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A Deontic Logic of Action

Abstract. The formal language studied in this paper contains two categories of expressions, terms and formulas. Terms express events, formulas propositions. There are infinitely many atomic terms and complex terms are made up by Boolean operations. Where α and β are terms the atomic formulas have the form $\alpha = \beta$ (α is the same as β), *Forb* α (α is forbidden) and *Perm* α (α is permitted). The formulae are truth functional combinations of these. An algebraic and a model theoretic account of validity are given and an axiomatic system is provided for which they are characteristic.

The 'closure principle', that what is not forbidden is permitted is shown to hold at the level of outcomes but not at the level of events. In the two final sections some other operators are considered and a semantics in terms of action games.

The present paper, which concentrates on formal issues, elaborates the deontic aspects of some ideas first presented in [4]. The modelling is inspired by both Kamp [1] and von Wright [6], and the resulting analysis is no doubt related to the system in the last section of von Wright [7]. Except in the concluding section, familiarity with [4] is not presupposed, but the reader who would like some motivation for the kind of modelling studied here is referred to that paper. The best background reference for the algebra done here is Rasiowa & Sikorski [2].

1. Formal syntax

The formal language studied in this paper will contain two categories of expressions, terms and formulas. To understand the motivation for this language, note that under the formal semantics to be developed in the next section terms will express events, formulas propositions.

There are infinitely many *atomic terms*: denumerably many *event letters* and the two *constants* **0** and **1**. The atomic terms are also *terms simpliciter*; and if α and β are *terms*, then so are $\alpha \frown \beta$, $\alpha \smile \beta$, $\bar{\alpha}$. The latter may be read "α and β", "α or β", "not α". The *atomic formulas* are of three kinds, $\alpha = \beta$, *Forb* α, *Perm* α, where α and β are terms. They may be read "α is the same as β", "α is forbidden", and "α is permitted". The atomic formulas are also *formulas simpliciter*; and if A and B are *formulas*, then so are $A \wedge B$, $A \vee B$, $\neg A$, $A \to B$, $A \leftrightarrow B$. The connectives involved here are the ones familiar from classical logic. In particular, $\to$ and $\leftrightarrow$ are material implication and material equivalence, respectively.

This completes our definition of the formal language to be studied in this paper. (Some banal modifications are considered at the end of it.) As a general convention let us use π for arbitrary event letters, α, β, γ for arbitrary terms, and $\boldsymbol{A}$, $\boldsymbol{B}$ for arbitrary formulas. We write $\alpha \neq \beta$ for $\neg(\alpha = \beta)$.

2. Formal semantics

Let us say that a structure $\mathfrak{A} = \langle A, \frown, \smile, -, 0, 1, F, P\rangle$ is a *deontic action algebra* if (i) $\langle A, \frown, \smile, -, 0, 1\rangle$ is a Boolean algebra of which (ii) F and P are ideals, and (iii) $F \cap P = \{0\}$. (Intuitively think of $\mathfrak{A}$ as an algebra of events, F and P the sets of forbidden, respectively, permitted events.) A *valuation for* $\mathfrak{A}$ is a function v from the set of atomic terms to A such that

$$v(\mathbf{0}) = 0,$$
$$v(\mathbf{1}) = 1.$$

Any valuation v can be uniquely extended to a function defined on the set of all terms and formulas in three steps as follows. (We continue to write just v, even though a careful treatment would require a more elaborate notation, for example, v', v'', and v''' for the three steps.)

Primary extension:

$$v(\alpha \frown \beta) = v(\alpha) \frown v(\beta),$$
$$v(\alpha \smile \beta) = v(\alpha) \smile v(\beta),$$
$$v(\bar{\alpha}) = -v(\alpha).$$

Secondary extension:

$$v(\alpha = \beta) = \begin{cases} \mathsf{T}, & \text{if } v(\alpha) = v(\beta), \\ \mathsf{F}, & \text{otherwise;} \end{cases}$$

$$v(\mathit{Forb}\,\alpha) = \begin{cases} \mathsf{T}, & \text{if } v(\alpha) \in F, \\ \mathsf{F}, & \text{otherwise;} \end{cases}$$

$$v(\mathit{Perm}\,\alpha) = \begin{cases} \mathsf{T}, & \text{if } v(\alpha) \in P, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

Tertiary extension: The value of $v(\boldsymbol{A})$ of non-atomic formulas $\boldsymbol{A}$ is determined by truth-tables in the usual way. (Here you are of course invited to read "true" for T and "false" for F.)

