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Algebraization of Quantifier Logics, an Introductory Overview

Abstract. This paper is an introduction: in particular, to algebras of relations of various ranks, and in general, to the part of algebraic logic algebraizing quantifier logics. The paper has a survey character, too. The most frequently used algebras like cylindric–, relation–, polyadic–, and quasi-polyadic algebras are carefully introduced and intuitively explained for the nonspecialist. Their variants, connections with logic, abstract model theory, and further algebraic logics are also reviewed. Efforts were made to make the review part relatively comprehensive. In some directions we tried to give an overview of the most recent results and research trends, too.

Contents

Research supported by Hungarian National Foundation for Scientific Research grant No. 1810.

1. Introduction

The research reviewed in this paper has been going on for approximately 130 years, but here we will almost completely ignore the historical aspects, because instead we wanted to concentrate on making the paper easily readable for the beginners. The historical aspects are treated e.g. in [HMTII], [TG87], Anellis–Houser [AH89], [Gi89], and [JSL]. The subject of this paper could be called, perhaps, first (and higher) order algebraic logic, or perhaps trans–propositional algebraic logic (or algebra of relations of various ranks).

This paper was written for a variety of readers ranging from the beginner to the specialist. In particular, *no* familiarity with algebraic logic is presupposed, moreover, we tried to avoid relying on familiarity with logic itself as much as we could. The reader not familiar with logic should be able to understand the paper modulo ignoring certain remarks. He/she is advised to view the paper as a study of relations of various ranks, and as that of *algebras whose elements are such relations*. Familiarity with the basics of naïve set theory and some basic concepts of universal algebra should suffice.

A pleasant, richly motivated introduction to the subject is found in Halmos [Ha85] pp. 206–215, which seems to be complementary to the present paper.[1]

When discussing the algebraizations of logics, we treated the semantic aspects (models, meanings of formulas in models etc.) of logics as primary, and the syntactic aspects (provability etc.) as secondary. This does not represent value judgment, nor does it mean that the syntactic aspects will be ignored, but instead it has two purposes. (1) We had to choose a unifying principle making the paper coherent and easy to read for the nonspecialist. (2) This way the main bulk of the paper will be understandable and appreciable for readers not too familiar with logic, we hope.

In §§2–5 we gradually introduce, and carefully explain the most frequently used algebras of this area: Cylindric algebras, relation algebras, and (quasi-) polyadic algebras (not necessarily in this order). At the same time, but in less detail, we discuss other algebras suitable for algebraizing unusual quantifier logics, e.g. fragments[2] of $L_{\omega\omega}$, higher order logic, many-sorted logic etc. In §6 we begin to speed up, and discuss algebraic counterparts of infinitary logics, and other abstract model theoretic logics with an emphasis on polyadic algebras. §7 is a quick overview of other approaches, where the change comes either from the method of algebraization, or from the choice of logic to be algebraized (roughly speaking). This overview also touches upon connections with computer science, and category theoretic logic. §7 is also intended to be a continuation of the similar overview at the end of [HMTII]. §8 (Appendix) deals with the algebraic counterparts of the syntactic aspects of logic. (These are the very abstract cylindric (etc.) algebras which need not be isomorphic with any kind of concrete algebras of relations.) In the paper, there is a further item (describing a special topic, and) running across the various sections, beginning with Remark 2 in §3, and recurring in each subsequent section. This item is devoted to an area full of intriguing famous problems which might be rather rewarding to work on. The area is known as "finitizability problem" or "finitization" for short, and became quite active recently. (The beginner may safely omit the parts dealing with finitization, at a first reading.)

Finally we should point out that because of the introductory character of this paper, there are several central areas of trans–propositional algebraic logic which are not even mentioned here. Therefore, having read this paper, it is advisable to leaf through [HMTII] and then [TG87]. From now on,

[1] For completeness, we note that the up-to-date basic source book for "first-order (and related) algebraic logic" (i.e. for the subject matter of this paper) is [HMTII] supplemented with [HMT] and [TG87]. ([HMTII] serves also as the standard reference monograph.)
[2] E.g. the positive fragment also called coherent fragment in category theoretic logic.

we will simply write algebraic logic instead of trans–propositional algebraic logic.

2. Getting acquainted with the subject

The algebraization of classical propositional logic, yielding Boolean algebras (BA's), was immensely successful. It is fairly well understood how to extend this algebraization to other propositional logics (Blok–Pigozzi [BP89]). These new algebraizations were also very successful. What happens then if we want to extend the original algebraization yielding BA's to first-order logic?

What are BA's? They are algebras of unary relations. I.e. the elements of a BA $\mathfrak{B}$ are unary relations and the operations of $\mathfrak{B}$ are the natural operations on unary relations e.g. intersection, complementation. The problem of extending this approach to predicate logics boils down to the problem of expanding the natural algebras of unary relations to natural algebras of relations of higher ranks, i.e. of relations in general. The reason for this is, roughly speaking, the fact that the basic building blocks of predicate logics are predicates, and the meanings of predicates can be relations of arbitrary ranks. Indeed, already in the middle of the last century, when De Morgan wanted to generalize algebras of propositional logic in the direction of what we would call today predicate logic, he turned to algebras of binary relations.[3] That was probably the beginning of the quest for algebras of relations in general. Returning to this quest, the new algebras will, of course, have more operations than BA's, since between relations in general there are more kinds of connections than between unary relations (e.g. one relation might be the converse, sometimes called inverse, of the other).

The *framework* for the quest for the natural algebras of relations *is universal algebra*. The reason for this is that universal algebra is the field which investigates classes of algebras in general, their interconnections, their fundamental properties etc. Therefore universal algebra can provide us for our search with a "map and a compass" to orient ourselves. There is a further good reason for using universal algebra. Namely, universal algebra is not only a unifying framework, but it also contains powerful theories. E.g. if we know in advance some general properties of the kinds of algebras we are going to investigate, then universal algebra can reward us with a powerful

[3]De Morgan illustrated the need for expanding the algebras of unary relations (i.e. BA's) to algebras of relations in general (the topic of the present paper) by saying that the scholastics, after two millennia of Aristotelian tradition, were still unable to prove that if a horse is an animal, then a horse's tail is an animal's tail. ("v_0 is a tail of v_1" is a binary relation.)

machinery for doing these investigations. Among the special classes of algebras concerning which universal algebra has powerful theories are the so called *discriminator varieties* and the *arithmetical varieties* (cf. the textbooks [BS81], [MMT], [W78]). At the same time, algebras originating from logic turn out to fall in one of these two categories, in most cases. More concretely, more than half of these algebras are in discriminator varieties and almost all are in arithmetical ones. Certainly, all the algebras studied in the present paper are in arithmetical varieties. Therefore, awareness of these recent parts of universal algebra can be rewarding in algebraic logic. In order to preserve the introductory character of this paper, we will not assume familiarity with these theories of universal algebra. From time to time we will point out places where these theories can be usefully applied but these parts can safely be ignored.

Let us return to our task of moving from BA's of unary relations to expanded BA's of relations in general. What are the elements of a BA? They are sets of "points". What will be the elements of the expanded new algebras? One thing about them seems to be certain, they will be sets of sequences. Why? Because relations in general are sets of sequences. These sequences may be just pairs if the relation is binary, they may be triples if the relation is ternary, or they may be longer — or more general kinds of sequences. (There is another consideration pointing in the direction of sequences. Namely, the semantics of quantifier logics is defined via satisfaction of formulas in models, which in turn is defined via evaluations of variables, and these evaluations are sequences. The meaning of a formula in a model is the *set of* those *sequences* which satisfy the formula in that model. So we arrive again at sets of sequences.) So, one thing is clear at this point, namely that the elements of our expanded BA's of relations will be sets of sequences. Indeed, this applies to all known algebraizations of predicate logics or quantifier logics, e.g. for cylindric algebras, quasi-polyadic algebras, polyadic algebras, Craig's algebras, modal-cylindric algebras, algebraic counterparts of nonclassical predicate logics, higher order logics, abstract model theoretic logics, see [HMTII] §5.6 pp. 263–271.

At this point it might be useful to point out that the most obvious approach based on the above observation (that the elements of the algebra are sets of sequences) does not seem to work, at least not without some fine-tuning. So, what is the most obvious approach? Consider some set U; let ${}^{<\omega}U$ denote the set of all finite sequences over U, and consider the BA $\mathcal{P}({}^{<\omega}U)$ (the powerset of ${}^{<\omega}U$ conceived as a BA the standard way). Now if we are given any finitary relation, say, $R \subseteq U \times U$ over U, then $R \in \mathcal{P}({}^{<\omega}U)$. So $\mathcal{P}({}^{<\omega}U)$ contains all relations over U independently of their ranks. Therefore it might be a candidate for being the universe of an algebra of relations. Before thinking about what the new, so called extra-Boolean operations on

$\mathcal{P}({}^{<\omega}U)$ should be, let us have another look at its Boolean structure: If R is a binary relation, we would like to obtain its complement $(U \times U) \setminus R$ as a result of applying a Boolean operation to R. However, in our algebra $\mathcal{P}({}^{<\omega}U)$ the complement of R is not $(U \times U) \setminus R$ but something infinitely bigger, so this approach does not seem to work very smoothly without putting some extra effort into it. With further work it can be turned into a fruitful approach to algebraizing logic, see §7 (2–4) and the section containing Facts 2, 3 at the end of §4 herein; see also §5.6.(A survey).3 on p. 265 of [HMTII] and the references therein. The approach originates with Craig, but already the algebras in Quine [Q36] consist of sets of finite sequences.

3. Binary relations

The above difficulty with $\mathcal{P}({}^{<\omega}U)$ motivates our concentrating first on the simplest nontrivial case, namely that of the *algebras of binary relations* (BRA's). Actually, BRA's will be strong enough to be called a truly first-order (as opposed to propositional) algebraic logic, namely [TG87] §5.3 shows that the logic captured by BRA's is strong enough to serve as a vehicle for set theory and hence for ordinary metamathematics.

The full BRA over a set U is defined to be the algebra

$$\langle \mathfrak{P}(U \times U), \circ, {}^{-1} \rangle$$

where $\mathfrak{P}(U \times U)$ is the BA with universe $\mathcal{P}(U \times U)$, $R \circ S$ is the usual composition of the binary relations R and S, while $R^{-1} = \{\langle b, a \rangle : \langle a, b \rangle \in R\}$ is the converse of R. Throughout this paper, $\mathcal{P}(V)$ denotes the powerset of V, and $\mathfrak{P}(V)$ denotes the BA with universe $\mathcal{P}(V)$ for any set V. By a full BRA we understand the full BRA over *some* set U. By a BRA we understand a subalgebra of a direct product of full BRA's up to isomorphism. Formally,

$$\mathsf{BRA} = \mathbf{SP}\{\langle \mathfrak{P}(U \times U), \circ, {}^{-1} \rangle : U \text{ is a set}\}.$$

Throughout, we use abbreviations like BRA also for denoting the corresponding class itself, e.g. BRA also denotes the class of all BRA's. Throughout, $\mathbf{S}$ and $\mathbf{P}$ are the operations of taking isomorphic copies of subalgebras and direct products respectively (as usual in universal algebra). It is not hard to see that to any BRA $\mathfrak{A}$ there is an equivalence relation E (over some set) such that $\mathfrak{A} \cong \mathfrak{A}^+ \subseteq \langle \mathfrak{P}(E), \circ, {}^{-1} \rangle$ for some[4] $\mathfrak{A}^+$. Whenever $\mathfrak{A}$ is embeddable into $\langle \mathfrak{P}(E), \circ, {}^{-1} \rangle$ then we say that $\mathfrak{A}$ is *representable by* $\mathcal{P}(E)$ or *is*

[4]We note that $\langle \mathfrak{P}(E), \circ, {}^{-1} \rangle$ would be a subalgebra of $\langle \mathfrak{P}(U \times U), \circ, {}^{-1} \rangle$ with E an equivalence relation on U if we would disregard the Boolean operations "$-$" and "1" (complementation and top element).

representable with top E. Note that if $\mathfrak{A}$ is representable this way, then $\mathfrak{A}$ is isomorphic to a Boolean algebra consisting of binary relations and closed under the operations $\circ$ and $^{-1}$. So, *every* BRA *is representable by an algebra consisting of binary relations.*

The reason for this representability of the direct products is that, if we think about the Stone duality theory of BA's, then we will realize that a product $\mathfrak{A} \times \mathfrak{B}$ of set algebras corresponds to taking the disjoint union $1^{\mathfrak{A}} \dot{\cup} 1^{\mathfrak{B}}$ of their top elements $1^{\mathfrak{A}}$ and $1^{\mathfrak{B}}$. I.e.: $\mathfrak{A} \times \mathfrak{B}$ is embeddable into $\mathfrak{P}(1^{\mathfrak{A}} \dot{\cup} 1^{\mathfrak{B}})$. But then if $\mathfrak{A}$ and $\mathfrak{B}$ are representable with tops $U \times U$ and $V \times V$ then $\mathfrak{A} \times \mathfrak{B}$ will be representable with top $(U \times U) \dot{\cup} (V \times V)$ which is an equivalence relation. This simple fact, that direct products preserve representability, is used throughout all branches of algebraic logic, and accordingly it will be used here. So *when defining new kinds of algebras of relations, we will first define the simplest version* (e.g. the one with top element $U \times U \times \cdots \times U$), *and then take all subalgebras of all direct products of these.*

REMARK 1. Let us return to universal algebra as a unifying framework. If $\mathfrak{A} \subseteq \langle \mathfrak{P}(E), \circ, ^{-1} \rangle$ as above, then the similarity type or signature of $\mathfrak{A}$ consists of the function symbols $\vee, -, \circ, ^{-1}$ where the first two are Boolean join and complementation. Homomorphisms, equations etc. are defined accordingly; e.g. homomorphisms should preserve all four operations, and $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$, $(x \vee y)^{-1} = x^{-1} \vee y^{-1}$ are typical equations. The same convention applies to algebras of relations introduced later in the paper. The important thing to remember is that if $\mathfrak{A} = \langle \mathfrak{B}, f_i \rangle_{i \in I}$ is a BA $\mathfrak{B}$ *expanded* with additional operators f_i (a BAO from now on), then the algebraic language of $\mathfrak{A}$ is that of BA's expanded with the extra-Boolean operation symbols f_i $(i \in I)$. So in particular, a homomorphism $h : \mathfrak{A} \longrightarrow \mathfrak{A}$ is a Boolean homomorphism preserving all the f_i's. The same applies for equations, subalgebras, and other universal algebraic concepts. The literature of BAO's is quite extensive, see e.g. [HMT] §2.7, Jónsson–Tarski [JT51], Jónsson [J84], Henkin [H70], Goldblatt [G88, G89], Sain [S84], Andréka–Jónsson–Németi [AJN], and §7 **(10)** herein.

Every variety of BAO's is arithmetical by [MMT] Thm.4.43 or [BS81] Thm.II.12.5. Hence the powerful theory of *arithmetical varieties*, cf. e.g. [MMT], [BS81] is applicable to practically all the algebras discussed in the present paper. Moreover, BRA's generate a *discriminator variety.*[5] This is proved in the proof of Thm.3 in §4 below. ∎

[5]Moreover, BRA's, as well as all the other discriminator algebras studied in this paper, generate a doubly pointed discriminator variety in the sense of [BP89b]. This is useful e.g. because in any doubly pointed discriminator variety V, to any universally quantified formula φ there is an equation e_φ such that φ and e_φ are equivalent in the subdirectly irreducible members of V. Hence this is true for the varieties BRA, BRA^0, RA, CA_n, RPA_α etc. in this paper, see Theorem 14 below.

Having a fresh look at our BRA's with an abstract algebraic eye, we notice that they should be very familiar from the abstract algebraic literature. Namely, a BRA

(3.1) $\mathfrak{A}$ consists of two well known algebraic structures, an involuted semigroup $\langle A, \circ, ^{-1} \rangle$ and a BA $\langle A, \vee, - \rangle$ sharing the same universe A. Further, the semigroup operations $\circ$ and $^{-1}$ distribute over the Boolean join $\vee$.

Property (3.1) defines a nice variety V_+ containing BRA and is a reasonable starting point for an axiomatic study of the algebras of relations. Postulates in (3.1) already appear in De Morgan [D1864], and since then investigations of relation algebras have been carried on for almost 130 years.

THEOREM 1 (Tarski). BRA *is a variety, i.e. is definable by a set of equations.*

For the **proof** see that of Theorem 3 in §4 herein.

We will return to historical remarks and references for this theorem and its proofs in this section, below the end of Remark 2.

Theorem 1 indicates that BRA is indeed a promising start for developing a nice algebraization of (at least a part of) first-order logic, or to put it more plainly, for developing an algebraic theory of relations. After Theorem 1, the question comes up naturally if we can strengthen the postulates in (3.1) to obtain a finite set Σ of equations describing the variety BRA, i.e. such that BRA $= Mod(\Sigma)$ would be the case. The answer is

THEOREM 2 (Monk). BRA *is not finitely axiomatizable, i.e. for no finite set* Σ *of first-order formulas is* BRA $= Mod(\Sigma)$.

See Monk [M64], and also [HMTII] 5.1.57, 4.1.3 for **proofs** (in slightly different settings).

Theorem 2 was strengthened by Andréka, Jónsson, and Maddux. Maddux [Ma89] proved that the set of equations containing only one variable and valid in BRA is not finitely axiomatizable, either. This result is reported in the book [TG87] where it turns out to be essential for applications in algebraizing metamathematics. Jónsson [J84] proved that no set Σ of equations containing only finitely many variables can axiomatize BRA. Andréka proved that no subreduct of BRA containing $\vee$ and $\circ$ among its operations can be axiomatized by a set of universally quantified formulas containing finitely many variables, or by a finite set of first-order formulas ([A89a]); where by a *subreduct* of a class K we mean the class of subalgebras of a reduct of K (to some language).

As a consolation for Theorem 2, by Monk [M69], one can obtain an example of a recursive (i.e. decidable) infinite set Σ of equations characterizing the variety BRA. Lyndon [Ly56] outlines another recipe for obtaining a different such Σ which may work for BRA. However, the structures of these Σ's are rather involved. Cf. [HMTII] pp. 112–119 for an overview. In this connection, we note that the following is still one of the most important open problems of algebraic logic:

PROBLEM 1. *Find simple, mathematically transparent, decidable sets Σ of equations characterizing* BRA. *(A solution for this problem has to be considerably simpler than, or at least markedly different from the Σ's discussed above.)*

Compare Problem 4.1 in [HMTII] p. 179, Henkin–Monk [HM74] Probl.5, [M77] p. 85_3, [TG87] p. 240_{16-13}, [AGN77] p. 21, [W88], Venema [V89] Def.1.4.1, Thms 1.4.16, 3.3.5, Simon [Si90].

One can get very far in doing algebraic logic (for quantifier or predicate logics) via BRA's. If we want to investigate nonclassical quantifier logics, we can replace the Boolean reduct $\mathfrak{B}$ of $\mathfrak{A} = \langle \mathfrak{B}, \circ, ^{-1} \rangle \in$ BRA with the algebras (e.g. Heyting algebras) corresponding to the propositional version of the nonclassical logic in question. By Andréka's above quoted result, if $\mathfrak{B}$ is (an expansion of, like Heyting algebras) a distributive lattice, then Theorem 2 carries over. We will return to this direction later. At a first reading, some parts of Remark 2 below might be too technical for the *nonspecialist*. They may be skipped safely, but it is advisable to come back to them, sometime.

Throughout, ω is the set of natural numbers, and it coincides with the least infinite ordinal.

REMARK 2 (*Finitization*). The efforts of trying to get rid of Theorem 2 became known as *finitizability investigations*. Why would we want to get rid of Theorem 2? The class of BA's of unary relations, i.e. $\mathbf{SP}\{\mathfrak{P}(U) : U \text{ is a set}\}$ admits a nice finite axiomatization. Theorem 2 says that the same is not possible for BRA's of binary relations. Since we are in the middle of the search for the right notion of algebras of (not only unary) relations, the question naturally comes up whether Theorem 2 was perhaps only a consequence of an unfortunate choice of the basic operations $\circ$ and $^{-1}$ of BRA's.[6]

Apparently, it is hard to get rid of Theorem 2 by moving to reducts of BRA's (cf. Andréka's result mentioned below Theorem 2, Schein [Sc89],

[6]Henkin pointed out during the 1987 Algebraic Logic conference in Asilomar that a positive solution for this problem would imply positive results for pure (non-algebraic) logic too; see Sain [S87a] §4, largely based on Henkin's suggestions. According to Henkin, on his own part, this was the main reason for including this "goal" as Problem 1 in [HM74].

Andréka [A89, A89a]). It is a second central open problem[7] of algebraic logic to find out whether we can get rid of Theorem 2 by moving in the direction opposite to taking reducts. This would mean to expand BRA's by new, "set-theoretically defined" operations on relations such that the new class would become finitely axiomatizable. This quest is motivated by the fact that there are many known examples for a non-finitely axiomatizable class K of structures such that an expansion K^+ of K is finitely axiomatizable. An example for such a K and K^+ are the elementary classes generated by the structures $\langle\omega, 0, succ\rangle$ and $\langle\omega, 0, succ, \leq\rangle$ respectively. Another example is set theory. ZF is not finitely axiomatizable but its conservative extension known as Gödel–Bernays set theory is such.[8] A drawback of these examples is that representability of members of K^+ (as some kinds of algebras of relations) do not enter the picture. To alleviate this, we quote an example due to Bredikhin: Subalgebras of $\langle\mathcal{P}(U \times U), \circ, ^{-1}\rangle$ for all U form a non-finitely axiomatizable quasivariety while substructures of their expansions $\langle\mathcal{P}(U\times U), \circ, ^{-1}, \subseteq, dom\rangle$ where $dom(R) = \mathrm{Id} \restriction Domain(R) = \{\langle a,a\rangle : (\exists b)(\langle a,b\rangle \in R)\}$ form a finitely axiomatizable universal Horn class (Bredikhin [Br77, Br77a]).[9] (Both classes are understood up to isomorphisms, of course.)

The problem was raised in various forms by Jónsson, Henkin–Monk [HM74] Problem 1, Monk [M70] p. 20, Tarski–Givant [TG87] lines 9–11 of p. 62 (the first sentence of the paragraph preceding 3.5.(ix)), and the first page of [TG87] §3.5 (p. 56), whether

(3.2) one can add to the operations of BRA finitely many new set-theoretically defined operations $f_1, \ldots, f_n$ on relations such that $K^+ = \{\langle\mathfrak{P}(U\times U), \circ, ^{-1}, f_1, \ldots, f_n\rangle : U \text{ is a set}\}$ would generate a finitely axiomatizable variety.

A perhaps even more important version of the problem asks whether

$(3.2)^+$ statement (3.2) is true in such a strong form that for K^+ in it, its closure $\mathbf{SP}K^+$ is a variety, or at least is a finitely axiomatizable class.

[7]This is not really a single problem; it is rather a large circle of problems motivated (and "punctuated") by many deep results. In a sense it is a rich theory built around a deep, open, almost philosophical question.

[8]Fact 1 below is a third, rather general motivating example. Actually Fact 1 seems to imply that what we want to do is *always* possible. The only question is whether we can do it in such a way that the new operations remain "representable as natural set-theoretic operations on relations (like $\cap$, $^{-1}$, and $\circ$ were)". It is this latter (vague) requirement which was referred to above as "set-theoretically definable".

[9]Bredikhin's $dom(x)$ is the same as Craig's $T_{[1/0]}(x)$ to be discussed in §6 here.

What does it mean in (3.2) that f_i should be set-theoretically defined? The intuitive idea is that it should be somehow "concretely specified" like the operations of BRA. Namely, in the latter e.g. $R \cap S$ or $R \circ S$ are defined by set-theoretical means in a very strong sense, which entails, among other things, that the results of these operations depend only on the choice of R and S, independently of the choice of the algebra in which we execute them. So in a sense, these operations are "concrete" (as opposed to the abstract algebraic operations of fields for example). The most generally accepted precise version of this condition says that f_i, $i \leq n$, should be *invariant under* the *permutations* of U. This means that whenever $p : U \rightarrowtail\!\!\!\twoheadrightarrow U$ is a permutation then $p(f_i(R)) = f_i(p(R))$ where[10] $p(R)$ is defined the natural way for $R \subseteq U \times U$. The following is still open.

PROBLEM 2. *Is* $(3.2)^+$ *above, or if not then at least* (3.2) *true, when "set-theoretically defined" means permutation invariant in the above sense?*

The problem in this form was raised among others by Bjarni Jónsson sometime before 1983; see e.g. [S87, S87a], the part of [Bi89] coming after Corollary 10, and the parts of [TG87] indicated just before formulating (3.2) above. Biró [Bi89] proved that $f_1, \ldots, f_n$ cannot all be "first-order definable" in a certain sense of the term we will not recall here[11]. Sain [S87a] contains a positive solution but for a different kind of algebras of relations to be discussed later. [Bi89] and [Ma89b] showed that the requirement of permutation invariance is essential (i.e. without this requirement there is a solution but the logical counterpart of this solution does not satisfy the axioms of abstract model theory or general theory of logics).

Possible choices of the f_i's are: the identity relation Id on U as a new constant (distinguished element), transitive closure $trc(R) = R \cup (R \circ R) \cup (R \circ R \circ R) \cup \ldots$, and a choice function picking out a singleton $\{\langle a, b \rangle\} \subseteq R$ from R, i.e. $ch(R) \subseteq R$ such that $|ch(R)| = min(1, |R|)$. Of these, Id is first-order definable, trc is not, but both are permutation invariant; ch is not permutation invariant (hence it is not allowed in Problem 2). Actually, Maddux

[10] In §7(9) herein we will describe a category theoretic (functorial) version of this condition excluding the operation $f(R)$="$\emptyset$ if $\emptyset \in U$ else R". According to the present (non-categorical) definition, this f is permutation invariant, but from the point of view of algebraizing arbitrary quantifier logics in the abstract model theoretic sense, f violates one of the axioms of abstract model theory; cf. Barwise–Feferman [BF85]. Cf. §7(9) herein.

[11] The definition can be found in Jónsson [J84] or Biró [Bi89], but using cylindric algebras to be introduced later, basically, an operation on relations is "first-order definable" iff it is term definable in cylindric algebras (of ω-ary relations, i.e. in RCA_ω). We note that cylindric algebras are basically expansions of BRA's as shown e.g. in [HMTII] §5.3, and sketched in §4 below.

