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Computation of Aristotle's and Gergonne's Syllogisms

Abstract. A connection between Aristotle's syllogistic and the calculus of relations is investigated. Aristotle's and Gergonne's syllogistics are considered as some algebraic structures. It is proved that Gergonne's syllogistic is isomorphic to closed elements algebra of a proper approximation relation algebra. This isomorphism permits to evaluate Gergonne's syllogisms and also Aristotle's syllogisms, laws of conversion and relations in the "square of oppositions" by means of regular computations with Boolean matrices.

Introduction

The aim of the present article is to establish a connection between Aristotle's syllogistic and the calculus of relations.

For this purpose I define a new algebraic structure called a proper approximation relation algebra (*PARA*) that is a combination of the proper relation algebra (see [2, p. 345]) and the approximation space (see [8]). *PARA* may be considered as a generalization of the classical calculus of relations proposed by Tarski (see [10], the bibliography of subsequent works see in [6]).

To determine the connection between *PARA* and Aristotle's syllogistic I consider the later as an algebraic structure. Such a treatment of Aristotle's syllogistic goes back to Lorenzen [5]. Then I introduce one more algebraic structure called Gergonne's syllogistic and show that Aristotle's syllogistic is in a sense a substructure of the Gergonne's syllogistic. Finally I point out a *PARA*, closed elements algebra of which is isomorphic to Gergonne's syllogistic. This isomorphism permits to evaluate Gergonne's syllogisms and also Aristotle's syllogisms, laws of conversion and relations in the "square of oppositions" by means of regular computations with Boolean matrices.

The algebraic model of Aristotle's syllogistic proposed in this article seems to be more satisfactory in comparison with Lorenzen's model that he built in [5] using the classical relation algebra. The distinguishing feature of my model is the introduction of approximation operators into the classical relation algebra. This permits to formalize Aristotle's syllogistic by using mood "Barbara" as the sole axiom (Lorenzen used 6 axioms in his model) and to compute effectively Aristotle's and Gergonne's syllogisms. Moreover, by distinguishing actual and potential syllogisms (see the end of §4 of this article) it seems to be possible to model Aristotelian "ecthesis" algebraically.

§1. Aristotle's syllogistic as an algebraic structure

Let us consider the algebraic structure

$$Ar = \langle \alpha_{AR}, \cdot, {}^{-1}, \leqslant, CR, CD, SC \rangle,$$

where $\alpha_{AR} = \{A, \tilde{A}, \mathrm{O}, \tilde{\mathrm{O}}, I, E, 1\}$; $\cdot$ is the binary operation on α_{AR} presented in Table 1.1 (empty squares correspond to the case $r \cdot s = 1$ for some r, $s \in \alpha_{AR}$); ${}^{-1}$ is such a unary operation on α_{AR} that $A^{-1} = \tilde{A}, \tilde{A}^{-1} = A, \mathrm{O}^{-1} = \tilde{\mathrm{O}}, \tilde{\mathrm{O}}^{-1} = \mathrm{O}, I^{-1} = I, E^{-1} = E, 1^{-1} = 1$; and finally $\leqslant$, CR, CD, SC are the binary relations on α_{AR} presented by the graph in Figure 1.1. This graph shows that CR, CD, SC are symmetric relations. It should be noted that Table 1.1 for the operation $\cdot$ was for the first time proposed by Lorenzen in [5] (He used the symbol **O** instead of the symbol 1). Then arbitrary elements of Ar will be designated as r, s, t.

Being intuitively interpreted the elements of the structure Ar correspond to the Aristotle's functors:

A	All — — are — —,
$\tilde{A}$	— — belongs to all — —,
O	Some — — is not — —,
$\tilde{\mathrm{O}}$	— — does not belong to some — —,
I	Some — — is — —,
E	No — — is — —;

unequalities $r \cdot s \leqslant t$ for $r \neq 1$, $s \neq 1$, $t \neq 1$ correspond to the syllogisms $r(x, z)$, $s(z, y) \vdash t(x, y)$; equalities $r \cdot s = 1$ for $r \neq 1$, $s \neq 1$ mean that the premisses $r(x, z), s(z, y)$ do not entail necessarily some categorical proposition; unequalities $r \leqslant s^{-1}$ for $r \neq 1$, $s \neq 1$ correspond to the laws of conversion $r(x, y) \vdash s(y, x)$; and finally relations $r \leqslant s$, $CR(r, s)$, $CD(r, s)$, $SC(r, s)$ for $r \neq 1$, $s \neq 1$ correspond accordingly to the superaltern relation, contrary relation, contradictory relation and subcontrary relation between the functors r and s. It is easy to verify that the algebraic structure Ar contains full information about all Aristotle's syllogisms, laws of conversion and relations in the "square of oppositions". Then in this article Aristotle's syllogistic will be meant as the algebraic structure Ar.

§2. Gergonne's syllogistic

In this paragraph I define an algebraic structure called Gergonne's syllogistic which may be considered as an algebraic model of the so-called

•	A	Ã	O	Õ	I	E	1
A	A			Õ		E	
Ã	I	Ã	O		I	O	
O		O					
Õ	Õ						
I	I					O	
E	Õ	E			Õ		
1							

Table 1.1.

