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A Finite Model Theorem for the Propositional μ-Calculus *

Abstract. We prove a finite model theorem and infinitary completeness result for the propositional μ-calculus. The construction establishes a link between finite model theorems for propositional program logics and the theory of well-quasi-orders.

1. Introduction

L_μ is a propositional μ- or least fixpoint-calculus related to systems of Scott and DeBakker [11] and Pratt [8]. L_μ was introduced in [2], where an exponential-time decision procedure and complete finitary deductive system were given for a restricted class of formulas. In [3], a nonelementary decision procedure was given for full L_μ. In this paper we prove that every satisfiable formula of L_μ is satisfied in a finite model. We also give a complete infinitary deductive system.

Finite model theorems are useful in obtaining efficient decision procedures. In general, the smaller the model (as a function of the size of the formula), the more efficient the decision procedure. The standard technique for obtaining finite models in propositional program logics is *filtration*, a technique borrowed from modal logic. It was first used in propositional program logics to obtain a finite model theorem for Propositional Dynamic Logic [1], thereby giving a nondeterministic exponential-time decision procedure. Filtration does not work for L_μ [2, 9], thus a new technique is needed.

We prove the result by showing that the size of a minimal model for a given satisfiable L_μ formula is related to the size of a maximal set of pairwise incomparable elements in a particular ordered structure involving sets of ordinals. This establishes a connection between finite model theorems for propositional program logics and the theory of well-quasi-orders.

Basic definitions are given in §2. In §3 we define a partial order $\preccurlyeq$ on formulas and extend it to a quasi-order on collections of formulas. In §4 we consider models whose states are labeled with sets of formulas, and give local conditions (involving $\preccurlyeq$) on labelings which insure that a state satisfies all formulas in its label. The results of this section may be of more general use in performing surgery on models. In §5 we show that a certain quasi-order $\preccurlyeq$ on sets of ordinals is a well-quasi-order [4], therefore has a finite base. In §6 we combine the results of §4 and §5 to obtain a finite model theorem. In §7 we show how the finite model theorem gives a complete infinitary deductive system. §8 contains conclusions and directions for further work.

* Supported by NSF grant DCR-8602663

2. Definition of L_μ and L_μ^+

The systems L_μ and L_μ^+ were defined in [2]. We review the definitions briefly, referring the reader to [2] for a more detailed presentation.

2.1. Syntax

The basic nonlogical symbols of L_μ and L_μ^+ consist of
1. propositional constants $P, Q, \ldots$
2. propositional variables $X, Y, \ldots$
3. program constants $a, b, \ldots$

Formulas $p, q, \ldots$ are defined inductively:
1. X
2. P
3. $p \vee q$
4. $\neg p$
5. $\langle a \rangle p$
6. $\alpha X . pX$, α an ordinal
7. $\mu X . pX$.

In (6) and (7), pX is a formula with a distinguished variable X, all of whose free occurrences are *positive* (occur in the scope of an even number of negations $\neg$). Intuitively, $\alpha X . pX$ represents the α-fold composition of the operator $\lambda X . pX$ applied to *false*. L_μ^+ is the language defined by (1−7). L_μ is the countable sublanguage obtained by deleting (6).

The operators $\wedge$, $\rightarrow$, *false*, *true*, and $[a]$ are defined as usual. In addition, we define

$$\nu X . pX = \neg \mu X . \neg p \neg X .$$

The operator ν is the *greatest fixpoint operator*.

The usual quantifier scoping rules, as well as the definitions of bound and free variables, apply to μX, νX, and αX. A formula with no free variables is called *closed*.

An L_μ^+ formula is said to be in *positive form* if it is built from the operators $\vee$, $\wedge$, μ, ν, $\langle\rangle$, $[]$, and $\neg$, with $\neg$ applied to atomic subformulas only. Every closed L_μ formula is equivalent to a formula in positive form.

