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A mapping method for numerical evaluation of two-dimensional integrals with $1/r$ singularity

Anantharaman Nagarajan and **Subrata Mukherjee**
Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, 14853, USA

Abstract. Singular integrals occur commonly in applications of the boundary element method (BEM). A simple mapping method is presented here for the numerical evaluation of two-dimensional integrals in which the integrands, at worst, are $O(1/r)$ (r being the distance from a source to a field point). This mapping transforms such integrals over general curved triangles into regular 2-D integrals. Over flat and curved quadratic triangles, regular line integrals are obtained, and these can be easily evaluated by standard Gaussian quadrature. Numerical tests on some typical singular integrals, encountered in BEM applications, demonstrate the accuracy and efficacy of the method.

Introduction

It is generally accepted that the accuracy of numerical solutions, obtained from the boundary element method (BEM), is very good if the numerical implementation of the method is carried out with care (Mukherjee 1982). One of the key ingredients of a proper implementation of the method is the accurate evaluation of singular integrals. These integrals can be weakly, Cauchy or hyper singular and can be one, two or three dimensional. Integrals over flat two-dimensional elements can sometimes be evaluated analytically while integrals over curved elements usually must be evaluated numerically. For some BEM applications, such as axisymmetric problems or shape design sensitivity analysis, even integrals over flat elements cannot be evaluated analytically and numerical integration must be employed.

It has been the experience of many researchers in the BEM, including the authors of this paper, that "brute force" evaluation of these integrals by standard numerical quadrature is extremely risky. While such methods may work for some problems, they may lead to unacceptable numerical errors in others. Accuracy demands on the computed values of such integrals are quite stringent for nonlinear problems (e.g. Rajiyah and Mukherjee 1987). The situation becomes even more critical if one tries to obtain shape sensitivities of nonlinear problems in solid mechanics (Zhang et al. 1992a, b).

The focus of this paper is the numerical evaluation of integrals where the integrands are $O(1/r)$ singular (r being the distance between a source and a field point) in two-dimensional regions. Such integrals occur in many problems. The integrals are usually of the form

$$I = \int_B (\text{kernel}) \times (\text{physical variable})\, dS \tag{1}$$

where the kernel is singular and B is a two-dimensional region. Usually the region B is discretized into elements. If the physical variable is unknown, it is expressed in terms of shape functions over each element. If the physical variable is known, its numerical value can be used directly in Eq. (1), or it can be expressed in terms of shape functions. The elements can be flat or curved and have straight or curved sides.

There are many situations in BEM applications where the integrand in Eq. (1) is $O(1/r)$ as r, the distance between a source and a field point, approaches zero. One example is 3-D linear elasticity

(Cruse 1969) where the kernel is

$$U_{ij}=\frac{1}{16\pi(1-\nu)Gr}[(3-4\nu)\delta_{ij}+r_{,i}r_{,j}] \tag{2}$$

and the physical variable is the surface traction vector with components τ_i. In the above, G and ν are the shear modulus and Poisson's ratio, respectively, of the material, δ_{ij} is the Kronecker delta and

$$r_{,i}=\frac{x_i(q)-x_i(p)}{r} \tag{3}$$

in terms of the field and source point coordinates $x_i(q)$ and $x_i(p)$, respectively.

Other examples come up in the determination of domain integrals in 2-D elasto-plasticity. The domain integral for the rate of displacement has the kernel (Mukherjee 1982)

$$U_{ij,k}=-\frac{1}{8\pi(1-\nu)Gr}[(3-4\nu)r_{,k}\delta_{ij}-\delta_{ik}r_{,j}-\delta_{jk}r_{,i}+2r_{,i}r_{,j}r_{,k}] \tag{4}$$

with the plastic strain rate $\dot{\varepsilon}_{ik}^{(n)}$ as the physical quantity.

