Andrzej W. Jankowski

# Universality of the Closure Space of Filters in the Algebra of All Subsets

**Abstract.** In this paper we show that some standard topological constructions may be fruitfully used in the theory of closure spaces (see [5], [4]). These possibilities are exemplified by the classical theorem on the universality of the Alexandroff's cube for $T_0$-closure spaces. It turns out that the closure space of all filters in the lattice of all subsets forms a "generalized Alexandroff's cube" that is universal for $T_0$-closure spaces. By this theorem we obtain the following characterization of the consequence operator of the classical logic: If $\mathscr{L}$ is a countable set and $C\colon \mathscr{P}(\mathscr{L}) \to \mathscr{P}(\mathscr{L})$ is a closure operator on $X$, then $C$ satisfies the compactness theorem iff the closure space $\langle \mathscr{L}, C\rangle$ is homeomorphically embeddable in the closure space of the consequence operator of the classical logic.

We also prove that for every closure space $X$ with a countable base such that the cardinality of $X$ is not greater than $2^{\omega}$ there exists a subset $X'$ of irrationals and a subset $X''$ of the Cantor's set such that $X$ is both a continuous image of $X'$ and a continuous image of $X''$.

We assume the reader is familiar with notions in [5].

## 1. Product in the category of $\langle \alpha, \delta\rangle$-closure spaces and continuous functions

Let $\alpha$ be a regular cardinal number or the "power" of a proper class and let $\delta$ be an infinite regular cardinal number or the "power" of a proper class.

Note that:

PROPOSITION 1. *Assume that:*

— *$X$ is a nonempty set,*
— *for every $i \in I$, $X_i = \langle X_i, \mathscr{F}_i\rangle$ is an $\langle \alpha, \delta\rangle$-closure space,*
— *for every $i \in I$, $f_i$ is a function from the set $X$ into the set $X_i$.*

*Then there is an $\langle \alpha, \delta\rangle$-closure space $X = \langle X, \mathscr{F}\rangle$ such that:*

— *for every $i \in I$ the function $f_i$ is continuous,*
— *if $X' = \langle X, \mathscr{F}'\rangle$ is an $\langle \alpha, \delta\rangle$-closure space such that for every $i \in I$ the function $f_i$ is continuous, then $\mathscr{F} \subseteq \mathscr{F}'$.*

PROOF. Note that if a family $\mathscr{B}$ is such that:

$$\mathscr{B} = \{Z \subseteq X \mid (\exists i)_I (\exists F)_{\mathscr{F}_i} \quad Z = \overleftarrow{f}(F)\},$$

then the closure space $X = \langle X, \mathscr{F}\rangle$ with an $\langle \alpha, \delta\rangle$-base satisfies Proposition 1. □

From Proposition 1 we have that for every family $\{X_i\}_{i\in I}$ of $\langle\alpha, \delta\rangle$-closure spaces there is a *product of the family* $\{X_i\}_{i\in I}$ *in the category of all* $\langle\alpha, \delta\rangle$*-closure spaces and continuous maps*, denoted by

$$\mathsf{P}^{\alpha,\delta}_{i\in I} X_i.$$

If for every $i\in I$ the $\langle\alpha, \delta\rangle$-closure space $X_i = \langle X, \mathscr{F}\rangle = X$ and $\overline{\overline{I}} = \mathfrak{m}$, then let

$$[X]^{\mathfrak{m}}_{\alpha,\delta} = \mathsf{P}^{\alpha,\delta}_{i\in I} X_i.$$

Note that:

PROPOSITION 2. *For every infinite cardinal number* $\mathfrak{m}$ *and for every* $\langle\alpha, \delta\rangle$*-closure space* $X$ *the closure space* $[X]^{\mathfrak{m}}_{\alpha,\delta}$ *is homeomorphic to the closure space* $[[X]^{\mathfrak{m}}_{\alpha,\delta}]^{\mathfrak{m}}_{\alpha,\delta}$. □

PROPOSITION 3. *Let* $\{X_i\}_{i\in I}\cup\{X\}$ *be a family of* $\langle\alpha, \delta\rangle$*-closure spaces and let* $\{f_i\}_{i\in I}$ *be a family of continuous functions such that for every* $i\in I$

$$f_i\colon X\to X_i.$$

*Then the diagonal function* $\mathop{\Delta}\limits_{i\in I} f_i\colon X\to \mathsf{P}^{\alpha,\delta}_{i\in I} X_i$ *is continuous, where*

$$\mathop{\Delta}\limits_{i\in I} f_i(x) = \langle f_i(x)\rangle_{i\in I}. \quad \square$$