2.2. Semantics

A *model* is a structure $\mathscr{M} = (S, I)$, where S is a set of *states* and I is an *interpretation function* interpreting the propositional and program constants, such that $I(P) \subseteq S$ and $I(a) \subseteq S \times S$. A formula $p(\bar{X})$ with free variables among $\bar{X} = X_1, \ldots, X_n$ is interpreted in $\mathscr{M}$ as an operator $p^{\mathscr{M}}$ which maps any valuation $\bar{A} = A_1, \ldots, A_n$ of $\bar{X}$ over subsets of S to a subset $p^{\mathscr{M}}(\bar{A})$ of S. The operator $p^{\mathscr{M}}$ is defined by induction as follows:

(2.1) $\quad X_i^{\mathscr{M}}(\bar{A}) = A_i$

(2.2) $\quad P^{\mathcal{M}}(\bar{A}) = I(P)$

(2.3) $\quad (p \vee q)^{\mathcal{M}}(\bar{A}) = p^{\mathcal{M}}(\bar{A}) \cup p^{\mathcal{M}}(\bar{A})$

(2.4) $\quad (\neg p)^{\mathcal{M}}(\bar{A}) = S - p^{\mathcal{M}}(\bar{A})$

(2.5) $\quad (\langle a \rangle p)^{\mathcal{M}}(\bar{A}) = \langle a^{\mathcal{M}} \rangle (p^{\mathcal{M}}(\bar{A}))$

where in (2.5),

$$\langle a^{\mathcal{M}} \rangle (B) = \{s \mid \exists t \in B(s,\, t) \in I(a)\}.$$

In order to give the semantics of $\alpha X.pX$ and $\mu X.pX$, let pX be a formula with distinguished free variable X occurring only positively. Let $\bar{X}$ denote the other free variables in p. Thus $pX = p(X, \bar{X})$. We assume by induction hypothesis that the operator $p^{\mathcal{M}}$ has already been defined.

(2.6) $\quad 0X.pX^{\mathcal{M}}(\bar{A}) = \mathit{false}^{\mathcal{M}} = \emptyset$

(2.7) $\quad (\alpha+1)X.pX^{\mathcal{M}}(\bar{A}) = p^{\mathcal{M}}(\alpha X.pX^{\mathcal{M}}(\bar{A}), \bar{A})$

(2.8) $\quad \delta X.pX^{\mathcal{M}}(\bar{A}) = \bigcup_{\beta < \delta} \beta X.pX^{\mathcal{M}}(\bar{A})$, δ a limit ordinal

(2.9) $\quad \mu X.pX^{\mathcal{M}}(\bar{A}) = \bigcup_{\beta} \beta X.pX^{\mathcal{M}}(\bar{A})$

where in (2.9), the union is over all ordinals β. Taking $\mu > \alpha$ for any ordinal α, (2.6–2.9) reduce to the single definition

(2.10) $\quad \alpha X.pX^{\mathcal{M}}(\bar{A}) = \bigcup_{\beta < \alpha} p^{\mathcal{M}}(\beta X.pX^{\mathcal{M}}(\bar{A}), \bar{A})$

where α is either an ordinal or μ.

Because p is positive in the variable X, $\lambda X.p^{\mathcal{M}}(X, \bar{A})$ is a monotone set operator, and

(2.11) $\quad \alpha < \beta \rightarrow \alpha X.pX^{\mathcal{M}}(\bar{A}) \subseteq \beta X.pX^{\mathcal{M}}(\bar{A})$.

There exists a least ordinal $\varkappa$ such that $\varkappa X.pX^{\mathcal{M}}(\bar{A}) = (\varkappa+1)X.pX^{\mathcal{M}}(\bar{A})$, and it follows that $\mu X.pX^{\mathcal{M}}(\bar{A}) = \varkappa X.pX^{\mathcal{M}}(\bar{A})$. The ordinal $\varkappa$ is called the *closure ordinal* of the operator $\lambda X.p^{\mathcal{M}}(X, \bar{A})$, and $\mu X.pX^{\mathcal{M}}(\bar{A})$ is the *least fixpoint* of $\lambda X.p^{\mathcal{M}}(X, \bar{A})$.

If p is closed, then $p^{\mathcal{M}}$ is a constant function, i.e., $p^{\mathcal{M}}(\bar{A})$ is a fixed set of states independent of $\bar{A}$. In this case, we say s *satisfies* p if $s \in p^{\mathcal{M}}(\bar{A})$ and write $\mathcal{M}, s \models p$ or $s \models p$ when $\mathcal{M}$ is understood. We write $\models p$ if $\mathcal{M}, s \models p$ for all $\mathcal{M}$ and s.