A formula $\boldsymbol{A}$ is *true in* $\mathfrak{A}$ *under* v if $v(\boldsymbol{A}) = \mathsf{T}$, *false in* $\mathfrak{A}$ *under* v if $v(\boldsymbol{A}) = \mathsf{F}$. $\boldsymbol{A}$ is *algebraically valid* if $\boldsymbol{A}$ is true in every deontic action algebra under every valuation. More generally, a set Γ *algebraically implies* $\boldsymbol{A}$ if, for every deontic action algebra $\mathfrak{A}$ and every valuation v for $\mathfrak{A}$, if every formula of Γ is true in $\mathfrak{A}$ under v, then so is $\boldsymbol{A}$.

Turning now to model theory, let us say that a structure $\mathfrak{F} = \langle U, \mathit{Ill}, \mathit{Leg}\rangle$ is a deontic action frame if (i) U is a set, (ii) $\mathit{Ill}, \mathit{Leg} \subseteq U$, and

(iii) $Ill \cap Leg = \varnothing$. (Thus a deontic action frame is more succinctly characterized as a set U with two disjoint subsets Ill and Leg. Infinitively think of U as a set of possible outcomes, Ill and Leg as the sets of illegal and legal outcomes, respectively. An event is then a set of outcomes.) A *valuation for* $\mathfrak{F}$ is a function V from the set of atomic terms to the set of subsets of U such that

$$V(\mathbf{0}) = \varnothing,$$
$$V(\mathbf{1}) = U.$$

As in the case of algebra, any valuation V can be uniquely extended to a function defined on the set of all terms and formulas in three steps:

Primary extention:

$$V(\alpha \frown \beta) = V(\alpha) \cap V(\beta),$$
$$V(\alpha \smile \beta) = V(\alpha) \cup V(\beta),$$
$$V(\bar{\alpha}) = U - V(\alpha).$$

Secondary extention:

$$V(\alpha = \beta) = \begin{cases} \mathsf{T}, & \text{if } V(\alpha) = V(\beta), \\ \mathsf{F}, & \text{otherwise;} \end{cases}$$

$$V(Forb\,\alpha) = \begin{cases} \mathsf{T}, & \text{if } V(\alpha) \subseteq Ill, \\ \mathsf{F}, & \text{otherwise} \end{cases}$$

$$V(Perm\,\alpha) = \begin{cases} \mathsf{T}, & \text{if } V(\alpha) \subseteq Leg, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

Tertiary expansion: By truth-tables.

Let us call $\mathfrak{M} = \langle U, Ill, Leg, V\rangle$ a *deontic action model* (*based on* the frame $\langle U, Ill, Leg\rangle$). A formula $\boldsymbol{A}$ is *true in* $\mathfrak{M}$ if $V(\boldsymbol{A}) = \mathsf{T}$, *false in* $\mathfrak{M}$ if $V(\mathfrak{A}) = \mathsf{F}$; in symbols we may render this $\mathfrak{M} \models \boldsymbol{A}$, respectively, $\mathfrak{M} \not\models \boldsymbol{A}$. We say that $\boldsymbol{A}$ is *model theoretically valid* if $\boldsymbol{A}$ is true in all deontic action models. More generally, a set Γ *model theoretically implies* $\boldsymbol{A}$ if, for every deontic action model, if every formula of Γ is true in this model, then so is $\boldsymbol{A}$.

3. The basic open deontic logic of urn model action

In the preceding section we introduced semantic concepts of validity and implication. Now we shall introduce syntactic counterparts by presenting an axiomatic system. As our only *inference rule* we appoint the time-honoured rule of *modus ponens*. Of axioms we will have many more. First of all we want a set of *axioms adequate for Boolean algebra*. Second, we want some *axioms for equality*: all instances of the following schemata will do:

(E1) $\alpha = \alpha$;
(E2) $\alpha = \beta \to (\boldsymbol{A} \to \boldsymbol{A}')$,

where $\boldsymbol{A}$ and $\boldsymbol{A}'$ differ only in that $\boldsymbol{A}$ contains an occurrence of α in some place where $\boldsymbol{A}'$ contains an occurrence of β. Third, we want a set of *axioms adequate for classical propositional logic*. Fourth, we want the following *special axioms for the deontic operators*: all instances of the following schemata:

(D1) $Forb(\alpha \cup \beta) \leftrightarrow (Forb\, \alpha \wedge Forb\, \beta)$;
(D2) $Perm(\alpha \cup \beta) \leftrightarrow (Perm\, \alpha \wedge Perm\, \beta)$;
(D3) $\alpha = \mathbf{0} \leftrightarrow (Forb\, \alpha \wedge Perm\, \alpha)$.