([Ma89b] Thm's 8, 9) showed that adding *ch* makes BRA finitely axiomatizable. A further example for a permutation invariant but not first-order definable operation is $fin(R)$ = "$\emptyset$ if R is finite, else R". (This operation will be relevant in §6.) Adding all the permutation invariant ones of these four operations to BRA does not make the class finitely axiomatizable. (The case of $\{trc, \mathrm{Id}\}$ is in [M89b] Thm.3, but see also [AGN87], Ng [N84], [N81], and Prob.7 in [M77] for investigations of *trc*.)[12]

Throughout, $\mathbf{Rd}$ is taken to be the operation forming appropriate reducts. In particular, $\mathsf{K}_1 = \mathbf{Rd}\mathsf{K}_2$ means that we can identify the operation symbols of K_1 with some of those of K_2 such that after forgetting the rest of the operations of K_2, the two classes become identical, up to isomorphisms. (Beginning with §4 below, we will use $\mathbf{Rd}$ in a slightly more general sense, namely we will allow the operation symbols of K_1 to correspond to term functions, i.e. derived operations of K_2 instead of only operation symbols of K_2. However, in the present section the narrower definition suffices.) $\mathsf{K}_1 = \mathbf{SRd}\mathsf{K}_2$ is defined in the same fashion. Now, (3.2) above implies that $\mathsf{BRA} = \mathbf{SPRd}(\mathsf{K}^+)$ for K^+ as described in (3.2). The following is a variant of $(3.2)^+$:[13]

(3.3) There are set-theoretically defined operations $f_1, \ldots, f_n$ on binary relations, such that

$$\mathsf{K}^+ = \mathbf{S}\{\langle \mathfrak{P}(E), f_1, \ldots, f_n\rangle : E \text{ is an equivalence relation}\}$$

<u>*is*</u> a finitely axiomatizable <u>*variety*</u> and $\mathsf{BRA} = \mathbf{SRd}(\mathsf{K}^+)$.

The main difference between (3.3) and $(3.2)^+$ is the replacement of $U \times U$ in $(3.2)^+$ by an equivalence relation E; without this change we cannot expect K^+ to be a variety (closure under $\mathbf{P}$ is the problem). This $U \times U \mapsto E$ substitution was used to the same effect immediately below the definition of BRA.

Variants of Problem 2 ask if (3.3) is true but again in formulating these problems, one has to make the adjective "set-theoretically defined" concrete and precise.

The condition $\mathsf{K}^+ = \mathbf{S}\mathsf{K}^+$ in (3.3) is important because, by [HMTII] 3.2.5, there is a finitely axiomatizable K^+ such that $\mathsf{BRA} = \mathbf{S}\mathsf{K}^+$. (Cf. also

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[12] [M89b] proves the "BRA + $\{trc, \mathrm{Id}\}$" case for equations containing only one variable, too. Further, Andréka proved that (i) "RRA + *trc*" is not finitely axiomatizable *over* RRA, and (ii) *no* expansion of "BRA + *trc*" is finitely axiomatizable. So if in (3.2) we would replace "$\circ, {}^{-1}$" with "$\circ, {}^{-1}, trc$" then the answer to all parts of Problem 2 would be NO, even if the condition "set-theoretically definable" were completely omitted.

[13] To avoid misunderstandings, we are not saying that (3.3) would be true; we only formulate it as a "statement" to be discussed later.

Jónsson–Tarski [JT51] Thm.4.31.) This also yields a kind of a characterization of the equations valid in BRA, namely all the equations derivable from the single formula axiomatizing K^+ in [HMTII] 3.2.5. This illustrates what kinds of answers are no longer acceptable as solutions for Problem 1, making the vague adjectives like "simple", "transparent" in the formulation, hopefully, more tangible.

In (3.2) and (3.3) above (hence in Problem 2), the condition saying that $f_1, \ldots, f_n$ should be "set-theoretically defined" (whatever concrete meaning the latter will receive later) is very important, because without this condition a version of the problem seems to receive a trivial solution by a result of Kleene from 1951 improved by Craig and Vaught [CV58]. Namely:

FACT 1. *For any quasivariety* K *axiomatizable by a recursively enumerable set* Σ *of first-order formulas, there is a finitely axiomatizable* $\mathsf{K}^+ = \mathbf{S}\mathsf{K}^+$ *such that* $\mathsf{K} = \mathbf{SRd}(\mathsf{K}^+)$.

PROOF. We will show how Fact 1 follows from Thm.2.1 of [CV58], which says that if $Mod(\Gamma)$ consists of infinite models and Γ is recursively enumerable, then $Mod(\Gamma) = \mathbf{Rd}(\mathsf{K}^+)$ for some finitely axiomatizable class K^+. Let Σ be as above. Form Σ_∞ by adding to Σ postulates stating that "if there are at least two elements, there are at least n different elements" for each $n \in \omega$. Then $\mathsf{K} = \mathbf{S}Mod(\Sigma_\infty)$ and each model of Σ_∞ is infinite (except for the one-element trivial one which may be safely ignored). By Thm.2.1 of [CV58], there is a finitely axiomatizable K^+ with $Mod(\Sigma_\infty) = \mathbf{Rd}(\mathsf{K}^+)$. But then $\mathsf{K} = \mathbf{SRd}(\mathsf{K}^+)$. Finally, the predicates in K^+ are easily coded by their characteristic functions, so K^+ can be transformed into a class of algebras. By introducing Skolem functions, we can achieve $\mathsf{K}^+ = \mathbf{S}\mathsf{K}^+$ (note that only finitely many Skolem functions are needed because only finitely many axioms are needed for describing K^+). Fact 1 has been proved. ∎

We conjecture that Fact 1 can be improved to conclude that K^+ is a finitely axiomatizable quasivariety.

Fact 1 immediately yields that version of (3.3) from which the condition "set-theoretically defined" is omitted, and in which "variety" is replaced by "universally axiomatizable class".

Sain [S87] proved that version of (3.3) in which "set-theoretically defined" is made precise as follows: f_i is set-theoretically defined iff we can write down a concrete set-theoretical formula $\varphi_i(x_1, \ldots, x_n, y)$ containing bounded quantifiers only such that for any binary relations $X_1, \ldots, X_n, Y$ we have $[f_i(X_1, \ldots, X_n) = Y \iff \varphi_i(X_1, \ldots, X_n, Y)]$. This immediately ensures that $f_i(R)$ depends only on R and not on the choice of the algebra in which $f_i(R)$ is computed.

We note that, in a sense, Sain's result improves Maddux's and Biró's choice function *ch* because *ch* is not definable in general in set theory, especially not by an absolute[14] formula like φ_i above. All these leave Problem 2 open since there "set-theoretically defined" is interpreted as meaning invariant under permutation.

In passing we note that Tarski and Givant [TG87] p. 62 raise a perhaps easier version of Problem 2. Namely, they ask for a finitely axiomatizable variety K_0 containing K^+ of (3.2) such that $\mathsf{BRA} = \mathbf{SRd}(\mathsf{K}_0)$. Here, the point is that not all members of K_0 have to be "representable" by $\mathbf{SP}\mathsf{K}^+$ or even by $\mathbf{HSP}\mathsf{K}^+$ (but still $\mathsf{K}^+ \subseteq \mathsf{K}_0$ *is* required and f_i of K^+ has to be permutation invariant)[15]. This problem, at least in principle, is indeed easier than Problem 2, by the following example. Andréka proved that if $\Omega = \{\vee, \circ\}$ and $\Gamma = \Omega \cup \{\wedge\}$ then $\mathbf{SRd}_\Omega\mathsf{BRA}$ and $\mathbf{SRd}_\Gamma\mathsf{BRA}$ are not finitely axiomatizable, but there is a finitely axiomatizable variety $\mathsf{K}_0 \supseteq \mathbf{SRd}_\Gamma\mathsf{BRA}$ with $\mathbf{SRd}_\Omega\mathsf{BRA} = \mathbf{SRd}_\Omega(\mathsf{K}_0)$. Here $\mathbf{Rd}_\Omega$ denotes taking reducts to the similarity type Ω. This "easier" version of Problem 2 is also open.

END of Remark 2

The natural logical counterpart of BRA's is classical first-order logic restricted to three individual variables v_0, v_1, v_2 and without equality. As shown in §5.3 of [TG87], this system is an adequate framework for building up set theory and hence metamathematics in it. One can illustrate most of the main results, ideas and problems of algebraic logic by using only BRA's. (An illustration of this is the result just quoted from [TG87] §5.3.) We do not know how far BRA's can be simplified without losing this feature. In this connection, let the class BSR of Boolean semigroups of relations be defined as

$$\mathsf{BSR} = \mathbf{SP}\left\{\langle \mathfrak{P}(U \times U), \circ \rangle : U \text{ is a set}\right\}.$$

[14] The φ_i's are known to be absolute in a very strong sense used in set theory, see Barwise [B75]. Note, however, that Sain's result does *not* give us a permutation invariant operation. Actually, the two properties are independent: there are absolute operations which are not permutation invariant, and also permutation invariant operations which are not absolute. Sain has a permutation invariant result too, but for that, relations of higher ranks are also needed as elements of the algebra. Therefore this result will be stated as Theorem 16 in §6.

[15] Their wording of the problem is different from the present algebraic one, since they formulate the problem in its logical form. (The same remark applies to our quoting [TG87] in connection with Problem 2 above.) In logical form, the presently discussed easier question does not require the expanded logic L^+ corresponding to K^+ to be complete, but L^+ still has to be *sound*. (And of course, L^+ has to be complete w.r.t. the old formulas.) Strangely enough, this soundness requirement (together with permutation invariance) seems to keep the problem on the tough side. [M89b] proved that adding $\{\mathrm{Id}, trc\}$ does not solve this easier problem either.

So we require only one extra-Boolean operation "$\circ$". Further, we do not require closure under any other operations. The question is, how far could BSR replace BRA as the simplest, "introductory" example of Tarskian algebraic logic. We conjecture that the answer will be "very far".

We know that BSR is a *discriminator variety*, and is not finitely axiomatizable. Thus Theorems 1, 2 remain true if BRA is replaced with BSR in them. Further, the equational theory of BSR is undecidable. (BRA's being a discriminator variety implies e.g. that the simple members of BSR form a universally axiomatizable class.) We conjecture that, following the lines of [TG87] §5.3, set theory can be built up in BSR instead of BRA with basically the same positive properties (e.g. finitely many axioms) as the present version [TG87] has. (Perhaps here [N85a], [N86] can be useful, because an analogous task was carried through there. The last 12 lines of Jónsson [J82] p. 276 seem to be also useful here.) It would be nice to know if this conjecture is true, and, more generally, to see a variant of algebraic logic elaborated on the basis of BSR. We do not know what natural fragment of first-order logic with three variables corresponds to BSR (if any). It certainly is difficult to simulate substitution of individual variables using only $\circ$. The converse operation, $^{-1}$, is the algebraic counterpart of substitution because, intuitively, $R(v_0,v_1)^{-1} = R(v_1,v_0)$. One can simulate quantification by $\circ$, and it is easily seen that $\circ$ is stronger than quantification but without $^{-1}$ it is not clear exactly how much stronger. Curiously enough, these issues are better understood in the case of cylindric algebras to be discussed later.

BRA's also play an important rôle in theoretical computer science (cf. e.g. Hennessey [H80], Bednarek–Ulam [BU77], Imieliński–Lipski [IL84], Berghammer–Zierer [BZ86], Hoare–Jifeng [HJ86], [SS89], van Benthem [vB90a, vB90], and §7 (7) herein). Jónsson [J89], [J90] call a finitely axiomatizable variety BRA^0 approximating BRA Program Specification Algebras. Here, by saying that the finitely axiomatizable BRA^0 approximates the not finitely axiomatizable BRA we mean that $\mathsf{BRA}^0 \supseteq \mathsf{BRA}$ and BRA^0 is fairly close to[16] BRA.

If we want to algebraize first-order logic with equality, we have to add an extra constant Id representing equality to the operations. RRA denotes the class of subalgebras of direct products of algebras of the form

$$\langle \mathfrak{P}(U \times U), \circ, ^{-1}, \mathrm{Id} \rangle$$

[16] These BRA^0-like approximations of BRA-like "non-finitizable" classes are quite characteristic of algebraic logic, and they are relevant e.g. to algebraizations of syntactic aspects of logic like proof theory. For more on this see the Appendix (§8) and the subsection "The axiomatic approach..." in the middle of §4.

where Id = Id $\upharpoonright U = \{\langle u,u\rangle : u \in U\}$ is a constant of the expanded algebra.[17] As in the case of BRA's, the definition of RRA is understood to be up to isomorphisms. RRA abbreviates representable relation algebras. RRA's have been investigated more thoroughly than BRA's; actually, Theorems 1,2 above were proved first for RRA's. For historical notes on Theorem 1 for RRA's see Thm.8.3(v) on p. 240 of [TG87]. A detailed, fairly simple, direct proof is Case 1 in the proof of Thm.3.1.103 on p. 43 of [HMTII] (elaborated for a slightly different but basically equivalent form of the statement); but see also the proof of Theorem 3 here.

We note that BRA = **SRd**RRA, where **Rd** is the operator of taking appropriate reducts (cf. Remark 2). We also note that RRA is *finitely axiomatizable over* BRA, which means that by adding finitely many equations to the equational theory of BRA, we can obtain an axiomatization of RRA.[18] Moreover, RRA is finitely axiomatizable over its $^{-1}$ *and* Id-free subreduct BSR, where a subreduct of K is a class of the form **SRd**K, cf. the convention for using **SRd** above item (3.3) (in Remark 2). This is in sharp contrast with the cylindric algebraic situation; see Andréka's negative solution for Problem 5.4 of [HMTII] to be mentioned later in this paper.

In connection with algebraization of logic, we note that the so called Leibniz law of equality which says that equals cannot be distinguished, translates into the algebraic condition that Id is the neutral element of the semigroup operation "$\circ$".

Some special classes of RRA's (RRA's with pairing function elements, algebras in which below every relation there is a nonempty one which is a function, pair-dense RRA's) turn out to be finitely axiomatizable, cf. Maddux [Ma78a, Ma87], Givant [Gi88].[19] The elegant, purely algebraic proofs in these papers of Maddux are examples for significant applications of algebra

[17] In Pratt [Pr90], the class RBM of representable Boolean monoids is obtained from our BSR's by adding Id as an extra distinguished constant. So BSR's are the Id-free subreducts of RBM's. All the results mentioned above for BSR's carry over to RBM's; e.g. RBM is a discriminator variety, hence the simple RBM's form a universally axiomatizable class, Theorems 1, 2 above apply to RBM etc. Some of the proofs (both for RBM and BSR) are analogous to that of Theorem 3 below. (These properties of RBM do not seem to be mentioned in [Pr90]).

[18] For issues concerning finite axiomatizability of a class over another one cf. [HMTII] p. 273 (Problems 5.4, 5.8). As a contrast we note that, letting $_\infty$RRA denote the class of the infinitely representable RRA's (i.e. where in $\langle\mathfrak{P}(U\times U)\ldots\rangle$, $|U|\geq\omega$ is required), Andréka proved that $_\infty$RRA is not axiomatizable over RRA by any set of universal formulas containing finitely many variables, solving Problem 8 from [Gi89]. The notation $_\infty$RRA is taken from [HMTII] p. 6.

[19] The possibility of extending Maddux's approach to algebras of relations of higher ranks was investigated in [AN88]. Further, Givant generalized the positive results concerning pair-dense algebras to a broader class he calls locally small RA's, cf. [Gi88].

to logic, via connections between algebra and logic indicated in [TG87], in the Thm.13 through Prop. 16/a in §II.1 of [ANS84], and [HMTII] 4.3.29 (p. 161). Another fruitful direction for obtaining finite axiomatizability results for algebras of relations was initiated in cylindric algebra theory by Diane Resek in 1969; the binary relation oriented counterpart of which are the finite axiomatizability results for "relativized" RRA's in [Ma82] and in Kramer [K89]. For Resek's fundamental result a useful reference is Andréka–Thompson [AT88]; but see also [HMTII] pp. 101, vi, 300.

The relevance of RRA's to predicate logic has been elaborated in the book [TG87], but see also [Ma83] and §5.3 in conjunction with §4.3 of [HMTII].

RRA's are also relevant to what is known as *category theoretical logic*. Namely, every RRA $\mathfrak{A} \subseteq \langle \mathfrak{P}(U \times U), \ldots, \mathrm{Id} \rangle$ is equivalent with a category **C** whose objects are the subsets of U and whose morphisms are the binary relations between these sets. This **C** is actually more than a "plain" category, it is an enriched category, but it is exactly the enriched categories which play a central rôle in category theoretical logic, cf. e.g. Daigneault [D69], Zlatos [Z83,Z84]. The objects of **C** can be recovered from $\mathfrak{A}$ as those elements of $\mathfrak{A}$ which are below the constant Id. The morphisms of **C** are the elements of $\mathfrak{A}$ in general, the domain of R is computed by the term $\mathrm{Id} \wedge (R \circ 1)$ where 1 is the greatest element (in our case $U \times U$) of any BA or BAO. Further, any homomorphism between two RRA's $\mathfrak{A}$ and $\mathfrak{A}'$ gives rise to a functor between the corresponding enriched categories **C** and **C**$'$ (and vice versa).

4. Algebras for logics with equality

By this point we might have developed some vague picture of how algebras of binary relations are introduced, investigated etc. One might even sense that they give rise to a smooth, elegant and very exciting, powerful theory. However, our original intention was to develop algebras of relations in general, which should surely incorporate not only binary but also ternary, and in general n-ary relations.

Let us see how to generalize our RRA's and BRA's to relations of higher ranks. As we said, we would like the new algebras to be expansions of RRA's (and BRA's), or something like this. However, defining composition of n-ary relations for $n > 2$ is complicated. Therefore the following sounds like a more attractive idea: We single out the simplest basic operations on n-ary relations, and hope that composition will be derivable as a term-function from these. Let us see how we could generalize our generic or full RRA's $\langle \mathfrak{P}(U \times U), \circ, ^{-1}, \mathrm{Id} \rangle$ to relations of rank n, for $n \in \omega$. The obvious part is that these algebras will begin with $\langle \mathfrak{P}(U \times U \times \cdots \times U), \mathrm{Id}, \ldots \rangle$, where $\mathrm{Id} = \{\langle u, u, \ldots, u \rangle : u \in U\}$ is the n-ary identity relation. Again, Id is a constant, just as it was in the RRA case. Let ${}^nU$ denote $U \times U \times \cdots \times U$, e.g. ${}^3U = U \times U \times U$. The new operations (besides the Boolean ones and Id) we will need are the algebraic counterparts of quantification $\exists v_i$, for $i < n$. So, we want an operation that sends the relation defined by $R(v_0, v_1)$ to the one defined by $\exists v_0 R(v_0, v_1)$, and similarly for $\exists v_1$. For $R \subseteq U \times U$ let $Dom(R)$ and $Rng(R)$ denote the usual domain and range of R. For $n = 2$ we define $c_0(R) = U \times Rng(R)$ and $c_1(R) = Dom(R) \times U$. Now $\langle \mathfrak{P}(U \times U), c_0, c_1, \mathrm{Id}\rangle$ is the *full cylindric set algebra* of binary relations over U, for short the full Cs_2.

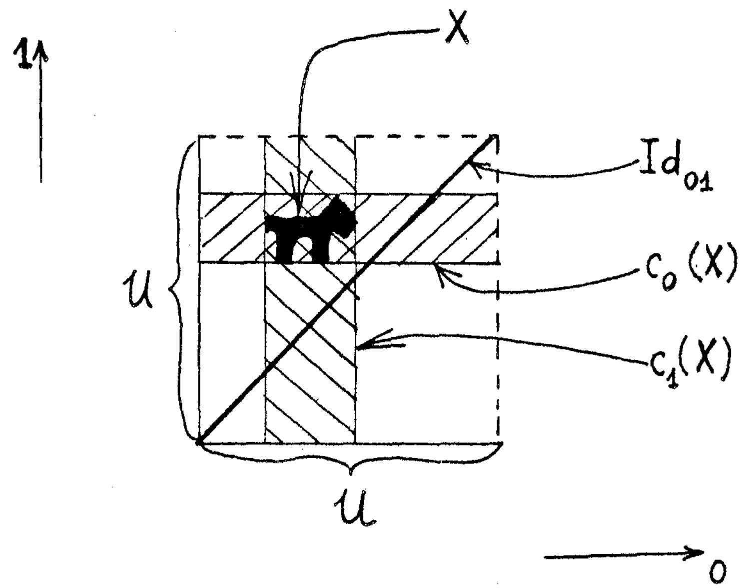

FIGURE 1

Before turning seriously to n-ary relations, we need the following:

CONVENTION. *Throughout we will pretend that Cartesian products and Cartesian powers are associative such that e.g.:* ${}^2U \times {}^3U = {}^5U, \quad {}^nU \times$

${}^{m}U = {}^{n+m}U$, and if $R \subseteq {}^{3}U$ then
${}^{2}U \times R \subseteq {}^{5}U \supseteq R \times {}^{2}U$.

The full Cs_n i.e. the full *cylindric set algebra of n-ary relations* is the natural generalization of Cs_2 as follows.

Let $R \subseteq {}^{n}U$. If $Rng(R) = \{\langle b_1 \dots b_{n-1}\rangle : \langle b_0 b_1 \dots b_{n-1}\rangle \in R$ for some $b_0\}$ then $\mathsf{c}_0(R) = U \times Rng(R)$ considered as a set of n-tuples. Similarly, let $Dom(R) = \{\langle b_0 \dots b_{n-2}\rangle : \langle b_0 \dots b_{n-2} b_{n-1}\rangle \in R$ for some $b_{n-1}\}$, and let $\mathsf{c}_{n-1}(R) = Dom(R) \times U$. Generalizing this to c_i with $i < n$ arbitrary, we obtain

$$\mathsf{c}_i(R) = \{\langle b_0, \dots, b_{i-1}, a, b_{i+1}, \dots, b_{n-1}\rangle \ : \ \langle b_0, \dots, b_{n-1}\rangle \in R \text{ and } a \in U\} \ .$$

c_i is one of the most natural operations on relations. It simply forgets the i-th argument of the relation, or in other words, deletes the i-th column. However, since deleting the i-th column would leave us with an $(n-1)$-ary relation, $Dom(R)$ if $i = n-1$, we replace the i-th column with a dummy column i.e. in the $i = n-1$ case we represent $Dom(R)$ with the "pseudo n-ary relation" $Dom(R) \times U$. The "real rank" of an $R \subseteq {}^{n}U$ is always easy to recover, namely it is $\Delta(R) = \{i < n : \mathsf{c}_i(R) \neq R\}$. So c_i is the natural operation of removing i from the (real) rank of a relation.

For example, $\mathsf{c}_{\text{father}}$ when applied to the "father, mother, child" relation gives back the "mother, child" relation coded as "anybody, mother, child" (in which the anybody argument carries no information i.e. is dummy). If $U = \{a, b, c\}$ then $\mathsf{c}_1\{\langle a, b, c\rangle\} = \{\langle a, a, c\rangle, \langle a, b, c\rangle, \langle a, c, c\rangle\}$ and $\mathrm{Id} = \{\langle a, a, a\rangle, \langle b, b, b\rangle, \langle c, c, c\rangle\}$. By a full Cs_n we understand an algebra

$$\mathfrak{Rel}_n(U) \stackrel{\text{def}}{=} \langle \mathfrak{P}({}^{n}U), \mathsf{c}_0, \dots, \mathsf{c}_{n-1}, \mathrm{Id}\rangle$$

for some set U. By a Cs_n we understand a subalgebra of a full Cs_n. By a *representable cylindric algebra of n-ary relations* (an RCA_n) we understand a subalgebra of a direct product of full Cs_n's (up to isomorphism), formally: $\mathsf{RCA}_n = \mathbf{SP}(\text{full } \mathsf{Cs}_n) = \mathbf{SP}\mathsf{Cs}_n$. By the same argument as in the case of BRA's, every RCA_n is directly representable as an algebra of n-ary relations (with the greatest relation a disjoint union of Cartesian spaces). RCA_n is one of the "leading candidates" for being the natural algebra of n-ary relations. The abstract algebraic picture is simple, an RCA_n is a BA together with n closure operations (in the usual abstract algebraic sense defined on the Boolean ordering) and an extra constant. Recall that a closure operation c on a partially ordered set $\langle P, \leq\rangle$ is an order preserving and idempotent function $\mathsf{c} : P \longrightarrow P$ such that $\mathsf{c}(x) \geq x$. Boolean orderings with closure operations on them are one of the central concepts of abstract algebra, for example topological spaces or subalgebras of an algebra are often represented as such (e.g. if $\mathfrak{A}$ is an algebra, consider $\langle \mathcal{P}(A), \mathrm{c}\rangle$, where $\mathrm{c}(X)$ is the subalgebra of $\mathfrak{A}$ generated by X for $X \subseteq A$). A natural question comes up: Can these simple RCA_n's recapture the power of RRA's? To answer this question, consider the dummy representation $Dr : \mathcal{P}(U \times U) \rightarrowtail \mathcal{P}(U \times U \times U)$ of binary relations with ternary ones discussed above, i.e. Dr sends R to $R \times U$. Then it is not hard to define terms τ° and τ^{-1} in the language of RCA_3's such that $Dr(R \circ S) = \tau^\circ(Dr(R), Dr(S))$ and $Dr(R^{-1}) = \tau^{-1}(Dr(R))$ (see [HMTII] §5.3)[20]. So in a sense RRA's form a kind of a reduct of RCA_n's for $n \geq 3$. Thus the answer is that RCA_n's, $n > 2$, recapture the power of RRA's. (On the other hand, RCA_2's do not.) This was a natural requirement we expected to meet, namely that the theory of n-ary relations should be an extension of that of binary relations.

At this point we can state the counterparts of Theorems 1, 2.

THEOREM 3 (3.1.103 on p. 43 of [HMTII]). *RCA_n is a variety for all n.*

ON THE PROOF. Our Theorems 1, 3 are proved in [HMTII] without relying too much on universal algebra. Therefore it is not mentioned there that these theorems are almost immediate consequences of a well known universal algebraic property of the so called *discriminator varieties* [BS81] §IV.9. Namely for any class $\mathsf{K} = \mathbf{S}\mathsf{K}$ of algebras with a discriminator term, if K is closed under ultraproducts then $\mathbf{SP}\mathsf{K}$ is a variety. Now, the choice $\mathsf{K} = \mathsf{Cs}_n$ is easily seen to satisfy these conditions as follows. Clearly $\mathsf{Cs}_n = \mathbf{S}\mathsf{Cs}_n$. To see that Cs_n has a discriminator term, let $\tau(x,y) = \mathrm{c}_0 \ldots \mathrm{c}_{n-1}(x \oplus y)$ where $\oplus$ is Boolean symmetric difference. Let $t(x,y,z) = (x \wedge \tau(x,y)) \vee (z - \tau(x,y))$, where $(x - y)$ abbreviates $(x \wedge -y)$. Then t is a discriminator term in Cs_n. See e.g. Example 2 below Def. 9.3 in [BS81] §IV.9 and [AJN88] for more on CA's being discriminator algebras. Cs_n is closed under ultraproducts (up to isomorphisms of course) because it is a pseudo-elementary class, i.e. if we add the base sets U as an extra sort to our Cs_n's obtaining two sorted structures like $\langle \mathfrak{A}, U, \mathrm{c}_i, \mathrm{Id}, \in\rangle$ where $\mathfrak{A} \subseteq \mathfrak{P}({}^nU)$ and $\in \subseteq {}^nU \times A$ is an $n+1$-ary relation between sorts U and A, then this enriched class becomes first-order axiomatizable, as it is easy to see. This proves Theorem 3.