2.3. Closure

Let p be an L_μ formula in positive form. The *closure* $CL(p)$ of p was defined in [2]. It corresponds to the Fischer-Ladner closure of *PDL* [1]. It is the smallest set of formulas such that:

1. $p \in CL(p)$
2. if $\neg P \in CL(p)$ then $P \in CL(p)$
3. if $q \vee r \in CL(p)$ then both $q, r \in CL(p)$
4. if $q \wedge r \in CL(p)$ then both $q, r \in CL(p)$
5. if $\langle a \rangle q \in CL(p)$ then $q \in CL(p)$
6. if $[a]q \in CL(p)$ then $q \in CL(p)$
7. if $\sigma X.qX \in CL(p)$ then $q(\sigma X.qX) \in CL(p)$,

where σ is either μ or ν. $CL(p)$ is finite, and is in fact no larger than the number of symbols of p [2].

3. A Partial Order on L_μ^+ Formulas

Let $\bar{\alpha} = \alpha_1, \ldots, \alpha_n$ and $\bar{\beta} = \beta_1, \ldots, \beta_n$ be n-tuples of ordinals or μ, and define $\bar{\alpha} \leqslant \bar{\beta}$ if $\alpha_i \leqslant \beta_i$, $1 \leqslant i \leqslant n$. Let p be a formula of L_μ^+ in positive form. Let $\alpha_1 X_1.q_1 X_1, \ldots, \alpha_n X_n.q_n X_n$ be a list of all occurrences of subformulas of p of the form $\alpha X.qX$, where each α_i is either μ or an ordinal, listed in the order in which they occur in p. We denote this by writing

$$p = p(\alpha_1, \ldots, \alpha_n) = p(\bar{\alpha}).$$

Replacing α_i in p with β_i, $1 \leqslant i \leqslant n$, results in a well-formed L_μ^+ formula $p(\bar{\beta})$. We define $p(\bar{\alpha}) \preccurlyeq p(\bar{\beta})$ if $\bar{\alpha} \leqslant \bar{\beta}$. That is, $p \preccurlyeq q$ if p and q are identical except for the ordinals appearing in subformulas of the form $\alpha X.rX$, and the ordinals of p are not greater than the corresponding ordinals of q. We define $p^\mu = p(\mu, \ldots, \mu)$, the L_μ formula obtained by replacing all ordinals by μ, and observe that $p \preccurlyeq p^\mu$ for all L_μ^+ formulas p. By (2.11) and the fact that p is positive,

(3.12) if $p \preccurlyeq q$ then $\models p \to q$.

If Σ, Γ are sets of L_μ^+ formulas, define

(3.13) $\Sigma \preccurlyeq \Gamma$ if $\forall q \in \Gamma\ \exists p \in \Sigma\ p \preccurlyeq q$.

From (3.12) we have that

(3.14) if $\Sigma \preccurlyeq \Gamma$ then $\models \bigwedge \Sigma \to \bigwedge \Gamma$.

Finally, if Σ is a set of L_μ^+ formulas, define

$$\Sigma^\mu = \{p^\mu \mid p \in \Sigma\}.$$

4. Annotated Models

Let $\mathscr{M} = (S, I)$ be any model, and let Θ be a function labeling each state $s \in S$ with a class Θ_s of closed positive L_μ^+ formulas. The labeling Θ is called an *annotation* of $\mathscr{M}$, and the triple (S, I, Θ) is called an *annotated model.*

The following definition of *well-annotation* gives local syntactic conditions that insure that states of an annotated model satisfy their labels (Lemma 4.2).

This is useful in performing surgery on models, because in practice it is easily checked that these local conditions are preserved by certain cutting and pasting operations.

DEFINITION 4.1. An annotation Θ is called a *well-annotation* if the following conditions hold:
1. if $P \in \Theta_s$, then $s \models P$
2. if $\neg P \in \Theta_s$, then $s \models \neg P$
3. if $p \vee q \in \Theta_s$, then either $p \in \Theta_s$ or $q \in \Theta_s$
4. if $p \wedge q \in \Theta_s$, then both $p, q \in \Theta_s$
5. if $\alpha X.pX \in \Theta_s$, then $\exists \beta < \alpha\ p(\beta X.pX) \in \Theta_s$, α an ordinal or μ
6. if $\nu X.pX \in \Theta_s$, then $p(\nu X.pX) \in \Theta_s$
7. if $\langle a \rangle p \in \Theta_s$, then $\exists t (s, t) \in I(a)$ and $\exists p' \preccurlyeq p\ p' \in \Theta_t$
8. if $[a] p \in \Theta_s$, then $\forall t (s, t) \in I(a) \rightarrow \exists p' \preccurlyeq p\ p' \in \Theta_t$. □

