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Bounded Contraction and Gentzen-style Formulation of Łukasiewicz Logics

Abstract. In this paper, we consider multiplicative-additive fragments of affine propositional classical linear logic extended with n-contraction. To be specific, n-contraction ($n \geq 2$) is a version of the contraction rule where $(n+1)$ occurrences of a formula may be contracted to n occurrences. We show that expansions of the linear models for $(n+1)$-valued Łukasiewicz logic are models for the multiplicative-additive classical linear logic, its affine version and their extensions with n-contraction. We prove the finite axiomatizability for the classes of finite models, as well as for the class of infinite linear models based on the set of rational numbers in the interval $[0, 1]$. The axiomatizations obtained in a Gentzen-style formulation are equivalent to finite and infinite-valued Łukasiewicz logics.

Key words: Łukasiewicz logics, Gentzen-style formulation, completeness, finite axiomatizability.

1. Introduction

This paper found its origin while investigating the effects of substituting bounded contraction for full contraction in classical Gentzen systems.

We here consider two versions of bounded contraction differing in the number of occurrences of a formula that may be contracted. For any $n \geq 2$, n-contraction and n-copy contraction are versions of the contraction rule where $(n+1)$ and $2n$ occurrences of a formula respectively may be contracted to n occurrences. In the presence of weakening the respective rules for n-contraction and n-copy contraction are interderivable. It will be seen, that the latter rules are convenient to formulate and prove the central lemmas for establishing completeness theorems in section 4.

Substituting n-contraction for full contraction in Gentzen systems already results in splitting of logical operations familiar from linear logic, see Girard [6]. Hence, the logics obtained are multiplicative-additive fragments of affine linear logic extended with n-contraction. Since the systems with n-contraction considered do not have the cut-elimination property (witness the counter-example in section 2) it is of vital importance to provide models for them.

In section 3, we specify the class of models based on many-valued semantics as models for multiplicative-additive fragment of propositional classical

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linear logic, its affine variant and their extensions with n-contraction. We emphasize that our models are just particular examples of MV-algebras introduced by Chang [4]

In section 4, we give an axiomatization for each of the classes of finite models considered. To prove the corresponding completeness theorems we use a suitable generalization of Kalmár method, well-known from the completeness proof of ordinary propositional logic with respect to 2-valued semantics. However, due to the fact that $A \sqcup B$ and $\sim (\sim A + B) + B$ (with $\sqcup$ and $+$ denoting the additive and the multiplicative disjunction respectively) coincide in the proposed models, the logics obtained are, in fact, equivalent to finite-valued Łukasiewicz logics. In other words Łukasiewicz logics in a linear Gentzen-style formulation. Thus it is now our interest and duty to remind the reader of the previous results obtained in this field and compare them with our own contribution.

The problem of finite axiomatizability of n-valued Łukasiewicz logics has attracted many mathematicians ever since Łukasiewicz introduced these logics (for $n = 3$ in 1920, others in 1922) by the well-known matrix method [9]. The best-known solutions of that question, are given by: Lindenbaum in [9], Rosser and Turquette in [12], Tokarz in [13] and Grigolia in [7]. In particular, Grigolia obtained elegant Hilbert-style axiomatizations of n-valued Łukasiewicz logics. Moreover, his completeness proof is based on a subclass of Chang's MV_n-algebras making use of a version of bounded contraction and specific conditions for primitive connectives depending on n.

Two more results by Rosser and Turquette are listed in the literature, namely in [10] and in [11]. However, the completeness proof given in the former article is somewhat obscure due to a defect in the definition of a notion of an S-sum. On the other hand, the more elegant axiomatization presented in the latter paper is not adequate for Łukasiewicz logics, as observed by the authors themselves. We emphasize that this observation also follows from the fact, that full contraction is derivable in the axiomatic systems considered by Rosser and Turquette. As we shall show below, full contraction is not admissible in any of many-valued Łukasiewicz logics. Let us conclude our brief survey with two more contributions on the finite axiomatizability in question. In [3] Cignoli gives an algebraic completeness proof based on proper n-valued Moisil algebras. And finally, we emphasize that Tuziak in [15] obtained elegant Hilbert-style axiomatizations where the connectives resemble to our linear ones and an equivalent for our n-contraction is present. However, his proof is of a different kind, namely by means of the Lindenbaum algebra.

The intuition behind our own method of axiomatization of finite-valued

Łukasiewicz logics shares an underlying basic idea with some of the publications mentioned above, namely that of encoding in a formula a certain "truth-value" (see [10] and [12]). However, here we consider different primitive connectives and introduce a new construction of such a formula. Moreover, due to the derivable n-copy contraction rules in our systems we can present a simpler proof yielding different, more perspicuous axiomatization in a Gentzen-style formulation with additional axioms. In fact, to the initially given affine classical linear logic with n-contraction we add the axiom-schemes which encode point-wise definitions of the operations in models corresponding to $+$ and $\sqcup$ respectively. Moreover, an additional axiom-scheme is an n-valued analogue of the 2-valued classical tautology $P \vee {\sim}P$.

In section 5, we consider the intersection of all the systems that are complete for the respective classes of finite models. We show that the intersection is complete for the class of linear models based on the set of rational numbers in the interval $[0, 1]$. We also prove the finite axiomatizability of the intersection considered, resulting in a Gentzen-style formulation of logic equivalent to $\aleph_0$- Łukasiewicz logic.