Gergonne's relations (see [3]). Moreover, here I propose a general method for building "the syllogistic of a theory over given situations system". The core of this notion is in the following. Let T be a first-order theory containing a finite set σ of binary predicate symbols. If the definite conditions are hold then σ is called a situations system of the theory T. It is designated in short as $T(\sigma)$. In the theory $T(\sigma)$ some set of its formulas

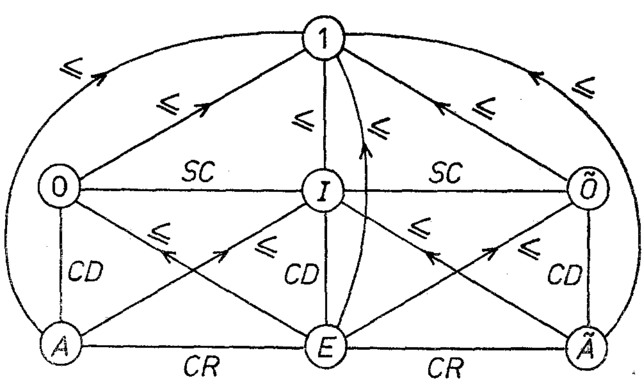


Figure 1.1.

called categorical propositions of the theory $T(\sigma)$ and some set of its theorems called syllogistic theses of the theory $T(\sigma)$ are selected in accordance with the definite rules. Moreover, the syllogistic theses are considered as some operations and relations on the set of categorical propositions. So, some algebraic structure called the syllogistic of the theory T over the situations system σ may be uniquely put into accordance with the theory $T(\sigma)$. Gergonne's syllogistic is one of the syllogistics of the transitive relation theory. Another interesting syllogistics of this theory and others were built by me in [4].

Then speaking about first-order theories I will follow Mendelson's textbook [7]. The designation $i = \overline{1, n}$ will mean that i is changed from 1 to n.

DEFINITION 2.1. Let $\sigma = \{\pi_1, \ldots, \pi_n\}$ be a set of binary predicate symbols of a theory T. σ is called a situations system of the theory T iff the following axioms are hold (here $i, j = \overline{1, n}$):

(1) $\vdash_T \pi_1(x, y) \vee \ldots \vee \pi_n(x, y)$,

(2) $\vdash_T \neg[\pi_i(x, y) \;\&\; \pi_j(x, y)]$ iff $i \neq j$,

(3) for any i there exists j such that
$\vdash_T \pi_i(x, y) \equiv \pi_j(y, x)$. □

Then I shall make use of $\pi, \pi_i, \ldots$ as designations for situations from σ. The expression $T(\sigma)$ or $T(\pi_1, \ldots, \pi_n)$ will designate the abbreviation for the phrase "a theory T containing a situations system $\sigma = \{\pi_1, \ldots, \pi_n\}$".

Let us consider the transitive relation theory T_A containing the single binary predicate symbol A and the single axiom $A(x, z) \;\&\; A(z, y) \Rightarrow \Rightarrow A(x, y)$. Let us define the additional binary predicate symbols $\approx, \subset, \supset, N, \text{ϖ}, |$ in the theory T_A as follows:

(1) $x \approx y \equiv A(x, y) \;\&\; A(y, x)$,

(2) $x \subset y \equiv A(x, y) \;\&\; \neg A(y, x)$,

(3) $x \supset y \equiv y \subset x$,

(4) $N(x, y) \equiv \neg A(x, y) \;\&\; \neg A(y, x)$,

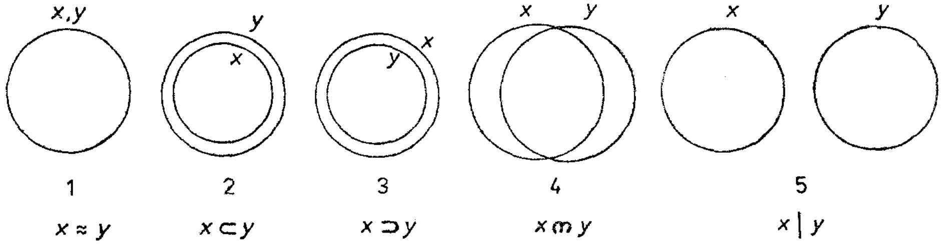
(5) $x \,\text{ϖ}\, y \equiv N(x, y) \;\&\; \exists z[(x \supset z) \;\&\; (z \subset y)]$,

(6) $x \mid y \equiv N(x, y) \;\&\; \neg\exists z[(x \supset z) \;\&\; (z \subset y)]$.

THEOREM 2.1. *The set of predicate symbols* $\{\approx, \subset, \supset, \text{ϖ}, |\}$ *is a situations system of the theory* T_A.

PROOF. By the axioms of Definition 2.1. □

Let us consider the structure $\mu_G = \langle \mathbf{O}, 1, 2, 3, 4, 5 \rangle$, where $\mathbf{O}$ is the set of all circles on the Euclide plane and $1, \ldots, 5$ are binary relations on $\mathbf{O}$ that is called Gergonne's relations (see [3]) and is defined in the following way:



The structure μ_G is a model of the theory T_A if the predicate symbols $\approx$, $\subset$, $\supset$, ω, $|$ are interpreted by the relations 1, ..., 5 as it is shown in the Figure.

To simplify the next discussion it is convenient to enrich first-order theories by the special binary predicate symbol $\emptyset$ in the following way: $\vdash_T \emptyset(x, y) \equiv F(x, y)$ iff $\vdash_T \neg F(x, y)$, where F is an arbitrary formula of a theory T that contains exactly two free variables x and y.

DEFINITION 2.2. Categorical propositions of a theory $T(\pi_1, \ldots, \pi_n)$ are formulas of this theory of the kind $\pi_1(x, y) \vee \ldots \vee \pi_m(x, y)$, where $m = \overline{1, n}$, $\pi_i \neq \pi_j$ for $i \neq j$; and also the formula $\emptyset(x, y)$. The categorical propositions $\pi_1(x, y) \vee \ldots \vee \pi_n(x, y)$ and $\emptyset(x, y)$ are called the unity proposition and the null proposition respectively. □

For example the following formulas are categorical propositions of the theory T_A ($\approx$, $\subset$, $\supset$, ω, $|$): $(x \subset y)$, $(x \,\omega\, y) \vee (x | y)$, $(x \approx y) \vee (x \subset y) \vee \vee (x \supset y)$. Then I shall refer to categorical propositions simply as to "propositions" and use u, v, w only for designation arbitrary propositions of a theory $T(\sigma)$. The expression $u(x, y)$ will mean that u contains exactly two free variables x and y; x is the first variable and y is the second variable.