LEMMA 4.2.
(i) *If Θ is a well-annotation, then $s \models \Theta_s$ for all $s \in S$.*
(ii) *If Θ is an annotation such that $\forall s\ s \models \Theta_s$, then Θ can be extended to a well-annotation $\hat{\Theta}$ satisfying the property*

$$(4.1) \qquad \bigcup_{s \in S} \hat{\Theta}_s^{\mu} \subseteq \bigcup_{t \in S, q \in \Theta_t} CL(q^{\mu}).$$

PROOF. (i) Suppose Θ is a well-annotation. For any closed positive L_μ^+ formula q, let $q^\Theta = \{s \mid \exists q' \preccurlyeq q\ q' \in \Theta_s\}$. If $\bar{q} = q_1, \ldots, q_n$ is a list of closed formulas, let $\bar{q}^\Theta$ denote the list $q_1^\Theta, \ldots, q_n^\Theta$. Let $p(\bar{X})$ be an L_μ^+ formula in positive form with free variables among $\bar{X} = X_1, \ldots, X_n$. We prove by induction on the structure of p and by transfinite induction on ordinals that

$$p(\bar{q})^\Theta \subseteq p^{\mathcal{M}}(\bar{q}^\Theta).$$

In particular, $p^\Theta \subseteq p^{\mathcal{M}}$ for any closed p. This will establish (i).
1. $P^\Theta \subseteq P^{\mathcal{M}}$ by Definition 4.1(1).
2. $\neg P^\Theta \subseteq \neg P^{\mathcal{M}}$ by Definition 4.1(2).
3. $X_i(\bar{q})^\Theta = q_i^\Theta = X_i^{\mathcal{M}}(\bar{q}^\Theta)$.
4. $(p \vee q)(\bar{q})^\Theta \subseteq p(\bar{q})^\Theta \cup q(\bar{q})^\Theta$ by Definition 4.1(3)
$\subseteq p^{\mathcal{M}}(\bar{q}^\Theta) \cup q^{\mathcal{M}}(\bar{q}^\Theta)$ by induction hypothesis
$= (p \vee q)^{\mathcal{M}}(\bar{q}^\Theta)$.
5. $(p \wedge q)(\bar{q})^\Theta \subseteq p(\bar{q})^\Theta \cap q(\bar{q})^\Theta$ by Definition 4.1(4)
$\subseteq p^{\mathcal{M}}(\bar{q}^\Theta) \cap q^{\mathcal{M}}(\bar{q}^\Theta)$ by induction hypothesis
$= (p \wedge q)^{\mathcal{M}}(\bar{q}^\Theta)$.
6. $\langle a \rangle p(\bar{q})^\Theta \subseteq \langle a^{\mathcal{M}} \rangle p(\bar{q})^\Theta$ by Definition 4.1(7)
$\subseteq \langle a^{\mathcal{M}} \rangle p^{\mathcal{M}}(\bar{q}^\Theta)$ by induction hypothesis and monotonicity of $\langle a^{\mathcal{M}} \rangle$
$\subseteq (\langle a \rangle p)^{\mathcal{M}}(\bar{q}^\Theta)$.
7. $[a] p(\bar{q})^\Theta \subseteq [a^{\mathcal{M}}](p(\bar{q})^\Theta)$ by Definition 4.1(8)
$\subseteq [a^{\mathcal{M}}] p^{\mathcal{M}}(\bar{q}^\Theta)$ by induction hypothesis and monotonicity of $[a^{\mathcal{M}}]$
$\subseteq ([a] p)^{\mathcal{M}}(\bar{q}^\Theta)$.

