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Interpolation Properties of Superintuitionistic Logics

Abstract. A family of propositional logics is considered to be intermediate between the intuitionistic and classical ones. The generalized interpolation property is defined and proved is the following.

Theorem on interpolation. For every intermediate logic $\boldsymbol{L}$ the following statements are equivalent:

(i) Craig's interpolation theorem holds in $\boldsymbol{L}$,

(ii) $\boldsymbol{L}$ possesses the generalized interpolation property,

(iii) Robinson's consistency statement is true in $\boldsymbol{L}$.

There are just 7 intermediate logics in which Craig's theorem holds.

Besides, Craig's interpolation theorem holds in $\boldsymbol{L}$ iff all the modal companions of $\boldsymbol{L}$ possess Craig's interpolation property restricted to those formulas in which every variable is preceeded by necessity symbol.

1. Interpolation properties of logical theories are of interest for logicians. W. Craig [1] stated interpolation theorem for the classical predicate logic in 1957. Interpolation theorem for intuitionistic predicate logic was proved by K. Schütte [19], D. M. Gabbay [3, 4] investigated interpolation properties of some extensions of intuitionistic predicate logic. Interpolation theorems are obtained for some modal logics in [2, 5, 18] and for many-valued predicate calculi in [15].

We consider a family of propositional logics to be intermediate between the intuitionistic and the classical logics.

It is proved in [11, 12] that there exist just 7 intermediate logics in which Craig's interpolation theorem holds; there are only 3 positive logics with the interpolation property. There are no more than 38 modal logics which are normal extensions of S4 and have the interpolation property [13].

We take &, $\vee$, $\supset$, $\neg$, $\mathbf{1}$ as primitive logical symbols of languages $\mathscr{L}_i$ of propositional logic.

Craig's interpolation theorem in a logic $\boldsymbol{L}$ is formulated as follows:

"If $(A \supset B)$ is in $\boldsymbol{L}$, then there exists a formula C such that $(A \supset C) \in \boldsymbol{L}$ and $(C \supset B) \in \boldsymbol{L}$ and all the variables of C are contained in both A and B".

This formula C is called an interpolant of A and B in $\boldsymbol{L}$. In classical logic $\boldsymbol{Cl}$ Craig's interpolation theorem has a number of equivalent formulations. For instance, interpolation theorem for disjunctions:

"If $(A \vee B)$ is in $\boldsymbol{L}$, A and B are formulae of languages $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, then there exists a formula C of the language $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$, such that both $(A \vee C)$ and $(C \supset B)$ are in L".

It is easy to get this sentence from Craig's theorem replacing $(A \vee B)$

by its equivalent (in Cl) $(\neg A \supset B)$. In intermediate logics different from Cl formulae $(A \vee B)$ and $(\neg A \supset B)$ are not equivalent. Of course, the abovementioned theorem for disjunctions holds in intuitionistic logic Int due to the well-known Gödel's theorem on disjunctions. However, Int is the only intermediate logic in which both Craig's interpolation theorem and Gödel's theorem hold.

Consider now formulae of the kind $(F_1 \supset F_2)$, where F_1 and F_2 are produced from $A_i \in \mathscr{L}_1$, $B_j \in \mathscr{L}_2$ by means of & and $\vee$. Such formulae are reduced in Int to conjunctions of formulas $((A_1 \,\&\, B_1) \supset (A_2 \vee B_2))$. The usual Craig's interpolation theorem seems not to be useful for the latter formulas. Only in Cl we can reduce such a formula to $((A_1 \,\&\, \neg A_2) \supset (\neg B_1 \vee B_2))$ and then apply Craig's theorem. In the present article we prove that any intermediate logic L with Craig's interpolation theorem possesses the generalized interpolation property:

"Let $(A_1 \supset A_2)$, $(B_1 \supset B_2)$ be formulas of languages $\mathscr{L}_1$, $\mathscr{L}_2$, respectively. If $((A_1 \,\&\, B_1) \supset (A_2 \vee B_2))$ is in L, then there exist an n and formulas $C_{11}, C_{21}, C_{22}, \ldots, C_{n1}, C_{n2}$ of the language $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$, such that L contains

$$(A_1 \supset (A_2 \vee C_{n1})),\ (A_1 \,\&\, C_{(k+1)2}) \supset (A_2 \vee C_{k1})),$$
$$((C_{11} \,\&\, B_1) \supset B_2),\ ((C_{(k+1)1} \,\&\, B_1) \supset (C_{(k+1)2} \vee B_2))$$

for any $k = 1, \ldots, n-1$".