DEFINITION 2.3. Syllogistic theses of a theory $T(\sigma)$ are its theorems of the following kind:

(1) $u(x, y) \vee v(x, y) \equiv w(x, y)$,

(2) $u(x, y) \& v(x, y) \equiv w(x, y)$,

(3) $u(x, y) \equiv \neg v(x, y)$,

(4) $u(x, y) \Rightarrow v(x, y)$,

(5) $u(x, y) \equiv v(y, x)$,

(6) $u(x, z) \ \& \ v(z, y) \Rightarrow w(x, y)$ and if for some proposition w_1, $\vdash_T u(x, z) \& v(z, y) \Rightarrow w_1(x, y)$, then $\vdash_T w(x, y) \Rightarrow w_1(x, y)$.

If the syllogistic thesis (6) takes place for some propositions u, v, w then the triplet (u, v, w) is said to form an (actual) syllogism of the theory $T(\sigma)$.

For example the following formulas of the theory $T_A(\approx, \subset, \supset, \omega, |)$ are syllogistic theses of this theory: $(x \,\omega\, y) \equiv (y \,\omega\, x)$, $[(x \subset y) \vee (x \,\omega\, y)] \& \& [(x \supset y) \vee (x \,\omega\, y)] \equiv (x \,\omega\, y)$, $(x \supset z) \ \& (z \subset y) \Rightarrow (x \approx y) \vee (x \subset y) \vee \vee (x \supset y) \vee (x \,\omega\, y)$. Indeed, it is easy to show that the first two formulas are theorems of T_A, the latter formula forms a syllogism of $T_A(\approx, \subset, \supset, \omega, |)$ as it is proved in the Appendix.

DEFINITION 2.4. An algebraic structure

$$\Psi[T(\sigma)] = \langle a, \cup, \cap, -, \leqslant, \emptyset, 1, {}^{-1}, \cdot \rangle,$$

where $\alpha = 2^\sigma$; $\cup$, $\cap$, $\cdot$ are binary and $-$, $^{-1}$ are unary operations on α; $\leqslant$ is a binary relation on α; $\emptyset$, 1 are designated elements of α, is called the (actual) syllogistic of the theory $T(\sigma)$ over the situations system σ iff the operations, the relations and the designated elements are defined as follows:

(1) $u' \cup v' = w'$ iff $\vdash_T u(x, y) \vee v(x, y) \equiv w(x, y)$,
(2) $u' \cap v' = w'$ iff $\vdash_T u(x, y) \,\&\, v(x, y) \equiv w(x, y)$,
(3) $u' = \overline{(v')}$ iff $\vdash_T u(x, y) \equiv \neg v(x, y)$,
(4) $u' \leqslant v'$ iff $\vdash_T u(x, y) \Rightarrow v(x, y)$,
(5) $u' = (v')^{-1}$ iff $\vdash_T u(x, y) \equiv v(y, x)$,
(6) $u' \cdot v' = w'$ iff (u, v, w) forms a syllogism,
(7) $\emptyset$ is the empty set and 1 is the set σ.

In this Definition u' designates the set of all predicate symbols contained in the proposition u. □

THEOREM 2.2. *Let $\Psi^*[T(\sigma)] = \langle \alpha, \cup, \cap, -, \leqslant, \emptyset, 1 \rangle$ be the reduct of $\Psi[T(\sigma)]$, and $|\sigma| = n$. Then $\Psi^*[T(\sigma)]$ is a finite n-atomic Boolean algebra, where $\cup$, $\cap$, $-$, $\leqslant$ are the usual set-theoretical operations and the relation on $\alpha = 2^\sigma$.*

PROOF. By the axioms of Definition 2.1. □

The symbols r, s, t will be used only for designation arbitrary elements of a syllogistic $T(\sigma)$.

THEOREM 2.3. *The following identities take place in every syllogistic $(\Psi[T(\sigma)]$:*

(1) $(r \cup s) \cdot t = (r \cdot t) \cup (s \cdot t)$,
(2) $r \cdot (s \cup t) = (r \cdot s) \cup (r \cdot t)$,
(3) $(r \cup s)^{-1} = r^{-1} \cup s^{-1}$,
(4) $r \cdot \emptyset = \emptyset \cdot r = \emptyset$,
5) $(r \cdot s)^{-1} = s^{-1} \cdot r^{-1}$,
(6) $(r^{-1})^{-1} = r$, $\overline{(r^{-1})} = \overline{(r)}^{-1}$.

PROOF. By the definition of the corresponding syllogistic operations. □

Let $\Psi[T(\pi_1, \ldots, \pi_n)]$ be a syllogistic. The reduct $\langle \alpha, \cdot \rangle$ of $\Psi[T(\pi_1, \ldots$ $\ldots, \pi_n)]$ will be called the groupoid of this syllogistic and the sets $\{\pi_1\}, \ldots$ $\ldots, \{\pi_n\}$ will be called atoms of this syllogistic.

Gergonne's syllogistic denoted then by G is the syllogistic $\Psi[T_A(\approx, \subset,$ $\supset, \infty, |)]$. Let us agree upon the representation of G. It is not necessary to represent specially Boolean operations of G because they are usual set-theoretical operations on $2^{\{\approx, \subset, \supset, \varpi, \mid\}}$. As it follows from identities (1) – (3) of Theorem 2.3, to determine values of the operations $\cdot$ and $^{-1}$ it is sufficient to point out its values on the atoms. Values of the operation $^{-1}$ on the atoms are: $\approx^{-1} = \approx$, $\subset^{-1} = \supset$, $\supset^{-1} = \subset$, $\varpi^{-1} = \varpi$, $\mid^{-1} = \mid$. Table 2.1 is the multiplication table of the smallest subgroupoid of the G's groupoid that contains the atoms. The following agreements are made in this Table:

(1) If the equality $r \cdot s = 1$ is true for some elements r, s of G then the corresponding square of the Table is empty.