The proofs for BRA and RRA in place of RCA_n are practically the same since c_0 and c_1 are term definable in BRA. This proves Theorem 1. ∎

[20] E.g. $\mathrm{c}_2[\mathrm{c}_1(x \wedge \mathrm{c}_0\mathrm{Id}) \wedge \mathrm{c}_0(y \wedge \mathrm{c}_1\mathrm{Id})]$ is one possible choice for $\tau^\circ(x,y)$.

THEOREM 4 (4.1.3 on p. 111 of [HMTII]). *The variety* RCA_n *is not finitely axiomatizable if* $n > 2$.

Theorem 4 above, due to Monk [M69], was refined by Andréka and Comer (but see also [Bi89], [Ma89], [J69], [ST89]). Andréka proved the following three results for $n \geq 3$. RCA_n cannot be axiomatized by any set Σ of universally quantified formulas if Σ contains only finitely many variables.[21] RCA_n is not finitely axiomatizable over (the full first-order theory of) its Id-free subreduct (i.e. over the variety generated by the $\langle \mathfrak{P}({}^nU), \mathsf{c}_0, \dots, \mathsf{c}_{n-1}\rangle$'s). Neither is it finitely axiomatizable over its RRA subreduct. To formalize the latter, let the terms τ° and τ^{-1} discussed above be chosen such that their effect is the following: for $R, S \subseteq U \times U \times U$, $\tau^\circ(R,S) = [Dom(R) \circ Dom(S)] \times U$ and $\tau^{-1}(R) = (Dom(R)^{-1}) \times U$ (where recall that $Dom : \mathcal{P}(U \times U \times U) \longrightarrow \mathcal{P}(U \times U)$). Then RCA_3 is not finitely axiomatizable over the universally quantified theory of the algebras of the form $\langle \mathfrak{P}(U \times U \times U), \tau^\circ, \tau^{-1}, \mathsf{c}_2, \mathrm{Id}\rangle$. The latter of course is the reduct obtained from RCA_3's by adding the term functions τ° and τ^{-1} as new basic operations, and forgetting c_0 and c_1. A similar result applies for $n \geq 3$. As a by-product, Andréka obtained a rather simple proof for Theorem 4. Comer ([Co89a]) proved that no generalized subreduct K of RCA_n containing all the cylindric algebraic operations except complementation is finitely axiomatizable for $n > 2$.[22] (The point in talking about *generalized* reducts is that derived operations of RCA_n like $\mathsf{c}_i^\partial = -\mathsf{c}_i-$ may occur in K.) The question is open for $n = 2$.

CONVENTION. An algebra $\mathfrak{R}$ is defined to be a *generalized reduct* of $\mathfrak{A}$ iff their universes coincide (R=A), and the operations of $\mathfrak{R}$ are term functions of $\mathfrak{A}$. I.e. iff $\mathfrak{R}$ is of the form $\langle A, \tau_i\rangle_{i\in I}$, where τ_i is a term function of $\mathfrak{A}$. A class K_1 is a generalized reduct of K_2 (formally $\mathsf{K}_1 = \mathbf{Rd}\mathsf{K}_2$) iff there is a uniform choice of the term functions $\langle \tau_i : i \in I\rangle$ along which each member of K_1 is a generalized reduct of one of K_2, and all such reducts from K_2 are in K_1. A *generalized subreduct* is defined to be a subalgebra of a generalized reduct. From now on we will drop the adjective "generalized". So *from now on*, by a reduct we will understand a generalized reduct (unless otherwise specified explicitly). More importantly, by a subreduct we will understand a generalized subreduct. We will use the convention for $\mathbf{SRd}$,

[21] This solves a problem on p. 342_{8-4} of Monk [M69] (there the problem was credited to W. Craig). In passing we note that Andréka also solved the problem preceding the quoted one in [M69] asking if $\mathbf{SNr}_n\mathsf{CA}_{n+k}$ is finitely axiomatizable. The rest of the problems in [M69] are still open.

[22] As a contrast, Andréka proved that the equational theories of RCA_n and RRA without complementation are decidable. (Using the terminology of [HMTI] 2.7.14 p. 439, validity of positive equations in RCA_n or RRA is decidable.) Cf. Andréka [A90a].

SPRd as introduced in Remark 2, but with the understanding that **Rd** means the *generalized* reduct. In particular, $\mathbf{Rd}\mathsf{K}_2$ is ambiguous because we must rely on context to specify the language (or similarity type) of $\mathbf{Rd}\mathsf{K}_2$. An example for such a context is $\mathsf{K}_1 = \mathbf{Rd}\mathsf{K}_2$, as explained in Remark 2.

EXAMPLES: $\langle \mathfrak{P}({}^3U), \tau^\circ, \tau^{-1}, \mathsf{c}_2, \mathrm{Id}\rangle$ discussed above is a reduct of the full Cs_3 $\langle \mathfrak{P}({}^3U), \mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \mathrm{Id}\rangle$ in the new sense, but it *was not* such in the old sense. Thus RRA's are subreducts of RCA_3's in the new sense (but were not such in the old one). Further, RCA_2's became subreducts of RRA's now, but they were not such before (hint: $\mathsf{c}_1(x) = x \circ 1$, where $1 = U \times U$). A third important example of generalized subreducts of RCA_3's will be Pinter's substitution-cylindric algebras obtained by introducing the term functions $\mathsf{s}_1^0(x) = \mathsf{c}_0(x \wedge \mathsf{c}_2(\mathrm{Id}))$ (and similarly for $\mathsf{s}_j^i(x)$ with $i, j < 3$, $i \neq j$), and considering the reducts $\langle \mathfrak{P}({}^3U), \mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \mathsf{s}_j^i : i, j < 3 \text{ and } i \neq j\rangle$. These algebras will be suitable for algebraizing logic without equality (so we will meet them in §5).

The ambiguous (or "context sensitive") nature of the notation **SRd** is illustrated by $\mathsf{RCA}_2 = \mathbf{SRd}\mathsf{RRA}$ and $\mathsf{RRA} = \mathbf{SRd}\mathsf{RCA}_3$ (especially when compared with $\mathsf{RCA}_2 = \mathbf{SRd}\mathsf{RCA}_3$ and $\mathsf{RCA}_2 \neq \mathsf{RRA}$). ∎

Recursive sets of equations characterizing RCA_n are given and discussed on pp. 112–119 of [HMTII], the first of these originating with Monk [M69]. Venema [V89] Def.3.3.3 & Thm.3.3.5 contain a new result in this line. The situation is analogous to the BRA case described below Theorem 2. Finding mathematically transparent, simple sets of equations defining RCA_n is one of the central open problems, cf. Problems 4.1, 4.16 of [HMTII]. We should point out the important result of Diane Resek that a slight generalization of RCA_n denoted by $\mathsf{CA}_n \cap \mathbf{I}\mathsf{Crs}_n$ in [HMTII] p. 101 is finitely axiomatizable; see Andréka–Thompson [AT88] for a simple proof.

([Ma89d] and Resek–Thompson [RT89] are also in this line.)

The following is an important digression.

The axiomatic approach, "abstract" (not necessarily representable) cylindric algebras (CA's), relation algebras (RA's) etc.

By Theorems 2, 4, the varieties RCA_n, BRA, and RRA, whose elements consist of concrete relations (up to isomorphism of course) are not finitely axiomatizable (non-finitizable, for short). Let us take the example of, say, RCA_n. Because of the above non-finitizability property of RCA_n, there is a conventional agreement in the literature for studying a fixed finitely axiomatizable variety CA_n *approximating* RCA_n. The members of CA_n are called *"(n-ary) cylindric algebras"*. By saying that CA_n approximates RCA_n we mean that $\mathsf{CA}_n \supseteq \mathsf{RCA}_n$ and that CA_n is as close to RCA_n as we can get without loosing attractiveness and mathematical transparency of our axioms describing CA_n. For example, the equations expressing that c_i is a closure operator are transparent enough, so we include them into the axiomatization of CA_n. Similarly for the equation $\mathsf{c}_i\mathsf{c}_j = \mathsf{c}_j\mathsf{c}_i$ expressing that the c_i's commute. On the other hand, consider the following statement $(*)$ about relations. For any $R \subseteq {}^nU$, let $Dom(R) = \{s_0 : s \in R\}$, i.e. $Dom(R)$ is the smallest H such that $R \subseteq H \times {}^{n-1}U$. Now,

$(*)$ If $R \subseteq {}^nU$ and $0 < |Dom(R)| = m < n$ then it is impossible to have disjoint relations $R_0, \ldots, R_m \subseteq R$ such that $(\forall k \leq m)\, \mathsf{c}_0 R_k = \mathsf{c}_0 R$.

$(*)$ is translatable into a cylindric algebraic equation $e_{(*)}$ using $m+2$ variables (standing for $R_0, \ldots, R_m$, and R). This translation is based on the fact that $|Dom(R)| = m$ can be expressed by using Id. This equation $e_{(*)}$ has been considered not elegant enough for being added as an axiom of CA_n's.[23]

If besides RCA_n we would like to see another example, then (3.1) is a transparent enough property of RRA such that it is postulated as an axiom of the finitary variety RA (*relation algebras*) approximating RRA. (A "not transparent enough" equation can be obtained from the above $e_{(*)}$ by choosing $n = 6$ and instead of expressing $|Dom(R)| < 6$ by Id which is impossible in RRA, letting ${}^2(Dom(R)) = S \dot\cup T \dot\cup \mathrm{Id}$ with S, T symmetric, antireflexive relations such that $T = (S \circ S) \setminus \mathrm{Id}$ and $S = (T \circ T) \setminus \mathrm{Id}$.)

The above circumscription of when we say that a finitizable variety V approximates a non-finitizable one, say RV, is not a precise mathematical definition, but we hope it may have some heuristic value. The name of the approximated variety begins with an R (like RV, RCA_n, or RRA) to remind the reader that these are the members of V which can be *represented* as algebras of real relations (it is left to the reader to decide which of the three "r"-s should the R in RV stand for).

To each of the distinguished varieties like RRA, RCA_n, RPA_n etc. discussed in this paper, there corresponds in the literature a finitizable approximation RA, CA_n, PA_n etc. respectively, analogously to the approximation of RCA_n

[23] Actually, from time to time some hesitation comes up concerning the status of *this* particular equation $e_{(*)}$, and perhaps sometime in the future some reinforced hyper-strong subvariety HCA_n of CA_n, satisfying $e_{(*)}$, might be introduced, but there are more complicated versions of $e_{(*)}$ which, very probably, will not be added to HCA_n either.: Let $|Dom(R)| = m$ and assume $n < m < \omega$. Then there are more complicated (than using Id) ways of equationally "forcing" $|Dom(R)| = m$ by introducing new relations which "count" the size of $Dom(R)$. The so obtained variant of $e_{(*)}$ is a typical example of a known validity of RCA_n which does not count as transparent enough to be added as an axiom of the approximating variety CA_n.

by CA_n. We will discuss these approximating varieties CA_n, RA etc. in the Appendix (§8), where we will also give references for their further study.

While the concrete algebras of relations like RCA_n or RRA provide the central tool for algebraizing the semantic aspects of logics, their finitary approximations like CA_n or RA are useful in algebraizing proof theoretic concepts, see [HMTII] §4.3 (and implicitly in [TG87]). From a different angle, the relationship between RCA_n, RRA etc. and CA_n, RA etc. seems to be somewhat analogous to that between the standard model $\mathfrak{N} = \langle \omega, +, \times, 0, 1 \rangle$ of arithmetic and Peano's Arithmetic. E.g. both the theory of Peano's Arithmetic and CA_n (or RA) devote considerable effort to developing methods for constructing nonstandard models (i.e. non-representable algebras), cf. e.g. [HMTII] pp. 85–100. Since much of the intuition behind the theories of Peano's Arithmetic, CA_n and RA derives from the standard models (i.e. $\mathfrak{N}$, RCA_n and RRA), we will concentrate on the representable algebras RCA_n, RRA etc., and will deal with their finitary approximations CA_n, RA etc. only very briefly, in the present paper. For completeness, full definitions of CA_n, RA etc. as well as further information on these abstract (or "approximating") classes are available in the Appendix (§8) of this paper.

* * * * *

Let us return to the connections between RCA_n and <u>algebraization of logic</u>.

Let $\mathfrak{M}$ be a model of a finite language (or signature). We may treat all symbols of the language as relation symbols as it is well known, cf. Bell–Slomson [BS69] §2.10 pp. 97–100. So $\mathfrak{M}$ is of the form $\mathfrak{M} = \langle M, R_0, \ldots, R_k \rangle$. Let n be a strict upper bound of the ranks of $R_0, \ldots, R_k$. We may safely treat each R_i as if it were of rank n by using our dummy representation $Dr(R_i) = R_i \times U \times U$ if the rank of R_i is, say, $n-2$. Now, $R_0, \ldots, R_k$ are elements of the full Cs_n $\mathfrak{Rel}_n(M) = \langle \mathfrak{P}({}^nM), \mathsf{c}_0, \ldots, \mathrm{Id} \rangle$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{Rel}_n(M)$ generated by $\{R_0, \ldots, R_k\}$. Now, $\mathfrak{C}$ is the algebraic counterpart (consisting of n-ary relations) of $\mathfrak{M}$ as a result of the standard algebraization of logic described in §4.3 of [HMTII] (where $\mathfrak{C}$ was denoted by $\mathfrak{Cs}_n^{(\mathfrak{M})}$). One can check that $\mathfrak{C}$ consists of exactly those relations over M which are definable in $\mathfrak{M}$ with using at most n variables. It is natural that the algebraic counterparts of models (i.e. of the semantic part of first-order logic) should be algebras consisting of those relations that are definable in the original models. Indeed, it is these definable relations which provide meanings for the formulas of first-order logic; hence by using them we can arrive at Lindenbaum–Tarski algebras of equivalence classes of formulas (which are the so called formula algebras of theories), i.e. algebraic counterparts of syntactic notions of logic. Moreover, along these lines we can arrive at algebraic counterparts of not only syntactical and semantical notions of logic, but also at an algebraic counterpart of the *syntax–semantics duality* in logic, including the notion of axiomatizable classes of models.[24] Of course, one can repeat the same process of algebraization with the same key ideas (algebras consisting of general relations conceived of as meanings of formulas) for different quantifier logics, e.g. first-order modal logics, higher order logics, logics with new quantifiers like "for many v_i φ", abstract model theoretic logics from Barwise–Feferman [BF85]. The details are worked out explicitly in §4.3 of [HMTII] for classical first-order logic in particular, and in §5.6 for logics in general. See also Appendix C of Blok–Pigozzi [BP89] and Németi [N89].

Let us have a concrete look at RCA_n. The logic naturally corresponding to RCA_n is first-order logic L_n restricted to the first n individual variables. Besides [HMTII], see also [TG87] §§3.10, 7.3, where L_n is denoted as $\mathcal{L}_n$, and sometimes as $\mathcal{L}_n^+$, $\mathcal{P}_n$. To illustrate the correspondence with logic, we note that Theorem 4 above implies a strong incompleteness theorem for L_n. Roughly speaking, this logical corollary of Theorem 4 says that no inference system given by a finite set of schemata of axioms and derivation rules[25] can be complete and sound for L_n. Similarly, an algebraic theorem in [ACN] implies that L_n does not enjoy the Beth definability property. In passing we note that following suggestions and questions from Leon Henkin, Sain [S89]

[24] In this connection we note that, roughly speaking, the algebraic counterparts of the basics of logical syntax (i.e. of the sets of logical formulas with the logical connectives acting on them) are the free algebras, cf. e.g. [AS78]. Accordingly, the free algebras of RCA_n, RRA, and of their finitizable approximations CA_n and RA (discussed above and in §8) have already received some attention. See e.g. Pigozzi's results concerning free algebras in [HMT, HMTII]; [N86], [N85a] for free cylindric algebras (and free RCA_n's), and Andréka–Jónsson–Németi [AJN88] for free relation algebras, free RRA's, and free algebras in other forms of algebraic logic (but see also Blok [Bl76], Urquhart [U73] for the latter). The field of free algebras is difficult to develop, and therefore this area is far from being closed, there are many open problems, and unexplored territories here.

[25] Here we use these words in the traditional sense, which was formalized e.g. on p. 5 of Blok–Pigozzi [BP89]. Inference systems of this kind are sometimes called Hilbert-style ones. An almost Hilbert-style complete inference system for L_n was recently obtained in Venema [V91]. This is given by a finite schema (but the notion of an inference rule is different from ours or from that of [BP89]). For completeness we note the following. Let $\vdash_n$ denote provability by one of the standard Hilbert style axiomatizations of L_n (cf. e.g. [HMTII] or [TG87]). Basically, $\vdash_n$ is obtained by restricting the usual axiomatization of $L_{\omega\omega}$ to L_n. If we take the syntactic version of the Lindenbaum–Tarski algebra of L_n *induced by* $\vdash_n$ then we obtain the free CA_n discussed in the preceding subsection ("The axiomatic approach...") and in §8. If we take all $\vdash_n$-consistent theories of L_n and their Lindenbaum–Tarski algebras (modulo $\vdash_n$ and not semantically) then we get exactly the members of CA_n.

discovered that the weak Beth property (considered the more important in Barwise–Feferman [BF85]) still might hold for L_n, because the corresponding algebraic question is still open for RCA_n's (and also for RRA's). This algebraic question asks whether for every epimorphism $f : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ w.r.t. which all full Cs_n's are injective, f is onto $\mathfrak{B}$ (here $\mathfrak{A}$, $\mathfrak{B}$ are arbitrary RCA_n's). Recently, an even weaker version of the Beth property was proved to hold for L_n in a collaboration of K. A. Kearnes and I. Sain using algebraic logic. (For more on the algebraic counterparts of logical properties like Beth's definability and Craig's interpolation see e.g. Maksimova [Ms77, Ms88], [HMTII] pp. 259–260, [HMT] pp. 356–357, p. 178, [ANS84], [Co84], [TR87], [W84], [O86], Goranko [G85].)

How far did we get in obtaining algebras of relations in general (binary, ternary, ..., n-ary, ...)? RCA_n is a smooth and satisfactory algebraic theory of n-ary relations. So, can our theory handle all finitary relations? The answer is both yes and no. Namely, since $n \in \omega$ is arbitrary, in a sense, we can handle all finitary relations. But, we cannot have them *all* in the same algebra or in the same variety. For any finite family of relations, we can pick n such that they are all in RCA_n. But this does not extend to infinite families of relations. To alleviate this, we could try working in the system $\langle \mathsf{RCA}_n : n \in \omega \rangle$ of varieties instead of using just one of these. To use them all together, we need a strong coordination between them. This coordination is easily derivable from the embedding function Dr sending R to $R \times U$ for $R \subseteq {}^nU$ defined above. Let $\mathfrak{A} \subseteq \mathfrak{Rel}_n(U) = \langle \mathfrak{P}({}^nU) \cdots \rangle$ be a Cs_n and let $\mathfrak{B}$ be the Cs_{n+1} generated by the Dr image of $\mathfrak{A}$, i.e. $\mathfrak{B} \subseteq \mathfrak{Rel}_{n+1}(U) = \langle \mathfrak{P}({}^{n+1}U) \cdots \rangle$ is generated by $\{Dr(R) : R \in A\}$. The biggest $\mathfrak{A}$ yielding the same $\mathfrak{B}$ is called the n-ary *neat-reduct* of $\mathfrak{B}$, formally $\mathfrak{A} = \mathbf{Nr}_n(\mathfrak{B})$. It is easy to extend $\mathbf{Nr}_n$ in a natural way to all elements of RCA_{n+1}. Intuitively, $\mathbf{Nr}_n(\mathfrak{B})$ is the algebra of n-ary relations "living in" the algebra $\mathfrak{B}$ of $n+1$-ary relations. It is not hard to see that $\mathbf{Nr}_n : \mathsf{RCA}_{n+1} \longrightarrow \mathsf{RCA}_n$ is a functor, in the category theoretical sense, for every n. Now, we can use the collection of varieties RCA_n for all $n \in \omega$, synchronized via the functors $\langle \mathbf{Nr}_n : n \in \omega \rangle$, as a single mathematical entity containing all finitary relations.

Another possibility is to insist that we want all finitary relations over U represented as elements of a *single* algebra. In other words, this goal means that instead of a system of varieties we want to consider a single variety that in some sense incorporates all the original varieties taken together. Indeed, each RCA_n can be viewed as incorporating all the RCA_k's for $k \leq n$, since the latter can be recovered from RCA_n by using the functors $\mathbf{Nr}_{n-1}$, $\mathbf{Nr}_{n-2}$ etc. So as n increases, RCA_n gets closer and closer to the variety we want. Indeed, we take the limit of this sequence. There are two ways of doing this, the naïve way we will follow here and the category theoretical way we only briefly mention. It is shown in the textbook Adamek–Herrlich–Strecker [AHS] that the system or "diagram"

$$\mathsf{RCA}_1 \overset{\mathbf{Nr}_1}{\longleftarrow} \mathsf{RCA}_2 \overset{\mathbf{Nr}_2}{\longleftarrow} \cdots \mathsf{RCA}_n \overset{\mathbf{Nr}_n}{\longleftarrow} \mathsf{RCA}_{n+1} \cdots$$

is "convergent" in the category theoretic sense, i.e. that it has a limit L. Indeed, it is this class L of algebras that we will construct below in a naïve way that does not use category theoretic tools or concepts.

We first extend our Convention, stated at the beginning of the present section concerning associativity of Cartesian products and powers. *In the sequel* ${}^{\omega}U$ is the set of ω-sequences over U. Furthermore, ${}^nU \times {}^{\omega}U = {}^{\omega}U$, and if $R \subseteq {}^nU$ then $R \times {}^{\omega}U \subseteq {}^{\omega}U$, for $n < \omega$. We will also have to distinguish the constant Id of RCA_3 from that of RCA_4. Therefore we let $\mathrm{Id}_n \overset{\text{def}}{=} \{\langle a, \dots, a\rangle : a \in U\}$ denote the n-ary identity relation on U.

How do we obtain an algebra containing all finitary relations over U? If R is binary, but we want to treat it together with a 5-ary relation then we represent R by $R \times U \times U \times U = R \times {}^3U$ in a Cs_5. Similarly, if we want to have R together with $(n+2)$-ary relations then we represent R with $R \times {}^nU$ in a Cs_{n+2}. Taking this procedure to the limit, if we want to treat R together with relations of arbitrary high ranks, then we can represent R with $R \times {}^{\omega}U$. This way we can embed all finitary relations into relations of rank ω, and relations of different ranks become "comparable" and "compatible"; in particular we avoid the problem we ran into at the end of §2 in connection with the Boolean algebra $\mathfrak{P}({}^{<\omega}U)$. Instead of trying to tame the Boolean-like algebra $\{R : R \subseteq {}^nU \text{ for some } n\} \subseteq \mathcal{P}({}^{<\omega}U)$, we simply represent $R \subseteq {}^nU$ by $R \times {}^{\omega}U$ which *is* an element of the BAO (Boolean algebra with operators) $\langle \mathfrak{P}({}^{\omega}U), \mathsf{c}_0, \dots, \mathsf{c}_n, \dots\rangle_{n<\omega}$, where $\mathfrak{P}({}^{\omega}U)$ *is* a BA, and c_i is defined exactly the same way as in the Cs_n case. We still haven't obtained the definition of Cs_ω's from that of Cs_n's because we do not know what to do with the constant Id. More specifically, we want to be able to use the neat reduct functor $\mathbf{Nr}_n$, as the inverse of $R \mapsto R \times {}^{\omega}U$ for $R \subseteq {}^nU$, in order to recover the original Cs_n's from the new Cs_ω. This means that for $\mathrm{Id}_n \subseteq {}^nU$ we want $\mathrm{Id}_n \times {}^{\omega}U$ to be a derived constant (distinguished element) in our algebra. Adding $\mathrm{Id}_\omega = \{\langle a, \dots, a, \dots\rangle : a \in U\}$ as an extra constant does *not* ensure this any more. One of the most natural solutions is letting $\mathrm{Id}_{ij} = \{q \in {}^{\omega}U : q_i = q_j\}$ and defining a full Cs_ω as

$$\mathfrak{Rel}_\omega(U) \overset{\text{def}}{=} \langle \mathfrak{P}({}^{\omega}U), \mathsf{c}_i, \mathrm{Id}_{ij}\rangle_{i,j<\omega} ,$$

where the Id_{ij}'s are constants. The price we had to pay for replacing the finite bound n on the ranks of relations we can treat with the infinite bound ω

is that we had to break up our single constant Id to infinitely many constants Id_{ij} $(i, j \in \omega)$.

RCA_ω is defined to consist of all subalgebras of direct products of full Cs_ω's (up to isomorphisms). Again, as it was the case with BRA's and RCA_n's, RCA_ω's are directly representable as algebras whose elements are ω-ary relations.

THEOREM 5 ([HMTII] 3.1.103, p. 43). *RCA_ω is a variety.*

ON THE PROOF. The proof is harder than those of Theorems 1, 3. The reason for this might be the fact that RCA_ω is not a discriminator variety unlike RCA_n, RRA, and BRA, which are discriminator varieties. In this connection we note that despite of this, many of the results of the theory of discriminator varieties carry over to RCA_ω because of the following. RCA_ω is "reduct-locally discriminator" in the sense that for any finite part t_0 of the similarity type t_1 of RCA_ω, there is a $t_0 \le t \le t_1$ such that $\mathbf{SRd}_t\mathsf{RCA}_\omega$ is a discriminator variety.

Let $\mathfrak{B} \in \mathsf{RCA}_\omega$ and $\mathfrak{A} = \mathfrak{B}/\Theta$ for some congruence Θ of $\mathfrak{B}$. For $n \in \omega$, let $\mathbf{Rd}_n\mathfrak{A}$ be the reduct of $\mathfrak{A}$ to the language of RCA_n's. It is not hard to see that $\mathbf{Rd}_n\mathfrak{B} \in \mathsf{RCA}_n$. By Theorem 3 then $\mathbf{Rd}_n\mathfrak{A} = (\mathbf{Rd}_n\mathfrak{B})/\Theta \in \mathsf{RCA}_n$. To prove Theorem 5, it is enough to prove that

$$(*) \qquad (\forall n \in \omega)\mathbf{Rd}_n\mathfrak{A} \in \mathsf{RCA}_n \text{ implies } \mathfrak{A} \in \mathsf{RCA}_\omega$$

for any $\mathfrak{A} \in \mathbf{HSP}\mathsf{RCA}_\omega$.

It is not very easy to prove $(*)$, but it is not hopelessly hard either. E.g. one can use ultraproducts in "pasting together" the representations of the $(\mathbf{Rd}_n\mathfrak{A})$'s in order to represent $\mathfrak{A}$. For this the basic lemma [HMTII] 3.1.92 (p. 36) discussing ultraproducts of Cs_ω's and their generalizations Gs_ω's is sufficient. ∎

For completeness, we note that [HMTII] 3.1.103 p. 43 gives a direct proof for Theorem 5 not using discriminator varieties or other special tools of universal algebra. Further, it is useful to keep in mind that though RCA_ω is not a discriminator variety, it is an arithmetical (congruence distributive and congruence permutable) one. For the rather strong theory of the latter see [MMT] p. 247, [BS81]. Actually, every variety of BAO's is arithmetical by [MMT] Thm.4.143 or [BS81] Thm.II.12.5 due to Pixley.