The domain integral for strain rate calculations in 2-D elasto-plasticity has a Cauchy singular kernel $U_{ij,kl}$ which is $O(1/r^2)$ (Zhang et al. 1992a), together with the physical quantity (after regularization) $[\dot{\varepsilon}_{ik}^{(n)}(q)-\dot{\varepsilon}_{ik}^{(n)}(p)]$. Since this physical quantity is $O(r)$ as $p\to q$, the integrand is $O(1/r)$ as $r\to 0$.

The general form of these singular integrals, obtained from the kernels such as those in Eqs. (2) or (4) (with the source point at the origin) is

$$f(x_1,x_2)/r^n$$

where $f(x_1,x_2)$ is a polynomial of order m, n is an integer $\geqq 1$, and $m-n=\beta=-1$. For example, Eq. (4) gives terms like

$$\frac{x_1}{r^2},\ \frac{x_1x_2^2}{r^4}\ \text{etc.}$$

There have been many papers published in the BEM literature that deal with the numerical evaluation of singular integrals. In the interest of brevity, attention is focussed below only on integrals with integrands of the type $O(1/r)$ (weakly singular) and $O(1/r^2)$ (Cauchy singular) on two-dimensional surfaces (planar or curved). Also, the papers cited below are intended to be a representative sample, rather than an exhaustive list.

One could classify such papers into various categories, depending on the method used for the evaluation of such integrals. A possible way to categorize them follows.

a) Gaussian integration methods with adjusted weights

Examples of this approach are Cristescu and Loubignac (1978); Pina et al. (1981) (see also, Brebbia et al. 1984) and Aliabadi and Hall (1987a).

Cristescu and Loubignac (1978) use a two-dimensional weighting function which is the inverse of the local planar distance from an integration point to a fixed point. One problem with this approach is that it is based on the assumption that the BEM integrands are of the form $h(x_1,x_2)/r$, where h is an analytic function of its arguments. In reality, however, h is typically of the form

$$h(x_1,x_2)=\frac{\text{Polynomial of order } n}{r^n}$$

e.g. $(x_1x_2^2)/r^3$, which is not defined (and therefore not analytic) at the origin. As a result, when the Gauss points and weights from Cristescu and Loubignac (1978) were employed in a very demanding BEM application, that of calculation of shape sensitivities for nonlinear solid mechanics problems (Zhang et al. 1992b) the results were not acceptably accurate. Aliabadi and Hall (1987a) also report serious problems when the method of Cristescu and Loubignac (1978) is used for integration of $O(1/r)$ functions over parallelograms other than perfect rectangles. They (i.e. Aliabadi and Hall 1987a) report success in integrating over parallelograms by using a weighting function $1/R_1$, where R_1 is an approximation of the real spatial distance between a source and a field point. It should also be mentioned here that the work of Pina et al. (1981) is also based on the use of a weighting function of the same type as Cristescu and Loubignac (1978). No numerical results, using their proposed Gauss points and weights, are given in the paper by Pina et al.

In summary, it is fair to say that this class of methods do not guarantee the exact cancellation of the $1/r$ singularity in all cases, and should be used with caution.

b) Subtraction and addition methods

Examples of this approach are Huang and Du (1988), Rajiyah and Mukherjee (1987) and Aliabadi, Hall and Phemister (1985). The idea here is to first subtract a term from the singular integral to make it regular and easily integrable by numerical methods. This term is then added back on and usually integrated analytically. Huang and Du (1988) subtract and add the physical variable (see Eq. (1)) while Rajiyah and Mukherjee use the leading term of the Taylor Series expansion of the kernel, which, in this case, was $O(1/r^2)$. Aliabadi et al. also apply the Taylor series idea, but to the entire integrand.