(2) The multiplications by 1 and by $\approx$ are not included in the Table because for any element r of G, $r \cdot 1 = 1 \cdot r = 1$, $r \cdot \approx \; = \; \approx \cdot r = r$.

Table 2.1 is given without proof. But using Theorem 6.1. the reader may easily control any syllogism of this Table.

$\bullet$	$\subset$	$\supset$	ϖ	$\mid$	$\subset\varpi$	$\supset\varpi$	$\subset\varpi\mid$	$\supset\varpi\mid$	$\approx\subset\supset\varpi$
$\subset$	$\subset$		$\subset\varpi\mid$	$\mid$	$\subset\varpi\mid$		$\subset\varpi\mid$		
$\supset$	$\approx\subset\supset\varpi$	$\supset$	$\supset\varpi$	$\supset\varpi\mid$	$\approx\subset\supset\varpi$	$\supset\varpi$		$\supset\varpi\mid$	$\approx\subset\supset\varpi$
ϖ	$\subset\varpi$	$\supset\varpi\mid$		$\supset\varpi\mid$					
$\mid$	$\subset\varpi\mid$	$\mid$	$\subset\varpi\mid$		$\subset\varpi\mid$	$\subset\varpi\mid$			$\subset\varpi\mid$
$\subset\varpi$	$\subset\varpi$			$\supset\varpi\mid$					
$\supset\varpi$	$\approx\subset\supset\varpi$	$\supset\varpi\mid$		$\supset\varpi\mid$					
$\subset\varpi\mid$	$\subset\varpi\mid$								
$\supset\varpi\mid$		$\supset\varpi\mid$							
$\approx\subset\supset\varpi$	$\approx\subset\supset\varpi$			$\supset\varpi\mid$					

Table 2.1.

The G's operation $\cdot$ is interpreted in the model μ_G as follows. For example, $\subset \cdot \varpi = \{\subset, \varpi, \mid\}$ (see Table 2.1). It means that for all circles x, z, y, if circles x and z are in the relation $\subset$ and circles z and y are in the relation ϖ, then circles x and y may be either in the relation $\subset$ or in the relation ϖ or in the relation $\mid$ and may be neither in the relation $\approx$ nor in the relation $\supset$.

DEFINITION 2.5. A syllogistic $\Psi[T(\sigma)]$ is a proper syllogistic of the theory $T(\sigma)$ iff each axiom of the theory T is derived from syllogistic theses of this theory. □

Evidently, G is a proper syllogistic of the theory $T_A(\approx, \subset, \supset, \varpi, \mid)$.

§3. Connection between Aristotle's and Gergonne's syllogistics

Aristotle's syllogistic Ar is in a sense a substructure of Gergonne's syllogistic G.

THEOREM 3.1. *If $\varphi: Ar \to G$ is the map*

r	A	$\tilde{A}$	0	$\tilde{0}$	I	E	1
$\varphi(r)$	$\{\approx, \subset\}$	$\{\approx, \supset\}$	$\{\supset, \text{m}, \mid\}$	$\{\subset, \text{m}, \mid\}$	$\{\approx, \subset, \supset, \text{m}\}$	$\{\mid\}$	1

then for all elements r, s of Ar,

(1) $\varphi(r \cdot s) = \varphi(r) \cdot \varphi(s)$,

(2) $\varphi(r^{-1}) = [\varphi(r)]^{-1}$,

(3) $r \leqslant s$ *iff* $\varphi(r) \leqslant \varphi(s)$,

(4) $CR(r, s)$ *iff* $\varphi(r) \cap \varphi(s) = 0$ *and* $\varphi(r) \cup \varphi(s) \neq 1$,

(5) $CD(r, s)$ *iff* $\varphi(r) \cap \varphi(s) = 0$ *and* $\varphi(r) \cup \varphi(s) = 1$,

(6) $SC(r, s)$ *iff* $\varphi(r) \cap \varphi(s) \neq 0$ *and* $\varphi(r) \cup \varphi(s) = 1$ *and* $r \neq 1$, $s \neq 1$.

PROOF. Immediately by Table 1.1 and Table 2.1. □

§4. *PARA* and its closed elements algebra

The aim of this paragraph is to point out a *PARA* closed elements algebra of which is isomorphic to Gergonne's syllogistic G. Potential syllogisms are cursorily discussed in the end of the paragraph.

DEFINITION 4.1. A proper approximation relation algebra (*PARA*) is an algebraic structure

$$\Gamma_M(\sigma^*) = \langle B, \cup, \cap, {}^-, \leqslant, \emptyset, 1, \odot, {}^{-1}, \sigma^* \rangle,$$

where $B = 2^{M^2}$ is the set of all binary relations on a set M; $\cup$, $\cap$, $^-$ are Boolean operations over relations; $\leqslant$ is Boolean inclusion of relations; $\emptyset$ is the zero relation and $1 = M^2$ is the unit relation on M; $^{-1}$ is the conversion of relations; $\odot$ is the composition or the relative product of relations; and finally σ^* is a system of relations on M which is a partition of the set M^2 into equivalence classes. □

Then r^*, s^*, t^* will be used for designation of arbitrary relations on M and π_i^* — for designation of the relations belonging to σ^*.