Let us finally mention that Troelstra's notation [14] for the linear logic connectives will be used in this paper.

2. Systems of affine classical linear logic with bounded contraction

For any $n \geq 2$, a classical system of affine propositinal linear logic with n-contraction, $\mathbf{PL}_n$, is given by the following axioms and rules.

Axiom scheme

$$A \Longrightarrow A$$

Logical rules

- left and right negation rules

$$L{\sim}\ \frac{\Gamma \Longrightarrow A, \Delta}{\Gamma, {\sim}A \Longrightarrow \Delta} \qquad \frac{\Gamma, A \Longrightarrow \Delta}{\Gamma \Longrightarrow {\sim}A, \Delta}\ R{\sim}$$

- left and right disjunction rules

$$L\sqcup\ \frac{\Gamma, A \Longrightarrow \Delta \qquad \Gamma, B \Longrightarrow \Delta}{\Gamma, A \sqcup B \Longrightarrow \Delta}$$

$$R\sqcup\ \frac{\Gamma \Longrightarrow A_i, \Delta}{\Gamma \Longrightarrow A_1 \sqcup A_2, \Delta} \quad \text{for } i = 1, 2$$

- left and right par rules

$$L+\ \frac{\Gamma_1, A \Longrightarrow \Delta_1 \qquad \Gamma_2, B \Longrightarrow \Delta_2}{\Gamma_1, \Gamma_2, A+B \Longrightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma \Longrightarrow A, B, \Delta}{\Gamma \Longrightarrow A+B, \Delta}\ R+$$

Structural rules

- left and right weakening rules

$$LW\ \frac{\Gamma \Longrightarrow \Delta}{\Gamma, A \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow A, \Delta}\ RW$$

- left and right n-contraction rules

$$LC_n\ \frac{\Gamma, A^{(n+1)} \Longrightarrow \Delta}{\Gamma, A^{(n)} \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow A^{(n+1)}, \Delta}{\Gamma \Longrightarrow A^{(n)}, \Delta}\ RC_n$$

where $A^{(k)} = A, A, \ldots, A$, i.e. k copies of formula A.

- cut rule

$$CUT\ \frac{\Gamma_1 \Longrightarrow A, \Delta_1 \qquad \Gamma_2, A \Longrightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Longrightarrow \Delta_1, \Delta_2}$$

Throughout the above rules $\Gamma, \Delta, \Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ denote finite multisets of formulas.

COMMENT: Due to the restriction on contraction rules a sequent $A + B \Longrightarrow A \sqcup B$ is not generally derivable in $\mathbf{PL}_n$ for any $n \geq 2$. In other words, the usual propositional logic connective $\vee$ has been split into the additive and multiplicative one, i.e. $\sqcup$ and $+$ respectively. The following connectives can be defined

- $A \sqcap B := \sim(\sim A \sqcup \sim B)$
- $A \star B := \sim(\sim A + \sim B)$
- $A \rightsquigarrow B := \sim A \sqcup B$
- $A \multimap B := \sim A + B$

The respective rules for each of these connectives, being the same as in classical linear logic, are derivable in $\mathbf{PL}_n$ for $n \geq 2$. We shall also use those connectives and the corresponding rules.

Consider now another possible restriction on contraction rules, namely for given natural number $n \geq 2$

$$\frac{\Gamma, A^{(n)}, A^{(n)} \Longrightarrow \Delta}{\Gamma, A^{(n)} \Longrightarrow \Delta} \qquad\qquad \frac{\Gamma \Longrightarrow A^{(n)}, A^{(n)}, \Delta}{\Gamma \Longrightarrow A^{(n)}, \Delta}$$

We shall call these rules left and right n-copy contraction respectively. In the presence of weakening these rules are interderivable with those introduced earlier. Thus, we shall, in what follows, use n-copy contraction or n-contraction as convenient to show some meta-properties of the system $\mathbf{PL}_n$.

To start with

FACT 2.1 *The cut rule can not be eliminated from the system* $\mathbf{PL}_n$*, for any* $n \geq 2$.

PROOF. The following is a counter-example for cut-elimination in the system $\mathbf{PL}_2$. Consider the sequent

$$u, z{-\!\circ}u,\ (z{-\!\circ}u){-\!\circ}((z{-\!\circ}u){-\!\circ}((z{-\!\circ}u){-\!\circ}x)) \Longrightarrow x,$$

where u, z, x are atomic formulas.
It is easy to check that this sequent does not have a cut-free derivation in $\mathbf{PL}_2$. On the other hand, witness the derivation of the same sequent obtained, as follows:
First, applying the left rule for $-\!\circ$ to

$$z{-\!\circ}u \Longrightarrow z{-\!\circ}u \quad \text{and} \quad x \Longrightarrow x$$

yields:

$$z{-\!\circ}u,\ (z{-\!\circ}u){-\!\circ}x \Longrightarrow x.$$

Further, applying twice the left rule for $-\!\circ$ to the last obtained sequent and to the axiom

$$z{-\!\circ}u \Longrightarrow z{-\!\circ}u$$

gives:

$$(z{-\!\circ}u)^{(3)},\ (z{-\!\circ}u){-\!\circ}((z{-\!\circ}u){-\!\circ}((z{-\!\circ}u){-\!\circ}x)) \Longrightarrow x.$$