The converse to the generalized interpolation property is the statement in any intermediate logic since

$$Int \vdash ((A_1 \supset (A_2 \vee C_{n1})) \,\&\, \mathop{\&}_{k=1}^{n-1} ((A_1 \,\&\, (C_{(k+1)2}) \supset (A_2 \vee C_{k1})) \,\&\, ((B_1 \,\&C_{11}) \supset B_2) \,\&\, \mathop{\&}_{k=1}^{n-1} ((B_1 \,\&\, C_{(k+1)1}) \supset (C_{(k+1)2} \vee B_2)) \supset ((A_1 \,\&\, B_1) \supset ((A_2 \vee B_2)).$$

Thus, (see [12]), there are just 7 intermediate logics with the generalized interpolation property:

$L_1 = Int$,
$L_2 = KC = Int + (\neg A \vee \neg\neg A)$,
$L_3 = Int + (A \vee (A \supset (B \vee \neg B)))$,
$L_4 = L_3 + ((A \supset B) \vee (B \supset A) \vee ((A \supset \neg B) \,\&\, (\neg B \supset A))$,
$L_5 = L_2 + L_3$,
$L_6 = LC = Int + ((A \supset B) \vee (B \supset A))$,
$L_7 = Cl = Int + (A \vee \neg A)$.

Obviously, one can take $n = 1$ in the generalized interpolation property for $L = Cl$ which is the only intermediate logic of the first slice [6]. One can take $n = 2$ for three logics of the second slice with the interpolation property. It is impossible to limit n for the remaining logics LC, KC and Int.

REMARK. It is easy to derive Craig's theorem from the generalized interpolation property. Take $B_1 = \mathbf{1}$, $A_1 = \mathbf{0} = \mathbf{1}$. For instance,

$$C = C_{n1} \,\&\, \mathop{\&}_{k=1}^{n-1} (C_{(k+1)2} \supset C_{k1})$$

is an interpolant of A_2 and B_2 in $\boldsymbol{L}$.

Craig's interpolation theorem is equivalent to Robinson's consistency statement [17] in theories based on the classical predicate logic. Robinson's consistency statement for intermediate logic $\boldsymbol{L}$ can be formulated as follows:

"Let $\langle T_1, F_1\rangle$, $\langle T_2, F_2\rangle$ be consistent $\boldsymbol{L}$-theories in languages $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, and let $\langle T_1 \cap T_2, F_1 \cap F_2\rangle$ be a complete $\boldsymbol{L}$-theory in the common language $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$. Then $\langle T_1 \cup T_2, F_1 \cup F_2\rangle$ is a complete $\boldsymbol{L}$-theory".

Remember, [3], that a consistent L-theory in a language $\mathscr{L}$ is a pair $\langle T, F\rangle$ of sets of formulas of $\mathscr{L}$ which satisfies the condition: there are no formulas $A_1, \ldots, A_k \in T$; $B_1, \ldots, B_l \in F$ such that $L \vdash \big((A_1 \,\&\, \ldots \,\&\, A_k) \supset \supset (B_1 \vee \ldots \vee B_l)\big)$. L-theory $\langle T, F\rangle$ is complete, if $T \cup F = \mathscr{L}$.

Gabbay [3, 4] has stated that in the family of extensions of intuitionistic predicate logic Robinson's statement implies Craig's theorem but the converse is not true, in particular, Robinson's statement does not hold in intuitionistic predicate logic. On the contrary, for propositional logics we prove

THEOREM 1 (on interpolation). *For any intermediate logic* $\boldsymbol{L}$ *the following statements are equivalent*:

1) *Craig's interpolation theorem in* $\boldsymbol{L}$,
2) *the generalized interpolation property of* $\boldsymbol{L}$,
3) *Robinson's consistency statement for* $\boldsymbol{L}$.

PROOF. $1\to 2$ is a consequence of theorems 2–4, which will follow, $3\to 1$ can be stated in the same way as in [3]. Now, we prove $2\to 3$. Let L possess the generalized interpolation property. Let $\langle T_1, F_1\rangle$ and $\langle T_2, F_2\rangle$ be consistent $\boldsymbol{L}$-theories in languages $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, and $\langle T_1 \cap T_2, F_1 \cap F_2\rangle$ be a complete $\boldsymbol{L}$-theory in the common language $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$.