Let $\mathscr{R}$ be a family of subsets of a nonempty set $X$ and for $i\in I$ let

$$f_i\colon X\to X_i.$$

Moreover we assume that for $i\in I$, $C_i$ is a proper closure operator for a closure space $X_i = \langle X_i, \mathscr{F}_i\rangle$ and $C$ is a proper closure operator for a closure space $\langle X, \mathscr{F}\rangle$. We will say that the *family* $\{f_i\}_{i\in I}$ *separates the family* $\mathscr{R}$ provided that for every $x\in X$, if for every non-empty set $R\in\mathscr{R}$ and for every $i\in I$ we have:

$$f_i(x)\in C_i\big(\vec{f_i}(R)\big).$$

then $x\in C(R)$.

The *family* $\{f_i\}_{i\in I}$ *separates points* provided that for every $x, y\in X$, if for every $i\in I$

$$f_i(x) = f_i(y),$$

then $x = y$.

Note that:

LEMMA 1. *A family $\{f_i\}_{i\in I}$ separates points iff $\Delta_{i\in I} f_i$ is a one-to-one mapping.* □

LEMMA 2. *A function $f$ is a homeomorphic embedding from a closure space $X_1$ into a closure space $X_2$ iff $f$ is a continuous and one-to-one mapping such that the family $\{f\}$ separates all closed sets in $X_1$.* □

Moreover we can prove the following:

LEMMA 3. *Let $\mathscr{B}$ be an $\langle\alpha, \delta\rangle$-base for the closure space $X$ and let $\{f_i\}_{i\in I}$ be a family of functions such that for every $i \in I$*

$$f_i\colon X \to X_i,$$

*where $X_i$ is an $\langle\alpha, \delta\rangle$-closure space. If the family $\{f_i\}_{i\in I}$ separates the family $\mathscr{B}$, then the family $\{\Delta_{i\in I} f_i\}$ separates the family of all closed sets in $X$.*

PROOF. Put $f = \Delta_{i\in I} f_i$ and $X_1 = \mathsf{P}^{\alpha,\delta}_{i\in I} X_i$. Moreover we assume that:

(i) $\mathscr{B}' = \{F \mid (\exists \mathscr{R})(\overline{\overline{\mathscr{R}}} < \alpha \,\&\, \mathscr{R} \subseteq \mathscr{B} \,\&\, F = \cup \mathscr{R}) \vee F \in \{\emptyset, X\}\}$,

(ii) $\mathscr{B}'' = \{F \mid (\exists \mathscr{R})(\mathscr{R} \subseteq \mathscr{B}' \,\&\, F = \cap \mathscr{R})\}$,

(iii) $\mathscr{B}''' = \{F \mid (\exists \mathscr{R})(\mathscr{R} \subseteq \mathscr{B}'' \,\&\, \mathscr{R}$ is a $\delta$-directed family $\&\, F = \cup \mathscr{R})\}$

By Theorem 1 (see [4]) $\mathscr{B}'''$ is the set of all closed sets in the $\langle\alpha, \delta\rangle$-closure space $X$. To prove Lemma 3 it is sufficient to prove that if $\mathscr{R} \in \{\mathscr{B}', \mathscr{B}'', \mathscr{B}'''\}$, then the family $\{f\}$ separates the family $\mathscr{R}$.

1° Let $\mathscr{R} = \mathscr{B}'$. Assume that $x \notin F = \bigcup_{k\in K} F_k$, where $\overline{\overline{K}} < \alpha$ and for every $k \in K$ we have $F_k \in \mathscr{B}$.