We note that RCA_ω is the variety generated by the class L which was the limit of the RCA_n's above, see [AHS]. Can we actually recover the algebras of finitary relations from the huge full Cs_ω's?

Let $\mathsf{Rf}(U) = \{R \times {}^\omega U : R \subseteq {}^n U \text{ for some } n\}$. Then $\mathsf{Rf}(U) \subseteq \mathcal{P}({}^\omega U)$; moreover it is a subalgebra of the full Cs_ω $\mathfrak{Rel}_\omega(U)$ with universe $\mathcal{P}({}^\omega U)$;

see [HMTII] Def.3.1.130 on p. 55, where this subalgebra was denoted by $\mathfrak{R}f(U,\omega)$. We will use the notation $\mathfrak{R}\mathsf{f}(U)$. (The letters Rf (and $\mathfrak{R}\mathsf{f}$) refer to "finitary relations".) Now, for the above mentioned class L we have $\mathsf{L} = \mathbf{S}\{\mathfrak{R}\mathsf{f}(U) : U \text{ is a set}\}$.[26] In a sense, L is the narrowest reasonable class of algebras of finitary relations. The class L and its relationship with RCA_ω was systematically investigated in Andréka [A73], Andréka–Gergely–Németi [AGN73], [AGN77], [HMTAN], [HMTII]. In the first three works the class was denoted by Lv or Lr, while in the last two by $\mathsf{Cs}_\omega^{reg} \cap \mathsf{Lf}_\omega$, the latter being the standard notation today. Cf. e.g. Shelah [Sh89], Monk [M89], Serény [Se86], for recent results not in [HMTII]. $\mathsf{RCA}_\omega = \mathbf{HSP}\mathsf{L}$ is Thm.3.1.123 of [HMTII]. [HMTAN] proves that the smallest quasivariety containing L is also RCA_ω but $\mathsf{RCA}_\omega \neq \mathbf{SP}\mathsf{L}$, i.e. the infinitary quasivariety generated by L is strictly smaller than RCA_ω.

We return briefly to <u>algebraization of logic</u>: RCA_ω is the true algebraic counterpart of full first-order logic (with equality). Within RCA_ω, the algebraic counterparts of models of first-order logic are the members of L. Indeed, let $\mathfrak{M} = \langle M, R_i\rangle_{i\in I}$ be a first-order model. Then the subalgebra of $\mathfrak{R}\mathsf{f}(M)$ generated by $\{R_i \times {}^\omega M : i \in I\}$ is the algebraic counterpart of $\mathfrak{M}$.[27] This is not very surprising after having seen that the logical counterpart of RCA_n was first-order logic L_n restricted to the first n variables. It is natural to expect that removing the bound n on the ranks of the relations will result in removing the bound on the number of variables, in the corresponding first-order logic.

The algebraic counterparts of the logical connectives are operations of RCA_ω. (This is standard in algebraic logic; see [HMTII] §5.6, or Blok–Pigozzi [BP89], or Andréka–Sain [AS78]). The algebraic counterpart of $\exists v_i$ is c_i, and that of $v_i = v_j$ is Id_{ij}. With these, substitution of variables is expressible; namely the result of substituting v_j for all free occurrences of v_i in the formula φ is equivalent to $\exists v_i(v_i = v_j \wedge \varphi)$; cf. [HMTII] §4.3. In particular, $R(v_1 v_1)$ is $\exists v_0(v_0 = v_1 \wedge R(v_0 v_1))$, $R(v_1 v_0)$ is $\exists v_2\left[v_2 = v_0 \wedge \exists v_0\left(v_0 = v_1 \wedge \exists v_1[v_1 = v_2 \wedge R(v_0 v_1)]\right)\right]$. Therefore we do not need to introduce an algebraic operation to represent substitutions of individual variables as we did in the case of BRA's, by means of $^{-1}$.

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[26] For this equality to be literally true, when forming the category theoretic limit L, instead of the varieties RCA_n we have to start out from their subdirectly irreducible members, which are nothing but Cs_n's. So L is the limit of the sequence $\mathsf{Cs}_1, \dots, \mathsf{Cs}_n, \dots$.

[27] For certain purposes the algebraic counterparts of models are not elements of the class L, but rather homomorphisms from free algebras (namely, free RCA_ω's over various generator sets) into elements of L; cf. [HMTII] pp. 256, 257_9; [AS78] pp. 46–48, where connections with Initial Algebra Semantics of Computer Science are explained; and Halmos [Ha85] p. 208 lines 6–7.

So, we know what the algebraic counterparts of logical connectives and of individual models are. The algebraic counterparts of *classes of models* (especially of axiomatizable classes) are certain elements[28] of **SP**L, cf. p. 168 of [HMTII] and Németi [N78]. The algebraic counterpart of a class K of models was denoted by $\mathfrak{Gs}^{\mathsf{K}}$ in [HMTII].

Moving in the opposite direction, the logical counterparts of cylindric equations (i.e. equations in the language of RCA_ω's) are not concrete first-order formulas, as one would expect, but instead they are schemata of formulas, see Németi [N87][29]. A typical example for a schema is $\varphi \to \exists v_1 \varphi$, where φ is a variable ranging over formulas. The corresponding cylindric equation is $x \le \mathsf{c}_1 x$. Examples for logical investigations where schemata of formulas play a central rôle are the recent Bernarducci [B89], [B89a], Vardanyan [V86], Smorynski [Sm85], [Sm84].

In more detail, the connection between first-order logic (also some of its generalizations) and RCA_ω theory (including its generalizations) is elaborated in §4.3 of [HMTII], Németi [N89], [N78], Blok–Pigozzi [BP89] Appendix C. (M. Rubin has results concerning decidability etc. of equational theories of RCA_ω's associated to various theories of logic.)

The theory of RCA_ω is extensively developed (cf. [HMT] Chapters 1–4, and [HMTAN]). The few results we can mention here are very far from forming even a representative sample. The lattice of subvarieties of RCA_ω together with their decision problems is investigated in [N87], [N85]. Open problems (about subvarieties etc.) that might be rewarding to work on are in [HMTAN], and [HMTII] pp. 179–180. E.g. Problem 4.2 in the latter asks if there are $2^{|\alpha|}$ many subvarieties of RCA_α for $|\alpha| > \omega$, where RCA_α is the natural generalization of RCA_ω to be defined soon. (A list indicating the status of the problems raised in [HMT], [HMTII], [HMTAN] is available from Monk, and has been published in [AMN].) Sain [S89] contains, besides recent results, important and natural open questions. One of them asks for a universal algebraic (or category theoretic) characterization of the full Cs_ω's as members of the variety RCA_ω. The analogous problem is also open for RCA_n and RRA. An important recent direction is taken in Monk [M89]. There is a Galois theory of RCA's, see e.g. Comer [Co84], Daigneault [D64], Driessel [Dr68], Reyes [R70], [M89], Plotkin [Pl88, Pl89]. We should also

[28] This claim might puzzle some readers because classes of models are "big" when compared to single models (so how come they both translate to single algebras). In this connection we would like to point out that in a certain algebraic sense the elements of **SP**L are big when compared to those of L. (The elements of L are subdirectly irreducible while those of **SP**L are not.)

[29] At least this is the case in many situations. A case when the logical counterparts can be treated as concrete formulas is Thm.4.3.57 of [HMTII].

mention the works of Serény [Se85], [Se86], Ferenczi [F89, F89a], Shelah [Sh89], Biró–Shelah [BiS88].

Theorems 2 and 4 above,which say that BRA, RRA, and RCA_n are not finitely axiomatizable, carry over to RCA_ω too, but to avoid triviality, instead of non-finite axiomatizability we have to state something stronger, because RCA_ω has infinitely many operations and finitely many axioms can speak about only finitely many operations anyway. Taking this into account, when trying to axiomatize RCA_ω, one could still hope for a finite "schema" (in some sense) of equations treating the infinity of the RCA_ω-operations uniformly. A possible example for a finite schema is (E0–7) in the proof of Proposition 8.3 in §8 (Appendix). The following theorem, generalizing a very important result of Monk ([HMTII] 4.1.7), implies that it will be hard to find such a schema, and that certain kinds of schemata are ruled out to begin with.

THEOREM 5.1 (Andréka). *The variety RCA_ω is not axiomatizable by any set Σ of universally quantified formulas if Σ contains only finitely many variables.*

This theorem was proved for RCA_α, to be introduced soon, for any ordinal $\alpha > 2$. Andréka found a rather simple proof for Theorem 5.1 in January 1986. This way she also found a simple proof for Monk's important non-finitizability theorem ([HMTII] 4.1.7). Andréka's simple proof is available in [A91], [M91].

Returning to the problem of finitization first discussed in Remark 2, the infinite number of basic operations of RCA_ω presents itself as a new kind of obstacle. Of course, one could try to alleviate this by changing the language of RCA_ω such that all the old operations $\{\mathrm{c}_i, \mathrm{Id}_{ij} : i, j \in \omega\}$ would become term definable in a new finite language and *then* one would search for a finite axiomatization. However, this approach to the problem belongs to a relatively big and central project, called true finitization, of mainstream algebraic logic. This project is in the spirit of Remark 2 way above, cf. Sain [S87a], and we will return to it at the end of this section, and in §§6,7. In the next paragraph, we will aim for axiomatizability with a finite *schema* only.[30] We will need the following operations on relations expanding RCA_ω. Let $R \subseteq {}^\omega U$ and $\tau : \omega \longrightarrow \omega$. Then $\mathrm{S}_\tau(R) \stackrel{\text{def}}{=} \{q \in {}^\omega U : q \circ \tau \in R\}$. The logical counterpart of the operation S_τ is substitution of variables along τ, e.g. $\mathrm{S}_\tau\big(R_1(v_0 v_1)\big)$ is intuitively $R_1\big(v_{\tau(0)}, v_{\tau(1)}\big)$.

[30] Here finite schema means something like (E0–7) in §8 (Appendix) herein. There are precise definitions of a finite schema in [HMTII] pp. 110, 261 but it is not necessary to look these up in order to understand the present paper; it is enough to know that a finite schema may contain only finitely many variables, in the sense of [HMTII].

Advanced topics in RCA-theory

Now we can turn to *finite schema axiomatizability* as a continuation of Remark 2. From now on, a finite schema is allowed to contain only finitely many variables. By Theorem 5.1, RCA_ω is not axiomatizable by a finite schema. W. Craig expanded RCA_ω with the two operations S_{succ} and S_{pred}, where the indices are the usual successor and predecessor functions on ω. Craig proved that the equational theory of the so expanded version of RCA_ω is axiomatizable by a finite schema. However, the price for this is that Craig's algebras do not form a variety; in fact they are not closed under ultraproducts, so they do not form an axiomatizable class, cf. [HMTII] p. 265 or Craig [Cr74]. The most active proponents of Craig's approach at present are, among others, I. Sain and R. J. Thompson, but see also the references on p. 265 of [HMTII], and §7 (2–4) herein. Continuing Craig's work, Sain has obtained a class of algebras which is an adequate algebraic counterpart of first-order logic without equality. (In Sain's algebras, for every finite transformation τ of ω, and for every $i \in \omega$, S_τ and c_i are term definable, but Id_{ij} is not.) Sain proved that her variant of RCA_ω is a variety axiomatizable by a finite schema of equations.[31] (This is related to but not identical with the results in the current version of Sain [S87a].) There are several intriguing open problems in Sain's works on Craig's approach, the solutions of which could provide considerable insight into the basic questions of algebraic logic. Most of these problems admit very simple purely semigroup theoretic formulations. (The main bulk of these is stated either in the June 1987 version or in the current version of Sain [S87a].)

CONVENTION. *Throughout, both ω and $n \in \omega$ are regarded as ordinals. Further, $n = \{0, \dots, n-1\}$ for $n \in \omega$, and more generally $\alpha = \{\beta :$ β is an ordinal and $\beta < \alpha\}$ for any ordinal α. For any ordinal α, ${}^\alpha U$ is the set of α-sequences of members of U. Formally,*
${}^\alpha U = \{f : f$ is a function mapping α into $U\}$.
The definition remains the same if α stands for an arbitrary set (and is not necessarily an ordinal).

To treat RCA_n and RCA_ω in a unified manner, we replace ω in the definition of RCA_ω with an arbitrary but fixed ordinal α, obtaining RCA_α (here $\alpha = n$ and $\alpha = \omega$ are of course permitted). This generalization will also be

[31] This result appears as Thm.1 on p. 3 of the June 1987 version of Sain [S87a] (a version of [S87a] that differs substantially from the current one). Thm.1 is proved on pp. 34–37 in §2, and Remark 10.1 on p. 37 contains a simplified finite schema defining her version of RCA_ω. This schema is indeed simple. Page and item numbers in this footnote refer to parts of op. cit.

useful in algebraizing various quantifier logics different from classical first-order logic $L_{\omega\omega}$. In particular, a full Cs_α is of the form $\langle \mathfrak{P}({}^\alpha U), \mathsf{c}_i, \mathrm{Id}_{ij}\rangle_{i,j<\alpha}$ defined analogously to RCA_n and RCA_ω.

An important and natural generalization of RCA_α is obtained by relaxing the condition that the top (i.e. largest) element of a full Cs_α is of the form ${}^\alpha U$. Namely, let $V \subseteq {}^\alpha U$. By a full *relativized* Cs_α, or a full Crs_α, we understand $\langle \mathfrak{P}(V), \mathsf{c}_i, \mathrm{Id}_{ij}\rangle_{i,j<\alpha}$, where the operations c_i and Id_{ij} are now relativized to V, e.g. $\mathrm{Id}_{ij} = \{q \in V : q_i = q_j\}$. (In the abbreviation Crs, the "r" refers to "relativized".) Subalgebras of full Crs_α's are called Crs_α's, and the class of isomorphic copies of Crs_α's is denoted by $\mathbf{I}\mathsf{Crs}_\alpha$ (see [HMTII] §5.5). The elements of Crs_α are still natural algebras of α-ary relations. The elements of a Crs_α $\mathfrak{A}$ are obviously α-ary relations, the only difference with RCA_α's is that now the greatest relation (i.e. the top element of $\mathfrak{A}$) is an arbitrary α-ary relation and not necessarily a Cartesian space or a disjoint union of such spaces. This flexibility will be useful in algebraizing quantifier logics different from $L_{\omega\omega}$ (like higher order, many-sorted, and nonclassical logics); cf. e.g. [N78]. Before looking into different quantifier logics, we mention some algebraic results.

THEOREM 6. $\mathbf{I}\mathsf{Crs}_\alpha$ *is a variety whose equational theory is not axiomatizable by a finite schema (for $\alpha > 2$), but is decidable.*

For a **proof** see Thm's 5.5.10, 5.5.13 of [HMTII] and [N86].

Related results for RRA's are in Maddux [Ma82] and Németi [N87a]; see also Kramer [K89]. The equational theory of BRA is already undecidable, moreover so is that of any variety containing as a subreduct either the Id-free subreduct of RCA_3 or the ${}^{-1}$-free subreduct of BRA (a result of Maddux; see [HMTII] 5.1.66). As a contrast, the equational theory of RCA_2 is both decidable and finitely axiomatizable ([HMTII] Thm's 4.2.9, 3.2.65 pp. 136, 84).

An important positive result concerning the *finitization problem* for algebras of relations (see Remark 2, and the discussion not too far above) is based on the following breakthrough by Diane Resek. In the sequel, 1 is the Boolean constant denoting the top element of the algebra.

THEOREM 7 (Resek–Thompson). *The elements of* Crs_α *satisfying* $\mathsf{c}_i\mathrm{Id}_{ij} = 1$ *(for all $i, j \in \alpha$) form a variety* V_α *axiomatizable by a finite schema of equations. (If α is finite,* V_α *is finitely axiomatizable.)*

Note that a Crs_α $\mathfrak{A}$ satisfies $\mathsf{c}_i\mathrm{Id}_{ij} = 1$ iff its top element satisfies a very simple set theoretic condition. For a relatively simple **proof** of Theorem 7 see Andréka–Thompson [AT88]. (Improvements and discussions of some of the conditions in [AT88] and similarly simple proofs of results related to Theorem 7, with an emphasis on relativization, are in [Ma89d].)

The subvariety V_α of $\mathbf{I}\mathsf{Crs}_\alpha$ in Theorem 7 has a decidable equational theory if $\alpha < \omega$ (Németi [N86]). It is an open problem whether this carries over to $\alpha = \omega$.

Let us turn to applying Crs_α to algebraizations of different quantifier logics. Let us consider *many-sorted logic* first (as shown in Barwise–Feferman [BF85] or [S87b], many-sorted logic is a good unifying framework for abstract model theoretic logics, nonclassical, and various unusual logics). Let U and W be two disjoint sorts in the logic we want to algebraize. Let $V = {}^{\omega}U \times {}^{\omega}W$. Then, according to our Convention concerning associativity of $\times$ and Cartesian power, $V \subseteq {}^{\omega+\omega}(U \dot{\cup} W)$. Then $\mathfrak{A} = \langle \mathfrak{P}(V), \mathsf{c}_i \ldots \rangle_{i,j<\omega+\omega}$ is a $\mathsf{Crs}_{\omega+\omega}$. Here c_i corresponds to quantifying over variables of sort U or W, depending on whether or not $i < \omega$, and similarly for Id_{ij} when i and j are "in the same copy of ω". The fact that $\mathrm{Id}_{0\omega} = 0$ corresponds to the logical fact that equality is not defined between elements of different sorts. Now, we could repeat, with appropriate changes, what we said before about the connection between RCA_ω and $L_{\omega\omega}$. The idea extends to an arbitrary number of sorts, in the obvious way. See Németi [N78], LeBlanc [L59, L62], [AGN75], Plotkin [Pl89, Pl88], for more detail. Now, many-sorted logic is a natural stepping stone for higher order logic.

To treat *second-order logic*, for example, we can take $V = {}^{\omega}U \times {}^{\omega}\mathcal{P}(U)$ and expand the $\mathsf{Crs}_{\omega+\omega}$ $\langle \mathfrak{P}(V), \mathsf{c}_i \ldots \rangle$ with the new constant $E = \{q \in V : q_0 \in q_\omega\}$ to represent algebraically the "element of" relationship between the first-order and second-order objects. One can do the same for higher order logics, see e.g. [AGN75, AGN75a], LeBlanc [L59], Venne [V65, V66], Salibra [Sa89].

It is also indicated in Németi [N78], Freeman [F76], Georgescu [G79a], Monk [M60] how Crs_α can be used for algebraizing nonclassical quantifier logics like *first-order modal logic* or *first-order temporal logic*. The use of Crs_α's as a unifying framework is more explicit in the first citation, while more details are available in the others, especially in [F76]. Roughly speaking, a Kripke model for first-order modal logic consists of a universe U of individuals, and a set W of possible states or "worlds" or time instances together with some further structure. In this case we choose the top element V of our Crs_α to be $W \times {}^{\omega}U$. Intuitively, $\langle w, q_0, \ldots, q_i, \ldots \rangle \in R$ means that $\langle q_0 \ldots q_i \ldots \rangle$ is in the relation R when viewed from the possible world or state w.

The point we are trying to make here is that algebraizing non-classical (say modal) first-order logics again leads naturally to an algebra of finitary relations. The choice of the logic introduces some peculiarities into the relations, e.g. in the case of modal or temporal logics the first argument of the relations comes from a different "universe" (namely, that of possible states or time instances) than the rest of the arguments. This, however, does not seem to diminish the importance of the fact that we are dealing with relations. This is why Crs_α seems to be a reasonable unifying framework for the algebraizations of these logics.

* * * * *

Let us return briefly to RCA_ω, and to the quest for finding natural algebras of finitary relations (we mean the quest we tried to illustrate by the "obvious" considerations in connection with $\mathcal{P}({}^{<\omega}U)$ in §2, and which admitted one possible satisfactory solution by our defining $\mathfrak{Rf}(U)$, and the class L below Theorem 5).

Let $Frl(U) = \{R : R \subseteq {}^nU$ for some $n \in \omega\}$. That is, $Frl(U)$ is the collection of all finitary relations over the set U. A relation $R \subseteq {}^nU$ is *essential* if either $R = \emptyset$ or for no $S \subseteq {}^{(n-1)}U$ is $R = S \times U$. So, an n-ary relation is essential if it is not a "dummy embedding into rank n" of another relation of some smaller rank. We let $Ref(U) = \{R \in Frl(U) :$ R is essential$\}$. We define a partial ordering $\leq$ on $Frl(U)$ as follows: For any $R, S \in Frl(U)$ we let $R \leq S$ iff either $S \subseteq R \times {}^kU$, or $S \times {}^kU \subseteq R$ for some $k \in \omega$ (i.e. if we level out the rank differences between S and R by using "dummy embedding" then S becomes a subset of R). Note that for all R, $\emptyset \leq R$ and $R \leq {}^0U (= \{\emptyset\} = \{\langle\rangle\})$ because $R \subseteq {}^nU = {}^0U \times {}^nU$ for some n.

FACT 2. *$\leq$ is a complemented distributive lattice ordering on $Ref(U)$.*

We let $\langle Ref(U), \wedge, \vee, -\rangle$ be the Boolean algebra induced by this ordering $\leq$. Note that to every $R \in Frl(U)$, there is a smallest (according to $\leq$) upper bound $ess(R)$ of R in $Ref(U)$. For $R \subseteq {}^nU$ and $i < n$ let $\mathsf{c}_i(R) \subseteq {}^nU$ be defined as in RCA_n. Let $R \in Ref(U)$ and $i < n$. Then $\mathsf{c}_i^+(R) = ess\big(\mathsf{c}_i(R)\big)$. For $i \geq n$ we let $\mathsf{c}_i^+(R) = R$. Finally, for $j < i$, let $I_{ij} = I_{ji} = \{q \in {}^{i+1}U : q_i = q_j\}$. So in particular $I_{01} = \{\langle a, a\rangle : a \in U\}$, $I_{12} = U \times I_{01}$, and $I_{n,n+1} = {}^nU \times I_{01}$. Now, $\mathfrak{Ref}(U) = \langle Ref(U), \wedge, \vee, -, \mathsf{c}_i^+, I_{ij}\rangle_{i,j<\omega}$.

FACT 3.

(i) $\mathfrak{Ref}(U) \in \mathsf{RCA}_\omega$, *moreover* RCA_ω *is the variety generated by the class* $\mathsf{K} = \{\mathfrak{Ref}(U) : U \text{ is a set}\}$.

(ii) *For* L *as introduced below Theorem 5, we have* $\mathsf{K} = \mathsf{L}$ *up to isomorphisms, because* $\mathfrak{Ref}(U) \cong \mathfrak{Rf}(U)$.

The above ideas are described in more detail in Andréka [A77] §V and p. 37. We note that $\mathfrak{Ref}(U)$ seems to be related to Craig's algebras of sets of finite sequences, with universe $\mathcal{P}({}^{<\omega}U)$, in §7.(3,4) below and in [HMTII], p. 265. The nature of the relationship is not clear to us however (except that $Ref(U) \subseteq \mathcal{P}({}^{<\omega}U)$). We included the algebras $\mathfrak{Ref}(U)$ here to show that the generators of the variety RCA_ω can be built up as algebras whose elements are truly finitary relations in the most concrete possible sense. Further the extra-Boolean operator c_i^+ when applied to an $(i+1)$-ary relation R results in literally deleting the "last column" of R, which is one of the most natural and most widely used operations on n-ary relations. E.g. $\mathsf{c}_2^+\{\langle a,b,c\rangle\} = \{\langle a,b\rangle\}$.

* * * * *

As we mentioned above Theorem 6, in our first extension of Remark 2 (finitization) to RCA_ω's i.e. in our discussion of the search for *finite schema axiomatizable expansions* of RCA_ω's, there exists a more ambitious approach aiming at <u>*true finitization*</u> of expansions of RCA_ω's and their variants, see e.g. Monk [M70], Problem 1 in Henkin–Monk [HM74], and Sain [S87a]. Roughly speaking, we are searching for an expansion V_1 of RCA_ω's such that V_1 would be term definably equivalent (polynomially equivalent in sense of p. 125 of [HMTI]) to a finitely axiomatizable variety V_2. This search is one incarnation of the general question, playing a central rôle in algebraic logic, which asks *whether the choice of the fundamental operations* of the algebras RCA_ω of relations *was the best possible*, or whether a different choice of the fundamental operations could improve the situation. Certainly we made some very quick decisions when passing from RRA's to RCA_n's, e.g. we decided to drop "$\circ$" and $^{-1}$ from the list of fundamental operations and derive them as term functions. We will return to the issue of these decisions in general in §7, but the subject of finitization that recurs throughout the paper is also an investigation of possible alternatives to these decisions. Next we briefly return to finitization.

Generalizing our definition that preceded Problem 2, we now define, for an arbitrary ordinal α, an operation $f : \mathcal{P}({}^{\alpha}U) \longrightarrow \mathcal{P}({}^{\alpha}U)$ to be *permutation* *invariant* iff for any permutation p of U and $R \subseteq {}^{\alpha}U$, we have $p(f(R)) = f(p(R))$. Consider the following statement.

(4.1) There are permutation invariant operations $f_i : \mathcal{P}({}^{\omega}U) \longrightarrow \mathcal{P}({}^{\omega}U)$, $(i \leq n)$ for each set U, such that, if $\mathfrak{N}(U) = \langle \mathfrak{P}({}^{\omega}U), f_0, \ldots, f_n \rangle$ and $\mathsf{K}^+ = \{\mathfrak{N}(U) : U \text{ is a set}\}$, then the following holds. The RCA_ω operations c_i and Id_{ij} are term definable in K^+, i.e., there are terms γ_i and δ_{ij} $(i, j \in \omega)$ of K^+ such that in $\mathfrak{N}(U)$, γ_i and δ_{ij} define the usual c_i and Id_{ij}.

Note that (4.1) implies $\mathsf{RCA}_\omega = \mathbf{SPRd}\mathsf{K}^+$.

PROBLEM 2.1. *Is* (4.1) *above true in such a strong form that for* K^+ *in it,* $\mathbf{SP}(\mathsf{K}^+)$ *is a finitely axiomatizable variety or at least such a quasivariety?*

Problem 2.1 is the RCA_ω counterpart of the stronger, $(3.2)^+$-part of Problem 2 in Remark 2[32]. The variant of Problem 2.1 corresponding to the slightly weaker, (3.2)-part of Problem 2 was solved by Sain (compare [S87a]), who proved the following.

THEOREM 8.1 (Sain). (4.1) *is true in such a strengthened form that* K^+ *in it generates a finitely axiomatizable variety.*

This result indicates that there really is a difference between the two versions of Problem 2. Further, Sain exhibited easier (than the proof of Theorem 8.1) solutions for the counterpart of the weaker variant of Problem 2 quoted from [TG87] at the end of Remark 2. Namely, she provided easier proofs for the following:

PROPOSITION 8.2. (4.1) *is true in such a form that to* K^+ *in it there is a finitely axiomatizable variety* $\mathsf{K}_0 \supseteq \mathsf{K}^+$ *such that* $\mathsf{RCA}_\omega = \mathbf{SRd}\mathsf{K}_0$. *Furthermore,* $\mathbf{Rd}$ *is taken using the terms* γ_i *and* δ_{ij} *fixed in* (4.1).