Next, using left 2-contraction rule results in:

$$(z{-\!\circ}u)^{(2)},\ (z{-\!\circ}u){-\!\circ}((z{-\!\circ}u){-\!\circ}((z{-\!\circ}u){-\!\circ}x)) \Longrightarrow x.$$

And finally, applying cut rule to the latter sequent and to the, clearly, $\mathbf{PL}_2$-derivable sequent

$$u \Longrightarrow z \multimap u,$$

yields:

$$u, z \multimap u,\ (z \multimap u) \multimap ((z \multimap u) \multimap ((z \multimap u) \multimap x)) \Longrightarrow x.$$

However, the reader is invited to check, that for any $n \geq 2$ the sequent given below provides a counter-example for cut-elimination in $\mathbf{PL}_n$:

$$u, (z \multimap u)^{(n-1)},\ (z \multimap u) \multimap ((z \multimap u) \multimap \cdots \multimap (z \multimap u) \multimap x) \ldots) \Longrightarrow x,$$

where $(z \multimap u)$ occurs $(n+1)$-times in the indicated subformula of the antecedent. ■

3. Many-valued semantics

We now give a particular many-valued semantics for the systems considered in section 2 .

Given a natural number $n \geq 2$, we shall define a model for $\mathbf{PL}_n$, called $M_n(v)$, as follows.

1. Take the following set of values:

$$S_n = \{k/n \mid k = 0, 1, \ldots, n-1, n\}.$$

   A valuation function v assigns to each propositional letter an element of S_n.

2. Extend v to arbitrary $\mathbf{PL}_n$-formula inductively, as follows:

$$v(\sim A) = 1 - v(A)$$

$$v(A \sqcup B) = \max\{v(A), v(B)\}$$

$$v(A + B) = \min\{v(A) + v(B), 1\}$$

$$v(A \sqcap B) = \min\{v(A), v(B)\}$$

$$v(A \star B) = \max\{v(A) + v(B) - 1, 0\}$$

$$v(A \leadsto B) = \max\{1 - v(A), v(B)\}$$

$$v(A \multimap B) = \min\{1 - v(A) + v(B), 1\}.$$

3. v is extended to any $\mathbf{PL}_n$-sequent $A_1, \dots, A_m \Longrightarrow B_1, \dots, B_j$ putting:

$$v(A_1, \dots, A_m \Longrightarrow B_1, \dots, B_j) = v({\sim}A_1 + \dots + {\sim}A_m + B_1 + \dots + B_j).$$

We shall say that a given sequent $\Gamma \Longrightarrow \Delta$ is true in the model $M_n(v)$ if and only if $v(\Gamma \Longrightarrow \Delta) = 1$. Moreover, a sequent $\Gamma \Longrightarrow \Delta$ is n-valid if and only if $v(\Gamma \Longrightarrow \Delta) = 1$ for all valuation functions v, i.e. if the sequent under consideration is true in every model $M_n(v)$. The class of all models $M_n(v)$ will be denoted by M_n.

REMARK. Note that $v(A \sqcup B) = v({\sim}({\sim}A + B) + B)$.

PROPOSITION 3.1. *[Soundness] Given a natural number $n \geq 2$ and a $\mathbf{PL}_n$-sequent $\Gamma \Longrightarrow \Delta$, if $\mathbf{PL}_n \vdash \Gamma \Longrightarrow \Delta$, then $\Gamma \Longrightarrow \Delta$ is n-valid.*

PROOF. By induction on the length of a derivation.

As an example, we will consider the case where the last applied rule within a given derivation is right n-copy contraction:

$$\frac{\Gamma \Longrightarrow A^{(n)}, A^{(n)}, \Delta}{\Gamma \Longrightarrow A^{(n)}, \Delta}$$

First, observe that $v(A_1 + \dots + A_m) = \min\{v(A_1) + \dots + v(A_m), 1\}$. Hence, we get $v(A_1 + \dots + A_m) = 1$ if and only if $v(A_1) + \dots + v(A_m) \geq 1$. Now, by induction hypothesis the following holds for all v:

$$\sum_{i=1}^{p}(1 - v(\gamma_i)) + 2nv(A) + \sum_{j=1}^{r} v(\delta_j) \geq 1,$$

where $\Gamma = \gamma_1, \dots, \gamma_p$ and $\Delta = \delta_1, \dots, \delta_r$.

And we are going to show that, then also:

$$\sum_{i=1}^{p}(1 - v(\gamma_i)) + nv(A) + \sum_{j=1}^{r} v(\delta_j) \geq 1, \quad \text{for all } v.$$

In fact, one only has to observe, that for a given v:

- either $v(A) = 0$, then the two considered inequalities coincide and we are done;
- or $v(A) \geq 1/n$, thus, $nv(A) \geq 1$, and a fortiori $\sum_{i=1}^{p}(1 - v(\gamma_i)) + nv(A) + \sum_{j=1}^{r} v(\delta_j) \geq 1$.

Since v was arbitrary, our claim is justified. ■

LEMMA 3.2. *Given a natural number $n \geq 2$, neither left nor right $(n-1)$-copy contraction is admissible in* $\mathbf{PL}_n$.

PROOF. Let A_P denote a $\mathbf{PL}_n$-formula $P \sqcap \sim(n-1)P$, where P is a propositional letter and $(n-1)P = P + \cdots + P$ with $(n-1)$ copies of P.