For every $k \in K$ there is an $i_0 \in I$ such that

$$f_{i_0}(x) \notin C_{i_0}\big(\vec{f}_{i_0}(F_k)\big).$$

Hence for every $k \in K$

$$f(x) \notin C_1\big(\vec{f}(F_k)\big).$$

It means that

$$f(x) \notin \bigcup_{k\in K} C_1\big(\vec{f}(F_k)\big).$$

Note that:

$$\bigcup_{k\in K} \vec{f}(F_k) \subseteq \bigcup_{k\in K} C_1\big(\vec{f}(F_k)\big)$$

and the set

$$\bigcup_{k\in K} C_1\big(\vec{f}(F_k)\big)$$

is closed in the $\alpha$-closure space $X_1$. Thus

$$x \notin C_1\big(\bigcup_{k\in K} \vec{f}(F_k)\big).$$

$2^0$ Let $\mathscr{R} = \mathscr{B}''$. Assume that $x \notin F = \bigcap_{l\in L} F_l$, where for every $l \in L$ we have $F_l \in \mathscr{B}'$. By $1^0$ there is an $l_0 \in L$ and an $i_0 \in I$ such that:

$$f_{i_0}(x) \notin C_{i_0}\big(\vec{f}_{i_0}(F_{l_0})\big).$$

It means that:

$$f(x) \notin C_1\big(\vec{f}(F_{l_0})\big).$$

Hence

$$f(x) \notin \bigcap_{l\in L} C_1\big(\vec{f}(F_l)\big).$$

Of course

$$C_1\big(\vec{f}(\bigcap_{l\in L} F_l)\big) \subseteq \bigcap_{l\in L} C_1\big(\vec{f}(F_l)\big).$$

Thus

$$f(x) \notin C_1\big(\vec{f}(\bigcap_{l\in L} F_l)\big).$$

$3^0$ Let $\mathscr{R} = \mathscr{B}'''$. Assume that $\{F_p\}_{p\in P}$ is a $\delta$-directed family such that $x \notin \bigcup_{p\in P} F_p$ and for every $p \in P$ we have $F_p \in \mathscr{B}''$.

By $2^0$ we have that for every $p \in P$

$$f(x) \notin C_1\big(\vec{f}(F_p)\big).$$

Hence

$$f(x) \notin \bigcup_{p\in P} C_1\big(\vec{f}(F_p)\big).$$

$X_1$ is a $\delta$-inductive closure space and of course

$$C_1\big(\bigcup_{p\in P} \vec{f}(F_p)\big) \subseteq \bigcup_{p\in P} C_1\big(\vec{f}(F_p)\big).$$

Thus

$$f(x) \notin C_1\big(\bigcup_{k\in K} \vec{f}(F_k)\big) = C_1\big(\vec{f}(\bigcup_{k\in K} F_k)\big).$$

This ends the proof of Lemma 3. □

We assume that for a cardinal number $\mathfrak{n}$, the symbol $l(\mathfrak{n})$ denotes the least cardinal number $\mathfrak{m}$ such the one can find $\mathscr{R} \subseteq \mathscr{P}(\mathfrak{n})$ with $\overline{\overline{\mathscr{R}}} = \mathfrak{m}$ and such that for every $x, y \in \mathfrak{n}$ there exists an $R \in \mathscr{R}$ with $\overline{\overline{\{x, y\} \cap R}} = 1$. For example $l(3) = l(4) = 2$, $l(5) = l(6) = l(7) = l(8) = 3, \ldots$etc.

PROPOSITION 4. *For every cardinal number* $\mathfrak{m}$ *we have*

$$l(2^{\mathfrak{m}}) = \mathfrak{m}.$$

PROOF. For a family $\mathscr{R} \subseteq \mathscr{P}(X)$ let

$$\mathscr{F}_{\mathscr{R}} = \{\cap \mathscr{B} \mid \mathscr{B} \subseteq \mathscr{R}\}.$$

Of course $\overline{\overline{\mathscr{F}_{\mathscr{R}}}} \leqslant 2^{\overline{\overline{\mathscr{R}}}}$. Moreover if the family $\mathscr{R}$ is such that for every $x, y \in X$ there exists an $R \in \mathscr{R}$ with $\overline{\overline{\{x, y\} \cap R}} = 1$, then $\langle X, \mathscr{F}_{\mathscr{R}} \cup \{\emptyset, X\}\rangle$ is a $T_0$-closure space. In this case $\overline{\overline{X}} \leqslant \overline{\overline{\mathscr{F}_{\mathscr{R}}}} \leqslant 2^{\overline{\overline{\mathscr{R}}}}$. Thus for every cardinal number $\mathfrak{n}$ we have $\mathfrak{n} \leqslant 2^{l(\mathfrak{n})}$. If $\mathfrak{n} = 2^{\mathfrak{m}}$ we obtain that $\mathfrak{m} \leqslant l(2^{\mathfrak{m}})$.