This defines a logic which we may perhaps call the *basic deontic logic of urn model action* (B.O.D.): the notions of theoremhood and implication are implicit.

It may be instructive to compare the axiomatic system with the definition of a deontic action algebra. The three first classes of axioms need no comment. (D1) and part of (D3), respectively, (D2), and part of (D3) correspond to the condition that F and P be ideals. The remaining part of (D3) corresponds to the condition that F and P have only the null element in common. That this correspondence is not fortuitous is shown by the following result.

THEOREM 3.1. *Let Γ be any set of formulas and $\boldsymbol{A}$ any formula. Then the following three conditions are equivalent:*

(i) *Γ implies $\boldsymbol{A}$ in B.O.D.*
(ii) *Γ algebraically implies $\boldsymbol{A}$.*
(iii) *Γ model theoretically implies $\boldsymbol{A}$.*

We give the proof in the following section.

4. Completeness of B.O.D.

What follows is a variation on an age-old theme. Say that Σ is a given fixed maximal consistent set of formulas (consistent with respect to B.O.D.). The following defines a binary relation in the set Θ of terms:

$$\alpha \equiv \beta\ (\mathrm{mod}\, \Sigma) \quad \text{if and only if} \quad \alpha = \beta \in \Sigma.$$

This relation may be called *the equivalence relation induced by Σ*. The terminology is not arbitrary, for, with the aid of axioms (E1) and (E2), it is easy to show that $\equiv$ is reflexive, symmetric and transitive. Thus we may define:

$$\alpha/\Sigma = \{\beta\colon \alpha = \beta \in \Sigma\}$$

and know that $\alpha/\Sigma = \beta/\Sigma$ if and only if $\alpha = \beta \in \Sigma$. Similarly, it is easy to show that

$$\begin{aligned}
&\alpha \equiv \alpha' \text{ and } \beta \equiv \beta' \text{ only if } \alpha \frown \beta \equiv \alpha' \frown \beta',\\
&\alpha \equiv \alpha' \text{ and } \beta \equiv \beta' \text{ only if } \alpha \smile \beta \equiv \alpha' \smile \beta',\\
&\alpha \equiv \beta \text{ only if } \bar{\alpha} \equiv \bar{\beta},\\
&\alpha \equiv \beta \text{ only if } \mathit{Forb}\,\alpha \leftrightarrow \mathit{Forb}\,\beta \in \Sigma,\\
&\alpha \equiv \beta \text{ only if } \mathit{Perm}\,\alpha \leftrightarrow \mathit{Perm}\,\beta \in \Sigma.
\end{aligned}$$

Now define the *Lindenbaum algebra* (*for* Σ) as the structure

$$\mathfrak{A}_\Sigma = \langle \Theta/\Sigma, \frown, \smile, -, 0_\Sigma, 1_\Sigma, F_\Sigma, P_\Sigma \rangle$$

where

$$\begin{aligned}
\Theta/\Sigma &= \{\alpha/\Sigma \colon \alpha \in \Theta\},\\
\alpha/\Sigma \frown \beta/\Sigma &= \alpha \frown \beta/\Sigma,\\
\alpha/\Sigma \smile \beta/\Sigma &= \alpha \smile \beta/\Sigma,\\
-\alpha/\Sigma &= \bar{\alpha}/\Sigma,\\
0_\Sigma &= \mathbf{0}/\Sigma,\\
1_\Sigma &= \mathbf{1}/\Sigma,\\
F_\Sigma &= \{\alpha/\Sigma \colon \mathit{Forb}\,\alpha \in \Sigma\},\\
P_\Sigma &= \{\alpha/\Sigma \colon \mathit{Perm}\,\alpha \in \Sigma\}.
\end{aligned}$$

The preceding remarks guarantee that the definition is meaningful.