Note that:

$$v(A_P) = \begin{cases} 1/n & \text{if } v(P) = 1/n \\ 0 & \text{otherwise} \end{cases}$$

Further, consider $(\sim A_P)^n$, where $X^n = X \star \cdots \star X$ with n copies of X. Clearly,

$$v((\sim A_P)^n) = \begin{cases} 0 & \text{if } v(P) = 1/n \\ 1 & \text{otherwise} \end{cases}$$

And finally, the following sequent presents a counter-example for admissibility of right $(n-1)$-copy contraction rule:

$$\Longrightarrow A_P^{(n-1)}, A_P^{(n-1)}, (\sim A_P)^n.$$

More precisely, observe that the given sequent is n-valid. But, clearly, $v(P) = 1/n$ is a refutation valuation for n-validity of the sequent

$$\Longrightarrow A_P^{(n-1)}, (\sim A_P)^n,$$

obtained by applying the right $(n-1)$-copy contraction rule to the sequent above.

Similarly, one can find a counter-example for admissibility of the left $(n-1)$-copy contraction rule. ■

4. Axiomatic completeness for finite models

In this section, our intention is to find an axiomatization of all n-valid $\mathbf{PL}_n$-sequents, for any $n \geq 2$. We shall start with the case $n = 2$.

Consider the system $\mathbf{PL}_2$ extended by the following

axiom-schemes

(i) $2(B \sqcap \sim B) \sqcap (\sim C)^2 \Longrightarrow 2((B \sqcup C) \sqcap \sim(B \sqcup C))$

$2(B \sqcap \sim B) \sqcap 2(C \sqcap \sim C) \Longrightarrow 2((B \sqcup C) \sqcap \sim(B \sqcup C))$

(ii) $2(B \sqcap {\sim}B) \star 2(C \sqcap {\sim}C) \Longrightarrow B + C$

$2(B \sqcap {\sim}B) \star ({\sim}C)^2 \Longrightarrow 2((B + C) \sqcap {\sim}(B + C))$

(iii) $\Longrightarrow P^2 \sqcup (2(P \sqcap {\sim}P))^2 \sqcup ({\sim}P)^2$, for P atomic

where $2X = X + X$ and $X^2 = X \star X$ for any $\mathbf{PL}_2$-formula X. This system will be referred to as $\mathbf{CPL}_2$.

We are going to prove that $\mathbf{CPL}_2$ is complete for the considered class of models M_2. Later on, it will become clear to the reader that the axiom schemes stated in (i) and (ii) are, in fact, forced by the proof of lemma 4.2 and the axiom scheme in (iii) by the proof of lemma 4.3. For that purpose, we have to elaborate first the necessary prerequisites.

DEFINITION 4.1. Let FPL_2 denote the set of all $\mathbf{PL}_2$ formulas and V_2 the set of all extended valuations on FPL_2. We define a function $[\,.\,,\,.\,] : FPL_2 \times V_2 \longrightarrow FPL_2$, as follows:

$$[A, v] = \begin{cases} {\sim}A & \text{if } v(A) = 0 \\ 2(A \sqcap {\sim}A) & \text{if } v(A) = 1/2 \\ A & \text{if } v(A) = 1 \end{cases} \qquad A \in FPL_2, v \in V_2.$$

LEMMA 4.2. *Given $A \in FPL_2$ containing exactly $P_1, \ldots, P_m$ distinct propositional letters and given $v \in V_2$, then*

$$\mathbf{CPL}_2 \vdash [P_1, v]^{(2)}, \ldots, [P_m, v]^{(2)} \Longrightarrow [A, v].$$

PROOF. By induction on the complexity of A.

We shall here consider only one typical case for the connective $+$. Assume $A = B + C$, and $v(B) = v(C) = 1/2$. By induction hypothesis we have:

$$\mathbf{CPL}_2 \vdash [R_1, v]^{(2)}, \ldots, [R_k, v]^{(2)} \Longrightarrow 2(B \sqcap {\sim}B)$$

and

$$\mathbf{CPL}_2 \vdash [Q_1, v]^{(2)}, \ldots, [Q_n, v]^{(2)} \Longrightarrow 2(C \sqcap {\sim}C).$$

And from that by the right rule for $\star$:

$$\mathbf{CPL}_2 \vdash [R_1, v]^{(2)}, \ldots, [R_k, v]^{(2)}, [Q_1, v]^{(2)}, \ldots, [Q_n, v]^{(2)} \Longrightarrow 2(B \sqcap {\sim}B) \star 2(C \sqcap {\sim}C)$$

If $\{R_1, \dots, R_k\} \bigcap \{Q_1, \dots, Q_n\} \neq \emptyset$, then a number of left 2-copy contraction rule is to be applied, yielding:

$$\mathbf{CPL}_2 \vdash [P_1, v]^{(2)}, \dots, [P_m, v]^{(2)} \Longrightarrow 2(B \sqcap \sim B) \star 2(C \sqcap \sim C)$$

where $P_1, \dots, P_m$ are precisely all distinct propositional letters occurring in B and C, hence in A.