In general, such methods work very well but can be expensive, especially for nonlinear problems where such operations must be carried out repeatedly (Rajiyah and Mukherjee 1987).

c) Mapping methods

Examples of this approach are Sarihan and Mukherjee (1982) (see, also, Mukherjee (1982); Li et al. (1985); Lean and Wexler (1985); Aliabadi and Hall (1987b)). The idea here is to map the physical domain into a different domain in such a way that the integrand becomes regular. Polar coordinate mapping has been used by Sarihan and Mukherjee and by Li et al. While the polar approach is mathematically exact, it can, however, require several mappings, leading to a significant computational burden (especially for nonlinear problems) and accumulated round off error. Integration, using this approach, has been found to be of unacceptable accuracy (in some cases) in ongoing work of Mukherjee's group in the area of shape design sensitivity of nonlinear problems in solid mechanics.

The work of Lean and Wexler (1985) and Aliabadi and Hall (1987a) is based on the astute observation that the mapping of a curved quadratic triangle to a square leads to a Jacobian that exactly cancels the $1/r$ singularity of the integrand. The resulting two-dimensional regular integral can even be reduced to a regular line integral (Aliabadi and Hall 1987b). This approach is really nice and works for other mappings such as linear triangles as well.

d) Use of special solutions

Special solutions of the entire boundary integral equation, when available, have been employed by many researchers to avoid direct computation of singular integrals. The use of rigid body motion (Lachat and Watson 1976) is now well established. Use of other modes, for example, appear in Mukherjee (1982) (inflation mode for axisymmetric problems) and Nishimura and Kobayashi (1989).

Overall, special solutions are very useful, but lack generality.

e) Regularization of singular integral

Recently, Guiggiani and Gigante (1990) have proposed a general regularization algorithm for $O(1/r^2)$ integrals on planar and curved surfaces. The original Cauchy principal value (CPV) integral is transformed in this work, by rigorous mathematical operations, into an element by element sum of regular integrals. This method is sound but is yet to be evaluated, in terms of efficiency, in complicated nonlinear problems.

The new mapping method

This paper presents an idea for the numerical evaluation of integrals of $O(1/r)$, over two-dimensional regions, based on a mapping approach. This mapping transforms such integrals over linear flat triangles into regular line integrals that can be easily evaluated by standard Gaussian quadrature. Integrals over curved quadratic triangles are first converted into regular two-dimensional integrals. One integration can then be carried out in closed form, leaving the other (now a regular line integral) to be evaluated by standard Gaussian quadrature. The situation (for curved quadratic triangles) is analogous to that discussed by Aliabadi and Hall (1987b). For the more general case of integrals over curved triangles with cubic or higher order shape functions, regular 2-D integrals are obtained. One integration can sometimes be carried out in closed form, leaving the remaining line integral to be evaluated by Gaussian quadrature. Such situations must be examined on a case by case basis.

The integrand under consideration is of the form

$$\phi(x, y) = \frac{f(x, y)}{r^n} \tag{5}$$

where $f(x, y)$ is a homogeneous polynomial of degree m, n is an integer ≥ 1 and $\beta = m - n \geq -1$. Consideration of a homogeneous polynomial is sufficient since the integral of interest is a finite sum of homogeneous polynomials divided by r^n. Also, $f(x, y)$ includes the product of a kernel and a shape function for the physical variable under consideration, assuming that the physical variable is analytical and is written in terms of piecewise polynomials over each element. The notation (x, y), instead of (x_1, x_2) is chosen from here on for convenience.

The regions of interest here are triangles, as shown in Fig. 1. The source point P is assumed to lie either at the mid-point of a side or on the vertex of the triangle. In either case, the triangle can be divided into two parts to yield T_1 or T_2 in Fig. 1. Of course, triangles of general shape can be mapped onto T_1. Also, T_2 can be transformed to T_1 by the mapping given below Fig. 1. In what follows attention is focused on T_1.