On the other hand note that if for a set $T$ a family $\mathscr{R}$ is such that

$$\mathscr{R} = \{F_t \subseteq \mathscr{P}(T) | F_t = \{Z \subseteq T \mid t \in Z\} \ \& \ t \in T\},$$

Then $\overline{\overline{\mathscr{R}}} = \overline{\overline{T}}$ and for every $U, V \subseteq T$ if $t \in U - V$, then $U \in F_t$ and $V \notin F_t$. Thus for the set $\mathscr{P}(T)$ there exists a family $\mathscr{R} \subseteq \mathscr{P}(\mathscr{P}(T))$ with $\overline{\overline{\mathscr{R}}} = \mathfrak{m}$ and such that for every $U, V \in \mathscr{P}(T)$ there exists an $R \in \mathscr{R}$ with $\overline{\overline{\{U, V\} \cap R}} = 1$. It means that $l(2^{\mathfrak{m}}) \leqslant \mathfrak{m}$. □

Let $Y$ be the closure space such that (see [6] pp. 114):

$$Y = \langle\{0, 0', 1, 1'\}, \{\emptyset, \{1, 1'\}, \{0, 0', 1, 1'\}\}\rangle.$$

By Lemma 2 and Lemma 3 we have the following theorem:

THEOREM 1. *For every closure space* $X$ *and cardinal number* $\mathfrak{n}$ *the following conditions are equivalent*:

(i) $X$ *is an* $\langle \alpha, \delta\rangle$*-closure space such that* $w_{\alpha,\delta}(X) \leqslant \mathfrak{n}$ *and* $l(\overline{\overline{X}}) \leqslant \mathfrak{n}$,

(ii) $X$ *is homeomorphically embeddable into the closure space* $[Y]^{\mathfrak{n}}_{\alpha,\delta}$.

PROOF. By Corollary in [5] the implication (ii) $\Rightarrow$ (i) is obvious. To prove that (i) $\Rightarrow$ (ii) assume that $\mathscr{B}$ is an $\langle \alpha, \delta\rangle$-base for $X$ and $\mathscr{R} \subseteq \mathscr{P}(X)$ is such that for every $x, y \in X$ there is an $R \in \mathscr{R}$ such that $\overline{\overline{\{x, y\} \cap R}} = 1$ and $\overline{\overline{\mathscr{B}}}, \overline{\overline{\mathscr{R}}} \leqslant \mathfrak{n}$. Without loss of generality we can assume that $\mathscr{R} = \{R_i\}_{i \in I}$, $\mathscr{B} = \{B_i\}_{i \in I}$ and $\overline{\overline{I}} = \mathfrak{n}$.

For $i \in I$ let $f_i\colon X \to Y$ be such that

$$f_i(x) = \begin{cases} 1 & \text{if } x \in B_i \cap R_i, \\ 1' & \text{if } x \in B_i - R_i, \\ 0 & \text{if } x \in R_i - B_i, \\ 0' & \text{if } x \notin B_i \cup R_i. \end{cases}$$

By Lemma 3 and Lemma 2 the function $\mathop{\Delta}\limits_{i \in I} f_i$ is a homeomorphic embedding of $X$ into $[Y]^{\mathfrak{n}}_{\alpha,\delta}$. □

## 2. Closure space of filters in a lattice of sets

Let $Y_0$ be the closure space such that:

$$Y_0 = \langle \{0,1\}, \{\emptyset, \{1\}, \{0,1\}\} \rangle.$$

The closure space $[Y_0]^{\mathfrak{n}}_{\alpha,\delta}$ is homeomorphic to the contraction of the closure space $[Y]^{\mathfrak{n}}_{\alpha,\delta}$ (i.e. $[Y_0]^{\mathfrak{n}}_{\alpha,\delta} = [Y]^{\mathfrak{n}}_{\alpha,\delta}[\emptyset]$). Hence every $T_0$ $\langle \alpha, \delta \rangle$--closure space is homeomorphically embeddable into the respective closure space $[Y_0]^{\mathfrak{n}}_{\alpha,\delta}$. Let $B^{\mathfrak{n}}_{\alpha,\delta} = \langle \mathscr{P}(\mathfrak{n}), \mathscr{R} \rangle$ be the $\langle \alpha, \delta \rangle$-closure space such that:

$\mathscr{P}(\mathfrak{n})$ is the family of all subsets of $\mathfrak{n}$, the $\langle \alpha, \delta \rangle$-base for $\langle \mathscr{P}(\mathfrak{n}), \mathscr{R} \rangle$ is the set of all complete ultrafilters in the lattice $\langle \mathscr{P}(\mathfrak{n}), \subseteq \rangle$.