8. $\alpha X.pX(\bar{q})^\Theta \subseteq \bigcup_{\beta<\alpha} p(\beta X.pX(\bar{q}), \bar{q})^\Theta$ by Definition 4.1(5)
$\subseteq \bigcup_{\beta<\alpha} p^{\mathcal{M}}(\beta X.pX(\bar{q})^\Theta, \bar{q}^\Theta)$ by induction hypothesis on p
$\subseteq \bigcup_{\beta<\alpha} p^{\mathcal{M}}(\beta X.pX^{\mathcal{M}}(\bar{q}^\Theta), \bar{q}^\Theta)$ by induction hypothesis on β and monotonicity of $p^{\mathcal{M}}$
$= \alpha X.pX^{\mathcal{M}}(\bar{q}^\Theta)$.
9. $\nu X.pX(\bar{q})^\Theta \subseteq p(\nu X.pX(\bar{q}), \bar{q})^\Theta$ by Definition 4.1(6)
$\subseteq p^{\mathcal{M}}(\nu X.pX(\bar{q})^\Theta, \bar{q}^\Theta)$ by induction hypothesis on p; but $\nu X.pX^{\mathcal{M}}(\bar{q}^\Theta)$ is the greatest subset A of S such that $A \subseteq p^{\mathcal{M}}(A, \bar{q}^\Theta)$, therefore

$$\nu X.pX(\bar{q})^\Theta \subseteq \nu X.pX^{\mathcal{M}}(\bar{q}^\Theta).$$

(ii) Let Θ be any annotation such that $s \models \Theta_s$. We will add new formulas to Θ to satisfy the conditions of Definition 4.1, always preserving $s \models \Theta_s$, and making sure that for any new formula p, $p^\mu \in CL(q^\mu)$ for some q already appearing in some Θ_t.

1. If $p \vee q \in \Theta_s$, then $s \models p \vee q$, so either $s \models p$ or $s \models q$. If the former, set $\Theta_s := \Theta_s \cup \{p\}$, otherwise set $\Theta_s := \Theta_s \cup \{q\}$.
2. If $p \wedge q \in \Theta_s$, set $\Theta_s := \Theta_s \cup \{p, q\}$.
3. If $[a]p \in \Theta_s$, then for all t such that $(s, t) \in I(a)$, $t \models p$. Set $\Theta_t := \Theta_t \cup \{p\}$ for all such t.
4. If $\nu X.pX \in \Theta_s$, set $\Theta_s := \Theta_s \cup \{p(\nu X.pX)\}$.
5. If $\alpha X.pX \in \Theta_s$, then $s \models \alpha X.pX$, so by (2.10) there must exist a $\beta < \alpha$ such that $s \models p(\beta X.pX)$. Pick one such β and set $\Theta_s := \Theta_s \cup \{p(\beta X.pX)\}$.
6. If $\langle a \rangle p \in \Theta_s$, then there must be a state t such that $(s, t) \in I(a)$ and $t \models p$. Pick one such t and set $\Theta_t := \Theta_t \cup \{p\}$.

Let $\hat{\Theta}$ be the final value of Θ obtained by this procedure. Then $\hat{\Theta}$ satisfies the conditions of Definition 4.1 and the property (4.1). □

5. Well-Quasi-Orders

DEFINITION 5.1. A *quasi-order* is an ordered set $(Q, \leqslant)$ such that $\leqslant$ is reflexive and transitive. $(Q, \leqslant)$ is a *well-quasi-order* if any of the following five equivalent conditions hold:

1. Every set has a finite base: $\forall A \subseteq Q \exists A_0 \subseteq A$, A_0 finite, such that $\forall y \in A \exists x \in A_0 x \leqslant y$ (i.e., such that $A_0 \leqslant A$ in the sense of (3.13)).
2. $\leqslant$ is well-founded, and there is no infinite set of pairwise $\leqslant$-incomparable elements.
3. Every countable sequence $x_0, x_1, \ldots$ has $x_i \leqslant x_j$ for some $i < j$.
4. Every countable sequence $x_0, x_1, \ldots$ has a countable monotone subsequence $x_{i_0} \leqslant x_{i_1} \leqslant \ldots$
5. Any linear order on the quotient $Q/\equiv$ extending $\leqslant$ is a well-order, where $x \equiv y$ iff $x \leqslant y$ and $y \leqslant x$. □

Any well-order is a well-quasi-order, any subset of a well-quasi-order is a well-quasi-order, and the direct product of any finite collection of well-quasi-orders is a well-quasi-order. The proof of the equivalence of the above five conditions uses Ramsey's theorem. See [4] for further details and references.