Sain also suggested several attractive simple choices of K^+ and K_0, and raised the problem whether these choices also satisfy the statement of Proposition 8.2. (The only problem she left open is to check if the seven cylindric postulates, like $\mathsf{c}_i\mathsf{c}_j = \mathsf{c}_j\mathsf{c}_i$, in [HMT] p. 162 are satisfied in K_0.)[33] Compare the June 1987 version of [S87a].

The fact that Sain's proof for Proposition 8.2 is easier than her proof for Theorem 8.1 points in the direction that the variant of Problem 2 at the end of Remark 2 might turn out to be easier (than even the (3.2)-part of the

[32] Problem 2.1 is a concretized instance of the similar one in Monk [M70] and Henkin–Monk [HM74] mentioned in Remark 2. It is closer to the problem in these papers than Problem 2 in Remark 2 was.

[33] That this might be a nontrivial task is indicated by Corollary 3.6 on p. 654 of Demaree [D72], as will be discussed below Theorem 16 herein.

original Problem 2). A finite schema version of Proposition 8.2 above was obtained by Craig before Sain proved her result, cf. Craig [Cr74] and our discussion of finite schema axiomatizability above Theorem 6.

PROBLEM 2.2. *What is the answer to Problem 2.1 if we replace each occurrence of* ω *with* n *(for some* $n > 2$*) in it? This means replacing* RCA_ω, ${}^\omega U$ *with* RCA_n, ${}^n U$ *respectively everywhere in Problem 2.1 and in item* (4.1). *This amounts to asking the same question that was asked in Remark 2 but now for* RCA_n *instead of* BRA.

We will return to finitization in §6 beginning with Theorem 16, and also in §7 (9). (However, a rather strong negative finitization result, due to Andréka, will be mentioned in a footnote at the beginning of §6.)

5. Algebras for logics without equality

So far we have extended RRA's (with extra-Boolean operations $\circ$, ${}^{-1}$, Id) to algebras of relations of higher ranks. Let us see how to extend BRA's whose extra-Boolean operations were only $\circ$ and ${}^{-1}$ to higher ranks. When extending RRA's to higher ranks, we were able to drop the algebraic counterpart ${}^{-1}$ of substitution because by using c_i and Id one can express ${}^{-1}$ as a derived operation. (This was shown in the paragraph discussing the rôle of RCA_ω in the algebraization of first-order logic in the remarks following Theorem 5.) But when extending BRA's we do not have Id; hence we cannot express ${}^{-1}$, and thus shouldn't drop it. To discuss the counterpart of ${}^{-1}$ for n-ary relations, we will use the operations $\mathsf{S}_\tau : \mathcal{P}({}^\omega U) \longrightarrow \mathcal{P}({}^\omega U)$ for $\tau : \omega \longrightarrow \omega$ introduced in §4 immediately above the subtitle "Advance topics in RCA-theory". We use the notation $n = \{0, \ldots, n-1\}$. Let n be fixed, and let $i, j < n$. Then $[i,j] : n \rightarrowtail\!\!\!\!\rightarrow n$ denotes the permutation of n interchanging i and j, and leaving everything else fixed. $[i/j] : n \longrightarrow n$ sends i to j and leaves all else fixed. (To be unambiguous we should write $[i,j]_n$ and $[i/j]_n$ but we rely on context to make things clear.) For $\tau \in {}^n n$, $\mathsf{S}_\tau : \mathcal{P}({}^n U) \longrightarrow \mathcal{P}({}^n U)$ is defined the same way as it was for the $\tau \in {}^\omega\omega$ case. We let $\mathsf{p}_{ij} \overset{\text{def}}{=} \mathsf{S}_{[i,j]}$ and $\mathsf{s}^i_j \overset{\text{def}}{=} \mathsf{S}_{[i/j]}$. So in particular $\mathsf{p}_{01}(\{\langle a, b, c \ldots\rangle\}) = \{\langle b, a, c \ldots\rangle\}$ or more generally $\mathsf{p}_{01}(R) = \{\langle q_1 q_0 q_2 q_3 \ldots\rangle : \langle q_0 q_1 q_2 \ldots\rangle \in R\}$. Clearly for $n = 2$ we have $R^{-1} = \mathsf{p}_{01}(R)$, and hence we may consider p_{ij} as (an economical) generalization of ${}^{-1}$ to higher ranks. The logical counterpart of p_{01} is the "substitution" sending the formula $R(v_0 v_1 v_2 \ldots)$ to $R(v_1 v_0 v_2 \ldots)$. If we want to algebraize first-order logic, we also need the substitution sending $R(v_0 v_1)$ to $R(v_1 v_1)$. Strangely enough, we cannot express this by using the p_{ij}'s and the c_i's in themselves. Therefore we also need s^0_1. We could stop at this point, but for aesthetic reasons we add all the s^i_j's for $i, j < n$. These are not essential however, and the reader is invited to develop his/her version of algebraic logic having only one of the s^i_j's.

The *full polyadic set algebra* of n-ary relations (the full Ps_n) over U is defined to be

$$\langle \mathfrak{P}({}^nU), c_i, s^i_j, p_{ij}\rangle_{i,j<n}.$$

By a *representable polyadic algebra* of n-ary relations (an RPA_n) we understand a subalgebra of a direct product of full Ps_n's up to isomorphisms. Formally, $RPA_n = \mathbf{SP}(\text{full } Ps_n)$. (Cf. the sentence above Remark 1 in §3.)

THEOREM 9. *RPA_n is a variety axiomatizable by a decidable set of equations.*

The proof of Theorem 3 also proves that RPA_n is a variety. See also the references in §5.4 of [HMTII]. The second part of Theorem 9 is proved by the methods in [HMTII] pp. 112–118. ∎

THEOREM 10 (Johnson [J69]). *RPA_n is not finitely axiomatizable if $n > 2$.*

This result can, perhaps, be improved, since Andréka, has a construction that might be suitable for proving that RPA_n, for $n > 2$, is not finitely axiomatizable over its p_{ij}-free subreduct, i.e. over the variety[34] generated by the $\langle \mathfrak{P}({}^nU), c_i, s^i_j\rangle_{i,j<n}$'s.

We can put cylindric algebras and polyadic algebras together obtaining the so-called $RPEA_n$'s (representable polyadic equality algebras), and we can ask ourselves if we get essentially more than RCA_n's. In a sense, Andréka gave an affirmative answer, solving Problem 5.8 of [HMTII]. We give more detail: $RPEA_n$ is obtained by expanding RPA_n's by the constants Id_{ij} of RCA_n's. Then RCA_n's become subalgebras of reducts of $RPEA_n$'s. By the proof of Theorem 3, $RPEA_n$ is a variety.

[34]These are Pinter's substitution-cylindric algebras (of n-ary relations) (RSC_n's); they will recur a few more times below. Andréka also proved that RCA_n, for $n > 2$, is not finitely axiomatizable over RSC_n either (s^i_j is an RCA_n-term; hence RSC_n's are also subreducts of RCA_n's). Andréka systematically generalized these kinds of results of hers to schemas for $\alpha \geq \omega$, in analogy with Monk's non-finitizability result for RCA_ω quoted from [HMTII] 4.1.7 above Theorem 6. She further strengthened most of these to stating non-axiomatizability by any set Σ of universally quantified sentences such that Σ contains only finitely many variables. This way she does not have to introduce schemata.

THEOREM 11 (Andréka–Tuza, Andréka).

(i) RPEA_n *is not finitely axiomatizable over* RCA_n *for* $n > 2$.
(ii) RPEA_n *is not finitely axiomatizable over* RPA_n *either, for* $n > 2$.
(iii) *Both* (i) *and* (ii) *remain true if we replace finite axiomatizability by axiomatizability with a set* Σ *of universally quantified formulas such that* Σ *uses only finitely many variables.*

For (i) see Andréka–Tuza [ATu88]; the $n = 3$ case of (i) is their joint result. The rest of Theorem 11 is due to Andréka.

(ii) above is in contrast with the finite axiomatizability of RRA's over BRA's. Furthermore, Theorem 11 (i) is in contrast with the finite axiomatizability of RRA over its $^{-1}$-free subreduct mentioned just before beginning of §4.

In analogy with the relationship between RCA_n's and RRA's, BRA's are subreducts[35] of reducts of RPA_n's, for $n > 2$. In particular, if $R, S \subseteq U \times U$ and if $Dr(R) = R \times U \subseteq {}^3U$, then $Dr(R \circ S) = \mathsf{c}_2\left[\mathsf{p}_{12}\left(Dr(R)\right) \cap \mathsf{p}_{02}\left(Dr(S)\right)\right]$. So, analogously to the remark following our definition of RCA_n's, we can observe that the algebras of relations of higher ranks without equality (RPA_n's) are expansions of the analogous algebras of binary relations (BRA's). We already noted in §4 that RCA_2's are subreducts of RRA's. So RRA's are in between RCA_3's and RCA_2's. (When algebraizing proof theory as opposed to the present emphasis on model theory — i.e. when dealing with the "axiomatic" classes CA_n and RA approximating RCA_n and RRA, discussed in the middle of §4 and in §8 — these numbers increase by one, so instead of 2 and 3, we get 3 and 4; see Maddux [Ma78, Ma83], [HMTII] §5.3, [TG87].) Strangely enough, RPA_2's are not subreducts of BRA's because s^0_1 is not a derived operation of BRA's. It would be nice to see a theory of BRA expanded with s^0_1. (Probably it would be somewhere halfway in between BRA's and RRA's.)

In RPA_n, the operation $\mathrm{S}_\tau : \mathcal{P}({}^nU) \longrightarrow \mathcal{P}({}^nU)$ is term definable, for each $\tau : n \longrightarrow n$. These S_τ's are quite important in RPA_n-theory. In this connection, recall from the subsection in the middle of §4 (cf. also §8) that in the literature there is a traditional, finitely axiomatizable variety PA_n of n-ary *polyadic algebras* approximating RPA_n. The classical axiomatizations of PA_n use all the S_τ's and not only the s^i_j's and p_{ij}'s, see [HMTII] Def.5.4.1 p. 225. Therefore the universal algebraic approach in [HMTII] pp. 260–263 based on the concept of a finite schema as defined in [HMTII] was not applicable to PA-theory. However, in Sain–Thompson [ST89], a single natural finite schema Σ_0 of equations was found, such that Σ_0 involves only

[35] Recall our Convention that subreducts are subalgebras of *generalized* reducts.

the s^i_j's and p_{ij}'s (and does not involve the S_τ's in general) besides the Boolean operations. It is proved there that $PA_n = Mod(\Sigma_0)$. In particular, the traditional polyadic axioms like $S_{\tau\circ\sigma}(x) = S_\tau \circ S_\sigma(x)$ for all $\tau, \sigma \in {}^n n$ follow from Σ_0. In other words, Σ_0 is equivalent with the traditional axiomatization of PA_n, and at the same time Σ_0 has certain advantages. An important point here is that the same Σ_0 works for all n (and even for "$n = \omega$" but that comes later). The key part of their Σ_0 consists of Jónsson's seven defining relations in 3.2.17(B) on p. 68 of [HMTII] for the semigroup ${}^n n$ as generated by the $[i, j]$'s and $[i/j]$'s transcribed to the p_{ij}'s and s^i_j's. E.g. Jónsson's defining relation $[i, j] \circ [i, j] = \mathrm{Id}$ translates to the axiom $p_{ij}p_{ij}(x) = x$ in Σ_0.

From now on, we will freely use the derived operations or term functions S_τ of RPA_n's, for $\tau \in {}^n n$, without recalling them.

Generalizing RPA_n to RPA_α with α an arbitrary ordinal (especially for $\alpha = \omega$) goes exactly as in the RCA_α case. However, for historical reasons, the resulting algebras are called quasi-polyadic algebras instead of polyadic algebras (though it was these algebras which were introduced first, cf. Halmos [Ha54], and originally they were called polyadic; but later the terminology changed). In particular, for *finite* α, polyadic and quasi-polyadic algebras coincide[36].

$\langle \mathfrak{P}({}^\alpha U), c_i, s^i_j, p_{ij}\rangle_{i,j<\alpha}$ is called the full *quasi-polyadic set* algebra (full Qps_α) of α-ary relations over U for any ordinal α and set U. A *quasi-polyadic set algebra* (a Qps_α) is a subalgebra of a full Qps_α. Let $\alpha \geq \omega$. Then by a *representable quasi-polyadic algebra* of α-ary relations (an RQA_α) we understand an isomorphic copy of a Qps_α. (Note the difference with the previous definitions!)[37] $RQA_\alpha = \mathbf{I}Qps_\alpha$, where $\mathbf{I}$ is the operator of taking isomorphic copies. We define $RQA_n = RPA_n$, for $n \in \omega$.

THEOREM 12.

(i) RQA_α *forms a variety if* $\alpha \geq \omega$.
(ii) *For countable* α, RQA_α *is axiomatizable by a decidable set of equations.*

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[36] The distinction between polyadic and quasi-polyadic will be useful later at the algebraization of infinitary logics.
[37] BRA, RRA, RCA_α, RPA_n were all defined by applying **SP** to algebras of the form $\langle \mathfrak{P}({}^\alpha U), \cdots\rangle$.

The weaker form of Theorem 12 (i) saying that $\mathbf{SP}\mathsf{RQA}_\alpha$ is a variety is proved by repeating the proof of Theorem 5 with the obvious changes.

Note the striking difference with cylindric algebras: by Theorem 12 (i), $\mathbf{I}\mathsf{Qps}_\alpha$ is a variety for $\alpha \geq \omega$, but $\mathbf{I}\mathsf{Cs}_\alpha$ is not. This result was perhaps known earlier, but the proof can be found in Sain [S89], Thm.3.

$\tau \in {}^\omega\omega$ is called a *finite transformation of* ω iff $\{i \in \omega : \tau(i) \neq i\}$ is finite. For every finite transformation τ of α, the operation S_τ is term definable in RQA_α.

In connection with the elegant finite schema Σ_0 of equations in Sain–Thompson [ST89] mentioned between Theorems 11 and 12 above, we note that Σ_0 works for $\alpha \geq \omega$; namely Σ_0 implies all the distinguished quasi-polyadic axioms like $S_{\tau\circ\sigma} = S_\tau \circ S_\sigma$ for finite transformations τ, σ of α. For a list of all these axioms see [HMTII], item 9, p. 266; cf. also [HMTII] Def. 5.4.1. I.e. for $\alpha \geq \omega$, $\mathsf{QPA}_\alpha = Mod(\Sigma_0)$, where QPA_α is the finitizable variety of α-ary *quasi-polyadic algebras* approximating RQA_α in the sense outlined in the middle of §4 and in §8.

For any finite subset $\Gamma = \{i_1, \ldots, i_n\}$ of α we can introduce the derived operation (or term function) $C_{(\Gamma)}(x) = c_{i_1} \ldots c_{i_n}(x)$. To see that this works, it is enough to recall that $c_i c_j = c_j c_i$ is valid. An equivalent and frequently used way of introducing RQA_α's is to start out from the algebras

$\langle \mathfrak{P}({}^\alpha U), C_{(\Gamma)}, S_\tau \quad : \quad \Gamma \subseteq \alpha$ is finite, and $\tau \in {}^\alpha\alpha$ is a finite transformation$\rangle$.

Indeed, this approach was taken in [HMTII] p. 266, Halmos [Ha62], [DM63], Andréka–Gergely– Németi [AGN77] etc.[38]

* * * * *

The algebraic theory of RQA_α's for $\alpha \geq \omega$ is practically the same as that of RQA_ω's, so we will concentrate on the latter. Their algebraic theory is not as extensively developed as that of RCA_ω's, but see Halmos [Ha62], Daigneault–Monk [DM63], [AGN77], Sain [S89], Sain–Thompson [ST89], Pinter [P73], the end of Pinter [P73a]; R. J. Thompson also has results in this line. The next theorem is a generalization of a result of J. S. Johnson, namely of Theorem 10 above.

[38] The reason for the popularity of this more complicated similarity type containing all the S_τ's and $C_{(\Gamma)}$'s might have been that the simple schema Σ_0 quoted above from [ST89] implying all the distinguished axioms like $S_{\tau\circ\sigma} = S_\tau \circ S_\sigma$ was not available. (Another reason might be that when Halmos [Ha54] introduced quasi-polyadic algebras, awareness of universal algebra as a desirable unifying framework for all branches of abstract algebra in the sense mentioned in §2 and Remark 1 herein, was almost nonexistent. We note that the same problem seems to keep popularity below the deserved level in the case of the unbelievably rich treasure Jónsson–Tarski [JT51] of knowledge on BAO's.)

THEOREM 13 (Sain–Thompson [ST89]). RQA_ω *is not axiomatizable by any finite schema*[39] *of equations in the sense of* [HMTII].

As in the case of RCA_α, RRA, and BRA, it is an important *open problem* to find simple, mathematically transparent, decidable sets Σ of equations axiomatizing RQA_α ($\alpha > 2$).

We could define the RQA_ω-theoretic versions of the class L and algebras $\mathfrak{Rf}(U)$ as the algebraic counterparts of the models of first-order logic $L_{\omega\omega}$ *without* equality as in the RCA_ω case. Denoting these by Lp and $\mathfrak{Rfp}(U)$ (where p stands for polyadic), we have that RQA_ω is the variety generated by $\mathsf{Lp} = \mathbf{S}\{\mathfrak{Rfp}(U) : U \text{ is a set}\}$; see[40] [AGN77] §3.3 Coroll.3.18 p. 28.

We conjecture that the lattice of subvarieties of RQA_ω will turn out to be considerably simpler than that of RCA_ω described to some extent in Németi [N87] §1.1, p. 245. It appears to us that developing the theory of RQA_ω's would be quite an important and rewarding task.

In <u>*algebraizing quantifier logics*</u>, the rôle of RQA_α is entirely analogous to that of RCA_α. Namely, RQA_ω is the true algebraic counterpart of classical first-order logic without equality. The logical counterparts of s^i_j or p_{ij} send the formula φ to $\varphi(v_i/v_j)$ or $\varphi(v_i/v_j, v_j/v_i)$ obtained by replacing v_i by v_j or interchanging v_i and v_j as free variables of φ respectively. Perhaps the connection with logic on this syntactical level is not as elegant as in the cylindric case, because the operations $\varphi \longmapsto \varphi(v_i/v_j)$ are not logical connectives like "$\exists v_i$" and "$v_i = v_j$" were in the cylindric case. One possible way of avoiding this problem would be to introduce s^i_j and p_{ij} as logical connectives, restricting atomic formulas to $R(v_0, v_1, \ldots, v_n)$, and treating $R(v_1 v_0)$ as a convenient abbreviation of $\mathsf{p}_{01} R(v_0 v_1)$. Of course, this yields an *equivalent* formulation of $L_{\omega\omega}$ without equality. The connections between RQA_ω-theory and logic without equality, including Lp and $\mathfrak{Rfp}(U)$ as algebraic counterpart of models, can be worked out analogously to §4.3 of [HMTII] keeping in mind §5.6 therein, [AGN77], Monk [M71], and Johnson [J73]. At least this is what we conjecture, but this work has not been done yet, at least not in the detail of [HMTII], §4.3, and Németi [N89]. As in the case of cylindric algebras outlined in the remarks after Theorem 4, RPA_n ($=\mathsf{RQA}_n$) is the algebraic counterpart of the version of L_n *without* equality; see Monk [M71], [J73]. These algebraic counterparts proved rather useful in studying both versions of L_n, cf. e.g. the relevant parts in [TG87].

[39] A set of individual schemata like $\{\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x,\ \mathsf{c}_i \mathrm{Id}_{ij} = 1\}$ is called again a schema for simplicity. If the set is infinite then we say that this schema is infinite. Though here we use the definition of a schema in [HMTII] without recalling it, we note that it is the natural generalization of the schema (E0–7) in the present Appendix (§8).

[40] For some reason, RQA_α's were called representable *substitution algebras* (and denoted as $\mathcal{R}s_\alpha$) in [AGN77] and its 1974 version. Our present Lp was denoted by $\mathcal{L}s_\omega$ there.

Advanced topics

If we expand RQA_α with the constants Id_{ij} representing equality then we obtain RQA_α's with equality, RQEA_α's, exactly as RPEA_n's were obtained from RPA_n's. Actually, $\mathsf{RPEA}_n = \mathsf{RQEA}_n$ for $n \in \omega$. Similar observations are true for these algebras, in particular for RQEA_ω's, as were spelled out for the case of $\alpha = n(< \omega)$ at the beginning of the present section (§5). But in contrast to Theorem 12, we have to define RQEA_α as an $\mathbf{SP}$-closure of set-algebras even if $\alpha \geq \omega$. One can define the analogs of Crs_α's for RQEA_α and for RQA_α. The simple finite schema Σ_0 in Sain–Thompson [ST89] has an RQEA_α-theoretic version Σ_1, which is also found in [ST89]; Σ_1 is also finite, and just as simple as Σ_0. So, $\mathsf{QPEA}_\alpha = Mod(\Sigma_1)$ is the finitary variety of *quasi-polyadic equality algebras* approximating RQEA_α. For $n < \omega$, $\mathsf{QPEA}_n = \mathsf{PEA}_n$. An important theorem of Diane Resek implies then that every QPEA_α is isomorphic to one of the above mentioned RQEA_α-like analogs of Crs_α. We do not know whether a similar result holds for RQA_α-like analogs of Crs_α's. We note that Andréka generalized Theorem 11 above to RQEA_α and RQA_α for arbitrary α. By the proof of Theorem 5 one can easily see that RQEA_α is a variety. (If not specified otherwise, we always mean "for every ordinal α"). Andréka proved that RQEA_α is not axiomatizable over RCA_α with any set Σ of universally quantified formulas if Σ contains only finitely many variables and $\alpha > 2$. For lack of space we do not go into more detail in connection with the theory of RQEA_α, but we note that they are basically cylindric algebras enriched with the p_{ij}'s so the two theories should be very close, and the differences illuminating for both.

The joint reducts $\langle \mathfrak{P}({}^\alpha U), \mathsf{s}^i_j\rangle_{i,j<\alpha}$ and $\mathfrak{Psc}_\alpha(U) = \langle \mathfrak{P}({}^\alpha U), \mathsf{c}_i, \mathsf{s}^i_j\rangle_{i,j<\alpha}$ of RQA_α's and RCA_α's have been investigated in Pinter [P73a], Preller [Pr70]. Here we should note that the subreducts $\mathsf{RSC}_\alpha = \mathbf{S}\{\mathfrak{Psc}_\alpha(U) : U \text{ is a set}\}$ are *already sufficient for carrying through the algebraization of $L_{\omega\omega}$ without* equality in a parallel fashion as described above for the RQA_ω-case (Pinter [P73b]). Therefore this subreduct RSC_ω of RQA_ω is not of a completely negligible importance. An axiomatization, analogous to Σ_0 quoted above from [ST89], of the finitary variety SC_α approximating[41] RSC_α can be obtained on the basis of Thompson [T87]. Pinter [P73b] calls the members of SC_α *substitution-cylindric* algebras. The proof of Theorem 12 also yields that RSC_α is a variety for $\alpha \geq \omega$. Sain proved that RSC_α is not axiomatizable by any finite schema in the sense of [HMTII] if $\alpha > 2$. Her proof can be recovered from the proofs of Thm.2(i) in [ST89].

[41] This is the same kind of approximation as CA_α approximates RCA_α, cf. the subsection in the middle of §4 and cf. §8. We note that $\mathsf{SC}_\alpha = \mathbf{HSRd}\,\mathsf{QPA}_\alpha \supsetneq \mathsf{RSC}_\alpha$.

We note that $\mathsf{RSC}_\omega = (\mathbf{SRd}\mathsf{RCA}_\omega) \cap (\mathbf{SRd}\mathsf{RQA}_\omega)$ seems to be the weakest reasonable subreduct of RCA_ω's still completely suitable for the algebraization of full first-order logic (without equality). An algebraization of logic completely parallel to the one in [HMTII] §4.3 can be worked out in the same detail for RSC_ω in place of RCA_ω, with RSC_ω's of universes $\mathsf{Rf}(U)$ corresponding to models etc.

6. Algebras for logics extending $L_{\omega\omega}$ [42]

Most of the logics we have in mind are discussed in Barwise–Feferman [BF85]. Many of these new logics remain in the realm of finitary relations, for example the finitary logics with additional *exotic quantifiers* (like "there are uncountably many", or topological quantifiers of Georgescu [G82a] etc.). These logics are generally denoted by $L_{\omega\omega}(Q)$, or just $L(Q)$, where Q refers to the extra (or exotic) quantifier we are adding to $L_{\omega\omega}$. Some of these logics are summarized in Part B of Barwise–Feferman [BF85] (but some are scattered all over the book, see e.g. pp.9, 509.)

Many of these quantifiers are given by fixing a function F which associates with every set U a collection $F(U) \subseteq \mathcal{P}(U)$ of its subsets. Then $Qx\varphi(x)$ holds in a model $\mathfrak{M}$ if $\{a \in M : \varphi(a)$ is true in $\mathfrak{M}\} \in F(M)$. A special example is the classical quantifier "There exist infinitely many elements x such that $\varphi(x)$" suggested by Mostowski in 1957. Other examples are "there exist $|U|$-many (where U is the universe of the model in question)", and the standard $Q\alpha$ quantifiers saying "there are $\aleph_\alpha$-many". Note that when fixing such a logic $L(Q)$, we define the function F for all sets U simultaneously. To make $L(Q)$ satisfy the axioms of abstract model theory, one of which corresponds to permutation invariance in Remark 2 way above, we should place some restrictions on the "universal" function F, but that is not our main concern in this paper.

Let us recall that the universe $\mathsf{Rf}(U)$ consists of elements of the form $R \times {}^\omega U$ with $R \subseteq {}^n U$, for some $n \in \omega$. $\mathsf{Rf}(U)$ was the universe of the generic example $\mathfrak{Rf}(U)$ of cylindric algebras, i.e. RCA_ω's as well as that of RQA_ω's since $\mathsf{Rf}(U)$ is closed under p_{ij} for $i, j \in \omega$. $\mathsf{Rf}(U)$ consists of all the (dummy embeddings or dummy representations of) finitary relations on U, and as our logic remains finitary, $\mathsf{Rf}(U)$ remains adequate for the universe of the generic algebras of relations.

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[42] Some algebraizations of some of these logics were already discussed between Theorem 7 and Fact 2 in §4. (These were applications of relativized versions, Crs_α's, of RCA_α's.)

Let us fix F as above. Then the new operator $Q : \mathcal{P}({}^{\omega}U) \longrightarrow \mathcal{P}({}^{\omega}U)$ is defined as follows:

$$Q(x) \stackrel{\text{def}}{=} \{\langle b_0, b_1, \ldots, b_n, \ldots\rangle \in {}^{\omega}U : \{u \in U : \langle u, b_1, \ldots, b_n, \ldots\rangle \in x\} \in F(U)\}\,.$$

The full Cs_F over U is

$$\langle \mathfrak{P}({}^{\omega}U), Q, \mathsf{c}_i, \mathrm{Id}_{ij}\rangle_{i,j<\omega}$$

i.e. it is the expansion of the full Cs_{ω} with Q. The representable cylindric algebras with exotic quantifiers determined by F (RCA_F's) are the algebras embeddable into products of full Cs_F's. The generic example: $\mathfrak{Rf}_F(U)$ is the subalgebra of the full Cs_F with universe $\mathsf{Rf}(U)$.