DEFINITION 4.2. Approximation operators in $\Gamma_M(\sigma^*)$ are operators $\Diamond: B \to B$ and $\Box: B \to B$ defined as follows. Let $\sigma^* = \{\pi_1^*, \ldots, \pi_n^*\}$. Then

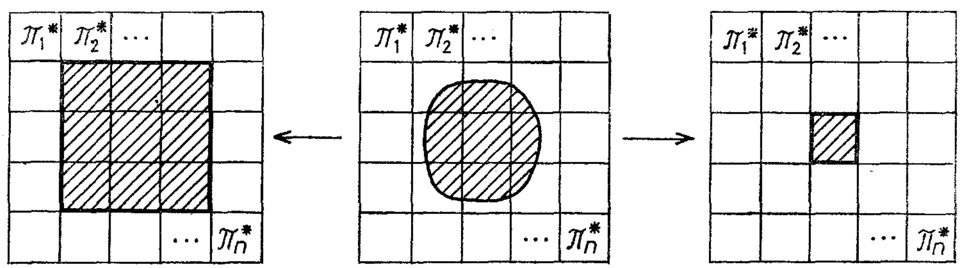
$$\Diamond r^* = \bigcup_{i=1}^{n} |\pi_i^*, r^*|, \quad \Box r^* = \bigcup_{i=1}^{n} \|\pi_i^*, r^*\|,$$

where

$$|\pi_i^*, r^*| = \begin{cases} \pi_i^*, & \text{if } \pi_i^* \cap r^* \neq \emptyset, \\ \emptyset, & \text{if } \pi_i^* \cap r^* = \emptyset; \end{cases}$$

$$\|\pi_i^*, r^*\| = \begin{cases} \pi_i^*, & \text{if } \pi_i^* \leqslant r^*, \\ \emptyset, & \text{if not } \pi_i^* \leqslant r^*. \end{cases} \qquad \square$$

For a given relation r^*, the relation $\Diamond r^*$ is called the closure or the upper approximation of r^*, and the relation $\square r^*$ is called the interior or the lower approximation of r^*. If for some r^*, $\Diamond r^* = r^*$ then r^* is called a closed relation. The following figure explains the sense of the operators $\Diamond$ and $\square$.



$\Diamond r^*$ $\quad$ r^* $\quad$ $\square r^*$

Here the checked squares picture a set M^2 with a partition $\sigma^* = \{\pi_1^*, \pi_2^*, \ldots, \pi_n^*\}$ and the hatched domains represent accordingly the relations $\Diamond r^*$, r^* and $\square r^*$.

It is evident, that *PARA* is a combination of the proper relation algebra (see [2, p. 345]) and the approximation space (see [8]). The choice of the symbols $\Diamond$ and $\square$ for designation of the approximation operators was motivated by the close connection between the approximation spaces and the propositional modal calculus *S*5 (this connection will not be discussed here).

Definition 4.3. The (actual) closed elements algebra of a *PARA* $\Gamma_M(\sigma^*)$ is an algebraic structure

$$\Gamma_M^{\Diamond}(\sigma^*) = \langle D, \cup, \cap, {}^{-}, \leqslant, \emptyset, 1, \cdot, {}^{-1}\rangle,$$

where D is the set of all closed relations on M; $\cup, \cap, {}^{-}, \leqslant, {}^{-1}$ are the reducts of the corresponding operations of $\Gamma_M(\sigma^*)$; $\emptyset, 1$ are the same relations as in $\Gamma_M(\sigma^*)$; the operation $\cdot$ is defined for all relations $r^*, s^* \in D$ as $r^* \cdot s^* = \Diamond(r^* \odot s^*)$. $\square$

Let us start building a *PARA*, closed elements algebra of which is isomorphic to the syllogistic G. For this purpose let us consider the structure

$\mu_0 = \langle M_0, A^* \rangle$, where $M_0 = \{1, 2, \ldots, 8\}$ and A^* is the following relation on M_0:

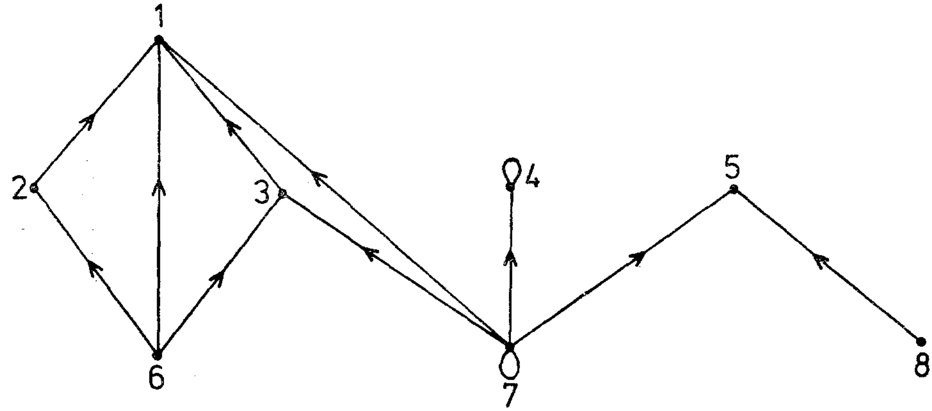


Figure 4.1.

If the predicate symbol A is interpreted by the relation A^* then μ_0 is a model of the theory T_A. Let us define additional relations on M_0 as follows:

$$\approx^* = A^* \cap (A^*)^{-1}, \qquad N^* = \overline{A^*} \cap (\overline{A^*})^{-1},$$
$$\subset^* = A^* \cap (\overline{A^*})^{-1} \qquad \omega^* = N^* \cap (\supset^* \odot \subset^*),$$
$$\supset^* = (\subset^*)^{-1}, \qquad |^* = N^* \cap \overline{(\supset^* \odot \subset^*)}.$$

The relations system $\sigma_0^* = \{\approx^*, \subset^*, \supset^*, \omega^*, |^*\}$ is evidently a partition of M_0^2. Let us define a map $f\colon G \to \Gamma_{M_0}(\sigma_0^*)$ as follows. For all $r, s, t \in G$,

(1) if r is an atom, then $f(\{\approx\}) = \approx^*, f(\{\subset\}) = \subset^*$ and so on;

(2) if $f(r)$ and $f(s)$ are already known and $t = r \cup s$ then $f(t) = f(r) \cup \cup f(s)$.