Note that:

PROPOSITION 5. $B^{\mathfrak{n}}_{\alpha,\delta} = [Y_0]^{\mathfrak{n}}_{\alpha,\delta}$. □

By Proposition 5, Lemma 2 and Lemma 3 we have

THEOREM 2 (see [3]). *For every closure space $X$ and cardinal number $\mathfrak{n}$ the following conditions are equivalent*:

(i) *$X$ is a $T_0$ $\langle \alpha, \delta \rangle$-closure space such that $w_{\alpha,\delta}(X) \leqslant \mathfrak{n}$,*

(ii) *$X$ is homeomorphically embeddable into the closure space $B^{\mathfrak{n}}_{\alpha,\delta}$.* □

Let $\mathfrak{m} \geqslant \omega$ and let $F_{\mathfrak{m}} = \langle \mathscr{L}, \mathscr{T} \cup \{\emptyset\} \rangle$ be the closure space such that:

$\mathscr{L}$ — is the set of all propositional sentences of the classical propositional calculus with a set of propositional variables of power $\mathfrak{m}$.

$\mathscr{T}$ — is the set of all theories in the language $\mathscr{L}$ — for the classical propositional calculus.

Observe that the proper closure operator for $F_{\mathfrak{m}}$ is the consequence operator for classical propositional calculus. Of course the contraction $F_{\mathfrak{m}}[\emptyset]$ is the closure space of all filters in the free Boolean algebra of power $\mathfrak{m}$. By Theorem 2 we have:

THEOREM 3 (see [3]). *For every closure space $X$ the following conditions are equivalent:*

(i) *$X$ satisfies the compactness theorem and $X$ is a countable closure space,*

(ii) *$X$ is homeomorphically embeddable in $F_\omega$.*

PROOF. Note that for every $a \in F^\omega$ the set

$$\{x \mid C(\{a\}) = C(\{x\})\}$$

(where $C$ is the proper closure operator for $F_\omega$) is infinite. Hence if $X[\emptyset]$ is homeomorphically embeddable into the closure space $F_\omega$, then $X$ is homeomorphically embeddable into the closure space $F_\omega$ too. Thus by the corollary in [3] if $X$ is a countable closure space which satisfies the $\omega$-compactness theorem then $X$ is homeomorphically embeddable in $F_\omega$. □

If $X$ is an uncountable closure space and $F_\mathfrak{m}$ is uncountable, then Theorem 3 is not true. For the proof of this remark let $D_{\omega_1}$ be the closure space such that:

$$D_{\omega_1} = \langle \omega_1, \{\{t\}, \{0, t\}\}_{t \in \omega_1 - \{0\}} \cup \{\emptyset, \omega_1\}\rangle.$$

REMARK 1. For every cardinal number $\mathfrak{m}$ the closure space $D_{\omega_1}$ is not homeomorphically embeddable into the closure space $F_\mathfrak{m}$.

PROOF. Assume that $f\colon D_{\omega_1} \to F_\mathfrak{m}$ is such that $D_{\omega_1}$ is homeomorphically embeddable in $F_\mathfrak{m}$ by $f$. Without loss of generality we can assume that $f(0)$ is an antitautology in $F_\mathfrak{m}$ (if $f(0)$ is not an antitautology in $F_\mathfrak{m}$, then we can put $f'(x) = f(x) - f(0)$).

By Theorem 5 (ii) (see [5]) for every $x, y \in \omega_1 - \{0\}$ such that $x \neq y$ the conjunction of $\{f'(x), f'(y)\}$ in $F_\mathfrak{m}$ is an antitautology in $F_\mathfrak{m}$. Thus the Boolean algebra $\mathfrak{a}'$ generated by $\vec{f'}(\omega_1)$ is isomorphic to the Boolean algebra of finite and cofinite subsets in the set $\omega_1$. It means that $\mathfrak{a}'$ is isomorphically embeddable in $F_\mathfrak{m}$. But the Stone space of $\mathfrak{a}'$ (i.e. the Alexandroff compactification for the discrete space of power $\omega_1$) is not a continuous image of the Stone space of $F_\mathfrak{m}[\emptyset]$ — i.e. the Cantor cube (see. [1]). □