Finally, an application of the rule cut to the last obtained sequent and to the $\mathbf{CPL}_2$ axiom $2(B \sqcap \sim B) \star 2(C \sqcap \sim C) \Longrightarrow B + C$ yields:

$$\mathbf{CPL}_2 \vdash [P_1, v]^{(2)}, \dots, [P_m, v]^{(2)} \Longrightarrow B + C,$$

what was to be shown, since $v(B + C) = 1$ for the given v. ■

LEMMA 4.3. *Given a* $\mathbf{PL}_2$*-formula* A, *if* $v(A) = 1$ *for all* $v \in V_2$, *then*

$$\mathbf{CPL}_2 \vdash \Longrightarrow A.$$

PROOF. Assume that $A \in FPL_2$ contains exactly $P_1, \dots, P_m$ distinct propositional letters and that $v(A) = 1$ for all $v \in V_2$. Take, now, arbitrary $v', v'', v''' \in V_2$ such that $v'(P_1) = 1$, $v''(P_1) = 1/2$, $v'''(P_1) = 0$ and $v' = v'' = v''' = v$ on $\{P_2, \dots, P_m\}$. By lemma 4.2 we get:

$$\mathbf{CPL}_2 \vdash P_1^{(2)}, [P_2, v]^{(2)}, \dots, [P_m, v]^{(2)} \Longrightarrow A,$$
$$\mathbf{CPL}_2 \vdash (2(P_1 \sqcap \sim P_1))^{(2)}, [P_2, v]^{(2)}, \dots, [P_m, v]^{(2)} \Longrightarrow A$$

and

$$\mathbf{CPL}_2 \vdash (\sim P_1)^{(2)}, [P_2, v]^{(2)}, \dots, [P_m, v]^{(2)} \Longrightarrow A.$$

Next, applying left rules for $\star$ and $\sqcup$ respectively yields:

$$\mathbf{CPL}_2 \vdash P_1^2 \sqcup (2(P_1 \sqcap \sim P_1))^2 \sqcup (\sim P_1)^2, [P_2, v]^{(2)}, \dots, [P_m, v]^{(2)} \Longrightarrow A.$$

Finally, an application of cut rule to the last obtained sequent and to the axiom:

$$\Longrightarrow P_1^2 \sqcup (2(P_1 \sqcap \sim P_1))^2 \sqcup (\sim P_1)^2$$

yields:

$$\mathbf{CPL}_2 \vdash [P_2, v]^{(2)}, \dots, [P_m, v]^{(2)} \Longrightarrow A,$$

where, clearly, v is arbitrary and the number of propositional letters in the antecedent of the sequent is reduced to $(m-1)$. Repeating the above strategy gives:

$$\mathbf{CPL}_2 \vdash \Longrightarrow A,$$

what was to be proved. ■

We need one more

FACT 4.4 $\mathbf{CPL_2} \vdash A_1, \dots, A_k \Longrightarrow B_1, \dots, B_m$ *if and only if*

$$\mathbf{CPL_2} \vdash \Longrightarrow {\sim}A_1 + \dots + {\sim}A_k + B_1 + \dots + B_m.$$

Finally, we are well equipped to prove the main

PROPOSITION 4.5. *[Completeness] If a* $\mathbf{PL_2}$*-sequent* $\Gamma \Longrightarrow \Delta$ *is 2-valid, then* $\mathbf{CPL_2} \vdash \Gamma \Longrightarrow \Delta$.

PROOF. Assume a $\mathbf{PL_2}$-sequent $A_1, \dots, A_k \Longrightarrow B_1, \dots, B_m$ to be 2-valid, i.e. $v({\sim}A_1 + \dots + {\sim}A_k + B_1 + \dots + B_m) = 1$ for all $v \in V_2$. Then, by lemma 4.3 we get:

$$\mathbf{CPL_2} \vdash \Longrightarrow {\sim}A_1 + \dots + {\sim}A_k + B_1 + \dots + B_m.$$

Using fact 4.4 yields: $\mathbf{CPL_2} \vdash A_1, \dots, A_k \Longrightarrow B_1, \dots, B_m$ and we are done. ■

In what follows, we are going to generalize the results just obtained. In other words, we shall below present an axiomatization of all n-valid $\mathbf{PL_n}$-sequents for an arbitrary natural number $n \geq 2$.

We first introduce

DEFINITION 4.6. Given a natural number $n \geq 2$, let A be a $\mathbf{PL_n}$- formula. We shall define a $\mathbf{PL_n}$-formula $\langle k : n \rangle A$ for $k = 1, \dots, (n-1)$ with the property:

$$v(\langle k : n \rangle A) = \begin{cases} 1 & \text{if } v(A) = k/n \\ 0 & \text{otherwise} \end{cases}$$

where $v \in V_n$.

The construction of $\langle k : n \rangle A$ is given inductively as follows:

(a) for $n = 2$ put $\langle 1 : 2 \rangle A = 2(A \sqcap {\sim}A)$.

(b) Assume, now, that $\langle k : i \rangle A$ has already been defined for all $i = 2, \dots, (n-1)$ and $k = 1, \dots, (i-1)$.

- Put $\langle 1 : n \rangle A = n(A \sqcap {\sim}(n-1)A)$ and $\langle (n-1) : n \rangle A = \langle 1 : n \rangle {\sim}A$.
- Assume, further, that $\langle m : n \rangle A$ has already been defined for all $m = 1, (n-1), 2, (n-2), \dots, (k-1), (n-(k-1))$, where $k \leq \lfloor n/2 \rfloor$. Now define

$$\langle k : n \rangle A = \begin{cases} (\langle 1 : m \rangle A)^k & \text{if } n = mk \\ \langle (n-l) : n \rangle (mA) & \text{if } n = mk + l,\ 1 \leq l \leq (k-1) \end{cases}$$

REMARK. It is a matter of patient, but straightforward, checking to see that the above construction is well-defined. Note however, that other possible constructions of $\mathbf{PL}_n$-formula with the distinguished property would yield equivalent formulations of the axiomatization in question.

