff L is a complete Heyting lattice. □

By Lemma 4.2 in [1] we have:

COROLLARY 3.1. *Assume that L is a poset such that the inclusion*

$$in\colon L \hookrightarrow \langle \mathscr{P}(\mathfrak{n}), \subseteq\rangle$$

is an $\langle \alpha, \delta\rangle$-monotone map and there exists an $\langle \alpha, \delta\rangle$-monotone retraction

$$r\colon \langle \mathscr{P}(\mathfrak{n}), \subseteq\rangle \to L.$$

Then L is an $\langle \alpha, \delta\rangle$-semidistributive lattice. □

Thus, by Theorem 2.1, we obtain:

PROPOSITION 3.1. *If L is an $\langle \alpha, \delta\rangle$-continuous lattice then L is an $\langle \alpha, \delta\rangle$-semidistributive lattice.* □

Moreover, we can show that:

THEOREM 3.2. *A lattice is an $\langle \alpha, \infty\rangle$-continuous lattice iff it is an $\langle \alpha, \infty\rangle$-semidistributive lattice.*

PROOF. By Proposition 3.1 we have that every $\langle \alpha, \infty\rangle$-continuous lattice is an $\langle \alpha, \infty\rangle$-semidistributive lattice.

Now let us assume that L is an $\langle \alpha, \infty\rangle$-semidistributive lattice and let

$$\{b_{k,p}\}_{k\in K, p\in P}$$

be a family of elements of L such that for every $k \in K$ the set $\{b_{k,p}\}_{p\in P}$ is a downward α-directed set in L (Remark 2.1 (i)).

We show that:

a) the family

$$\{\downarrow\{b_{k,f(k)}\}_{k\in K}\}_{f\in P^K}$$

is a downward α-directed set in $\langle \mathscr{P}(L), \subseteq\rangle$,

(b) for every φ: $P^K \to K$ there exists a ψ: $P^K \to K$ such that:

(b$_1$) $\{b_{\psi(f),f(\psi(f))}\}_{f\in P^K}$ is a downward α-directed set in L,

(b$_2$) $\{b_{\psi(f),f(\psi(f))}\}_{f\in P^K} \subseteq \uparrow\{b_{\varphi(f),f(\varphi(f))}\}_{f\in P^K}$.

For the proof of (a) let $F \subseteq P^K$ and $\overline{\overline{F}} < \alpha$. For every $k \in K$ $\{b_{k,p}\}_{p\in P}$ is a downward α-directed set. Hence for every $k \in K$ there exists a $\bar{f}(k) \in P$ such that for every $f \in F$

$$b_{k,\bar{f}(k)} \leqslant b_{k,f(k)}.$$

Thus

$$\downarrow\{b_{k,\bar{f}(k)}\}_{k\in K} \subseteq \bigcap\{\downarrow\{b_{k,f(k)}\} \mid f \in F\}.$$

What means that (a) holds.

For the proof of (b) first notice that for every φ: $P^K \to K$ there exists a $\bar{k} \in K$ such that

(c) $\{b_{\bar{k},p}\}_{p\in P} \subseteq \uparrow\{b_{\varphi(f),f(\varphi(f))}\}_{f\in P^K}$.

If it is not true then there exists a $\bar{\varphi}$: $P^K \to K$ such that for every $k \in K$ there is a $\bar{p} = \bar{f}(k) \in P$ such that for every $f \in P^K$

$$b_{\bar{\varphi}(f),f(\bar{\varphi}(f))} \nleqslant b_{k,\bar{p}}.$$

But

$$b_{\bar{\varphi}(\bar{f}),\bar{f}(\bar{\varphi}(\bar{f}))} \leqslant b_{\bar{\varphi}(\bar{f}),\bar{f}(\bar{\varphi}(\bar{f}))}$$

and if $k = \bar{\varphi}(\bar{f})$ then

$$b_{\bar{\varphi}(f),\bar{f}(\bar{\varphi}(\bar{f}))} \leqslant b_{k,\bar{p}}.$$

A contradiction.

Now let ψ: $P^K \to K$ be a function such that

$$\psi(f) = \bar{k}.$$

Because $\{b_{\bar{k},f(\bar{k})}\}_{f\in P^K} = \{b_{\bar{k},p}\}_{p\in P}$ is a downward α-directed set, by (c) we get (b$_2$). Thus

$$\inf_{f\in P^K}\sup_{k\in K} b_{k,f(k)} = \sup_{\varphi\in K^{(P^K)}}\inf_{f\in P^K} b_{\varphi(f),f(\varphi(f))}.$$

Moreover, by (c)

$$\sup_{\varphi\in K^{(P^K)}}\inf_{f\in P^K} b_{\varphi(f),f(\varphi(f))} \leqslant \sup_{k\in K}\inf_{p\in P} b_{k,p}.$$

Consequently,

$$\inf_{f\in P^K} \sup_{k\in K} b_{k,f(k)} \leqslant \sup_{k\in K} \inf_{p\in P} b_{k,p}.$$

By Theorem 3.1 it means that L is an $\langle \alpha, \infty\rangle$-continuous lattice. □

Note that for every complete lattice L the function

$$\downarrow : L \to \langle \mathscr{P}(L), \subseteq\rangle$$

(where $\downarrow(l) = \downarrow\{l\}$) preserves all meets in L.