THEOREM 4.1. *f is an isomorphism between G and $\Gamma^{\diamond}_{M_0}(\sigma_0^*)$.*

PROOF. It follows from Theorem 6.2 of the Appendix. □

So, the evaluation of Gergonne's syllogistic theses is reduced to computations in the algebraic structure $\Gamma_{M_0}(\sigma_0^*)$. From computational point of view it is convenient to represent the elements of $\Gamma_{M_0}(\sigma_0^*)$ (i.e. the binary relations on M_0) by means of Boolean matrices. I hope that the reader knows the way of the binary relations representation (else see for example [1, Ch. XIV, §§ 13-14]). The relation A^* under these conditions may be represented as follows: (empty squares denote zeros)

Let us consider as an example a computation of Gergonne's syllogism $\omega \cdot | = \{\supset, \omega, |\}$.

	1	2	3	4	5	6	7	8
1								
2	1							
3	1							
4				1				
5								
6	1	1	1					
7	1		1	1	1		1	
8					1			

A^*

1			1	1			
	1	1					
	1	1	1	1			
1		1		1			
1		1	1	1			

ω^*

$\odot$

							1
			1	1		1	1
							1
	1				1		1
	1				1		
			1	1	1	1	1
	1				1		1
1	1	1	1		1	1	1

$|^*$

$=$

	1				1		1
			1	1		1	1
	1		1	1	1	1	1
	1				1		1
	1				1		1

$\omega^* \odot |^*$

Let us designate $t^* = \omega^* \odot |^*$. In accordance with Definition 4.2, it may be computed that $\approx^* \cap t^* = \subset^* \cap t^* = \emptyset$, $\supset^* \cap t^* \neq \emptyset$, $\omega^* \cap t^* \neq \emptyset$, $|^* \cap t^* \neq \emptyset$. So, $\omega^* \cdot |^* = \Diamond(\omega^* \odot |^*) = \supset^* \cup \omega^* \cup |^*$.

By the use of Theorem 3.1, the method described above may be adopted to the evaluation of arbitrary Aristotle's syllogisms, laws of conversion and relations in the "square of oppositions".

Potential syllogisms may be defined in any theory $T(\sigma)$ dually in comparison with the actual syllogisms. Namely, some propositions (u, v, w) of a theory $T(\sigma)$ will be said to form a potential syllogism of $T(\sigma)$ iff $\vdash_T w(x, y) \Rightarrow \exists z[u(x, z) \ \& \ v(z, y)]$ and for any proposition w_1, if $\vdash_T w_1(x, y) \Rightarrow \exists z[u(x, z) \ \& \ v(z, y)]$, then $\vdash_T w_1(x, y) \Rightarrow w(x, y)$. By means of a suitable set of the potential syllogisms it seems to be possible to model Aristotelian "ecthesis". (See about this [9], [11]).

§5. Evaluation of n-premissed theses

The method considered in §4 allows to evaluate Gergonne's and Aristotle's binary syllogisms. In this paragraph I extend the received results to arbitrary n-premissed Gergonne's and Aristotle's theses. It will be supposed in this paragraph that variables of the theory T_A are numbers $1, 2, 3, \ldots$ and metavariables over the variables are the symbols i, j. I remind also that u' means the set of all predicate symbols containing in a proposition u.

Definition 5.1. A n-premissed Gergonne's wff is a T_A 's formula of the kind $u_1 \& \ldots \& u_n \Rightarrow v$ in which there are no propositions of the type $u(i, i)$. □

Gergonne's wffs will be called simply as "wffs". Arbitrary wffs will be denoted by Φ. The wffs containing exactly m different variables will be denoted as Φ^m. It always will be supposed (it does not lead to decreasing of generality) that the set of all variables contained in some wff Φ^m is $\{1, 2, \ldots, m\}$.

The following T_A's formula denoted then as Φ_0 is a 4-premissed wff containing four different variables $1, 2, 3, 4$.

$$[(2 \supset 1)] \& [(4 \subset 1) \vee (4 \infty 1)] \& [(2|3)] \\ \& [(4 \subset 3) \vee (4 \infty 3)] \Rightarrow [(2 \infty 4)]. \tag{5.1}$$

Definition 5.2. A wff $u_1 \& \ldots \& u_n \Rightarrow v$ is a n-premissed Gergonne's thesis iff

(1) $v' \neq 1$,

(2) $\vdash_{T_A} u_1 \& \ldots \& u_n \Rightarrow v$,

(3) if v contains variables i, j then the formula $u_1 \& \ldots \& u_n \Rightarrow \emptyset(i, j)$ is not derived in the theory T_A. □

Definition 5.3. The matrix of a wff

$$\Phi^m: u_1 \& \ldots \& u_n \Rightarrow v \text{ is a } m \times m \text{ matrix } D(\Phi^m),$$

elements of which are elements of Gergonne's syllogistic G. These elements of the matrix $D(\Phi^m)$ are defined as follows:

$$d_{ij} = \begin{cases} \{\approx\}, & \text{if } i = j, \\ \bigcap_{k \in \Delta ij} u'_k(i, j), & \text{if } \Delta_{ij} \text{ is not empty}, \\ 1, & \text{in all other cases.} \end{cases}$$

Here $k \in \Delta_{ij}$ iff $k \in \{1, \ldots, n\}$ and there exists a proposition $u_k \in \{u_1, \ldots, u_n\}$ such that $u_k = u_k(i, j)$, i.e. u_k contains exactly two free variables i and j and moreover i is the first variable and j is the second variable (see §2, Def. 2.2 and below). □

The matrix of the wff Φ_0 designated as D_0 is represented in Figure 5.1a.

Definition 5.4. The symmetrization of a matrix $D(\Phi)$ is a matrix $\overline{D(\Phi)}$ defined in the following way: $\overline{d_{ij}} = d_{ij} \cap d_{ij}^{-1}$. Here $\cap$ and $^{-1}$ are the operations of the syllogistic G. □

The symmetrization of the matrix D_0 is presented in Figure 5.1b.