Suppose for reduction ad absurdum that $\langle T_1 \cup T_2, F_1 \cup F_2\rangle$ is inconsistent. Then there exist formulas $A_1^1, \ldots, A_1^k \in T_1$; $B_1^1, \ldots, B_1^l \in T_2$; $A_2^1, \ldots, A_2^r \in F_1$; $B_2^1, \ldots, B_2^s \in F_2$, such that $\boldsymbol{L} \vdash \big((A_1 \,\&\, B_1) \supset (A_2 \vee B_2)\big)$ where $A_1 = A_1^1 \,\&\, \ldots \,\&\, A_1^k$, $B_1 = B_1^1 \,\&\, \ldots \,\&\, B_1^l$, $A_2 = A_2^1 \vee \ldots \vee A_2^r$, $B_2 = B_2^1 \vee \ldots \vee B_2^s$. Using the generalized interpolation property we have an n and $C_{11}, C_{21}, C_{22}, \ldots, C_{n1}, C_{n2} \in \mathscr{L}_0$, such that

(a1) $\boldsymbol{L} \vdash \big(A_1 \supset (A_2 \vee C_{n1})\big)$ (a2) $\boldsymbol{L} \vdash \big((A_1 \,\&\, C_{(k+1)2}) \supset (A_2 \vee C_{k1})\big)$

(b1) $\boldsymbol{L} \vdash \big((C_{11} \,\&\, B_1) \supset B_2\big)$ (b2) $\boldsymbol{L} \vdash \big((C_{(k+1)1} \,\&\, B_1) \supset (C_{(k+1)2} \vee B_2)\big)$

where $k = 1, \ldots, n-1$.

Due to (a1) we have $C_{n1} \notin F_1$ because of consistency of $\langle T_1, F_1\rangle$. So, $C_{n1} \notin F_0 = F_1 \cap F_2$ and $C_{n1} \in T_0 = T_1 \cap T_2$ as $\langle T_0, F_0\rangle$ is complete. Now, $C_{n1} \in T_2$, so (b2) and consistency of $\langle T_2, F_2\rangle$ imply $C_{n2} \notin F_2$ and $C_{n2} \in T_0$. Further, $C_{n2} \in T_1$ and $C_{(n-1)1} \notin F_1$ because of (a2), so $C_{(n-1)1} \in T_0$. Applying (b2) and (a2) again we receive $C_{ij} \in T_0$ for all i, j. In particular, $C_{11} \in T_0 \subseteq T_2$ and $\langle T_2, F_2\rangle$ is an inconsistent $\boldsymbol{L}$-theory because of (b1). Contradiction.

We adduce now two corollaries of Craig's interpolation theorem. In the same way as for classical logic one can prove

STATEMENT 1. *If in intermediate logic $\boldsymbol{L}$ Craig's interpolation theorem holds, then $\boldsymbol{L}$ satisfies Beth's definability theorem.*

There is one problem still open, if Beth's theorem implies Craig's theorem in intermediate logics.

In [14] there were formulated some principles of variables separation, in particular:

If $((A_1 \& B_1) \supset (A_2 \vee B_2)) \in \boldsymbol{L}$ and formulae $(A_1 \supset A_2)$ and $(B_1 \supset B_2)$ have no common variables, then $(A_1 \supset A_2) \in \boldsymbol{L}$ or $(B_1 \supset B_2) \in \boldsymbol{L}$.

STATEMENT 2. *If $\boldsymbol{L}$ is an intermediate logic in which Craig's interpolation theorem holds, then $\boldsymbol{L}$ satisfies the abovementioned separation principle. In particular, all intermediate logics with the interpolation property are Hallden-reasonable.*

One can prove it using the generalized interpolation property. All the formulas $C_{11}, C_{21}, C_{22}, \ldots, C_{n1}, C_{n2}$ are to belong to $\{\mathbf{0}, \mathbf{1}\}$. Let $(B_1 \supset B_2) \notin \boldsymbol{L}$. Then, $C_{11} = \mathbf{0}$ from $\boldsymbol{L} \vdash ((C_{11} \& B_1) \supset B_2)$. Further, we obtain $C_{(k+1)2} = \mathbf{0}$ and $C_{(k+1)1} = \mathbf{0}$ for $k = 1, \ldots, n-1$ from the conditions $\boldsymbol{L} \vdash ((A_1 \& C_{(k+1)2}) \supset (A_2 \vee C_{k1}))$, $\boldsymbol{L} \vdash ((C_{(k+1)1} \& B_1) \supset (C_{(k+1)2} \vee B_2))$. Consequently, we have $\boldsymbol{L} \vdash (A_1 \supset A_2)$ because $\boldsymbol{L} \vdash (A_1 \supset (A_2 \vee C_{n1}))$. Hallden--completeness is obtained at $A_1 = \mathbf{1}$, $B_1 = \mathbf{1}$.