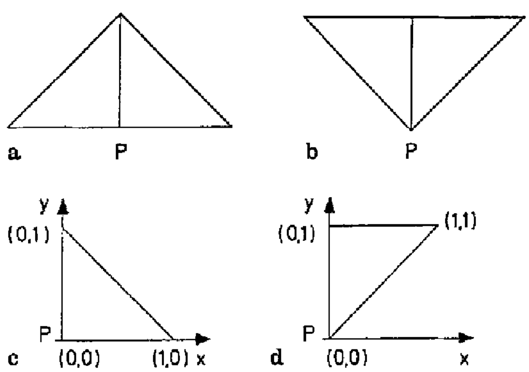
Consider the mapping

$$x = \rho \cos^2 \theta, \; y = \rho \sin^2 \theta \tag{6}$$

which transforms T_1 into a rectangle with sides 1 and $\pi/2$ as shown in Fig. 2.

Noting that

$$\phi(x, y) = \frac{f(x, y)}{r^n} = \rho^\beta g(\theta),$$



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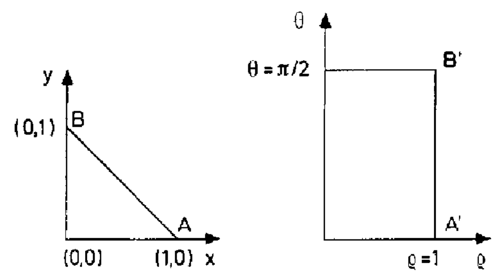

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Figs. 1 and 2. **1a, b** Triangle with source point at midside and at vertex. **c, d** Modelling half of the triangle from **a** and **b**. T_2 (Fig. d) can be transformed to T_1 (Fig. c) by the mapping $\bar{x} = x$, $\bar{y} = y - x$. **2** The mapping used in this paper $x = \rho \cos^2 \theta$, $y = \rho \sin^2 \theta$

where $g(\theta) = \dfrac{f(\cos^2\theta, \sin^2\theta)}{(\cos^4\theta + \sin^4\theta)^{n/2}}$ and using the Jacobian of the transformation $J = \rho \sin 2\theta$, one gets

$$I = \int_{\Delta} \frac{f(x,y)}{r^n} dx\,dy = \int_0^{\pi/2} d\theta \int_0^1 \rho^{\beta+1} g(\theta) \sin 2\theta\, d\rho = \frac{1}{\beta+2} \int_0^{\pi/2} g(\theta) \sin 2\theta\, d\theta \quad (7)$$

which is a 1-D integral. This integral can be evaluated using Gaussian quadrature.

Integrals of functions of $O(1/r)$ over general curved and flat triangles

Referring to Fig. 3, a general mapping from a curved triangle to T_1, with the origin (source point) mapped to the origin, can be expressed as

$$u = a_1 x + a_2 y + a_3 x^2 + \cdots$$
$$v = b_1 x + b_2 y + b_3 x^2 + \cdots \quad (8)$$

so that the Jacobian

$$J(x,y) = \det\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \text{higher order terms.}$$

Here, it is assumed that at least one of the coefficients a_1, a_2, b_1 or b_2 is non-zero. Further, the determinant above is assumed to be non-zero since, otherwise, there would be a cusp at the origin in physical space.

Now, $r^2 = u^2 + v^2 = g(x,y)$ where the leading term in $g(x,y)$ is quadratic. In the neighborhood of the origin

$$r \approx [(a_1 x + a_2 y)^2 + (b_1 x + b_2 y)^2]^{1/2}$$

so that asymptotically, as $Q \to P$ in Fig. 3, the $O(1/r)$ singularity is cancelled out when the mapping in Eq. (6) is used. Thus, this approach converts integrals of $O(1/r)$ over general curved triangles into regular integrals.

It can be demonstrated that, if the triangle in physical space is curved quadratic (see for example, Aliabadi and Hall 1987), one of the two regular integrations can be carried out in closed form and one is left with a regular line integral which can be evaluated by Gaussian quadrature. For higher order shape functions on curved triangles, it may also be possible to carry out the integral with respect to ρ in closed form. Such situations must be handled on a case by case basis.