We can give some modifications of Theorem 3 for uncountable cardinal numbers. In a way similar to the proof of Theorem 3 we can prove the following theorems:

THEOREM 4. *For every closure space $X$ and infinite cardinal number $\mathfrak{m}$ the following conditions are equivalent:*

(i) *$X$ satisfies the compactness theorem and the power of $X[\emptyset]$ is not greater than $\mathfrak{m}$,*

(ii) *there is a subset $T$ of $F_{\mathfrak{m}}$ such that $X[\emptyset]$ is homeomorphically embeddable in $F_{\mathfrak{m}}[T]$.*

PROOF. Of course every Boolean algebra is the image of a free Boolean algebra. □

THEOREM 5. *If we have GCH, then for every infinite cardinal number $\mathfrak{m}$ there is a subset $T$ in $F_{\mathfrak{m}}$ such that for every closure space $X$ the following conditions are equivalent:*

(i) *$X$ satisfies the compactness theorem and the power of $X[\emptyset]$ is not greater than $\mathfrak{m}$,*

(ii) *$X[\emptyset]$ is homeomorphically embeddable in $F_{\mathfrak{m}}[T]$.*

PROOF. If we have *GCH*, then we can prove that for every cardinal number $\mathfrak{m} \geqslant \omega$ there is a Boolean algebra $\mathfrak{a}$ such that every Boolean algebra of power not greater than $\mathfrak{m}$ is isomorphically embeddable in $\mathfrak{a}$ and the power of $\mathfrak{a}$ is $\mathfrak{m}$ (see [2]). □

REMARK 2. Observe that there are many examples of consequence operators $C$ for non-classical logics such that the consequence operator for the classical propositional logic is embeddable into $C$. For example $F_{\omega}$ is embeddable into the consequence operator for intuitionistic logic, modal logic, Post logic etc. (see [7]). Hence we can give many modifications of Theorems 3, 4 and 5. □

Let $D$ be the closure space such that

$$D = \langle\{0,1\}, \{\{0\}, \{1\}, \{0,1\}, \emptyset\}\rangle.$$

Of course $[D]^{\omega}_{\omega,\infty}$ is the Cantor space (see [1]) and if $a \leqslant a'$, $\delta \geqslant \delta'$, $\omega \leqslant \mathfrak{n} \leqslant \mathfrak{n}'$, then the closure space $[Y]^{\mathfrak{n}}_{a,\delta}$ is a continuous image of the closure space $[D]^{\mathfrak{n}'}_{a',\delta'}$. Thus by Theorem 1 we have:

THEOREM 6. *Assume that $a' \geqslant a$, $\delta' \leqslant \delta$ and $\omega \leqslant \mathfrak{n} \leqslant \mathfrak{n}'$. Then for every $\langle a, \delta\rangle$-closure space $Y$ such that $w_{a,\delta}(Y) \leqslant \mathfrak{n}$ and $l(\overline{\overline{Y}}) \leqslant \mathfrak{n}$ there is a subset $Y'$ in the closure space $[D]^{\mathfrak{n}'}_{a',\delta'}$, such that $Y$ is a continuous image of $Y'$. In particular for every closure space $X$ such that $w_{0,\infty}(X) \leqslant \omega$ and $\overline{\overline{X}} \leqslant 2^{\omega}$ there is a subset $X'$ of the Cantor space (irrational numbers, real numbers) such that $X$ is a continuous image of $X'$.* □

REMARK 3. Note that Theorem 6 is a modification of Theorem III.9.4 in [6]. Moreover let a *closure Boolean algebra* be a Boolean algebra $\mathscr{A}$ with an operation $I$ which, to every element $a \in \mathscr{A}$, assigns an element $Ia$ in such a way that the following axioms are satisfied:

$$Ia \leqslant a,$$
$$IIa = Ia,$$
$$I1_{\mathscr{A}} = 1_{\mathscr{A}}.$$
$$\text{if } a \leqslant b, \text{ then } Ia \leqslant Ib$$

We can give now a modification of the theory of topological Boolean algebras (see III in [6], VI in [7]) and its application in the algebraic semantics of modal logic (see *X* in [6], XIII in [7] and Supplement in [7]). □

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Institute of Mathematics
Warsaw University



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