Note that the principle of variables separation is valid, for example, in all extensions of $\boldsymbol{LC}$.

2. We use the interpretation of intermediate logics in extensions of modal logic $\boldsymbol{S4}$ to prove the generalized interpolation property of logics $\boldsymbol{L}_1 - \boldsymbol{L}_7$. Take $\&, \vee, \supset, \sim, \Box, \Diamond, \mathbf{1}$ as primitive logical symbols of modal logic.

Remember, [7], that one can accord to every formula A of intuitionistic logic its translation $\mathrm{T}(A)$ — a formula of modal logic which satisfies the following condition:

$$A \in \boldsymbol{Int} \Leftrightarrow \mathrm{T}(A) \in \boldsymbol{S4}.$$

The translation T is defined as follows:

$$\begin{aligned}
&\mathrm{T}(P) = \Box P \text{ if } P \text{ is a variable},\\
&\mathrm{T}(A \,\&\, B) = \mathrm{T}(A) \,\&\, \mathrm{T}(B),\\
&\mathrm{T}(A \vee B) = \mathrm{T}(A) \vee T(B),\\
&\mathrm{T}(A \supset B) = \Box(\sim \mathrm{T}(A) \vee \mathrm{T}(B),\\
&\mathrm{T}(\neg A) = \Box \sim \mathrm{T}(A),\\
&\mathrm{T}(\mathbf{1}) = \mathbf{1}.
\end{aligned}$$

A formula of modal logic is said to be special if each of its variables is preceded by the necessity symbol $\Box$.

TRANSLATION LEMMA. *For any special modal formula D there exists a formula D' of intuitionistic logic, such that*

$$\mathbf{S4} \vdash (\Box D \equiv \mathrm{T}(D')).$$

Proof by induction on the construction of D.

Let $\mathbf{NE}(\mathbf{S4})$ be a family of all the normal extensions of $\mathbf{S4}$. If M is in $\mathbf{NE}(\mathbf{S4})$, $\varrho(M)$ (see [8]) is its superintuitionistic fragment, i.e.,

$$\varrho(M) = \{A \mid \mathrm{T}(A) \in M\}.$$

Each M in $\mathbf{NE}(\mathbf{S4})$ is called a modal companion of $\varrho(M)$.

The weak Craig's theorem in modal logics is Craig's interpolation theorem with the additional condition that the (modal) formula $(A \supset B)$ is special.

THEOREM 2. *Let $M \in \mathbf{NE}(\mathbf{S4})$. Then the weak Craig's theorem holds in M iff $\varrho(M)$ possesses the generalized interpolation property.*

PROOF. Let the weak Craig's theorem hold in M and let $\varrho(M) \vdash$ $\vdash ((A_1 \,\&\, B_1) \supset (A_2 \vee B_2))$. Then, $M \vdash ((\mathrm{T}(A_1) \,\&\, \sim \mathrm{T}(A_2)) \supset (\sim \mathrm{T}(B_1) \vee$ $\vee\, \mathrm{T}(B_2))$. By the weak Craig's theorem there exists an interpolant C of $(\mathrm{T}(A_1) \,\&\, \sim \mathrm{T}(A_2))$ and $(\sim \mathrm{T}(B_1) \vee \mathrm{T}(B_2))$ in M. Replacing all variables P_i by $\Box P_i$ we obtain a special formula C', such that

(1) $\quad M \vdash ((\mathrm{T}(A_1) \,\&\, \sim \mathrm{T}(A_2)) \supset C'), \quad M \vdash (C' \supset (\sim \mathrm{T}(B_1) \vee \mathrm{T}(B_2)).$

The formula C' is a Boolean combination of subformulas $\Box C_j$. Therefore, by Lemma 5 of [10] there exists an n such that

(2) $\quad \mathbf{S4} \vdash (C' \equiv C_{11} \vee (C_{21} \,\&\, \sim C_{22})) \vee \ldots \vee (C_{n1} \,\&\, \sim C_{n2})),$

where $C_{11} = \Box C'$, $C_{(k+1)1} = \Box(C' \vee C_{(k+1)2})$, $C_{(k+1)2} = \Box(\sim C' \vee C_{k1})$.
Hence,

(3) $\quad \mathbf{S4} \vdash (C' \equiv C_{n1} \,\&\, (C_{(n-1)1} \vee \sim C_{n2}) \,\&\, \ldots \,\&\, (C_{11} \vee \sim C_{22})).$

By translation lemma there exist non-modal formulae $D_{11}, D_{21}, D_{22}, \ldots$ $\ldots, D_{n1}, D_{n2}$, such that

$S4 \vdash (C_{ij} \equiv \mathrm{T}(D_{ij}))$.