The situation for the linear flat triangle reduces to (see Fig. 4)

$$u = u_1 x + u_2 y, \quad v = v_1 x + v_2 y \quad (9)$$

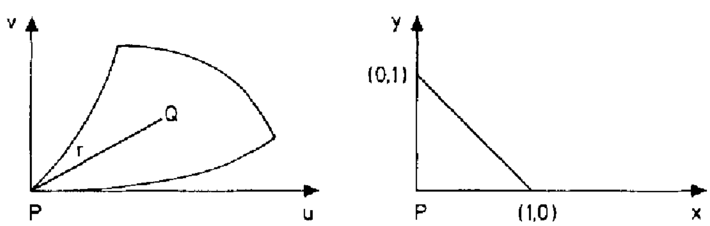

Fig. 3. Mapping of a general curved triangle to the standard triangle T_1

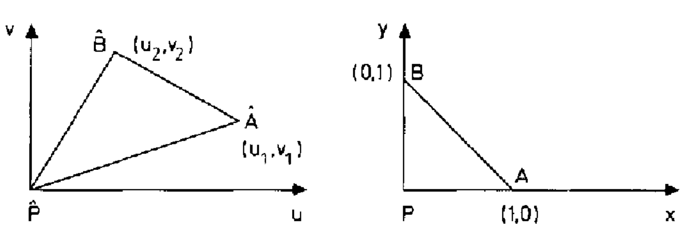

Fig. 4. Mapping a general flat linear triangle to T_1

with again, the source point P at the origin of the (u, v) plane being mapped to the origin in the (x, y) plane.

The distance r now is

$$r = (u^2 + v^2)^{1/2} = [(u_1 x + u_2 y)^2 + (v_1 x + v_2 y)^2]^{1/2}. \tag{10}$$

Next, the mapping from Eq. (6) is used to transform T_1 into the rectangle shown in Fig. 2. Since the mapping in Eq. (10) is globally homogeneous, ρ^n (see below (6)) is again a common factor in the denominator of the transformed integrand. Thus, the integration with respect to ρ for integrands that are, at worst, $O(1/r)$, can be carried out easily, and one is again left with a regular line integral. This is evaluated by Gaussian quadrature.

Numerical results

a) Integration over the standard triangle T_1 of Fig. 1: Numerical results have been obtained for integrals of the form

$$I_1 = \int_{\Delta} \frac{x^{m-p} y^p}{r^n} dA \tag{11}$$

where m, p, n and $m - p$ are non-negative integers with $\beta = m - n = -1$. The region of integration is the triangle T_1 in Fig. 1. The values of the integers are chosen so that the integrands evaluated are

$$\frac{1}{r}, \frac{x}{r^2}, \frac{y}{r^2}, \frac{x^2}{r^3}, \frac{xy}{r^3}, \frac{y^2}{r^3}, \frac{x^3}{r^4}, \frac{x^2 y}{r^4}, \frac{xy^2}{r^4} \quad \text{and} \quad \frac{y^3}{r^4},$$

which are the singular integrands from Eqs. (2) and (4). Since the method presented here reduces two-dimensional integrals with $\beta \geqq -1$ to one-dimensional regular integrals, there should be no problems in accurately evaluating other integrals of the type in Eq. (11) as long as $\beta \geqq -1$. Thus, products of singular kernels and regular shape functions can be taken care of easily provided that the resulting integrand satisfies the criterion $\beta \geqq -1$.