LEMMA 4.1. *$\mathfrak{A}_\Sigma$ is a deontic action algebra.*

PROOF. Considering that Σ is maximal consistent, this follows at once: from the axioms for Boolean algebra, that $\langle \Theta/\Sigma, \frown, \smile, -, 0_\Sigma, 1_\Sigma \rangle$ is a Boolean algebra; from (D1) – (D3), that F_Σ and P_Σ are ideals; and from (D3), that F_Σ and P_Σ have 0_Σ as their unique common element. ■

We define a valuation v_Σ for $\mathfrak{A}_\Sigma$ by requiring that, for every atomic term π,

$$v_\Sigma \colon \pi \mapsto \pi/\Sigma.$$

The proof of the following result is immediate:

LEMMA 4.2. *For all terms α and all formulas $\boldsymbol{A}$,*
(i) $v_\Sigma(\alpha) = \alpha/\Sigma$,
(ii) $v_\Sigma(\boldsymbol{A}) = \mathsf{T}$ *if and only if* $\boldsymbol{A} \in \Sigma$.

From this lemma, half of the Completeness Theorem follows (that is, Theorem 3.1 of the preceding section). That (i) of that theorem implies (ii) is clear. For the converse, assume that Γ does not imply $\boldsymbol{A}$ in our logic, for some particular Γ and $\boldsymbol{A}$. Then, by the wellknown Lindenbaum's Lemma, there is some maximal consistent set Γ^* such that $\Gamma \subseteq \Gamma^*$ and $\boldsymbol{A} \notin \Gamma^*$. Consider the Lindenbaum algebra for Γ^*. By Lemma 4.2, $v_{\Gamma^*}(\boldsymbol{A}) = \mathsf{F}$, and yet, for all $\boldsymbol{B} \in \Gamma$, $v_{\Gamma^*}(\boldsymbol{B}) = \mathsf{T}$. Hence Γ does not algebraically imply $\boldsymbol{A}$.

To obtain also the remaining half of the Completeness Theorem, we go from Lindenbaum algebras to Stone spaces in the usual way. Thus, with Σ as before, let U_Σ be the set of ultrafilters of $\mathfrak{A}_\Sigma$. By the *Stone map* let us understand the map

$$\phi_\Sigma\colon\ a/\Sigma \mapsto \{\nabla \in U_\Sigma\colon\ a/\Sigma \in \nabla\}.$$

The set $S_\Sigma = \{\phi_\Sigma(a/\Sigma)\colon\ a \in \Theta\}$ we call the *Stone field*. The Stone field has two properties which should be noted: (i) it is closed under set complement, and (ii) it is closed under finite intersection. It follows from (ii) that if T_Σ is the set of all subsets of U_Σ that can be written as the union of members of S_Σ, then T_Σ is a topology (of which S_Σ is a base). The topological space $\mathfrak{T}_\Sigma = \langle U_\Sigma, T_\Sigma\rangle$ is called the *Stone space*. Note that by (i), every subset of U_Σ of the form $\phi(a/\Sigma)$ is *clopen* in this topology (that is, closed as well as open). Moreover, from M. H. Stone's well known work we know that ϕ_Σ is an isomorphism from the Boolean algebra $\langle\Theta/\Sigma, \frown, \smile, -, 0, 1\rangle$ to the field of sets $\langle S_\Sigma, \cap, \cup, -, \varnothing, U\rangle$, and that $\mathfrak{T}_\Sigma$ is *compact* (that is, if $\{A_i\colon\ i \in I\}$ is a collection of open sets $A_i \subseteq U$, for any index set I, such that $\bigcup\{A_i\colon\ i \in I\} = U_\Sigma$, then there is a finite set $J \subseteq I$ such that $\bigcup\{A_i\colon\ i \in J\} = U_\Sigma$). Let the *Stone frame* (for Σ) be defined as the structure $\mathfrak{S}_\Sigma = \langle U_\Sigma, Ill_\Sigma, Leg_\Sigma\rangle$, where

$$Ill_\Sigma = \{\nabla \in U_\Sigma\colon\ \exists a(a/\Sigma \in \nabla\ \&\ Forb\, a \in \Sigma)\},$$
$$Leg_\Sigma = \{\nabla \in U_\Sigma\colon\ \exists a(a/\Sigma \in \nabla\ \&\ Perm\, a \in \Sigma)\}.$$

It is noteworthy that both Ill_Σ and Leg_Σ are open sets:

LEMMA 4.3. $Ill_\Sigma = \bigcup\{\phi_\Sigma(a/\Sigma)\colon\ a/\Sigma \in F_\Sigma\}$.