From (1) and (3)

$M \vdash ((\mathrm{T}(A_1) \mathbin{\&} \sim\mathrm{T}(A_2)) \supset C_{n1})$,

$M \vdash ((\mathrm{T}(A_1) \mathbin{\&} \sim\mathrm{T}(A_2)) \supset (C_{k1} \vee \sim C_{(k+1)2}))$ $(k = 1, \ldots, n-1)$

Then

$\varrho(M) \vdash (A_1 \supset (A_2 \vee D_{n1}))$,

$\varrho(M) \vdash ((A_1 \mathbin{\&} D_{(k+1)2}) \supset (A_2 \vee D_{k1}))$ $(k = 1, \ldots, n-1)$

Similarly, from (1) and (2)

$\varrho(M) \vdash ((D_{11} \mathbin{\&} B_1) \supset B_2)$,

$\varrho(M) \vdash ((D_{(k+1)1} \mathbin{\&} B_1) \supset (D_{(k+1)2} \vee B_2))$ $(k = 1, \ldots, n-1)$

So, $\varrho(M)$ has the generalized interpolation property.

Sufficiency. Let $\varrho(M)$ have the generalized interpolation property and let $M \vdash (A \supset B)$, where $(A \supset B)$ is a special formula. Reduce A to the form $\bigvee_i (A_{i1} \mathbin{\&} \sim A_{i2})$, and B – to the form $\mathop{\&}_j (\sim B_{j1} \vee B_{j2})$, where $A_{ik} = \Box A'_{ik}$, $B_{jk} = \Box B'_{jk}$. Then, the formula $(A \supset B)$ is equivalent in $S4$ to the conjunction of formulas $((A_{i1} \mathbin{\&} \sim A_{i2}) \supset (\sim B_{j1} \vee B_{j2}))$ for all i, j. By translation lemma

$S4 \vdash \Box((A_{i1} \mathbin{\&} \sim A_{i2}) \supset (\sim B_{j1} \vee B_{j2})) \equiv \mathrm{T}((A^*_{i1} \mathbin{\&} B^*_{j1}) \supset (A^*_{i2} \vee B^*_{j2}))$

for some non-modal A^*_{i1}, A^*_{i2}, B^*_{j1}, B^*_{j2}. Now we have

$$\varrho(M) \vdash (A^*_{i1} \mathbin{\&} B^*_{j1}) \supset (A^*_{i2} \vee B^*_{j2}).$$

Using the generalized interpolation property of $\varrho(M)$ we have

$$M \vdash (\mathrm{T}(A^*_{i1}) \mathbin{\&} \sim\mathrm{T}(A^*_{i2})) \supset C_{ij}, \quad M \vdash C_{ij} \supset (\sim\mathrm{T}(B^*_{j1}) \vee \mathrm{T}(B^*_{j2})),$$

where C_{ij} is a Boolean combination of formulae C^*_{ijk}, C^*_{ijk} – non-modal formulae containing only common variables of $(A^*_{i1} \supset A^*_{i2})$ and $(B^*_{j1} \supset B^*_{j2})$. Then, $C = \bigvee_j \mathop{\&}_i C_{ij}$ is an interpolant of A and B in M. Q.E.D.

Note that if M is a logic of the second slice [10], one can replace (2) in the proof by $M \vdash C' \equiv C_{11} \vee \sim C_{22}$. So, the generalized interpolation property can be formulated for intermediate logics of the second slice as follows:

If $L \vdash (A_1 \mathbin{\&} B_1) \supset (A_2 \vee B_2)$, there exist formulae C_1 and C_2 such that

$$L \vdash (A_1 \mathbin{\&} C_2) \supset (A_2 \vee C_1),$$
$$L \vdash (C_1 \mathbin{\&} B_1) \supset B_2, \quad L \vdash B_1 \supset (C_2 \vee B_2)$$

and all the variables of C_1 and C_2 are contained in both $(A_1 \supset A_2)$ and $(B_1 \supset B_2)$.

Due to theorem 2, Craig's interpolation theorem for M implies the generalized interpolation property of $\varrho(M)$. Craig's interpolation theorem was proved in ***S4, S4.2*** [5], and in ***S4.4, S5*** [18]. Hence, it follows that intermediate logics $L_1 = Int$, $L_2 = KC$, $L_5 = KC + \big(A \vee (A \supset (B \vee \neg B))\big)$, $L_7 = Cl$ have the generalized interpolation property and the weak Craig's theorem holds in all modal companions of L_1, L_2, L_5, L_7, i.e., in infinitely many logics. Remember, [13], that Craig's interpolation theorem is valid only for finitely many logics in $\mathbf{NE}(S4)$.