The power set of a well-quasi-order, ordered by (3.13), is not necessarily a well-quasi-order. However, Nash-Williams defined the concept of *better-quasi-order*, and showed that any better-quasi-order is a well-quasi-order, and the power set of a better-quasi-order is a better-quasi-order [6, 7]. Since the definition of better-quasi-order is rather involved, we refer the reader to [5, 6, 7] for the definitions and basic results, from which the following lemma is not hard to derive:

LEMMA 5.2. *$(P(O^n), \leqslant)$ is a well-quasi-order, where O^n is the class of n-tuples of ordinals, $P(O^n)$ is the class of all sets of such n-tuples, and $S \leqslant T$ iff $\forall \bar{\beta} \in T \exists \bar{\alpha} \in S\ \bar{\alpha} \leqslant \bar{\beta}$.* □

6. The Finite Model Property

THEOREM 6.1. *Every satisfable L_μ formula p_0 is satisfied in a finite model.*

PROOF. Suppose p_0 is satisfiable. Let $\mathscr{M} = (S^{\mathscr{M}}, I^{\mathscr{M}})$ be a model and $s_0 \in S^{\mathscr{M}}$ such that $s_0 \models p_0$. Label $\Theta_{s_0} = \{p_0\}$, $\Theta_s = \emptyset$ for $s \neq s_0$. By Lemma 4.2(ii), Θ extends to a well-annotation $\hat{\Theta}$ satisfying the property (4.1).

We wish to show that $(\{\hat{\Theta}_s | s \in S^{\mathscr{M}}\}, \preccurlyeq)$ is a well-quasi-order. Let $\{p_0, \ldots, p_k\} = CL(p_0)$. By (4.1), every $p \in \hat{\Theta}_s$ satisfies $p^\mu = p_i$ for some $1 \leqslant i \leqslant k$. In other words, every $p \in \hat{\Theta}_s$ is contained in some $\preccurlyeq$-ideal $(p_i) = \{q | q \preccurlyeq p_i\}$. Moreover, $p \preccurlyeq q$ only if $p^\mu = q^\mu$. Therefore

$$\begin{aligned}\hat{\Theta}_s \preccurlyeq \hat{\Theta}_t &\leftrightarrow \forall q \in \hat{\Theta}_t \exists p \in \hat{\Theta}_s p \preccurlyeq q \\ &\leftrightarrow \forall q \in \hat{\Theta}_t \cap (p_i) \exists p \in \hat{\Theta}_s \cap (p_i)\ p \preccurlyeq q,\ 1 \leqslant i \leqslant k \\ &\leftrightarrow \hat{\Theta}_s \cap (p_i) \preccurlyeq \hat{\Theta}_t \cap (p_i),\ 1 \leqslant i \leqslant k.\end{aligned}$$

Since any finite product of well-quasi-orders is again a well-quasi-order, it suffices to show that each of the k quasi-orders

$$\{\hat{\Theta}_s \cap (p_i) | s \in S^{\mathscr{M}}\},\ 1 \leqslant i \leqslant k$$

is a well-quasi-order. If $p_i = p_i(\mu, \ldots, \mu)$, this amounts to showing that each

$$\{\{\bar{\alpha} | p_i(\bar{\alpha}) \in \hat{\Theta}_s\} | s \in S^{\mathscr{M}}\},\ 1 \leqslant i \leqslant k$$

is a well-quasi-order. But this is immediate from Lemma 5.2.

Now by Definition 5.1(1), the set $\{\hat{\Theta}_s | s \in S^{\mathscr{M}}\}$ has a finite base under $\preccurlyeq$. Therefore there exists a finite set $S^{\mathscr{F}} \subseteq S^{\mathscr{M}}$ such that $\forall s \in S^{\mathscr{M}} \exists t \in S^{\mathscr{F}}\ \hat{\Theta}_t \preccurlyeq \hat{\Theta}_s$. Let $f\colon S^{\mathscr{M}} \to S^{\mathscr{F}}$ such that $\hat{\Theta}_{f(s)} \preccurlyeq \hat{\Theta}_s$.

Define a new annotated model $\mathscr{N}$ as follows. Take $S^{\mathscr{N}} = S^{\mathscr{M}}$ and $I^{\mathscr{N}}(P) = I^{\mathscr{M}}(P)$, but

$$I^{\mathscr{N}}(a) = \{(s, f(t)) | (s, t) \in I^{\mathscr{M}}(a)\}.$$

In other words, $\mathcal{N}$ is exactly the same as $\mathcal{M}$, except that we cut every edge (s, t) in $I^{\mathcal{M}}(a)$ and replace it with the edge $(s, f(t))$. The annotation $\hat{\Theta}$ on $\mathcal{N}$ is still a well-annotation: Definition 4.1(1−6) is still satisfied since no labels were changed, and Definition 4.1(7−8) is still satisfied since $\hat{\Theta}_{f(t)} \preccurlyeq \hat{\Theta}_t$.