The new operation $Q : \mathsf{Rf}(U) \longrightarrow \mathsf{Rf}(U)$ acting on our (embedded or "coded") finitary relations is (for most choices of F) not term definable in RCA_{ω}, in particular, is not such in $\mathfrak{Rf}(U)$, but does not lead out from $\mathsf{Rf}(U)$. It produces (codes of) finitary relations from (codes of) finitary relations. We could have introduced Q_i for each $i \in \omega$ by letting $Q_i = \mathsf{p}_{0i} \circ Q \circ \mathsf{p}_{0i}$, but this Q_i is expressible in $\mathfrak{Rf}_F(U)$ as defined above. The generalization of RCA_F's from ω-ary relations to α-ary relations for any ordinal α is straightforward. Analogously to the just described RCA case, the new operation Q can be added to RQA_{α}'s, and the other algebras with universe $\mathcal{P}({}^{\alpha}U)$ discussed so far; it can be added even to RRA's.) Pinter [P75], Georgescu [G82, G82a], Schwartz [Sw80a] are some of the references to algebras of relations of this kind.

We note the following in connection with Remark 2 (*finitization*). In practically all of the cases (relevant to logic), $F(U)$ is closed under all permutations of U. In the interesting cases, $Q : \mathsf{Rf}(U) \longrightarrow \mathsf{Rf}(U)$ is not term definable in $\mathfrak{Rf}(U)$. Thus expanding RCA_{ω} or RCA_n or even RRA with Q is a new possibility in the search outlined in Remark 2, especially because the negative or limiting result of Biró [Bi89] mentioned in Remark 2 above does not apply to Q. However, this observation is mainly of didactical interest, namely a result of Andréka seems to imply that adding such a Q will not result in a finitely axiomatizable variety.[43]

[43] Andréka proved that if Q defined by F is permutation invariant (for all $p : U \rightarrowtail\!\!\!\!\rightarrow U$, $p(F(U)) = F(U)$), then adding one or more of these Q's (each may be based on a different but invariant F) does not make RCA_n finitely axiomatizable for $n > 2$. Moreover, she proved the same for any new permutation invariant *unary* operator $f : \mathcal{P}({}^{n}U) \longrightarrow \mathcal{P}({}^{n}U)$ satisfying the equation $f(x \cup y) = f(x) \cup f(y)$ in place of Q. So adding such a new operation f to RCA_n does not make RCA_n finitely axiomatizable. Thus Problem 2.2 cannot be solved by adding such f's or Q's (to the old operations). She also proved that the weaker version (of the problem) quoted from [TG78] at the end of Remark 2 is not

Concerning the other kinds of quantifiers in Part B of Barwise–Feferman [BF85], like Henkin quantifiers, we do not know of any work on their algebraization. We feel that one should be able to stay inside the universe $\mathsf{Rf}(U)$ when algebraizing them, too.

Algebras for infinitary logics

Let α be an arbitrary ordinal. After having introduced and studied RQA_α's in [Ha54], Halmos expanded them to what are now called representable *polyadic algebras* of α-ary relations (RPA_α's), cf. [HMTII] §5.4, Halmos [Ha62]. The new operations are the natural infinitary generalizations of the s^i_j, p_{ij}, and c_i's of RQA_α's. Next we generalize the operation $\mathrm{C}_{(\Gamma)}$ from finite $\Gamma \subseteq \alpha$ to arbitrary $\Gamma \subseteq \alpha$. Let $x \subseteq {}^\alpha U$ and $\Gamma \subseteq \alpha$. Then

$$\mathrm{C}_\Gamma(x) \stackrel{\text{def}}{=} \mathrm{C}_{(\Gamma)}(x) \stackrel{\text{def}}{=} \{q \in {}^\alpha U \ : \ (\exists s \in x) q \restriction (\alpha \setminus \Gamma) = s \restriction (\alpha \setminus \Gamma)\}.$$

Note that $\mathsf{c}_i x = \mathrm{C}_{\{i\}} x$ and $\mathrm{C}_{\{i,j\}} x = \mathsf{c}_i \mathsf{c}_j x = \mathsf{c}_j \mathsf{c}_i x$, and the same for any finite set $\{i_1, \ldots, i_n\} \subseteq \alpha$. For arbitrary $\tau \in {}^\alpha\alpha$, the operation S_τ has already been defined in §4.

By an RPA_α we understand an algebra embeddable into a direct product of algebras of the form $\langle \mathfrak{P}({}^\alpha U), \mathrm{C}_\Gamma, \mathrm{S}_\tau \ : \ \Gamma \subseteq \alpha \text{ and } \tau \in {}^\alpha\alpha \rangle$.

Note that, though for $\alpha = n < \omega$ we have two different definitions of RPA_n, they are equivalent (up to term definitional equivalence), i.e. they differ only in the choice of the fundamental operators (like BA's of the form $\langle B, \wedge, \vee, -, 0\rangle$, or $\langle B, \wedge, \vee, -, 1\rangle$, or $\langle B, \wedge, -\rangle$).

For $\alpha < \omega$ we already discussed these algebras. So let $\alpha \geq \omega$ for a while.

THEOREM 14. *RPA_α is a discriminator variety in the standard universal algebraic sense (see [BS81], [AJN88]). So are BRA, RRA, RCA_n, and RPA_n if $n < \omega$.*[44]

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solvable either by adding such f's or Q's. I.e. she proved that no finitely axiomatizable variety K_0 containing the version K^+ of RCA_n ($n > 2$) expanded by such f's or Q's has the property $\mathsf{RCA}_n = \mathbf{SRd}\,\mathsf{K}_0$. (Sain proved that this last negative result does *not* extend to RCA_ω from RCA_n. Cf. [S87, 87a].) These negative results of Andréka generalize to polyadic algebras with equality, and remain true if we add the operation *trc* of taking transitive closure to RCA_n. Here, if $R \subseteq U \times U$ then $trc(R \times {}^{n-2}U) = (R^* \times {}^{n-2}U)$ where R^* is the usual transitive closure of the binary relation R. (These were proved by Andréka.) (The case of adding *binary* invariant operators remains wide open.)

[44] Discriminator varieties are getting more and more into the focus of attention in algebraic logic, see e.g. [AJN88], Blok–Pigozzi [BP89a], [BP89b], Pigozzi [P89]. (Every discriminator variety with two term definable constants is very close to being a variety of BAO's, see the last two references, and [BP89b].) As mentioned in the footnote of Remark 1 in §3, our classes are more than discriminator, they are doubly pointed discriminator, and hence stronger results are available.

We cannot speak about finite or even recursively enumerable or decidable axiomatizations in connection with RPA_α's, because they have at least *continuum* many basic operators. Intuitively, this large number of operations seems to be a drawback of RPA_α's when one is trying to apply them to finitary logics or to algebras of arbitrary finitary relations. The main bulk of these operations seems to be irrelevant for $L_{\omega\omega}$ or for $\{R : R \subseteq {}^nU \text{ and } n \in \omega\}$, and therefore seems to "pull" the theory in misleading directions. On the other hand, for infinitary logics these operations are useful and relevant, cf. e.g. Ex.5.6.3, 5.6.6, 5.6.9 in [HMTII]. Further, RPA_α's are *very* useful via their reducts as we will soon see.

One could argue that postulates like $\{\mathsf{S}_{\tau\circ\sigma}(x) = \mathsf{S}_\tau \circ \mathsf{S}_\sigma : \tau, \sigma \in {}^\omega\omega\}$ could be considered as some kind of generalized finite schemata because they look "schematic" in some sense. However, the intuitive value (or concreteness) of these generalized schemata is more problematic than that of the $\mathsf{c}_i\mathsf{c}_j = \mathsf{c}_j\mathsf{c}_i$ kind because if e.g. we are given three recursive functions $\tau, \sigma, \rho \in {}^\omega\omega$ by their recursive definitions then it might be an unsolvable problem to decide whether $\mathsf{S}_\rho(x) = \mathsf{S}_\tau\mathsf{S}_\sigma(x)$ is an instance of the above schema. In connection with Theorems 13, 10, 4 we note that Daigneault and Keisler independently proved the existence of a finite generalized schema of equations in the above sense axiomatizing the variety RPA_α (recall that $\alpha \geq \omega$), cf. [HMTII] items 5.4.41 and 5.4.1, Daigneault–Monk [DM63]. Pinter [P73b] simplified this generalized schema considerably. Pinter's simplified axioms for RPA_α proved rather useful in subsequent work (e.g. of R. J. Thompson).

Returning briefly to the <u>*connection with logic*</u>[45] as approached in the RCA_ω and RQA_ω cases, recall that $\mathfrak{Rfp}(U)$, the quasi-polyadic algebra of all finitary relations over U, was introduced below Theorem 13 in an analogous manner to $\mathfrak{Rf}(U)$ introduced below Theorem 5. Let L_p^∞ be the class of subalgebras of RPA_ω's with universes of the form $\mathsf{Rf}(U)$, that is the RQA_ω algebras $\mathfrak{Rfp}(U)$ of (embedded) finitary relations over U, but now expanded with the operations S_τ, $\mathsf{C}_{(\Gamma)}$ for $\tau \in {}^\omega\omega$ and $\Gamma \subseteq \omega$. Similarly to the previous cases, L_p^∞ consists of the polyadic algebras corresponding to models of $L_{\omega\omega}$. (Further $\mathbf{SP}\mathsf{L}_p^\infty$ contains all the algebras corresponding to other aspects, e.g. theories of $L_{\omega\omega}$)

The following is in sharp contrast with what we saw for RCA_ω's and RQA_ω's.

[45] We will return to this connection again with a greater emphasis on infinitary logics.

THEOREM 15. $\mathsf{RPA}_\omega \neq \mathbf{HSP}(\mathsf{L}_p^\infty)$. *That is, the variety generated by* L_p^∞ *is strictly smaller than* RPA_ω.

PROOF. [46] Let $\tau : \omega \rightarrowtail\!\!\!\!\rightarrow \omega$ be a permutation acting as predecessor on the even numbers, and as a successor on the odds. I.e. $\tau(0) = 1$ and $\tau(2n+2) = 2n$, $\tau(2n+1) = 2n+3$. Let $e(x)$ be the equation

$$\mathrm{C}_{(\alpha)}(x \oplus \mathrm{S}_\tau x) \geq \mathrm{C}_{(\alpha)}(x \oplus \mathrm{C}_{(\alpha)} x)$$

where $\oplus$ denotes symmetric difference. Then $\mathsf{L}_p^\infty \models e(x)$, but $\mathsf{RPA}_\omega \not\models e(x)$ because $e(x)$ fails for e.g. $x = \{\langle u, u, \ldots, u, \ldots\rangle : u \in U\}$, as well as for $x = \{q \in {}^\omega U : \{i : q_i \neq b\} \text{ is finite}\}$, for any fixed $b \in U$ if $|U| > 1$. ∎

The above theorem points in the direction of the limited relevance of RPA_ω-theory for finitary quantifier logics. Namely, there are polyadic equations valid in all algebraic counterparts of finitary logics which are not valid in RPA_ω, hence investigating RPA_ω in general might lead one to questions irrelevant for finitary logics[47]. (The above equation distinguishes the whole of RPA_ω from polyadic algebraic counterparts of aspects or "parts" other than just models of finitary logics like the Lindenbaum–Tarski algebras of formulas, of those of theories etc.) Of course, this does not diminish the value of RPA_ω for investigating infinitary logics.

There is a way, however, in which RPA_ω's seem to be very useful for all parts of algebraic logic, namely they provide us with a wealth of subreducts to study. Indeed, for any subsemigroup $\langle \mathcal{S}, \circ\rangle \subseteq \langle {}^\omega\omega, \circ\rangle$ of the full transformation semigroup ${}^\omega\omega$ of ω we have a potentially interesting "reduct" generated as a variety by the algebras of the form $\langle \mathfrak{P}({}^\omega U), \mathsf{c}_i, \mathrm{S}_\tau : \tau \in \mathcal{S}, i \in \omega\rangle$. RQA_ω's are the case when $\mathcal{S}_1$ = "all finite transformations", and Pinter's substitution-cylindric algebras obtained from $\langle \mathfrak{P}({}^\omega U), \mathsf{c}_i, \mathsf{s}^i_j\rangle_{i,j<\alpha}$ are the case when $\mathcal{S}_2$ = "subsemigroup of $\mathcal{S}_1$ generated by $\{[i/j] : i, j \in \omega\}$". Let $\mathsf{RPA}_\mathcal{S}$ be the variety associated above to the semigroup $\mathcal{S}$. For many interesting choices of $\mathcal{S}$ with an arbitrary $\alpha \geq \omega$ in place of ω, there are deep results in Daigneault–Monk [DM63] which seems to be one of the key sources for results on polyadic algebras and their generalizations like $\mathsf{RPA}_\mathcal{S}$.

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[46] The idea of this proof occurs in the proof of Thm.9(i) in the June 1987 version of Sain [S87a], and in Sain [S89].

[47] Since the opposite of Theorem 15 is true for RCA_ω and RQA_ω, this danger is not present there.

THEOREM 16. (Sain [S87a] Thm.0, Corollary 17) *There is a subsemigroup S of ${}^{\omega}\omega$ such that*

(i) RPA_S *is term definitionally (i.e. polynomially) equivalent with a variety* V_S *axiomatizable by finitely many equations.*
(ii) $\mathsf{RQA}_\omega = \mathbf{SRd}(\mathsf{RPA}_S)$ *i.e. the class of representable quasi-polyadic algebras coincides with the appropriate subreducts of the class* RPA_S.

The above result is a positive solution for the finitizability problem which we have discussed throughout this paper beginning with Remark 2. In particular, Theorem 16 solves positively the RQA_ω theoretic version of Problem 4.1 in §4. As we indicated in Remark 2, this problem is really a whole family of problems (or a theory built around problems driven by a fundamental desire for insights in connection with a mysterious pattern of recurring negative results which might be the symptoms of some uncovered property of first-order logics). Therefore a single positive result, even if as powerful as the above one, cannot be the final word about the finitization problem, rather the opposite is the case: it is a new beginning in finitizability theory. In our opinion, the above result is a breakthrough. Namely, all elements of RPA_S are representable as (i.e. are isomorphic to) algebras of ω-ary relations and *all* operations of RPA_S's are permutation invariant set theoretic ones (obviously, since they are S_τ's and c_i's); and at the same time, RPA_S is a finitely axiomatizable variety if we choose its basic operations properly. The latter of course implies that the clone (set of term functions) of RPA_S is finitely generated (as we mentioned at discussing the finitization problem of RCA_ω). Making the clone of RPA_S finitely generated is only the first step of the proof, the easiest one. Namely, it is easy to expand RCA_ω's, or RQA_ω's with S_τ's such that the clone of the expanded class becomes finitely generated, see Copeland algebras on p. 264 of [HMTII]. However, as indicated therein, it is very hard to force the new algebras to become finitely axiomatizable. Even if the original class was finite schema axiomatizable (which is not the case with RCA_ω or RQA_ω), this property tends to go away as a by-product of our coding the original infinity of operations by finitely many new ones, cf. Corollary 3.6 on p. 654 of Demaree [D72]. The hardest part, however, seems to be forcing the new operations to be representable, too (by adding the new operations, one forces the old ones to be representable; but then the problem is, roughly, that nothing forces the new ones to be representable).

In §4 (Application to logic) of Sain [S87a], she shows that RPA_S is indeed adequate for the algebraization for first-order logic without equality. Actually [S87a] works out the connections with logic in a way parallel with [HMTII] §4.3.

There are important open problems stated in Sain [S87a]; one of them is to simplify the finite set of equations describing RPA_S and another one is to simplify the class (or operations of) RPA_S itself. Actually [S87a] presents some very nice, simple and elegant candidates $\mathcal{S}$ to take the place of S, the only problem left open is to decide whether one of these $\mathcal{S}$'s is finitely presented as a semigroup. A positive answer would provide us with an intuitively illuminating, transparent and very attractive version of RPA_S in Theorem 16 above. In this connection we should mention that R. J. Thompson has a result which is also highly relevant to the finitization problem. His class RPA_T is also a finitely axiomatizable subreduct of RPA_ω; but RQA_ω or even the subreduct RSC_ω of RQA_ω containing only the s^i_j's and c_i's (or even the one containig the c_i's only) is *not* obtainable as a subreduct of RPA_T. So adequateness of RPA_T for algebraizing first-order logic seems to be problematic. (On the other hand, the RPA_T operations when applied to $\mathsf{Rf}(U)$ generate the same subalgebras as those of $\mathfrak{Rfp}(U)$.)

The above important results, in a sense, form a part of a relatively recent movement in algebraic logic connected with polyadic algebras. This movement looks at classes K of subreducts of RPA_ω's such that K contains all RQA_ω's as subreducts. Actually Pinter's simpler substitution algebras $\mathsf{RSC}_\omega = \mathbf{SRd}(\mathsf{RQA}_\omega)$ introduced at the end of §5 above are sufficient instead of RQA_ω's. For simplicity, we look at

$$(5.1) \qquad \mathsf{RQA}_\omega = \mathbf{SRd}\mathsf{K} \qquad \text{and} \qquad \mathsf{K} = \mathbf{SRd}(\mathsf{RPA}_\omega)\ .$$

Theorem 16 above shows that the equational theory of such a class K can be finitely axiomatizable. According to Thm's 7(i), 9(i) on pp. 14–15 (proved on p. 15) of the June 87 version of Sain [S87a], there are choices of K satisfying (5.1) such that K is only a quasivariety but not a variety. (It is also proved there that $\mathbf{H}\mathsf{K}$ is axiomatizable by a finite schema of equations for the same K. It is an interesting open problem whether that K is axiomatizable by a finite schema of quasi-equations.) This means that when looking for simpler solutions for the finitization problem than the one in Theorem 16, then it might be a good idea to look for finitely axiomatizable quasivarieties too, and not only for varieties. The works of Blok and Pigozzi (cf. [BP89]) indicate that quasivarieties are just as good vehicles for algebraic logic as varieties. Works concerning classes like K above are numerous, but we should point out Craig's semigroup oriented school (cf. earlier references to Craig's works, Howard [H65], Thompson's works), Demaree [D72], Daigneault–Monk [DM63] — without claiming that we have mentioned all the important ones.

In passing we note that expanding RCA_α's with the single operations $\mathsf{C}_{(\alpha)}$ yields a class RCA^+_α highly relevant to $L_{\omega\omega}$ and having a smoother theory in some respect than RCA_α's for $\alpha \geq \omega$. E.g. RCA^+_α is a discriminator variety, while RCA_α is not if $\alpha \geq \omega$. (The logical meaning of $C_{(\alpha)}$ is taking the universal closure of a formula.)

Let us return to algebraizing infinitary logics. For the rôle of RPA_α's ($\alpha \geq \omega$) and their reducts and expansions already mentioned in this paper, see Examples 5.6.2, 5.6.5, 5.6.8 of [HMTII], Keisler [K63], Daigneault–Monk [DM63].[48] Other but not unrelated algebraizations of infinitary logics are in Preller [Pr68a, Pr69, Pr69a], Lucas [L68].

Lucas [L68] is related to observing that if we want to treat infinitary logic with equality via RPA_α's then it is not enough to add the Id_{ij}'s as new constants, but we will also need constants denoting the intersections of infinitely many of the Id_{ij}'s.

The transformational algebras in Craig [Cr74a] p. 30 achieve this effect in a very elegant way, but the approach there is more ambitious than aiming for just this. Namely, one of the purposes of that work is unification of (a very large portion of) algebraic logic.[49] For any τ, besides S_τ, Craig introduces $T_\tau : \mathcal{P}({}^\alpha U) \longrightarrow \mathcal{P}({}^\alpha U)$ as follows. $T_\tau(x) = \{q \circ \tau : q \in x\}$, for any $x \subseteq {}^\alpha U$. We note that T_τ is a conjugate of S_τ in the sense of Jónsson–Tarski [JT51], and that the term function $t^i_j(x) = \mathrm{Id}_{ij} \wedge c_i x$ of CA_α's is the same as $T_{[i/j]}$. Let $\mathcal{S}$ be a subsemigroup of the transformation semigroup ${}^\alpha\alpha$. Then the class $\mathsf{TB}_\mathcal{S}$ of *$\mathcal{S}$-transformational algebras* consists of the algebras embeddable into products of algebras of the form $\langle \mathfrak{P}({}^\alpha U), S_\tau, T_\tau \rangle_{\tau \in \mathcal{S}}$. Observe that $c_i x = S_{[i/j]} T_{[i/j]}(x)$ if $j \neq i$, moreover, to each $\Gamma \neq \alpha$ there is $\tau \in {}^\alpha\alpha$ such that $C_{(\Gamma)} x = S_\tau T_\tau(x)$. But then, $C_{(\alpha)}(x) = C_{(\{0\})} C_{(\alpha \smallsetminus \{0\})}(x)$ is expressible, too. So the $\mathsf{TB}_\mathcal{S}$'s are more powerful than the $\mathcal{S}$-polyadic algebras[50], i.e. all $\mathcal{S}$-polyadic operations are expressible, moreover $\mathrm{Id}_{ij} = T_{[i/j]}(1)$ shows that even more are expressible in $\mathsf{TB}_\mathcal{S}$. If $\mathcal{S}$ is generated by a subset G then, of

[48] A problem in connection with algebraizing infinitary logic is the following. As mentioned in the discussion of RCA_ω's around the end of §4, the algebraic counterparts of models of $L_{\omega\omega}$ form the class L which admits an intrinsic characterization condensed into the notation $\mathsf{L} = \mathsf{Cs}^{reg}_\omega \cap \mathsf{Lf}_\omega$ in [HMTAN] and [HMTII]. In infinitary $L_{\kappa\lambda}$, it is customary to have at least $\alpha = |\kappa + \lambda|^+$ variables. This leads to the reducts of RPA_α's in Daigneault–Monk [DM63] in which $C_{(\Gamma)}$ is restricted to the case when $|\Gamma| < \alpha$. But then the class of algebraic counterparts of models (analogous to L) is not the polyadic version of Cs_α, nor the intersection of that of Cs^{reg}_α with any abstract class (generalizing Lf) as was shown in [N78] (based on Andréka [A73]). So a sleek (like Cs^{reg}) characterization of the algebraic counterparts of models in the infinitary case seems to need a new idea. Some nonconclusive experiments in this direction were reported in [N78].

[49] Different approaches for a similar kind of unification are in Schein [Sc70], Andréka–Németi [AN80], cf. [HMTII] p. 260 and Andréka [A77], Salibra [Sa89], and on a different, more universal algebraic level of abstraction [HMTII] pp.255–260, cf. Andréka–Sain [AS78] and [ANS84].

[50] These are basically the $\mathsf{RPA}_\mathcal{S}$'s, but sometimes C_Γ is also included for those choices of Γ for which $|\tau(\Gamma)| = 1$ for some $\tau \in \mathcal{S}$.

course, it is enough to consider $\langle \mathfrak{P}({}^\alpha U), \mathrm{S}_\tau, \mathrm{T}_\tau\rangle_{\tau\in G}$. If $G = \{[i/j] : i,j < \alpha\}$, then $\mathsf{TB}_\mathcal{S}$ is the same as RCA_α. If $\mathcal{S}$ consists of the finite transformations of α then $\mathsf{TB}_\mathcal{S}$ is RQEA_α i.e. quasi-polyadic algebras with equality. If $\mathcal{S} = {}^\alpha\alpha$ then $\mathsf{TB}_\mathcal{S}$ coincides with Lucas's expanded RPEA_α's. (The latter are basically RPA_α's with new constants Id_E for equivalence relations E on α.) There are further examples in Craig [Cr74a], we listed the above ones to illustrate the unifying power and elegance of this approach.

In passing we note that if K is a reduct of $\mathsf{TB}_\mathcal{S}$ ($\mathcal{S} = {}^\alpha\alpha$) containing $\mathrm{C}_{(\alpha)}$ and the Boolean operations as derived operations, then **HSP**K is a discriminator variety, cf. Theorem 14.

7. Other algebras of relations

We have no reason to believe that the decisions we made (at certain "branching points") in the development outlined so far were the best possible ones. (One aspect of this observation already appeared as a sequence of recurring remarks starting with Remark 2 addressing the finitization problem.) Many of the approaches listed below originate from making some of these decisions differently.

As mentioned in the introduction, this section was intended to be somewhat complementary to a similar overview at the end of [HMTII]. Therefore, in this section we tend to restrict ourselves to giving such references which were not listed in [HMTII] (e.g. because they appeared later than [HMTII]).

(1) Many-sorted cylindric algebras

In the middle of §4, our task was to glue the algebras

$$\mathfrak{Rel}_2(U) = \langle \mathfrak{P}({}^2U)\ldots\rangle, \quad \ldots, \quad \mathfrak{Rel}_n(U) = \langle \mathfrak{P}({}^nU)\ldots\rangle, \quad \ldots \qquad n\in\omega$$

belonging to the varieties $\mathsf{RCA}_2,\ldots,\mathsf{RCA}_n,\ldots$ respectively, together into a *single* algebra of relations of arbitrary finite ranks. This task admits an easy solution if we are willing to use many-sorted (in other words heterogeneous) algebras (cf. Burmeister [Bu86], Ehrig–Mahr [EM85], Lugowski [L76] for the many-sorted version of universal algebra). Many-sorted cylindric algebras, MsCA's have the $\mathfrak{Rel}_n(U)$'s as sorts for each $n < \omega$. So, in particular, there are ω-many sorts, and the old operations of $\mathfrak{Rel}_n(U)$ act on the sort or universe $\mathcal{P}({}^nU)$. We need operations connecting the different sorts too, so let $\mathrm{c}_n^+ : \mathcal{P}({}^nU) \longrightarrow \mathcal{P}({}^{n+1}U)$ be our old dummy embedding, $\mathrm{c}_n^+ : R \mapsto R\times U$ for $R\subseteq {}^nU$. Its "inverse" $\mathrm{c}_n^- : \mathcal{P}({}^{n+1}U) \longrightarrow \mathcal{P}({}^nU)$ deletes the last column of any $n+1$-ary relation, e.g. $c_2^-\{\langle a,b,c\rangle\} = \{\langle a,b\rangle\}$. So in particular $c_n^+ c_n^-(x) = c_n(x)$ for $x \subseteq {}^{n+1}U$. Now, a full MsCA is of the form

$$(7.1) \qquad \langle \mathfrak{Rel}_n(U), c_n^+, c_n^- \; : \; n < \omega \rangle \; .$$

Then one defines MsCA's by taking subalgebras of direct products. By using results from cylindric algebra theory (cf. 2.3.8 on p. 63 of [HMTII]), one can rearrange these algebras such that they form a many-sorted variety *axiomatizable* by a finite schema of equations, see [HMTII] p. 263, Bernays [B59]. Other works on this MsCA approach are e.g. in Börner [B88], Schwartz [Sw79]. In passing we note that this many-sorted approach seems to be connected with the universal algebraic theory of clones (as was made explicit in Börner [B88]). Namely, clones are sets of special finitary relations whose speciality is that they are functions. Therefore an algebra of clones over a set U is at the same time an algebra of (certain) finitary relations over U. The many-sorted algebraic approach to clones is one of the oldest, most traditional ones, cf. Cohn [C65] §III.3 and its Exercise 3 for example.