It remains to prove the weak Craig's theorem for modal companions of L_3, L_4, L_6. It is stated [13], that Craig's interpolation theorem in $M \in \mathbf{NE}(S4)$ is equivalent to the superamalgamation property of the corresponding variety $\mathfrak{M}_M$ of closure algebras defined by the identities $\{A = 1 \mid A \in M\}$. We prove that the weak Craig's theorem holds in M iff the class $\mathfrak{M}_M^S$ of all special closure algebras in $\mathfrak{M}_M$ has the superamalgamation property. Remember, [10], that closure algebra $\mathfrak{A}$ is special if it is generated by the set $\mathbf{G}(\mathfrak{A}) = \{\Box x \mid x \in \mathfrak{A}\}$.

To any pseudo-Boolean algebra $\mathfrak{A}$ there corresponds a special closure algebra $\mathbf{S}(\mathfrak{A})$, such that $\mathbf{G}\big(\mathbf{S}(\mathfrak{A})\big) = \mathfrak{A}$.

A class $\mathbf{K}$ is said to possess the superamalgamation property, if for any $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathbf{K}$, such that $\mathfrak{A}_0$ is the common subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, there exist $\mathfrak{A} \in \mathbf{K}$ and monomorphisms $\varepsilon_1\colon \mathfrak{A}_1 \to \mathfrak{A}$, $\varepsilon_2\colon \mathfrak{A}_2 \to \mathfrak{A}$, such that $\varepsilon_1 \restriction \mathfrak{A}_0 = \varepsilon_2 \restriction \mathfrak{A}_0$ and

$$(\forall x \in \mathfrak{A}_i)(\forall y \in \mathfrak{A}_j)[\varepsilon_j(x) \leqslant \varepsilon_j(y) \Leftrightarrow \\ \Leftrightarrow (\exists z \in \mathfrak{A}_0)(x \leqslant_i z \wedge z \leqslant_j y),$$

where $\{i, j\} = \{1, 2\}$.

THEOREM 3. *Let M be in $\mathbf{NE}(S4)$. Then the following statements are equivalent*:

a) *The weak Craig's theorem holds in M,*

b) *$\mathfrak{M}_M^S$ has the superamalgamation property.*

c) *For any $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}_M^S$, such that $\mathfrak{A}_0$ is the common subalgebra of $\mathfrak{A}_1, \mathfrak{A}_2$, and for any $a \in A_1$, $b \in A_2$, satisfying the condition $\neg(\exists z \in \mathfrak{A}_0)$ $(a \leqslant_1 z \wedge z \leqslant_2 b)$, there exist $\mathfrak{A} \in \mathfrak{M}_M$* and homomorphisms $h_1\colon \mathfrak{A}_1 \to \mathfrak{A}$, $h_2\colon \mathfrak{A}_2 \to \mathfrak{A}$, *such that $h_1(a) \nleqslant h_2(b)$ and $h_1 \restriction \mathfrak{A}_0 = h_2 \restriction \mathfrak{A}_0$.*

PROOF of a→b is analogous to lemma 2 of [13].

Obviously, b→c. We prove c→a.

Let A, B be special formulas and there be no interpolant of A and B in M. Let $\mathscr{F}_1, \mathscr{F}_2$ be the sets of all special formulas with variables of A and B, respectively, $\mathscr{F}_0 = \mathscr{F}_1 \cap \mathscr{F}_2$, and let $\mathscr{F}$ be the set of all the special formulas. Then the algebra $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathscr{F} / \sim_M$ is in $\mathfrak{M}_M^S$, where

$$A_1 \sim_M A_2 \leftrightharpoons M \vdash \Box(A_1 \equiv A_2).$$

Take a subalgebra $\mathfrak{A}_0$ of algebra $\mathfrak{A}_1$ with the universe $\{C/_{\sim_M} C \in \mathscr{F}_0\}$. Let $a = A/_{\sim_M} \in \mathfrak{A}_1$, $b = B/_{\sim_M} \in \mathfrak{A}_2$, then there is no $c \in \mathfrak{A}_0$, such that $a_0 \leqslant c$ and $c \leqslant b$. By condition c) there exist $\mathfrak{A} \in \mathfrak{M}_M$, $h_1\colon \mathfrak{A}_1 \to \mathfrak{A}$, $h_2\colon \mathfrak{A}_2 \to \mathfrak{A}$, such that $h_1(a) \nleqslant h_2(b)$ and $h_1 \restriction \mathfrak{A}_0 = h_2 \restriction \mathfrak{A}_0$.