Consider now the system $\mathbf{PL}_n$ extended by the following

axiom-schemes:

(i) for $0 < k \leq m < n$:

$$\langle k:n\rangle B \sqcap \langle m:n\rangle C \Longrightarrow \langle m:n\rangle(B \sqcup C)$$

(ii) for $0 < m < n$:

$$(\sim B)^n \sqcap \langle m:n\rangle C \Longrightarrow \langle m:n\rangle(B \sqcup C)$$

(iii) for $0 < k, m < n$:

$$\langle k:n\rangle B \star \langle m:n\rangle C \Longrightarrow \langle (k+m):n\rangle(B+C), \quad \text{if } k+m < n$$

$$\langle k:n\rangle B \star \langle m:n\rangle C \Longrightarrow B+C, \quad \text{if } k+m \geq n$$

(iv) for $0 < m < n$:

$$(\sim B)^n \star \langle m:n\rangle C \Longrightarrow \langle m:n\rangle(B+C)$$

(v) for P atomic formula:

$$\Longrightarrow \sqcup_{k=1}^{n-1}(\langle k:n\rangle P)^n \sqcup P^n \sqcup (\sim P)^n$$

The system just introduced will be called $\mathbf{CPL}_n$.

REMARK. The axiom-schemes given above are, again, forced by the proofs of lemmas analogous to those for the case $n = 2$. Moreover, to match the intuition given in the introduction, consider the axiom-schemes in (iii). Note, that the considered sequents are trivially true in all the models $M_n(v)$ except for those where $v(B) = k/n$ and $v(C) = m/n$. But, clearly in these cases, the valuation of the formula in succedent is 1 by definition. Hence, the axiom-schemes in (iii) correspond to the following statements:
if $v(B) = k/n$ and $v(C) = m/n$, then $v(B+C) = (k+m)/n$, when $k+m < n$ and $v(B+C) = 1$, otherwise.

In order to show that $\mathbf{CPL}_n$, for any $n \geq 2$, is indeed an axiomatization of all n-valid $\mathbf{PL}_n$-sequents, we proceed as follows.

DEFINITION 4.7. Given a natural number $n \geq 2$, we define a function $[\,.\,,\,.\,]_n : FPL_n \times V_n \longrightarrow FPL_n$ by:

$$[A, v]_n = \begin{cases} \sim A & \text{if } v(A) = 0 \\ \langle v(A)n : n \rangle A & \text{otherwise} \\ A & \text{if } v(A) = 1 \end{cases} \qquad A \in FPL_n, v \in V_n.$$

We state the central

LEMMA 4.8. *Given $A \in FPL_n$ containing exactly $P_1, \ldots, P_m$ distinct propositional letters and given $v \in V_n$, for some $n \geq 2$, then:*

$$\mathbf{CPL}_n \vdash [P_1, v]_n^{(n)}, \ldots, [P_m, v]_n^{(n)} \Longrightarrow [A, v]_n.$$

PROOF. By induction on the complexity of A. ■

REMARK. The rest of the story, omitted here, proceeds similarly to the case $n = 2$, establishing the following

PROPOSITION 4.9. *[Completeness] Let $n \geq 2$. If a $\mathbf{PL}_n$-sequent $\Gamma \Longrightarrow \Delta$ is n-valid, then $\mathbf{CPL}_n \vdash \Gamma \Longrightarrow \Delta$.*

5. Axiomatic completeness for infinite models

In this section, we consider the intersection of the systems $\mathbf{CPL}_n$, for $n \geq 2$, denoted by $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$. We emphasize that $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$ refers to the set of theorems, i.e. to the sequents derivable in every system $\mathbf{CPL}_n$, for $n \geq 2$. First, we are going to show soundness and completeness of $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$ with respect to the class of infinite models, based on the set of rational numbers Q in the interval $[0, 1]$. Models based on $Q \bigcap [0, 1]$ are defined in the same way as the finite models in section 3, only that the set of values, S_∞, is now $Q \bigcap [0, 1]$. Moreover, if a given sequent is true in every model based on $Q \bigcap [0, 1]$ we shall say that this sequent is $Q \bigcap [0, 1]$- valid.

PROPOSITION 5.1. *A given sequent $\Gamma \Longrightarrow \Delta$ is a theorem of $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$ if and only if $\Gamma \Longrightarrow \Delta$ is $Q \bigcap [0, 1]$-valid.*

PROOF.
(i) soundness
Assume that a sequent $\Gamma \Longrightarrow \Delta$ is a theorem of $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$. We want to prove that $v(\Gamma \Longrightarrow \Delta) = 1$ for every valuation v based on $Q \bigcap [0, 1]$.

Observe first, that there are only finitely many propositional letters in $\Gamma \Longrightarrow \Delta$, say: $P_1, \ldots, P_j$. Consider a valuation v based on $Q \bigcap [0,1]$ such that $v(P_i) = k_i/m_i$, for $i = 1, \ldots, j$. Let $m \geq 2$ be the least common multiple of $\{m_1, \ldots, m_j\}$. Then, for every $i = 1, \ldots, j$ there is some s_i such that $v(P_i) = s_i/m$. Now, we can see that $v|\{P_1, \ldots, P_j\}$ is in fact the restriction of some valuation $v_m \in V_m$. Since $\Gamma \Longrightarrow \Delta$ is by assumption a theorem of $\mathbf{CPL}_m$, for all $m \geq 2$, and moreover, soundness, see proposition 3.1, implies m-validity, we get:

$$v(\Gamma \Longrightarrow \Delta) = v_m(\Gamma \Longrightarrow \Delta) = 1.$$

And this was to be proved, after having realized that the remaining case for $m = 1$ is trivial.