Now define a finite annotated model $\mathcal{F} = (S^{\mathcal{F}}, I^{\mathcal{F}}, \hat{\Theta} \lceil S^{\mathcal{F}})$ by restricting $\mathcal{N}$ to $S^{\mathcal{F}}$, i.e., $I^{\mathcal{F}}(P) = I^{\mathcal{N}}(P) \cap S^{\mathcal{F}}$ and $I^{\mathcal{F}}(s) = I^{\mathcal{N}}(a) \cap S^{\mathcal{F}} \times S^{\mathcal{F}}$. The annotation $\hat{\Theta} \lceil S^{\mathcal{F}}$ is still a well-annotation: Definition 4.1(1−6) is still satisfied since no labels were changed, and Definition 4.1(7−8) is still satisfied, since any $(s, t) \in I^{\mathcal{F}}(a)$ is also in $I^{\mathcal{N}}(a)$.

Thus $\mathcal{F}$ is a finite well-annotated model. Moreover, $f(s_0) \models \hat{\Theta}_{f(s_0)}$ by Lemma 4.2(i), therefore $f(s_0) \models \hat{\Theta}_{s_0}$ by (3.14), and $p_0 \in \hat{\Theta}_{s_0}$, therefore $f(s_0) \models p_0$. □

7. An Infinitary Deductive System

The deductive system is the same as the one in [2], with the addition of the infinitary rule of inference

$$\text{(7.1)} \qquad \frac{nX.pX \to q, \text{ all } n < \omega}{\mu X.pX \to q}.$$

This deductive system can be shown complete by a straightforward adaptation of the completeness proof of [2]. The difficult part is showing that the system is sound, because the above rule is not valid if interpreted as an implication; in other words, it is not true in general that

$$nX.pX^{\mathcal{M}} \subseteq q^{\mathcal{M}}, \text{ all } n < \omega \to \mu X.pX^{\mathcal{M}} \subseteq q^{\mathcal{M}},$$

as easy counterexamples show (see [2]). However, it is the case that if $nX.pX^{\mathcal{M}} \subseteq q^{\mathcal{M}}$ for all $n < \omega$ *in all models* $\mathcal{M}$, then $\mu X.pX^{\mathcal{M}} \subseteq q^{\mathcal{M}}$ in all models $\mathcal{M}$. For, suppose there were a model $\mathcal{M}$ and state s with $s \in \mu X.pX^{\mathcal{M}} \cap \neg q^{\mathcal{M}}$, or in other words $s \models \mu X.pX \wedge \neg q$. By Theorem 6.1, there would be a finite model $\mathcal{F}$ and state t of $\mathcal{F}$ with $t \models \mu X.pX \wedge \neg q$. But since all closure ordinals in a finite model are finite, $t \models nX.pX \wedge \neg q$ for some $n < \omega$, thus $nX.pX^{\mathcal{F}} \not\subseteq q^{\mathcal{F}}$, a contradiction. We have established

THEOREM 7.1. *The deductive system of* [2], *augmented with the infinitary rule* (7.1), *is sound and complete for* L_μ. □

8. Conclusions and Directions for Further Work

Streett and Emerson [10] have recently given an elementary-time decision procedure for L_μ involving automata on infinite trees. As a corollary to their construction, they obtain a finite model property. Moreover, their construction gives elementary bounds on the size of the model (roughly four exponentials), whereas ours does not, at least in the current state of (the author's) knowledge. Nevertheless, the construction of the present paper has the advantage that it is more direct, and establishes a link between finite model theorems for program logics and the theory of well-quasi-orders. For example, the construction of Theorem 6.1 shows that the size of a minimal model for a given satisfiable formula is related to the size of a maximal set of pairwise $\preccurlyeq$-incomparable elements in a particular structure on sets of ordinals. We are currently refining this relationship in the hope that it may shed light on the complexity of the decision problem for $\underline{L}_\mu$.

Acknowledgment

We are grateful to Rick Statman for introducing us to the theory of well-quasi-orders.

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Received September 2, 1986