Define now a valuation v in $\mathfrak{A}$ as follows:

$$v(P) = \begin{cases} h_1(\Box P/\sim_M), & \text{if the variable } P \text{ is contained in } A \\ h_2(\Box P/\sim_M), & \text{if } P \text{ is in } B. \end{cases}$$

Note that if P is a common variable of A and B, then

$$h_1(\Box P/\sim_M) = h_2(\Box P/\sim_M)$$

and v is defined correctly. Therefore, we have $v(D) = h_i(D/\sim_M)$ for any $D \in \mathscr{F}_i$ $(i = 1, 2)$. Hence,

$$v(A \supset B) = v(A) \supset v(B) = h_1(a) \supset h_2(b) \neq 1.$$

Because of $\mathfrak{A} \in \mathfrak{M}_M$, we obtain $M \not\vdash A \supset B$. Q.E.D.

THEOREM 4. *All modal companions of $\boldsymbol{L}_1$–$\boldsymbol{L}_7$ satisfy condition* c) *of Theorem* 3.

PROOF. In [11, statements 1–6] it was in fact proved for $\boldsymbol{L} = \boldsymbol{L}_1, \ldots, \boldsymbol{L}_7$ that for any strongly compact pseudo-Boolean algebras $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}_{\boldsymbol{L}}$ with a common subalgebra $\mathfrak{A}_0$ there exist a strongly compact pseudo-Boolean algebra $\mathfrak{A} \in \mathfrak{M}_{\boldsymbol{L}}$ and monomorphisms $\varepsilon_1: \mathfrak{A}_1 \to \mathfrak{A}$, $\varepsilon_2: \mathfrak{A}_2 \to \mathfrak{A}$, such that $\varepsilon_1 \restriction \mathfrak{A}_0 = \varepsilon_2 \restriction \mathfrak{A}_0$.

Now, let $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ be special closure algebras in $\mathfrak{M}_M$, $x_0 \in \mathfrak{A}_1$, $y_0 \in \mathfrak{A}_2$, $\neg(\exists z \in \mathfrak{A}_0)(x_0 \leqslant_1 z \wedge z \leqslant_2 y_0)$. Then, by lemma 3 from [11] there exist ultrafilters Φ_1 on $\mathfrak{A}_1$ and Φ_2 on $\mathfrak{A}_2$, such that $x_0 \in \Phi_1$, $y_0 \notin \Phi_2$ and $\Phi_1 \cap \mathfrak{A}_0 = \Phi_2 \cap \mathfrak{A}_0$. Let

$$\nabla_1 = \{x \in \mathfrak{A}_1 \mid \Box x \in \Phi_1\}, \qquad \nabla_2 = \{x \in A_2 \mid \Box x \in \Phi_2\}.$$

Then ∇_1, ∇_2 are I-filters [16] on $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively, hence, there exist natural homomorphisms

$$g_1: \mathfrak{A}_1 \to \mathfrak{A}_1/\nabla_1 = \mathfrak{B}_1, \qquad g_2: \mathfrak{A}_2 \to \mathfrak{A}_2/\nabla_2 = \mathfrak{B}_2$$

onto special closure algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are strongly compact since

$$(\Box x \vee \Box y) \in \nabla_i \Rightarrow \Box x \in \nabla_i \quad \text{or} \quad \Box y \in \nabla_i.$$

Let $\nabla_0 = \nabla_1 \cap \mathfrak{A}_0 = \nabla_2 \cap \mathfrak{A}_0$, $\mathfrak{B}_0 = \mathfrak{A}_0/\nabla_0$. Then there exist monomorphisms

$$\delta_j: \mathfrak{B}_0 \to \mathfrak{B}_j \quad \text{where } j = 1, 2, \qquad \delta_j(z/\nabla_0) = z/\nabla_j \quad \text{for } z \in \mathfrak{A}_0.$$