Despite of the fact that this many-sorted approach leads to a natural positive solution of the *schema version* of the finitization problem (cf. Remark 2 in §3, Problem 4.1 at the end of §4 etc. for this problem), as far as we know, it hasn't been explored too much beyond proving (incarnations of) this result.[51] We do not know the reason for this.

(2) Partial cylindric algebras

We mentioned clones above as motivating examples for the many-sorted approach. Clones are also often treated as partial algebras (cf. McKenzie et al. [MMT] p. 143, Exercise 3 in §III.3 of Cohn [C65]). For the theory of partial algebras see Craig [Cr88], Burmeister [Bu86], Andréka–Németi [AN76]. Actually, many-sorted algebras are well known to be special cases of one-sorted partial algebras. Therefore an alternative way of doing **(1)** above is using partial algebras. (Boole originally introduced his algebras as partial ones, anyway.)

Let $Frl(U) = \{R : R \subseteq {}^nU \text{ for some } n < \omega\}$ be the collection of finitary relations over U (as above Fact 2 in §4). If $\emptyset \neq R \subseteq {}^nU$ then $-R = {}^nU \setminus R$, otherwise $-$ is undefined. $R \cup S$ and $R \cap S$ are defined only if the ranks of R and S coincide. Let $R \subseteq {}^nU$. For $i < n$, $c_i(R)$ is the usual, for $i \geq n$ we let $c_i(R) = R$. c_i^+ and c_i^- are as in subsection **(1)** above, and are defined only on $R \subseteq {}^nU$ and $R \subseteq {}^{n+1}U$ respectively. For each $n < \omega$, we add the

[51] Recall (from §4) that the schema version is the less ambitious version of the finitization projects.

constant $\mathrm{Id}_n = (\mathrm{Id} \restriction ({}^nU)) = \{q \in {}^nU : q_0 = q_i \text{ for all } i < n\}$. Now our generic algebras are of the form

$$\langle Frl(U), \cap, \cup, -, \mathsf{c}_n, \mathsf{c}_n^+, \mathsf{c}_n^- \rangle_{n<\omega} \ .$$

PaCA's are the algebras embeddable into direct products of these (as usual). Since ${}^nU = cc_0 \ldots c_{n-1}(\mathrm{Id}_n)$ is a derived constant, each sort $\mathfrak{Rel}_n(U)$ of the algebras in **(1)** above is recoverable in the present partial algebraic setting too.

We note that the point in using the new, partial versions of $\cup$, $\cap$, and $-$ is that they do not lead out from the circle of the usual finitary relations, i.e. $Frl(U)$ is closed under them. The main problem with the total versions of these operations was, noted immediately above the beginning of §3, that both total complementation and total union led out of $Frl(U)$, e.g. the union of relations of different ranks is a set of sequences of various length hence it is not a relation in the usual sense.

Despite of the availability of partial algebra theory, almost nothing is known about these algebras, as far as we know.

(3) Algebras of finite sequences

Let us recall the BA $\mathfrak{P}({}^{<\omega}U)$ of finite sequences from before §3. Let $succ : n \mapsto n+1$, $pred : n+1 \mapsto n$ with $pred(0) = 0$ be the usual transformations of ω. Then T_{succ}, T_{pred}, S_{succ}, S_{pred} are defined on subsets of ${}^{<\omega}U$ too, since for any $q \in {}^{<\omega}U$ we have $q \circ succ, q \circ pred \in {}^{<\omega}U$, too. E.g. $\langle a,b,c\rangle \circ succ = \langle b,c\rangle$ and $\langle a,b,c\rangle \circ pred = \langle a,a,b,c\rangle$. For $R \subseteq {}^{<\omega}U$, by definition $\mathrm{S}_{succ}(R) = \{q \in {}^{<\omega}U : q \circ succ \in R\}$ and similarly for S_{pred}. The definition of T_{succ}, T_{pred} is even more literally the same as it was around the end of §6 above.

By a finite sequence algebra (FSA) we understand an algebra embeddable into a direct product of algebras of the form

$$\langle \mathfrak{P}({}^{<\omega}U), \mathsf{c}_i, \mathrm{S}_{succ}, \mathrm{S}_{pred}, \mathrm{T}_{succ}, \mathrm{T}_{pred} \rangle_{i<\omega} \ ,$$

where c_i is the usual one, e.g. $\mathsf{c}_2\{\langle a,b,c\rangle\} = \{\langle a,b,u\rangle : u \in U\}$, $\mathsf{c}_3\{\langle a,b,c\rangle\} = \{\langle a,b,c\rangle\}$, and $\mathsf{c}_i(x \cup y) = (\mathsf{c}_i x) \cup (\mathsf{c}_i y)$. Recall that 1 denotes the top element ${}^{<\omega}U$ in BA language. We note that ${}^1U = \{\langle u\rangle : u \in U\} = -\mathsf{c}_0\mathrm{T}_{pred}(1)$, ${}^0U = \{\langle\rangle\} = \mathrm{T}_{succ}({}^1U)$, ${}^2U = \mathrm{S}_{succ}({}^1U)$, $\ldots$, ${}^{n+1}U = \mathrm{S}_{succ}({}^nU)$ for $n > 0$, finally $\mathrm{Id}_2 = \{\langle u,u\rangle : u \in U\} = \mathrm{T}_{pred}({}^1U)$ and $\mathrm{Id}_{n+1} = \mathrm{T}_{pred}(\mathrm{Id}_n)$ for $n > 0$. Since ${}^nU$ and Id_n are derivable constants, and the c_i's and the standard BA operations are built in, the cylindric algebra $\mathfrak{Rel}_n(U)$ is available as the set of elements below ${}^nU$.

So the many-sorted cylindric algebra with n-th sort $\mathcal{P}({}^nU)$ or $\mathfrak{Rel}_n(U)$ described in (7.1) in subsection (1) above is contained in a sense in our present FSA with universe $\mathcal{P}({}^{<\omega}U)$. Moreover, all the operations of the many-sorted algebra in (7.1) are definable in our present FSA. Despite of all these nice things, the algebraic theory of FSA cannot replace that of MsCA because e.g. the algebraic operators like direct products or ultraproducts behave rather differently in the two cases. For example, the direct power or ultrapower with exponent ω of the above FSA will contain elements x, y corresponding to the ω-sequences $\langle {}^2U, {}^3U, \dots, {}^nU, \dots \rangle$ and $\langle \mathrm{Id}_2, \mathrm{Id}_3, \dots, \mathrm{Id}_n, \dots \rangle$ respectively. Elements like x and y do not fit into any of the sorts of MsCA's, and indeed they do not appear in direct powers or ultrapowers of MsCA's.

Finally we point out that Craig's algebras as described in item (3) on p. 265 of [HMTII] are term definably equivalent with FSA's above. To see this, we note that, using the notation $\mathbf{Q}(x)$ and $\mathbf{P}(x)$ introduced therein, we have $\mathbf{Q}(x) = \mathrm{S}_{succ}(x) \setminus {}^0U$ and $\mathbf{P}(x) = \mathrm{T}_{succ}(x \setminus {}^0U)$. This is enough for one direction. In the other direction $\mathrm{T}_{pred}(x) = \big(\mathrm{Id}_{01} \cap \mathbf{Q}(x)\big) \cup (x \cap {}^0U)$, where ${}^0U = -\mathbf{Q}(1)$, and similarly for the rest of the FSA operations.

For interesting results concerning FSA see op. cit., and the references therein, e.g. Craig [Cr74a, Cr74], Monk [M70].

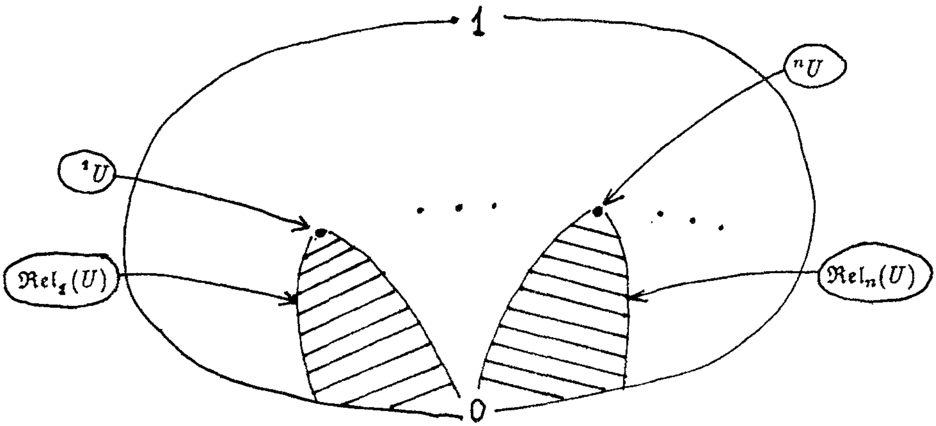


FIGURE 2

(4) Algebras of generalized finite sequences

One disadvantage of working with ${}^{<\omega}U$ as above is that these finite sequences impose a structure $\langle \omega, succ, < \rangle$ on the index set. I.e. the indices 1,2 and say 5 are not equivalent as they were in the RCA_n or RCA_ω or RQA_ω case. Another symptom of the same "asymmetry" is that $\mathsf{p}_{12} = \mathsf{S}_{[1,2]}$ cannot be applied to $\mathcal{P}({}^{<\omega}U)$ because it leads out of $\mathcal{P}({}^{<\omega}U)$. Namely, $\mathsf{S}_{[1,2]}\{\langle a,b\rangle\} = \{\langle a,b\rangle \circ [1,2]\}$ but $f = \langle a,b\rangle \circ [1,2]$ is not a finite sequence since $f(0) = a$, $f(1)$ is undefined, $f(2) = b$.

Let $\mathsf{Gfs}(U) = \bigcup\{{}^{H}U : H \subseteq \omega \text{ and } |H| < \omega\}$. That is, a *generalized finite sequence* is a function mapping a finite set of natural numbers into U. $\mathsf{Gfs}(U)$ is the set of these. Now, replacing ${}^{<\omega}U$ with $\mathsf{Gfs}(U)$ in subsection **(3)** above, we obtain the definition of GFSA's, i.e. generalized FSA's. The theory becomes smoother this way. Moreover, we may safely add $\mathsf{S}_{[i/j]}$, $\mathsf{S}_{[i,j]}$ to the operations. If after adding these we remove the ones associated with *pred* and *succ* then we obtain something very close to RQA_ω's or RCA_ω's, depending on whether we keep Id_2.

In many respect the latter version of GFSA's are very close to what are called weak cylindric set algebras in [HMTII]. See e.g. Howard [H65], Craig [Cr74], Andréka [A77] for GFSA's. As far as we know, they originate with Craig. In passing we note that this approach of generalized sequences allows for the fulfillment of an old dream of some algebraic logicians: If we want, we can introduce c_i^- as an operator of simply deleting i from the arguments of all the sequences, i.e. $\mathsf{c}_i^-(x) = \{f \restriction (Dom(f) \setminus \{i\}) : f \in x\}$. Then c_i^- really deletes (unmistakably abstracts away from) the i-th column of any relation (in algebras of relations coming from computer science this deletion operation is frequent). In general, this GFSA approach seems to be close to the way relations are often approached in computer science.

(5) Making function symbols of logics explicit

In subsection **(1)** of this section we already noted that the algebra of clones is a well investigated example of special algebras of finitary relations which happen to be functions, see the references in **(1)**. The operations of such algebras of finitary functions will have a different character from those discussed so far, because in choosing them, we have to make sure that they yield functions when applied to functions (so even the Boolean operations are excluded).

To any set U, we can associate two algebras, one, say $\mathfrak{Fn}(U)$, of finitary functions, and an RCA_ω, say $\mathfrak{Rf}(U)$, for finitary relations over U. Since their elements are strongly related, we can combine them into a single two-sorted algebra $\mathfrak{Ts}(U) = \langle \mathfrak{Fn}(U), \mathfrak{Rf}(U), f_0, \ldots, f_n \rangle$, where the f_i's are operations acting between the sorts. Algebras like $\mathfrak{Ts}(U)$ can be used in algebraizing quantifier logics *with function symbols* in such a way that terms (and term-like syntactic entities of the language) correspond to $\mathfrak{Fn}(U)$, and formulas correspond to $\mathfrak{Rf}(U)$. So when forming the Lindenbaum–Tarski algebra of a theory then (the equivalence classes of) terms form one sort of the algebra, and (those of) formulas form another. In general, when a logic has several syntactic categories often called types in computer science logics (see Salibra [Sa89]), then each syntactic category or type gives rise to a separate sort. Such an approach was elaborated in Andréka–Sain [AS78] and subsequent works reviewed on pp. 250–260 of [HMTII], see Examples 5.6.3, 5.6.6, 5.6.9 for the treatment of the "term sort" mentioned above. Feldman [Fe89], Cirulis [C89, C88, C88a], [MP88] represent a variety of approaches, and contain further results in connection with this topic.

(6) Algebraic logics for nonclassical quantifier logics

See e.g. item (4–6, 18) on pp. 265–270 of [HMTII], Freeman [F76], Monk [M60], Georgescu [G79, G79a], Schwartz [Sw80], Ferenczi [F89a], our §7 (7) below (Rasiowa's approach), and §4 of [M77]. This is a rather incomplete list. What item (11) below writes about nonclassical (or non-Boolean) generalizations of Boolean Algebras with Operators is relevant here.

(7) The Rasiowa–Sikorski approach

In principle, one could modify the approaches reviewed so far such that instead of adding new operations like c_i to BA's, this modified approach would only assume the existence of the potential values of these operators as elements of the BA. In a sense, this seems to be similar to defining groups as semigroups in which certain kinds of elements exist. Roughly speaking, the approach discussed in this subsection differs from the ones reviewed so far in that instead of adding new operations to BA's, one uses BA's satisfying certain existence properties. So, in a sense, instead of expanding BA's to BAO's, we move from BA's to a special subclass of BA's. Of course, when applied to a nonclassical logic, BA's in this approach are replaced with the algebraic counterpart of the propositional version of the logic under investigation, see e.g. Rasiowa [R51–R74a], [RS]. In this survey we cannot attempt covering the literature of this branch of algebraic logic, but see e.g. Maksimova [Ms77, Ms79, Ms88], [MsR74], Bloom and Brown [BB73], Brown and Suszko [BrS73], [G85], [GP89], [GgP88]. Font–Verdú [FV89–91] is a continuation of this approach, but they are also taking into account [BB73], [Br73], and [BP89–89d]. In Rasiowa [R74], the approach is traced back to a paper of Mostowski from 1948.

(8) Connections with, and approaches originating from, computer science

This area was sporadically mentioned throughout this paper, we do not repeat those references. For lack of space, here we restrict ourselves to a list of references the items of which are not listed in [HMTII]. Even if we restrict our attention to papers that appeared after [HMTII], our list is severely incomplete. Some of the recent works in this area are Cosmadakis [C87], Trnková–Reiterman [TR87], [MP88], [C88a, C86], Imielinski–Lipski [IL84], Venema [V88], [W88], [L87], [LM89], [GL90], [Ca88], Goldblatt [G87], Jónsson [J89, J90], Salibra [Sa89], Ng [N84], Volkov [Vo86–88a], [N80, N81], Sanders [Sa80], Knuth–Rónyai [KR83], [BZ86], [BKSS], van Benthem [vB88–90a], [AGN87], [C87], [O89, O89a], Schönfeld [Sö77], Düntsch [Dü89], the textbook [SS89], [BMP], Pratt [Pr90], [K90], [N90], [H80], [HJ86], [BU77], [AN86], [Pl88, Pl89], [Ro72], [Zi83].

(9) Connections with category theoretical logic

First we briefly return to the *finitization* problem using the language of category theory. Let α be an ordinal (as before). Let $\mathbf{Full\ Cs}_\alpha$ be the usual category of full Cs_α's (see §4) as objects and homomorphisms as morphisms between them. Let $\mathbf{SetIso}$ be the category of sets as objects and isomorphisms (bijections) as morphisms between them. Then there is a functor $F^+_\alpha : \mathbf{SetIso} \rightarrowtail \mathbf{Full\ Cs}_\alpha$ such that $F_\alpha(U) = \mathfrak{Rel}_\alpha(U) = \langle \mathfrak{P}(^\alpha U), \ldots \rangle$, and for any morphism $p : U \rightarrowtail\!\!\!\!\rightarrow W$, $F_\alpha(p) : \mathcal{P}(^\alpha U) \rightarrowtail\!\!\!\!\rightarrow \mathcal{P}(^\alpha W)$ is the natural one, i.e. $F_\alpha(p)(R) = \{p \circ q : q \in R\}$ for $R \subseteq {}^\alpha U$. It is not hard to check that F_α is really a functor (cf. Exercise 3P (Algebraic Logic) (c) in [AHS]). (In [HMTAN], [HMTII] p. 15, $F_\alpha(p)$ was called the *base isomorphism* induced by p, and $F_\alpha(p)$ was denoted by $\tilde{p}^\alpha$.)

Let K^+ be an expansion of the class full Cs_α, i.e. full $\mathsf{Cs}_\alpha = \mathbf{Rd}(\mathsf{K}^+)$, cf. Remark 2. We say that K^+ is *functorially invariant* if we can "expand" F_α defined above to a functor $F_\alpha : \mathbf{SetIso} \rightarrowtail \mathbf{K}^+$ such that $F_\alpha(U) = \mathbf{Rd}(F^+_\alpha(U))$, and $F_\alpha(p)$ "as a function" coincides with $F^+_\alpha(p)$. Here $\mathbf{K}^+$ is the category made from K^+ the usual way (adding the usual homomorphisms as morphisms).[52]

[52] Using the terminology of [HMTAN], [HMTII], K^+ is functorially invariant if its full members "admit" *base-isomorphisms* the same way as those of Cs_α do. In passing we note that restricting our attention to full algebras is important here. Namely, Andréka constructed a simple RRA $\mathfrak{A}$ and an automorphism h of $\mathfrak{A}$ such that in *no* representation of $\mathfrak{A}$ is h induced by a permutation of the base set U (of the algebra $\mathfrak{B} \subseteq \langle \mathfrak{P}(U \times U) \ldots \rangle$ representing $\mathfrak{A}$). I.e. for any embedding of $\mathfrak{A}$ into a full set RA $\langle \mathfrak{P}(U \times U) \ldots \rangle$, h does not extend to an automorphism of the full algebra. Her construction carries over to cylindric and polyadic algebras from RA's. This solves Problem 19a of Maddux [Ma89c]. Shelah

As noted in a footnote in Remark 2, functorial invariance is stronger than invariance (the latter quoted in Remark 2 from the literature), and conforms more to algebraizing abstract model theory. (The finitely axiomatizable expansion of RQA_ω in Sain [S87a] and in Theorem 16 in §6 is functorially invariant, too.)

PROBLEM 3. *Find functorially invariant* K^+ *such that* $\mathbf{SP}\mathsf{K}^+$ *is a finitely axiomatizable variety (or at least a finitely axiomatizable quasivariety) satisfying one of* (i)—(iii) *below.*

(i) full $\mathsf{Cs}_\alpha = \mathbf{Rd}(\mathsf{K}^+)$ *for* $\alpha = \omega$ *or for* $2 < \alpha < \omega$.
(ii) *The same as* (i) *but with* full set RQA_α *in place of* full Cs_α.
(iii) *The same as* (i) *but for Pinter's substitution-cylindric algebras (see §5).*

The above problem comprises 6 different problems, each one interesting on its own right, e.g. the $\alpha < \omega$ and $\alpha = \omega$ cases are two separate problems.

For $\alpha = \omega$, (ii), (iii) above were provided with one solution in Sain [S87a], *but* it would be still immensely interesting to see simplifications (both of K^+ and of the axiomatization of the variety generated by K^+) of that solution, and to see different solutions. It would be also very interesting to see a (preferably streamlined) finite axiomatization of the *quasivariety* generated by $\{F_\alpha^+(U) : U \text{ is a set}\}$ for the choice of K^+ given in [S87a] (as a solution of (ii), (iii) above), where the functor $F_\alpha^+ : \mathbf{SetIso} \rightarrowtail \mathsf{K}^+$ was introduced preceding Problem 4.1.

Connections with category theoretical logic were sporadically mentioned throughout this paper. E.g. it was pointed out that RRA's are actually categories. We should mention that so are the various kinds of algebras unifying all finitary relations (RCA_ω's, RQA_ω's etc.) discussed in this paper. The objects of these categories are the elements representing ${}^nU$ for $n < \omega$. This is the easiest to see in the case of many-sorted cylindric algebras in subsection **(1)** above. There the objects of the category are the sorts of the algebra. Then elements of the algebra can be considered as relations acting between sorts, and in this quality they are morphisms acting between the sorts as objects.

Another, more generally used approach (cf. [Z83], [Z84], [D69], [Mk87], [MkR77]) considers all elements of a many-sorted cylindric algebra $\mathfrak{A}$ (or a partial cylindric algebra, cf. items (1,2) above) as objects of the corresponding category $\mathbf{C}$. A pair $\langle R, S\rangle$ of elements R, S of $\mathfrak{A}$ is a morphism of $\mathbf{C}$ iff R can be considered as a function with range contained in S (i.e. whenever

proved that similar kinds of results can apply even for very innocent looking cylindric algebras [Sh89], [BiS88] (solving a problem from [GMTAN]).

S is an n-ary relation then R is an $n+k$-ary one for some k, and can be considered as associating n-tuples to certain k-tuples; the codomain of this morphism is S, while its domain is $\mathsf{c}_n^- \mathsf{c}_{n+1}^- \cdots \mathsf{c}_{n+k}^-(R)$). (This connection extends to ordinary cylindric algebras the obvious way, i.e. by transforming $\mathfrak{A}$ into such an algebra.) If we turn a cylindric algebra $\mathfrak{A}$ into a category $\mathbf{C}$ along these lines then any functor $F : \mathbf{C} \longrightarrow \mathbf{Set}$ preserving[53] finite limits and finite colimits will correspond to a model of the usual first-order theory represented by $\mathfrak{A}$ or, equivalently, F will correspond to a representation of (a subdirectly indecomposable factor of) $\mathfrak{A}$ as a (cylindric) set algebra or, still equivalently, F will correspond to a homomorphism from $\mathfrak{A}$ into a (cylindric) set algebra. Roughly speaking, the special categories called small pretoposes (see e.g. [Mk87], [MkR77]) in categorical logic correspond to ("abstract") cylindric algebras (or RQA_ω's), and pretopos functors from small pretoposes into $\mathbf{Set}$ correspond to homomorphisms from arbitrary cylindric algebras into cylindric set algebras (Cs_ω's or Qps_ω's). The large pretopos $\mathbf{Set}$ corresponds to the class of full Cs_ω's. The reason why there is no single Cs_ω (but instead a class of them) corresponding to $\mathbf{Set}$ is that $\mathbf{Set}$ is a proper class, and for "purely administrative" reasons, algebras are required to be sets. It is not hard to construct e.g. a single class size Qps_ω $\mathfrak{B}$ such that $\mathfrak{B}$ would play the rôle of $\mathbf{Set}$ in quasi-polyadic algebraic logic. In particular, the class of all models of a theory T would correspond to the class $Hom(\mathfrak{A}_T, \mathfrak{B})$ of homomorphisms from the RQA_ω $\mathfrak{A}_T$ corresponding to T into our single "class-algebra" $\mathfrak{B}$.

We are afraid that the remarks made so far do not convey enough of the subject of this item. For lack of space we restrict ourselves to giving an incomplete list of references.: Daigneault [D69], Monk [M78], Zlatos [Z83, Z84], Comer [Co72], Barthelemy [B74], Georgescu [G73], Gergely [Ge80], Jónsson [J88], Olivier–Serrato [OS80], Laita [L76], Ouellet [O82], Volger [V75], Topencarov [T74], Freyd–Scedrov [FS90].

The natural category theoretic investigations (which have already been carried through in great detail for groups, semigroups etc.) for cylindric and related algebras constitute an area where more work would be welcome. (E.g. the connections between RRA's and RCA_n's in §5.3 of [HMTII] are not functorial in their present form, and it would be very nice to have such a version.) For fragmentary results and open problems see e.g. Preller [Pr68], Adámek–Herrlich–Strecker [AHS], and Sain [S89].

[53] $\mathbf{Set}$ is the usual category of sets (as objects) and functions between them (as morphisms).

(10) Peano Arithmetic, Gödel's incompleteness etc.

The algebraic approaches (sometimes called multi-modal logic approaches) to this field use BAO's (Boolean algebras with operators) just as the rest of the directions reported herein. Usually at least one of the extra-Boolean operators is based on self-reference as the latter was used in Gödel's proof (definability of syntactic concepts in strong enough theories). This field became recently very active. See Solovay [So76], Smorynski [Sm85, Sm84], and the references in Chapter 2 of Bernarducci [B89]. See also Halmos [Ha60], and pp. 206–215 of Halmos [Ha85], [M77] Problems 1, 4, 5. (Works of Magari, Mangione, Hansoul, and Jeroslow quoted in [HMTII] are relevant to the present subject.)

(11) Boolean algebras with operators (BAO's)

As already mentioned e.g. in Remark 1 (§3), the theory of BAO's is an important one unifying not only practically all the approaches discussed in this paper, but a large portion of propositional algebraic logic, too. Besides in algebraizations of quantifier logics, BAO's play a fundamental rôle in the theories of modal, multimodal, and temporal logics, cf. e.g. Goldblatt [G76–G89]. To mention only one example, what are known today as Kripke-models, were first discovered in BAO-theory, cf. Jónsson–Tarski [JT48], [JT51].

Besides the fundamental works [HMT] §2.7 and [JT51], for a sample of works too recent for being quoted in [HMTII], on BAO's and their strong connections with multi-modal logics (temporal logics), see e.g. Jónsson [J84, J90], Henkin [H70], Goldblatt [G85, G87, G88, G89], Naturman–Rose [NR89], Bernardi–D'Aquino [BD88], Brink [Bn88], [Sha80], Blok [Bl76, Bl80, Bl80a], Blok–Pigozzi [BP89a, BP89b], [Ra79], [Te86], Sain [S84], [N80], works of Hansoul and Wu in the bibliography of [HMTII].

Non-classical (i.e. non-Boolean) generalizations of BAO's:
If we want to study e.g. intuitionistic multi-modal logic then we have to generalize Boolean Algebras with Operators accordingly. Indeed, if we replace the Boolean part of BAO's with the algebras of some nonclassical propositional logic we will obtain standard generalizations of BAO's like Heyting algebras with operators ([M60], reference Fischer–Servi [81'] of [HMTII], [FV89], [F87], [Ve86]), De Morgan algebras with operators, Hilbert algebras with operators, distributive lattices with operators ([Co89a], [IL84], [A89a], [A89]). Weak Brouwerian semilattices with operators and pseudo-interior algebras with operators were proved in [BP89b] to be of central importance for the theory of discriminator varieties. See also [Sw80] and references Georgescu [72'], Servi [79'] of [HMTII]. Some classes of Heyting algebras with operators suitable for studying certain intuitionistic modal logics were investigated in [FV89].