PROOF. For any ultrafilter ∇ of $\mathfrak{A}_\Sigma$,

$$\begin{aligned}
&\nabla \in Ill_\Sigma\\
&\quad \text{iff}\quad \exists a(a/\Sigma \in \nabla\ \&\ Forb\, a \in \Sigma)\\
&\quad \text{iff}\quad \exists a(\nabla \in \phi_\Sigma(a/\Sigma)\ \&\ a/\Sigma \in F_\Sigma)\\
&\quad \text{iff}\quad \nabla \in \bigcup\{\phi_\Sigma(a/\Sigma)\colon\ a/\Sigma \in F_\Sigma\}. \blacksquare
\end{aligned}$$

LEMMA 4.4. $Leg_\Sigma = \bigcup\{\phi_\Sigma(a/\Sigma)\colon\ a/\Sigma \in P_\Sigma\}$.

PROOF. Similarly. ■

LEMMA 4.5. *$\mathfrak{S}_\Sigma$ is a deontic action frame.*

PROOF. Suppose that there is some ∇ such that $\nabla \in Ill_\Sigma \cap Leg_\Sigma$. Then there are terms α and β such that α/Σ, $\beta/\Sigma \in \nabla$ and $Forb\, \alpha$, $Perm\ \beta \in \Sigma$. Consequently, since ∇ is a filter, $\alpha/\Sigma \frown \beta/\Sigma \in \nabla$, and so

(1) $\alpha \frown \beta/\Sigma \in \nabla$.

Moreover, $\alpha/\Sigma \in F_\Sigma$ and $\beta/\Sigma \in P_\Sigma$. Therefore, since F_Σ and P_Σ are ideals, $\alpha/\Sigma \frown \beta/\Sigma \in F_\Sigma \cap P_\Sigma$, and so $\alpha \frown \beta/\Sigma \in F_\Sigma \cap P_\Sigma$. But $\mathfrak{A}_\Sigma$ is a deontic action algebra, so this implies that

(2) $\alpha \frown \beta/\Sigma = 0_\Sigma$.

Since ∇ is an ultrafilter, (1) and (2) are in contradiction. ∎

We now define a valuation V_Σ for $\mathfrak{S}_\Sigma$ by requiring that, for every atomic term π,

$$V_\Sigma\colon\ \pi \mapsto \phi_\Sigma(\pi/\Sigma).$$

The model $\mathfrak{M}_\Sigma = \langle U_\Sigma, Ill_\Sigma, Leg_\Sigma, V_\Sigma\rangle$ we call the *canonical model* (*for* Σ).

LEMMA 4.6. *For all terms* α *and all formulas* A,

(i) $V_\Sigma(\alpha) = \phi_\Sigma(\alpha/\Sigma)$.

(ii) $V_\Sigma(A) = v_\Sigma(A)$.

PROOF. That (i) holds is immediate. The only non-trivial part of the proof of (ii) occurs in the basic step of the induction over the complexity of A, viz., when A is of the form $Forb\,\alpha$ or $Perm\,\alpha$. However, this case follows from the observation that, for all α,

$$\phi_\Sigma(\alpha/\Sigma) \subseteq Ill_\Sigma \quad \text{if and only if} \quad \alpha/\Sigma \in F_\Sigma,$$
$$\phi_\Sigma(\alpha/\Sigma) \subseteq Leg_\Sigma \quad \text{if and only if} \quad \alpha/\Sigma \in P_\Sigma.$$

We prove the former assertion, beginning with the argument from left to right. Assume that $\phi_\Sigma(\alpha/\Sigma) \subseteq Ill_\Sigma$. Evidently, $\big(U_\Sigma - \phi_\Sigma(\alpha/\Sigma)\big) \cup Ill_\Sigma = U_\Sigma$. By Lemma 4.3, then,

(1) $\big(U_\Sigma - \phi_\Sigma(\alpha/\Sigma)\big) \cup \bigcup\{\phi_\Sigma(\beta/\Sigma) : \beta/\Sigma \in F_\Sigma\} = U_\Sigma$.