Maps $\bar{\delta}_j = \delta_j \restriction \mathbf{G}(\mathfrak{B}_0)$ are monomorphisms of a pseudo-Boolean algebra $\mathbf{G}(\mathfrak{B}_0)$ into $\mathbf{G}(\mathfrak{B}_j)$. Since pseudo-Boolean algebras $\mathbf{G}(\mathfrak{B}_0)$, $\mathbf{G}(\mathfrak{B}_1)$, $\mathbf{G}(\mathfrak{B}_2)$ are strongly compact and belong to $\mathfrak{M}_{\varrho(M)}$, then by above mentioned property of $\mathfrak{M}_{\varrho(M)}$, there exist strongly compact $\mathfrak{C} \in \mathfrak{M}_{\varrho(M)}$ and monomorphisms $\varepsilon_1: \mathbf{G}(\mathfrak{B}_1) \to \mathfrak{C}$, $\varepsilon_2: \mathbf{G}(\mathfrak{B}_2) \to \mathfrak{C}$, such that $\varepsilon_1 \bar{\delta}_1 = \varepsilon_2 \bar{\delta}_2$. Monomorphisms ε_j of pseudo-Boolean algebras can be extended [8] to monomorphisms $\tilde{\varepsilon}_j: s(\mathbf{G}(\mathfrak{B}_j)) \to s(\mathfrak{C})$ of special closure algebras. If

(4) $\quad z = \bigvee_i (z_{i1} \,\&\, \sim z_{i2})$, where $z_{ik} \in \mathbf{G}(\mathfrak{B}_j)$, $j = 1, 2$,

then

$$\tilde{\varepsilon}_j(z) = \bigvee_i (\varepsilon_j(z_{i1}) \mathbin{\&} \sim\varepsilon_j(z_{i2})).$$

If z is of the form (4), where $z_{ik} \in \mathbf{G}(\mathfrak{B}_0)$, then

$$(5) \quad \tilde{\varepsilon}_1\delta_1(z) = \bigvee_i \big(\varepsilon_1\bar{\delta}_1(z_{i1}) \mathbin{\&} \sim\varepsilon_1\bar{\delta}_1(z_{i2})\big) = \bigvee_i \big(\varepsilon_2\bar{\delta}_2(z_{i1}) \mathbin{\&} \sim\varepsilon_2\bar{\delta}_2(z_{i2})\big) = \tilde{\varepsilon}_2\delta_2(z).$$

Since $\mathfrak{C}$ is strongly compact, $\{1_{\mathfrak{C}}\}$ is a prime filter on $\mathfrak{C}$. There exists an ultrafilter Φ on $\mathfrak{A} = S(\mathfrak{C})$, such that $\Phi \cap \mathfrak{C} = \{1_{\mathfrak{C}}\}$ (see lemma 5 in [9]). Now, let $j \in \{1, 2\}$, $z \in \mathfrak{A}_j$, $z = \bigvee_i (z_{i1} \mathbin{\&} \sim z_{i2})$, $z_{ik} \in \mathbf{G}(\mathfrak{A}_j)$. Prove that

$$(6) \qquad \tilde{\varepsilon}_j g_j(z) \in \Phi \Leftrightarrow z \in \Phi_j.$$

In fact, for any $z_{ik} \in \mathbf{G}(\mathfrak{A}_j)$:

$$\varepsilon_j g_j(z_{ik}) \in \Phi \Leftrightarrow \varepsilon_j g_j(z_{ik}) = 1 \Leftrightarrow g_j(z_{ik}) = z_{ik}/V_j = 1 \Leftrightarrow z_{ik} \in V_j \Leftrightarrow z_{ik} \in \Phi_j$$

Therefore,

$$\tilde{\varepsilon}_j g_j(z) \in \Phi \Leftrightarrow (\exists i \leqslant n)\big((\varepsilon_j g_j(z_{i1}) \mathbin{\&} \sim\varepsilon_j g_j(z_{i2})\big) \in \Phi) \Leftrightarrow$$
$$\Leftrightarrow (\exists i \leqslant n)(\varepsilon_j g_j(z_{i1}) \in \Phi \wedge \varepsilon_j g_j(z_{i2}) \notin \Phi) \Leftrightarrow$$
$$\Leftrightarrow (\exists i \leqslant n)(z_{i1} \in \Phi_j \wedge z_{i2} \notin \Phi_j) \Leftrightarrow z \in \Phi_j.$$

Now, for any $z \in \mathfrak{A}_j$ let

$$h_j(z) = \tilde{\varepsilon}_j g_j(z).$$

Due to (6) we have $h_1(x_0) \in \Phi$, $h_2(y_0) \in \Phi$, hence $h_1(x_0) \not\leqslant h_2(y_0)$. For $z \in A_0$ on account of (5)

$$h_1(z) = \tilde{\varepsilon}_1 g_1(z) = \varepsilon_1(z/V_1) = \tilde{\varepsilon}_1\delta_1(z/V_0) = \tilde{\varepsilon}_2\delta_2(z/V_0) = h_2(z). \quad \text{Q.E.D.}$$

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Received March 18, 1979.