(ii) completeness
Suppose that $v(\Gamma \Longrightarrow \Delta) = 1$ for any v based on $Q \bigcap [0,1]$. Given arbitrary $n \geq 2$, clearly any $v \in V_n$ is also a valuation based on $Q \bigcap [0,1]$, hence $v(\Gamma \Longrightarrow \Delta) = 1$ for all $v \in V_n$. This means that the sequent under consideration is n-valid and by completeness, see proposition 4.9, we get: $\mathbf{CPL}_n \vdash \Gamma \Longrightarrow \Delta$. Since n was arbitrary, it follows that $\Gamma \Longrightarrow \Delta$ is a theorem of $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$. This completes the proof. ∎

It is a routine of universal algebra to provide the following by-product

FACT 5.2 *A given sequent* $\Gamma \Longrightarrow \Delta$ *is* $Q \bigcap [0,1]$*-valid if and only if* $\Gamma \Longrightarrow \Delta$ *is valid for the class of models* $\prod_{n=2}^{\infty} M_n$.

A natural question, which arises at this point, is whether there exists a finite axiomatization of $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$. The answer is positive. Clearly, the maximal deductive subsystem contained in every $\mathbf{PL}_n$, and thus, also in every $\mathbf{CPL}_n$, for $n \geq 2$, will be taken as the base of the axiomatization in question. Since there are no contraction rules left in the considered subsystem, we shall call it **PL**.

Moreover, we claim that the system **PL** extended by the axiom-scheme:

$$\sim(\sim A + B) + B \implies A \sqcup B,$$

presents an axiomatization of $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$. We shall refer to this system as **CPL**.

In what follows, we reduce the proof of this assertion to the well-known result of Wajsberg [9] who presented a Hilbert-style axiomatization of $\aleph_0$-valued Łukasiewicz logic in the language $< \sim, \multimap >$, using our notation. (An algebraic proof of the considered axiomatization, based on linearly ordered MV-algebras, was obtained by Chang in [5].)

Wajsberg's axiomatization is as follows:

axiom-schemes:

(1) $A \multimap (B \multimap A)$

(2) $(A \multimap B) \multimap ((B \multimap C) \multimap (A \multimap C))$

(3) $((A \multimap B) \multimap B) \multimap ((B \multimap A) \multimap A)$

(4) $(\sim A \multimap \sim B) \multimap (B \multimap A)$

with the single rule of inference, modus ponens.
We shall refer to the above deductive system as **HW**.

From now on, we shall think of **HW** as a one-sided sequent calculus with axioms given by (1)-(4) and modus ponens having the form:

$$\frac{\Longrightarrow A \qquad \Longrightarrow A \multimap B}{\Longrightarrow B}$$

Further, we shall show that omitting the logical rules for the connective $\sqcup$ from **PL**, but adding to the resulting system the axiom-scheme:

$$\sim(\sim A + B) + B \Longrightarrow \sim(\sim B + A) + A,$$

yields a Gentzen-style formulation of **HW**, called **GW**.
In other words, we want to show that **GW** is complete for the same class of models as **HW**, i.e. for models based on $Q \bigcap [0,1]$. Witness the next

PROPOSITION 5.3. *The system* **GW** *is complete for the class of models based on* $Q \bigcap [0,1]$.

PROOF. Our proof will be based on the following simple observations:

(i) **HW** is a subsystem of **GW**;

(ii) **GW** is sound with respect to models based on $Q \bigcap [0,1]$;

(iii) in **GW** a given sequent $A_1, A_2, \ldots, A_k \Longrightarrow B_1, \ldots, B_{m-1}, B_m$ is provably equivalent with

$$\Longrightarrow A_1 \multimap (A_2 \multimap (\ldots (A_k \multimap (\sim B_1 \multimap (\ldots (\sim B_{m-1} \multimap B_m) \ldots) .$$

Assume now, that a given sequent $A_1, A_2, \dots, A_k \Longrightarrow B_1, \dots, B_{m-1}, B_m$ is $Q\bigcap[0,1]$-valid. Then, from (iii) and (ii) it follows, that also the sequent $\Longrightarrow A_1 \multimap (A_2, \multimap (\dots (A_k \multimap (\sim B_1 \multimap (\dots (\sim B_{m-1} \multimap B_m) \dots)$ is $Q\bigcap[0,1]$-valid. Thus, by completeness, the latter sequent is derivable in **HW** and hence, also in **GW** by (i). Finally, due to (iii), we get:

$$\mathbf{GW} \vdash A_1, A_2, \dots, A_k \Longrightarrow B_1, \dots, B_{m-1}, B_m ,$$

what was to be proved. ■

To continue with:

LEMMA 5.4. **GW** *is a subsystem of* **CPL**.