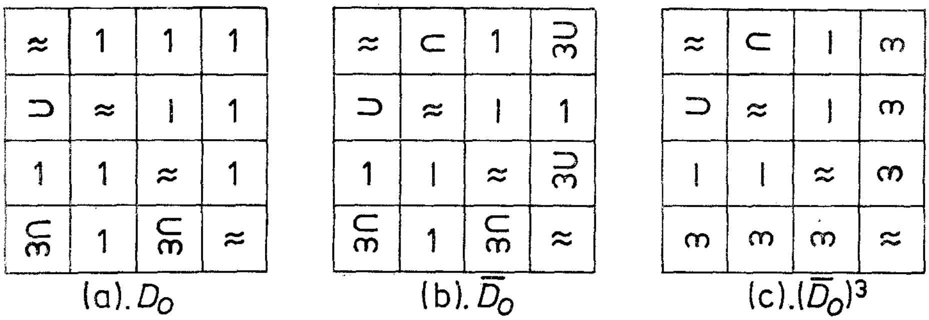


Figure 5.1.

A matrix multiplication is defined as $D''(\Phi^m) = D'(\Phi^m)\cdot D(\Phi^m)$ iff $d''_{ij} = \bigcap_{k=1}^{m} d'_{ik}\cdot d_{kj}$, where $\bigcap$, $\cdot$ are the operations of G. The r-power of a matrix $D(\Phi)$ for $r \geqslant 1$ is defined as

$$D^1(\Phi) = D(\Phi),$$
$$D^{r+1}(\Phi) = D^r(\Phi)\cdot D^r(\Phi).$$

DEFINITION 5.5. $D^r(\Phi)$ is the saturation of a matrix $D(\Phi)$ iff $D^{r+1}(\Phi) = D^r(\Phi)$. □

The saturation of the matrix $\overline{D}_0$ is presented in Figure 5.1c.

THEOREM 5.1. *Let $\Phi: u_1 \,\&\, \dots \,\&\, u_n \Rightarrow v$ be some Gergonne's wff and $v' \neq 1$. Let also $[\overline{D(\Phi)}]^r$ be the saturation of the matrix $\overline{D(\Phi)}$. Then the following two assertions A and B are equivalent:*

(A) *$u_1 \,\&\, \dots \,\&\, u_n \Rightarrow v$ is n-premissed Gergonne's Thesis;*

(B) *if v contains variables i, j then*

 (1) $\overline{d^r_{ij}} \leqslant v'(i, j)$,

 (2) $\overline{d^r_{ij}} \neq \varnothing$.

This theorem will be strictly proved in my another article. □

The element $\overline{d^3_{24}}$ of the matrix $(\overline{D_0})^3$ pictured in Figure 5.1c is equal to ω. In accordance with Theorem 5.1 it means that the wff Φ_0 is a 4-premissed Gergonne's thesis. So, Theorem 5.1 provides a method for evaluation of arbitrary n-premissed Gergonne's theses. By using Theorem 3.1 it is easy to adopt this method to evaluation of arbitrary n-premissed Aristotle's theses.

§6. Appendix

The Appendix consists of two parts. The deduction syllogisms Theorem and an example of its utilization are considered in Part 1. The syllogistics representation Theorem is proved in Part 2. This Theorem asserts that *PARA*, closed elements algebra of which is isomorphic to a syllogistic $\Psi[T(\sigma)]$ may be made from "a sufficiently rich model" of the theory $T(\sigma)$.

Part 1. The deduction (actual) syllogisms Theorem

THEOREM 6.1. *Let $\Psi[T(\sigma)]$ be the proper syllogistic of a theory $T(\sigma)$. For all propositions u, v, w of the theory $T(\sigma)$ the following assertions A and B are equivalent*:

(A) (u, v, w) *forms a syllogism of* $T(\sigma)$;

(B) (1) *for any predicate symbol π contained in the proposition w there exists a model μ_π of the theory T such that*

$$\mu_\pi \models \exists x\, \exists y\, \exists z[u(x, z)\ \&\ v(z, y)\ \&\ \pi(x, y)];$$

(2) *for any predicate symbol $\pi \in \sigma$ which is not contained in the proposition w*,

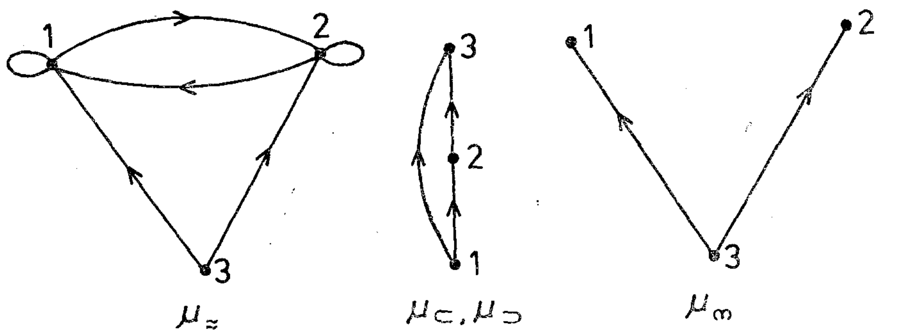
$$\vdash_T u(x, z)\ \&\ v(z, y) \Rightarrow \neg\pi(x, y).$$

The Theorem was proved in [4]. □

To illustrate the utilization of Theorem 6.1 let us consider a proof of the G's syllogism $\supset \cdot \subset\ = \{\approx, \subset, \supset, \between\}$. In accordance with item (1) of Theorem 6.1, for each predicate symbol π contained in the proposition $(x \approx y) \vee (x \subset y) \vee (x \supset y) \vee (x \between y)$ we should point out a model μ_π of the theory T_A such that

$$\mu_\pi \models \exists x\, \exists y\, \exists z[(x \supset z)\ \&\ (z \subset y)\ \&\ (x\pi y)]. \tag{6.1}$$

The following structures are for example the suitable models:



Indeed, these structures are transitive relations, hence they are models of the theory T_A. Condition (6.1) is also satisfied. This condition is satisfied for example for $\mu_{\subset}$ if we set $2 \to x$, $1 \to z$, $3 \to y$.