We shall say that a complete lattice L is an $\langle \alpha, \delta\rangle$-*pseudodistributive lattice* provided that the inequality

$$\inf_{t\in T} \sup_{s\in S} a_{t,s} \leqslant \sup_{f\in S^T} \inf_{t\in T} a_{t,f(t)}$$

holds for every family $\{a_{t,s}\}_{t\in T, s\in S}$ of elements of L fulfilling the following two conditions:

(a) $\overline{\overline{T}} < \delta$,

(b) the family $\{\downarrow\{a_{t,s}\}_{s\in S}\}_{t\in T}$ is a downward α-directed set in $\langle \mathscr{P}(L), \subseteq\rangle$.

REMARK 3.2.

(i) if L is an $\langle \alpha, \delta\rangle$-pseudodistributive lattice then L is an $\langle \alpha, \delta\rangle$-semidistributive lattice,

(ii) L is a $\langle 0, \delta\rangle$-pseudodistributive lattice iff L is a $\langle 0, \delta\rangle$-semidistributive lattice,

(iii) L is a $\langle 0, \infty\rangle$-pseudodistributive lattice iff L is a $\langle 0, \infty\rangle$-continuous lattice. □

Note that for every complete lattice L the function:

$$\downarrow : L \to \langle \mathscr{P}(L), \subseteq\rangle$$

(where $\downarrow(l) = \downarrow\{l\}$) preserves all meets in L.

THEOREM 3.3. *If L is an $\langle \alpha, \delta\rangle$-pseudodistributive lattice then the function*

$$r: \mathscr{P}(L) \to L$$

such that for $Z \subseteq L$

$$r(Z) = \sup\{l \mid \downarrow(l) \subseteq Z\}$$

is an $\langle \alpha, \delta\rangle$-monotone map and for every $l \in L$

$$r\downarrow(l) = l.$$

PROOF. Let $\{X_t\}_{t\in T}$ be a non-empty $\langle \alpha, \delta\rangle$-set in $\langle \mathscr{P}(L), \subseteq\rangle$. Moreover, for $t \in T$ let

$$\{a_{t,s}\}_{s\in S} = \{l \in L \mid \downarrow(l) \subseteq X_t\}.$$

Of course the family

$$\{\{a_{t,s}\}_{s\in S}\}_{t\in T}$$

is a downward α-directed set in $\langle \mathscr{P}(L), \subseteq\rangle$. Hence

$$\begin{aligned} r\Big(\bigcap_{t\in T} X_t\Big) &= sup\,\{l \mid \downarrow(l) \subseteq \bigcap_{t\in T} X_t\} \\ &= sup\,\{l \mid (\forall t)_T \downarrow(l) \subseteq X_t\} \\ &= \sup_{\varphi\in S^T} \inf_{t\in T} a_{t,\varphi(t)} \\ &= \inf_{t\in T} r(X_t). \qquad \square \end{aligned}$$

By Theorem 3.3 and Theorem 4.2 in [1] we have:

THEOREM 3.4. *If K is an $\langle \alpha, \delta\rangle$-pseudodistributive lattice then:*

(i) *K is an absolute extensor for the category of all $\langle \alpha, \delta\rangle$-semilattices and $\langle \alpha, \delta\rangle$-monotone maps,*

(ii) *if $P(\alpha, \delta, \overline{\overline{K}}) = 1$ then the closure space $F_{\alpha,\delta}(K)$ is an absolute extensor for the category of all $\langle \alpha, \delta\rangle$-closure spaces.* $\square$

By Corollary 3.1, Theorem 2.2, Remark 3.2 (ii) and Theorem 3.4 we have

THEOREM 3.5. *If $\alpha = 0$ or $\delta = \infty$ or $\alpha \geqslant \delta$, then:*

(i) *an $\langle \alpha, \delta\rangle$-semilattice L is an absolute extensor for the category of all $\langle \alpha, \delta\rangle$-semilattices and $\langle \alpha, \delta\rangle$-monotone maps iff L is an $\langle \alpha, \delta\rangle$-semidistributive lattice,*

(ii) *a closure space X is an absolute extensor for the category of all $\langle \alpha, \delta\rangle$-closure spaces iff a contraction of X is the closure space of all $\langle \alpha, \delta\rangle$-filters in an $\langle \alpha, \delta\rangle$-semidistributive lattice.* $\square$

By Remark 3.1 we obtain that:

COROLLARY 3.2. [2]. *A closure space X is an absolute extensor for the category of all closure spaces which satisfy the δ-compactness theorem iff a contraction of X is a closure space of all δ-multiplicative filters in a complete lattice L such that for every family $\{a_{t,s}\}_{t\in T, s\in S} \subseteq L$, if $\overline{\overline{T}} < \delta$ then*

$$\inf_{t\in T} \sup_{s\in S} a_{t,s} = \sup_{f\in S^T} \inf_{t\in T} a_{t,f(t)}. \qquad \square$$

COROLLARY 3.3 [2]. *A closure space X is an absolute extensor for the category of all closure spaces iff a contraction of X is a closure space of all principial filters in a completely distributive complete lattice.* $\square$

COROLLARY 3.4 [2]. *A closure space X is an absolute extensor for the category of all closure spaces which satisfy the compactness theorem iff a contraction of X is a closure space of all filters in a complete Heyting lattice.* $\square$

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