Upon substitution of Eq. (6), the integral in Eq. (11) becomes

$I_1 = 2^{n/2} I_2$ with

$$I_2 = \int_0^{\pi/2} \frac{(\cos\theta)^{2m-2p} (\sin\theta)^{2p} \sin 2\theta \, d\theta}{(1 + \cos^2 2\theta)^{n/2}}. \tag{12}$$

Gaussian integration can be used directly to evaluate I_2. It is more accurate, however, to first

Table 1. Numerical results for $O(1/r)$ BEM-integrals over the standard triangle of Fig. 1

Function	Percentage error			
	Gauss points			
	3	4	5	6
$1/r$	0.2582	0.0393	0.0061	0.0010
x/r^2	0.7981	0.1381	0.0239	0.0041
y/r^2	0.7981	0.1381	0.0239	0.0041
x^2/r^3	0.2582	0.0393	0.0061	0.0010
xy/r^3	4.0173	0.7945	0.1527	0.0287
y^2/r^3	0.2582	0.0393	0.0061	0.0010
x^3/r^4	1.7476	0.4437	0.1014	0.0217
x^2y/r^4	6.2500	1.3841	0.2922	0.0595
xy^2/r^4	6.2500	1.3841	0.2922	0.0595
y^3/r^4	1.7476	0.4437	0.1014	0.0217

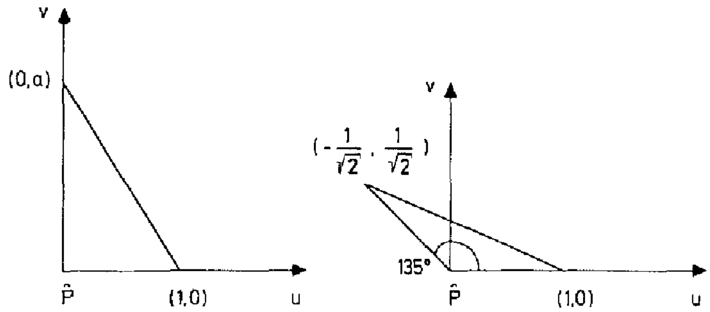

Fig. 5. Flat triangles

Table 2. Integration of $u/(u^2+v^2)$ over flat triangles of general shape

	Percentage error					
	Gauss points					
a	4	5	6	7	9	12
1	0.1381	0.0239	0.0041	0.0007		
2	0.0241	0.0603	0.0156	0.0013		
3	0.8386	0.0039	0.0496	0.0046		
4	0.3859	0.4316	0.0168	0.0341		
5	1.1794	0.5852	0.1978	0.0288		
Obtuse angled			1.3836		0.1244	0.0111

use the substitutions

$$t = \sin^2\theta, \quad dt = \sin 2\theta d\theta \tag{13}$$

and then to use standard Gaussian integration. This approach has been used to obtain the results that are presented next.

Results are presented in Table 1. Percentage errors are defined as follows:

$$\text{Percentage error} = \frac{\text{abs(Numerical value} - \text{Analytical value)}}{\text{Analytical value}} \times 100$$

with the analytical value being obtained by analytical integration using MACSYMA (Rand 1984). The numerical results, with six Gauss points, are seen to be uniformly accurate with the highest error less than 0.06%. The errors in Table 1 can be further reduced, if desired, by using more Gauss points.

b) Integrations over linear flat triangles of general shape

Numerical results are presented in Table 2 for integration over right angled triangles (for various values of 'a'), and an obtuse angled triangle (Fig. 5). In Table 2, the integrand is the function $u/(u^2+v^2)$ in the triangle in the physical domain. Once again, numerical results with seven Gauss points are very accurate for integrations over right angled triangles. The obtuse angled triangle requires twelve Gauss points to get very accurate results. Of course, an obtuse angled triangle can be broken up into two right angled triangles in a general BEM application.

Conclusions

A simple, efficient and accurate mapping method has been presented here for the numerical evaluation of two-dimensional integrals that are, at worst, $1/r$ singular, over triangular (curved or flat) domains. In general, this mapping transforms such integrals into regular two-dimensional ones. Integrals over curved quadratic or flat linear triangles are further reduced to regular line integrals that can be easily evaluated to desired accuracy by Gaussian quadrature.

It is expected that this method will be more accurate than polar mapping methods (Mukherjee 1982) because far fewer arithmetic operations are involved here. Investigations along these lines, for nonlinear BEM applications, are currently in progress.

Acknowledgments

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