(12) Logics without variables (combinatory logic)

This fairly active branch of logic is strongly related to algebraic logic. Both fields (logic without variables, LWV for short, and algebraic logic, AL) aim for a more refined analysis of the structure of formulas (together with their meanings, proofs etc.) than conventional logic does. Namely, conventional logic takes atomic formulas with all their substituted versions like $R(v_0 v_1)$, $R(v_1 v_0)$, $R(v_2 v_2)$ etc. for granted, i.e. they are treated as "primitives". A disadvantage of the conventional view is that these primitives are far from being independent of each other, and in particular they cannot be replaced by each other in formulas without affecting validity. A consequence of this conventional treatment of atomic formulas, often mentioned both in LWV and AL, is a certain complication in the axioms or proof rules when one has to talk about free and bound occurrences of individual variables when specifying an axiom or rule. A typical example is the axiom schema saying "$\exists v_i \varphi \to \varphi$ if v_i does not occur freely in φ". This kind of complication (reference to free variables) is not acceptable in propositional logics. Both AL and LWV eliminate this complication from quantifier logics. This is obviously true for LWV, concerning AL we mention that the inference system for predicate logic given in App C of [BP89] p. 69, no axiom or rule schema refers to individual variables. Actually, in conventional logic, talking about free occurrences of variables is needed to express the connection between the atomic formulas $R(v_0 v_1)$, $R(v_2 v_2)$ etc. mentioned above. Therefore, if we want to get rid of variables, we will have to express these connections by some other means, e.g. by new logical connectives. Indeed, both AL and LWV begin eliminating (explicit mention of) free individual variables by introducing new logical connectives binding the substitution instances of $R(v_0 v_1)$ to each other. Such a connective is $\mathsf{p}_{01} = \mathsf{S}_{[0,1]}$ recalled in §5 herein, and discussed throughout this paper. Since $\mathsf{p}_{01} R(v_0 v_1)$ is logically equivalent with $R(v_1 v_0)$, we may restrict our supply of atomic formulas to the ones of the form $R(v_0 v_1)$ i.e. to the ones in which the individual variables occur in a once for all fixed order $v_0, v_1, \ldots, v_n, v_{n+1} \ldots$ if we have all the p_{ij}'s and s^i_j's as logical connectives. But since the individual variables following predicate letters are uniquely determined, one can just as well dispense with variables altogether. Indeed, this is done in many of the LWV papers as reviewed at the end of §I in Kuhn [Ku80]. In many of the LWV papers, the logical connectives p_{ij}, s^i_j are called combinators or *combinatory predicate functors*, see item (2) on p. 249 of Došen [Do88]. In the AL literature, the above indicated way of eliminating variables is described in detail e.g. in [AGN77], [HMTII] (restricted formulas), [BP89] App C, Sain [S87a] §4, to mention a few recent sources; but this is a classical tool of AL.

To indicate why AL would want to get rid of the conventional machinery of free individual variables, observe that when translating an algebraic equation like $x \leq c_0 x$ to logic, we obtain the logical schema $\varphi \to \exists v_i \varphi$ in which the free variables of φ are not mentioned. The algebraic variable x translates to a formula φ or to a metavariable φ running over formulas. What it does not translate to is an individual variable. Somehow the individual variables of logic (sometimes indicated in logic as $\varphi(v_1 v_2)$ meaning that v_1 and v_2 are free variables in φ) do not show up on the algebraic side. Very roughly speaking, this is one of the reasons why AL and LWV are so strongly connected. (Another reason is that they both are ambitious analyses of logic, and they both are deeply concerned with finding the right concepts.)

A very incomplete list of references is: Došen [Do88], Quine [Q36, Q71, Q81], Kuhn [Ku80, Ku83], [Bc85], Venema [V89], Craig [Cr74], Tarski–Givant [TG87], [Gi89].

Further discussion of the connection between combinatory logic and AL is found in [Q71], and in the excellent recent book [TG87]. [TG87] is of central importance in understanding the deep connections between Tarskian AL and Tarski's approaches to other foundational fields. A very long time was devoted to the careful and ambitious preparation of that book. Indeed, it provides an illuminating synthesis of the various schools of thought (think of "Tarskian semantics" as an example) Tarski created or pioneered.

(13) Conncetions between algebras of logic and "pure logic"; universal algebraic logic

In a sense, these investigations concern the connections between algebraic logic (AL) in the narrower sense on the one hand, and logic (mathematical logic, philosophical logic, symbolic logic etc.) in itself on the other hand. (This sentence is somewhat paradoxical because it might seem to imply that AL does not contain these connections, and that logic in itself does not contain AL, but no such implication was intended here.) One of the main concerns is explicit applications and interpretations of results of AL, in "pure logic". (The other direction is not ignored either.)

Universal algebraic logic starts out from a very general definition of a logic L (of which the well known logics like classical logic, modal logics, temporal logics, intuitionistic logic, computer science logics etc. are special cases), and investigates the kinds of mathematical objects (e.g. classes of algebras like that of cylindric algebras, multi-modal algebras, Heyting algebras etc.) associated to L by AL. Associating such objects to L is called algebraization of L. One of the subjects investigated here is elaborating a general *method* for algebraizing L ([HMTII] pp. 255–260), another is finding an adequate *criterion for algebraizability* of L (Blok–Pigozzi [BP89]) which was called "adequateness criterion" in [AS78] and "semantical well presentability" in

[HMTII] p. 257, [ANS84]. The recent [FV91], [FV90] point in the direction of possible unifications of algebraizability in the sense of [BP89, BP89c, BP89d] and that in the sense of [HMTII] §5.6, [AN78], [ANS84]. A typical problem area of universal AL is proving that (independently of the choice of L) a certain logical property of L is equivalent with an algebraic property of the corresponding class of algebras cf. the works of Blok and Pigozzi, [HMTII] §5.6, [ANS84], [AGN73, AGN77], [AN75] §1, [O86], [Ma77], [W84], [S89].

Algebraic model theory concentrates on the algebraization of L from the point of view of the *model theory* of the logic L. (This again might sound somewhat paradoxical since the model theory of L might already have some algebraic flavor. In this area, however, algebraization is independent of *that* algebraic flavor.) Cf. [HMT] §4.3, [N89, 78], [S89], [Se85, 86], [Sh89], [J73]. For lack of space we cannot discuss the present subject (connections between AL and logic) adequately. Further references are e.g. [S87] §4, [Bl76–BP89], [Co72], [Cz79], [G76], [M71, 77, 78], [N85a, 87], [O89], [P89], [Ra90], [A77].

(14)

For algebraizations of quantifier logics not mentioned in this paper the reader is referred to Chapters 5, 6 of [HMTII] (cf. also §7 of [M77]).

8. APPENDIX: The traditional finitely axiomatizable approximations CA, RA etc. of our classes RCA, RRA etc. of algebras of relations

DEFINITION 8.1. *By an α-ary cylindric Kripke frame we understand a relational structure* $\mathfrak{W} = \langle W, \equiv_i, \mathrm{E}_{ij}\rangle_{i,j<\alpha}$ *where* $\mathrm{E}_{ij} \subseteq W$ *and* $\equiv_i$ *is an equivalence relation on* W *satisfying* (1–4) *below, for all* $i, j < \alpha$.

(1) $(\equiv_i \circ \equiv_j) = (\equiv_j \circ \equiv_i)$, *i.e. the equivalence relations commute;*

(2) $\mathrm{E}_{ij} = \mathrm{E}_{ji}$ *is a set of representatives for* $\equiv_i$, *i.e.* $(\forall u \in W)|(u/\equiv_i) \cap \mathrm{E}_{ij}| = 1$ *if* $i \neq j$, *else* $\mathrm{E}_{ii} = W$;

(3) $(\forall u \in \mathrm{E}_{ij})(u/\equiv_k) \subseteq \mathrm{E}_{ij}$ *for* $k \notin \{i, j\}$, *i.e.* E_{ij} *is a union of* $\equiv_k$-*equivalence classes;*

(4) $\mathrm{E}_{ij} \cap \mathrm{E}_{jk} \subseteq \mathrm{E}_{ik}$. ∎

Let $\mathfrak{W} = \langle W, \equiv_i, \mathrm{E}_{ij} \rangle_{i,j<\alpha}$ be as in Definition 8.1. The *powerset algebra* or complex algebra $\mathfrak{P}(\mathfrak{W})$ of $\mathfrak{W}$ is defined as follows:

$$\mathfrak{P}(\mathfrak{W}) = \langle \mathfrak{P}(W), \mathrm{C}_i, \mathrm{E}_{ij} \rangle_{i,j<\alpha}$$

where $\mathrm{C}_i(X) = \bigcup \{(u/\equiv_i) : u \in X\}$, i.e. $\mathrm{C}_i(X)$ consists of the $\equiv_i$-neighbors of elements of X.

DEFINITION 8.2. *The class* CA_α *of* α*-ary cylindric algebras is defined as*

$$\mathsf{CA}_\alpha = \mathbf{S}\{\mathfrak{P}(\mathfrak{W}) : \mathfrak{W} \text{ is an } \alpha\text{-ary cylindric Kripke frame}\}.$$

Recall that $\mathbf{S}$ *is understood up to isomorphism.* ∎

PROPOSITION 8.3. (i) CA_α *is a variety axiomatizable by a finite schema of equations.*

(ii) CA_n *is axiomatizable by a finite set of equations, for* $n < \omega$.

OUTLINE OF PROOF. Translating the conditions in Definition 8.1 (making $\mathfrak{W}$ a cylindric Kripke frame) into equations valid in $\mathfrak{P}(\mathfrak{W})$ is an easy exercise. E.g. transitivity of $\equiv_i$ translates to the equation $\mathsf{c}_i\mathsf{c}_i x \leq \mathsf{c}_i x$; or reflexivity of $\equiv_i$ translates to $x \leq \mathsf{c}_i x$. A system of equations obtained this way is the following, postulated for all $i, j, k < \alpha$.

(E0) the Boolean equations	(E4) $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$
(E1) $\mathsf{c}_i(x \vee y) = \mathsf{c}_i x \vee \mathsf{c}_i y$	(E5) $\mathrm{Id}_{ii} = 1$ and $\mathrm{Id}_{ij} = \mathrm{Id}_{ji}$
(E2) $x \leq \mathsf{c}_i x = \mathsf{c}_i\mathsf{c}_i x$	(E6) $\mathrm{Id}_{ik} = \mathsf{c}_j(\mathrm{Id}_{ij} \wedge \mathrm{Id}_{jk})$ if $j \notin \{i, k\}$
(E3) $\mathsf{c}_i - \mathsf{c}_i x = -\mathsf{c}_i x$	(E7) $\mathrm{Id}_{ij} \wedge \mathsf{c}_i(\mathrm{Id}_{ij} \wedge x) \leq x$ if $i \neq j$.

CLAIM 8.3.1 CA_α coincides with the variety defined by (E0–7).

To prove this: It is easy to check that $\mathsf{CA}_\alpha \models$ (E0–7); (E0–4) correspond to the first part of Definition 8.1 including (1), while (E5–7) correspond to (2–4). In particular, (E7) together with $\mathsf{c}_i\mathrm{Id}_{ij} = 1$ correspond to the main part of (2), where $\mathsf{c}_i\mathrm{Id}_{ij} = 1$ follows from (E5–6). Therefore, what remains to prove is that every algebra in which (E0–7) is valid is in CA_α. This goes by repeating the proof of the Stone representation theorem for Boolean algebras, but now paying attention to the non-Boolean operations $(\mathsf{c}_i, \mathrm{Id}_{ij})$ too. This is not very hard, and is described in detail e.g. in [HMT] 2.7.4–5, 2.7.40. ∎

Terminology: We say that CA_α is *finitizable* exactly because Proposition 8.3 holds; namely because both CA_n is finitely axiomatizable *and* CA_α is axiomatizable by a finite schema, for $\alpha \geq \omega$.

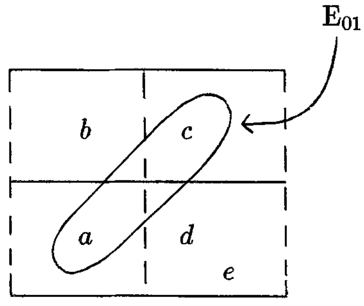
PROPOSITION 8.4. $\mathsf{RCA}_\alpha \subseteq \mathsf{CA}_\alpha$.

PROOF. By Proposition 8.3 it its enough to prove that $\mathrm{full}\mathsf{Cs}_\alpha \subseteq \mathsf{CA}_\alpha$. Let
$\mathfrak{A} = \langle \mathfrak{P}\left({}^\alpha U\right), \mathsf{c}_i, \mathrm{Id}_{ij}\rangle_{i,j<\alpha}$. For any $p, q \in {}^\alpha U$ let $p \equiv_i q$ iff $(\forall j \in \alpha \smallsetminus \{i\})p_j = q_j$. It is easy to check that $\mathfrak{W} = \langle {}^\alpha U, \equiv_i, \mathrm{Id}_{ij}\rangle$ is a cylindric Kripke frame. Since $\mathfrak{A} = \mathfrak{P}(\mathfrak{W})$, we are done. ∎

Summing up, we have a finite schema axiomatizable variety CA_α containing RCA_α as a finitary approximation of RCA_α. The next natural question asks how good this approximation is, how far CA_α is from RCA_α. The representation results in [HMTII] §3.2 seem to say that they are pretty close. For example a result says that if a CA_α is atomic with all atoms "rectangular" then it is already in RCA_α, where an atom x of a CA_2 is rectangular iff $x = \mathsf{c}_0(x) \wedge \mathsf{c}_1(x)$. Rectangularity for arbitrary α is the natural generalization of this (the intuitive idea being that the rectangular "relation" x is a Cartesian product), cf. [HMT] §1.10 p. 227. Another result says that if $\mathfrak{A} \in \mathsf{CA}_\alpha$ and $(\forall x \in A)(\exists$ infinite $H \subseteq \alpha)(\forall i \in H)\mathsf{c}_i x = x$ then $\mathfrak{A} \in \mathsf{RCA}_\alpha$.

By Monk's non-finitizability result (Theorem 4 here) the gap will always remain infinite between RCA_α and its finitary approximation CA_α. The same applies to BRA's, RRA's etc.

To illustrate the difference between RCA_α and CA_α, the figure below represents a cylindric Kripke frame $\mathfrak{W}$ for which $\mathfrak{P}(\mathfrak{W}) \in \mathsf{CA}_2 \smallsetminus \mathsf{RCA}_2$. W has 5 elements, the vertical, solid line represents $\equiv_1$, while the horizontal broken one represents $\equiv_0$. E.g. $d \equiv_0 e$ and $d \equiv_1 e$, further $d \equiv_0 a$ but $d \not\equiv_1 a$.



$\mathfrak{W}$

It is easy to check that $\mathfrak{W}$ satisfies the conditions in Definition 8.1, hence $\mathfrak{P}(\mathfrak{W}) \in \mathsf{CA}_2$. It is not hard to see that $\mathfrak{P}(\mathfrak{W}) \notin \mathsf{RCA}_2$ since $\mathfrak{P}(\mathfrak{W})$ is simple, hence if it were in RCA_2 then it were isomorphic to a Cs_2 etc. We omit the rest but the reader is invited to try to represent $\mathfrak{P}(\mathfrak{W})$ as a Cs_2.

* * * * *

The finitely axiomatizable variety RA of *relation algebras* approximating RRA is obtained the following way. Let $(3.1)^+$ be obtained from (3.1) above Theorem 1 in §3 by replacing "involuted semigroup $\langle A, \circ, ^{-1}\rangle$" in (3.1) with "involuted monoid $\langle A, \circ, ^{-1}, \mathrm{Id}\rangle$". Consider the equation

$$(8.1) \qquad x^{-1} \circ -(x \circ y) \leq -y.$$

An algebra is an RA iff it satisfies both $(3.1)^+$ and (8.1). It is easy to see that $\{(3.1^+), (8.1)\}$ is a finite set of equations, hence RA is a finitely based variety. Clearly, RA $\supseteq$ RRA.

Completely analogous observations apply to the RA–RRA relationship as were outlined for the CA–RCA relationship above.[54]

Remark (*in connection with defining* RA). Let RA^- be the variety defined by $(3.1)^+$. Then $\mathsf{RA}^- \supsetneq \mathsf{RA}$, moreover RA^- is *not* a discriminator variety (while RA, CA_n etc. are such). What is lost in RA^- is that if we define $\mathsf{c}_0 x$ as $1 \circ x$ then though c_0 remains a closure operator in RA^-, it is no more a complemented one, i.e. the complement of a c_0-closed element need not be c_0-closed.

Therefore, if we take [HMTII] Def.5.3.1 (p. 211) defining RA, and remove the last equation (R_7) then the variety defined by the remaining postulates $(R_1 - R_6)$ is not discriminator. (We note that (R_7) is our (8.1), and $(R_1 - R_6)$ is equivalent with our $(3.1)^+$.) A similar remark applies to other standard definitions of RA, e.g. to [JT52] Def.4.1.

If we add the following simple equation

$$(8.2) \qquad 1 \circ -(1 \circ x) \leq -x$$

to $(3.1)^+$ then we obtain a discriminator variety containing RA. (Actually the weaker equation $1 \circ - \big(1 \circ [x \vee x^{-1}]\big) \leq -x$ is sufficient in place of (8.2).)

It is interesting to know whether we are in a discriminator variety e.g. because in any discriminator variety V, quasi-equations and equations are equivalent in the sense that any quasi-equation q can be translated to an equation e such that for every subdirectly irreducible $\mathfrak{A} \in \mathsf{V}$, ($\mathfrak{A} \models q$ iff $\mathfrak{A} \models e$).

[54] See e.g. the results quoted from Maddux [Ma87a, 87] etc. close to the end of §3. Further, Givant proved that if an RA $\mathfrak{A}$ is generated by a chain $H \subseteq A$ of equivalence elements then $\mathfrak{A}$ is representable. This generalizes a result of Jónsson. Givant generalized parts of this to cylindric algebras.

So as soon as we know that we are in a discriminator variety, we may use quasi-equations in defining subvarieties.

Maddux [Ma78, Ma82, Ma87] introduced the important varieties $\mathsf{NA} \supset \mathsf{WA} \supset \mathsf{SA} \supset \mathsf{RA}$ containing RA. These are obtained by weakening associativity of "$\circ$" in $(3.1)^+$. These are important from the point of view of algebraizing syntactic aspects of logic (e.g. proof theory) as well as from the point of view of the so called relativized algebras of relations, cf. the discussions around Theorems 6, 7. Sain proved that in WA and NA, for any class K with

$$\text{"simple algebras"} \subseteq \mathsf{K} \subseteq \text{"subdirectly irreducible algebras"},$$

K is not first-order axiomatizable (actually, $\mathsf{K} \neq \mathbf{Up}\mathsf{K}$), solving a problem on p. 112 of Maddux [Ma78], §8, or [Ma89c] Problem 5.

(Sain proved the same for the cylindric algebraic counterpart of these classes [cf. e.g. the discussions around Theorems 6, 7] e.g. for $\mathbf{I}\mathsf{Crs}_\alpha$, too.) The proof uses methods in [S84, S89]. These results show that WA and NA are far from being discriminator, since in discriminator varieties every such K is axiomatizable by a single universal formula. (SA is a discriminator variety.)

* * * * *

The class RQPA_α, abbreviated as RQA_α in the present paper, of representable quasi-polyadic algebras of α-ary relations was discussed in §5. The finitizable variety QPA_α (of quasi-polyadic algebras) approximating RQPA_α is defined analogously to CA_α or RA by a finite schema Σ_0 of equational axioms. This approximation Σ_0 of QPA_α was discussed and outlined above and below Theorem 12 in §5.

So far we discussed the finitizable approximations CA_α, RA, and QPA_α of our classes RCA_α, RRA, and RQA_α of algebras of relations. The finitizable approximations, PA_n, PEA_n etc. of our other kinds RPA_n, RPEA_n of algebras can be defined analogously to our above definitions (of CA_α and RA), and more or less analogous observations can be made. There is a general notational convention: the finitary approximating class has the shorter name (like CA_α or RA), and the name of the approximated class is obtained by writing an "R" in front of the short name (like RCA_α or RRA). Here this extra R stands for representable.

The finitizable approximating varieties CA_α, RA, QPA_α, PA_n, QPA_n etc. are discussed in the two volume monograph [HMT], [HMTII], and in the references therein. CA_α is studied in especially great detail: almost the whole of the first volume [HMT] is devoted to CA_α.

Remark (*discriminator varieties*). CA_n, RA, PA_α, PEA_α, QPA_n, QPEA_n are all discriminator varieties ($n \in \omega$ and α an arbitrary ordinal). Actually, CA_n is contained in a discriminator variety V_n much larger than CA_n. Namely, let $c_{(n)}x \stackrel{\text{def}}{=} c_0 \dots c_{n-1}(x)$. Then V_n is defined by the following 4 equations:

$x \le c_{(n)}x, \qquad c_{(n)}0 = 0,$

$c_i(y \wedge c_{(n)}x) \wedge c_{(n)}x = c_i(y) \wedge c_{(n)}x,$

$c_i(y - c_{(n)}x) - c_{(n)}x = c_i(y) - c_{(n)}x,$ for all $i < n$; where $(x - y)$ abbreviates $(x \wedge -y)$.

Clearly $\mathsf{CA}_n \subsetneqq \mathsf{V}_n$. The discriminator term for V_n is the one given in the proof of Theorem 3. The idea of the proof is that the above equations ensure that "relativizing" both with $c_{(n)}x$ and $-c_{(n)}x$ are homomorphisms, hence in any subdirectly irreducible member of V_n we have $c_{(n)}x \in \{0, 1\}$. Note that the above equations do not tell us anything about the behaviors of the Id_{ij}'s. An immediate corollary of this is that the (fairly elaborate) axioms governing the Id_{ij}'s (recall Definition 8.1 (2–4)) do not contribute to CA_n's being a discriminator variety. (We note that we could achieve even bigger discriminator varieties by picking some cylindric algebraic term $\sigma(x)$ and writing $\sigma(x)$ in place of $c_{(n)}x$ everywhere in the above equations. Of course, one has to choose $\sigma(x)$ in such a way that these equations remain valid in CA_n.)

List of symbols

Classes of algebras (the index n or α like in Cs_n or Cs_α referring to ranks of relations is omitted):

Symbol	*Name*
BA	Boolean Algebras
BAO	Boolean Algebras with Operators
BRA^0	finitary approximation of BRA
BRA	Algebras of Binary Relations (without identity)
BSR	Boolean semigroups of relations
CA	Cylindric Algebras
Crs	Cylindric relativized set algebras
Cs	= (Cylindric set algebras (Cs_n=algebras of n-ary relations))
FSA	Finite Sequence Algebras
full Cs, full Crs, full Ps, etc.	= (full (or powerset) elements of Cs etc. i.e. algebras with universe $\mathcal{P}(V)$ for some V)
GFSA	Generalized Finite Sequence Algebras

Gs	Generalized cylindric set algebra
HCA	Hyper Cylindric algebras, $\mathsf{CA}_\alpha \subset \mathsf{HCA}_\alpha \subset \mathsf{RCA}_\alpha$
L	$= \left(\begin{array}{c} (\mathsf{Cs}_\omega^{reg} \cap \mathsf{Lf}_\omega) = \lim_{n\to\omega} \mathsf{Cs}_n \text{ i.e.} \\ \text{class of algebras of finitary relations} \end{array} \right)$
Lf	Locally finite dimensional cylindric algebras
Lp	same as L but for the quasi-polyadic case
$Mod(\Sigma)$	$= \left(\begin{array}{c} \text{class of all models} \\ \text{of the set } \Sigma \text{ of formulas} \end{array} \right)$
MsCA	Many sorted CA's
NA	Nonassociative relation Algebras
PA	Polyadic Algebras
PaCA	Partial CA's
PEA	Polyadic Equality Algebras
PK	class of Products of (members of) K
Ps	Polyadic set algebras
QPA	Quasi-Polyadic Algebras
QPEA	Quasi-Polyadic Equality Algebras
Qps	Quasi polyadic set algebras
QRA	Relation algebras with quasi-projective elements
RA	Relation Algebras
RCA	Representable Cylindric Algebras
RdK	class of Reducts of (members of) K
RPA	Representable Polyadic Algebras
RPEA	Representable Polyadic Equality Algebras
RQA	Representable Quasi-polyadic Algebras
RQEA	Representable Quasi-polyadic Equality Algebras
RQPA	=RQA
RRA	Representable Relation Algebras
RSC	Representable Substitution Cylindric algebras
RV	Representable members of the Variety V
SA	Semi-associative relation Algebras
SC	Substitution Cylindric algebras
SK	class of Subalgebras of (members of) K
$\mathsf{TB}_{\mathcal{S}}$	$\mathcal{S}$-Transformational Algebras
IK	class of isomorphic copies of algebras in K
WA	Weakly associative relation Algebras

Algebras and their universes:

Symbol	*Name*
$\mathfrak{Fn}$	algebra of Finitary functions
Frl	set of all Finitary relations
Gfs	set of Generalized finite sequences
$\mathbf{Nr}_n(\mathfrak{B})$	Neat reduct of $\mathfrak{B}$

$\mathfrak{Psc}$	substitution cylindric Powerset algebra (a reduct of $\mathfrak{Rel}$)
$\mathcal{P}(V)$	Powerset of V (= universe of $\mathfrak{P}(V)$)
$\mathfrak{P}(V)$	$= \left(\substack{=\langle \mathcal{P}(V), \cap, \cup, \emptyset, V, - \rangle \\ \text{Boolean algebra of subsets of } V} \right)$
Ref	set of essentially finitary Relations
$\mathfrak{Ref}$	algebra of essentially finitary Relations
$\mathfrak{Rel}_\alpha$	algebra of α-ary Relations
Rf	set of finitary Relations (universe of $\mathfrak{Rf}$)
$\mathfrak{Rf}$	algebra of finitary Relations
$\mathfrak{Rfp}$	quasi-polyadic version of $\mathfrak{Rf}$
$\mathfrak{Ts}$	Two sorted cylindric-like algebra

Other:

Symbol	*Name*
c_i	$\left\{ \substack{i\text{-th cylindrification} \\ \text{(forgetting the } i\text{-th argument)}} \right\}$
C_Γ	generalization of c_i
Id	identity relation (both binary and n-ary)
Id_{ij}	identity of i-th and j-th arguments
p_{ij}	permutation of i and j
s^i_j	substitution of j for i
S_τ	substitution along τ
ω	set of all natural numbers
$R \circ S$	$= \{\langle a, b\rangle : \exists c(aRc \text{ and } cSb)\}$
${}^nU$	$= U \times U \times \cdots \times U$ (n times), or equivalently
${}^HU$	set of all functions from H into U
${}^{<\omega}U$	set of all finite sequences over U

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Received October 23, 1989
Revised October 9, 1990