Then, in accordance with item (2) of Theorem 6.1, we should prove the theorem $(x \supset z) \,\&\, (z \subset y) \Rightarrow \neg(x|y)$ in the theory T_A. The proof of such theorems is very simple. Let us suppose that there are a, b, c such that (1) $(a \supset b)$,

(2) $(b \subset c)$, (3) $(a|c)$. Then we have:

(4) by (1) and (2), $\exists z[(a \supset z) \,\&\, (z \subset c)]$;

(5) by (3) and by the definition of the predicate symbol $|$, $\neg \exists z[(a \supset z) \,\&\, \&\, (z \subset c)]$,

but this contradicts to (4). So, the theorem is proved.

In practice it is advisable to use Theorem 6.1 only for deduction of a "sufficient set" of syllogisms.

DEFINITION 6.1. A sufficient set of syllogisms of a theory $T(\sigma)$ is such set of its syllogisms that for any pair of elements of the syllogistic $\Psi[T(\sigma)]$ the values of the operation $\cdot$ may be obtained from this set by means of the identities of Theorem 2.3. □

The following set is a sufficient set of syllogisms of the theory $T_A(\approx, \subset, \supset, \varpi, |)$:

$\approx \cdot \approx = \approx$, $\approx \cdot \subset = \subset$, $\approx \cdot \supset = \supset$, $\approx \cdot \varpi = \varpi$, $\approx \cdot | = |$, $\subset \cdot \subset = \subset$, $\subset \cdot \supset = 1$, $\subset \cdot \varpi = \{\subset, \varpi, |\}$, $\subset \cdot | = |$, $\supset \cdot \subset = \{\approx, \subset, \supset, \varpi)$, $\supset \cdot \varpi = \{\supset, \varpi\}$, $\supset \cdot | = \{\supset, \varpi, |\}$, $\varpi \cdot \varpi = 1$, $\varpi \cdot | = \{\supset, \varpi, |\}$, $| \cdot | = 1$.

Part 2. (Actual) syllogistics representation Theorem

DEFINITION 6.2. μ is a sufficiently rich model of a theory $T(\sigma)$ iff for any its syllogism (u, v, w) and for any predicate symbol π contained in the proposition w,

$$\mu \models \exists x \exists y \exists z[u(x, z) \,\&\, v(z, y) \,\&\, \pi(x, y)] \tag{6.2}$$

□

It is evident that verification of condition (6.2) may be restricted to a sufficient set of syllogisms. Using Figure 4.1 and the sufficient set of syllogisms of the theory $T_A(\approx, \subset, \supset, \varpi, |)$ which was presented above it is easy to see that μ_0 considered in §4 is a sufficiently rich model of this theory.

Let us take the following agreements up to the end of the article. Let $T(\sigma)$ be a fixed theory and $\Psi[T(\sigma)]$ its proper syllogistic. Then, let $\mu = \langle M, \sigma^* \rangle$, where M is a non-empty set and σ^*, $|\sigma^*| = |\sigma|$, is a system of binary relations on M, be a fixed sufficiently rich model of $T(\sigma)$. Let $I: \sigma \to \sigma^*$ be the corresponding interpretation map such that $I(\pi) = \pi^*$, where $\pi \in \sigma$, $\pi^* \in \sigma^*$.

By axioms of Definition 2.1, σ^* is a partition of the set M. Hence, we can determine the *PARA* $\Gamma_M(\sigma^*)$. Let us define the map $f\colon \Psi[T(\sigma)] \to \Gamma_M(\sigma^*)$ as follows. For all $r, s, t \in \Psi[T(\sigma)]$,

(1) if r is an atom then $f(r) = I(r)$;

(2) if $f(r)$ and $f(s)$ are already known and $t = r \cup s$, then $f(t) = f(r) \cup f(s)$.

THEOREM 6.2. *f is an isomorphism between $\Psi[T(\sigma)]$ and $\Gamma_M^{\Diamond}(\sigma^*)$.*

PROOF. f is evidently an injection and its image is the set of all closed elements of the *PARA* $\Gamma_M(\sigma^*)$. In another words, f is a bijection between $\Psi[T(\sigma)]$ and $\Gamma_M^{\Diamond}(\sigma^*)$. The proof of the theorem is trivial except the proof of the equality $f(r\cdot s) = \Diamond[f(r) \odot f(s)]$ for all elements r, s, t of $\Psi[T(\sigma)]$. But the proof of this equality is evidently reduced to the proof of the following theorem.

THEOREM 6.2.1. *If propositions (u, v, w) form a syllogism of the theory $T(\sigma)$, then*

(1) for any predicate symbol π contained in the proposition w, $(\tilde{u} \odot \tilde{v}) \cap \tilde{\pi} \neq \emptyset$;

2) for any predicate symbol $\pi \in \sigma$ which is not contained in the proposition w, $(\tilde{u} \odot \tilde{v}) \cap \tilde{\pi} = \emptyset$.

Here $\tilde{u}$ designates the relation in $\Gamma_M(\sigma^*)$, corresponding to a proposition u. More correctly, if u' is the set of all predicate symbols contained in u, then $f(u') = \tilde{u}$.

PROOF. (1) In accordance with the agreements above, μ is a sufficiently rich model of $T(\sigma)$, i.e. condition (6.2) is satisfied. This implies item (1) of the theorem.

(2) In accordance with Theorem 6.1, for any syllogism (u, v, w) and for any predicate symbol $\pi \in \sigma$, which is not contained in the proposition w,

$$\vdash_T u(x, z) \mathbin{\&} v(z, y) \Rightarrow \neg \pi(x, y),$$

i.e.

$$\mu \models \neg[u(x, z) \mathbin{\&} v(z, y) \mathbin{\&} \pi(x, y)].$$

This implies item (2) of the theorem. □

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Received April 10, 1986