PROOF. It is easy to see that:

$$\mathbf{CPL} \vdash \sim(\sim A + B) + B \Longrightarrow \sim(\sim B + A) + A.$$

■

DEFINITION 5.5. A mapping s from **CPL**-formulas into **GW**-formulas is given inductively, as follows:

(i) $s(P) := P$, for any propositional letter P

(ii) $s(\sim A) := \sim s(A)$

(iii) $s(A \sqcup B) := \sim(\sim s(A) + s(B)) + s(B)$

(iv) $s(A + B) := s(A) + s(B)$

LEMMA 5.6. *A given* **CPL***-formula A is equivalent to $s(A)$ in the following sense:* $\mathbf{CPL} \vdash A \Longrightarrow s(A)$ *and* $\mathbf{CPL} \vdash s(A) \Longrightarrow A$.

PROOF. By induction on the complexity of A.
To illustrate the proof we shall work out the crucial case:

$$\mathbf{CPL} \vdash s(C \sqcup D) \Longrightarrow C \sqcup D.$$

First, by induction hypothesis we have:

$$\mathbf{CPL} \vdash s(C) \Longrightarrow C$$

and

$$\mathbf{CPL} \vdash s(D) \Longrightarrow D.$$

Applying twice $R\sqcup$ we get:

$$\mathbf{CPL} \vdash s(C) \Longrightarrow C \sqcup D$$

and

$$\mathbf{CPL} \vdash s(D) \Longrightarrow C \sqcup D.$$

And from that by $L\sqcup$:

$$\mathbf{CPL} \vdash s(C) \sqcup s(D) \Longrightarrow C \sqcup D.$$

Finally, an application of CUT to the **CPL**-axiom

$$\sim(\sim s(C) + s(D)) + s(D) \Longrightarrow s(C) \sqcup s(D)$$

and to the last obtained sequent above yields:

$$\sim(\sim s(C) + s(D)) + s(D) \Longrightarrow C \sqcup D.$$

Since $s(C \sqcup D) = \sim(\sim s(C) + s(D)) + s(D)$ by definition, we are done. ■

Finally, we are prepared to prove the claim mentioned earlier, and now stated by

PROPOSITION 5.7. *A given* $\mathbf{PL}_n$*-sequent* $\Gamma \Longrightarrow \Delta$ *is a theorem of* $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$ *if and only if* $\mathbf{CPL} \vdash \Gamma \Longrightarrow \Delta$.

PROOF. First, assume that a sequent $A_1, \ldots, A_m \Longrightarrow B_1, \ldots, B_n$ is a theorem of $\bigcap_{n=2}^{\infty} \mathbf{CPL}_n$. Thus, by soundness, see proposition 5.1, the given sequent is $Q \bigcap [0,1]$-valid. Using the fact that $v(A \sqcup B) = v(\sim(\sim A + B) + B)$ for all v based on $Q \bigcap [0,1]$, we know that the sequent

$$s(A_1), \ldots, s(A_m) \Longrightarrow s(B_1), \ldots, s(B_n)$$

remains $Q \bigcap [0,1]$-valid. Hence, by proposition 5.3,

$$\mathbf{GW} \vdash s(A_1), \ldots, s(A_m) \Longrightarrow s(B_1), \ldots, s(B_n).$$

And therefore, by lemma 5.4, also

$$\mathbf{CPL} \vdash s(A_1), \ldots, s(A_m) \Longrightarrow s(B_1), \ldots, s(B_n).$$

Due to lemma 5.6, we can now successively apply CUT to the last obtained sequent and to one of the **CPL**-theorems $A_i \Longrightarrow s(A_i)$ and $s(B_j) \Longrightarrow B_j$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, which yields:

$$\mathbf{CPL} \vdash A_1, \ldots, A_m \Longrightarrow B_1, \ldots, B_n.$$

One direction of the proposition has thus been verified.
The other direction is left to the reader. ■

6. Conclusion

The time has come to offer the reader a final discussion. First, observe, that our last proposition has been established essentially due to the fact that $A \sqcup B$ and $\sim(\sim A + B) + B$ coincide in the proposed models. This, moreover, shows that $\sqcup$ can be defined in terms of $+$ and $\sim$ within the systems that are complete for the indicated class of models. Thus, the above introduced systems $\mathbf{CPL}_n$ $(n \geq 2)$ and **CPL**, in fact, present Gentzen-style formulation in a linear setting of logics equivalent to finite and infinite-valued Łukasiewicz logics respectively. As far as we know, only Grishin [8] and Avron [1] obtained a Gentzen-style formulation for the 4-valued and the 3-valued Łukasiewicz logics respectively. It might be interesting to recall that Grishin's axiomatization includes restricted contraction rules, as well. In particular, in his system, but in our notation, only the formulas in one of the following forms: X^3, $(X+X^2)^2$ or $X+X^2+2(\sim X)^2$ may be contracted. On the other hand, Avron's cut-free axiomatization is based on a calculus of hypersequents due to the fact that the usual full contraction rule is not valid for the consequence relation introduced there.

Let us mention a particular perspicuity of our formulation of the logics under consideration. In Łukasiewicz, Tarski [9], the authors especially emphasized that the introduced systems, nowadays referred to as Łukasiewicz logics, are only proper subsystems of propositional logic. From the formulations obtained in this paper and the fact that $\sqcup$ is definable in terms of $+$ and $\sim$, the following becomes evident. These logics are fragments of purely multiplicative propositional logic where essentially in the $(n+1)$-valued case n-contraction is substituted for full contraction, while the infinite-valued Łukasiewicz logic is contraction free.

To sum up, the main contribution of this paper is to present a simple proof of the finite axiomatizability of Łukasiewicz logics resulting in a Gentzen-style formulation.

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