As was noted above, any set of type $\phi_\Sigma(\gamma/\Sigma)$ is clopen in the Stone space. Thus the sets appearing in (1) are all open. Therefore, by compactness,

(2) $\big(U_\Sigma - \phi_\Sigma(\alpha/\Sigma)\big) \cup \phi_\Sigma(\beta_0/\Sigma) \cup \ldots \cup \phi_\Sigma(\beta_{n-1}/\Sigma) = U_\Sigma$,

for some $\beta_0, \ldots, \beta_{n-1}$ such that

(3) $\beta_0/\Sigma, \ldots, \beta_{n-1}/\Sigma \in F_\Sigma$.

By (2),

(4) $\phi_\Sigma(\alpha/\Sigma) \subseteq \phi_\Sigma(\beta_0/\Sigma) \cup \ldots \cup \phi_\Sigma(\beta_{n-1}/\Sigma)$.

Let ∇ be any ultrafilter such that $\alpha/\Sigma \in \nabla$. Then $\nabla \in \phi_\Sigma(\alpha/\Sigma)$, and so by (4) there is some $i < n$ such that $\nabla \in \phi_\Sigma(\beta_i/\Sigma)$, whence $\beta_i/\Sigma \in \nabla$. The filter property of ∇ then gives us $\beta_0/\Sigma \cup \ldots \cup \beta_{n-1}/\Sigma \in \nabla$. Thus in view of the Ultrafilter Theorem our argument has established that

(5) $\alpha/\Sigma \leqslant \beta_0/\Sigma \cup \ldots \cup \beta_{n-1}/\Sigma$.

From (3), (5), and the fact that F_Σ is an ideal we infer that $a/\Sigma \in F_\Sigma$, which is the desired conclusion.

For the converse, suppose that $a/\Sigma \in F_\Sigma$. Then

(6) $\quad Forb\, a \in \Sigma.$

Take any $\nabla \in \phi_\Sigma(a/\Sigma)$. Then

(7) $\quad a/\Sigma \in \nabla.$

From (6) and (7) it follows that $\nabla \in Ill_\Sigma$. Consequently, $\phi_\Sigma(a/\Sigma) \subseteq Ill_\Sigma$. ∎

The remaining half of the Completeness Theorem now follows from the fact that conditions (i) and (ii) have already been shown to be equivalent, for Lemma 4.6 shows that $\langle \mathfrak{A}_\Sigma, v_\Sigma \rangle$ and $\mathfrak{M}_\Sigma$ verify and falsify exactly the same formulas.

5. The basic closed deontic logic of urn model action

In normative theory, especially jurisprudence, there is a condition called the *Closure Principle* which is often aired — loosely speaking, the condition that what is not forbidden is permitted. A normative system is *closed* if it satisfies this condition, *open* otherwise. Let us see how this principle fares in the present context.

Let us begin with the observation that, in a certain weak sense, B.O.D. satisfies the Closure Principle. The sense intended is seen from the following result. Let us say that a deontic action frame $\langle U, Ill, Leg \rangle$, as well as any model based on it, is *closed* if $Ill \cup Leg = U$.

THEOREM 5.1. *Suppose that Γ is a set of formulas and A is a formula. Then the following conditions are equivalent:*

(i) *Γ implies A in B.O.D.*

(ii) *Γ implies A in every closed model.*

In other words: we might consider augmenting our formal semantics by adding a condition on frames to the effect that every outcome be either illegal or legal. But whether we do or not will have no effect on our logic.

This may seem surprising. It follows, however, from the following fact:

LEMMA 5.2. *Let $\mathfrak{M} = \langle U, Ill, Leg, V \rangle$ be any model. Then there is a model $\mathfrak{M}^* = \langle U^*, Ill^*, Leg^*, V^* \rangle$ such that*

(i) *$Ill^* \cup Leg^* = U^*$,*

(ii) *$\mathfrak{M}$ and $\mathfrak{M}^*$ are equivalent (that is, for all A, $\mathfrak{M} \models A$ iff $\mathfrak{M}^* \models A$).*

PROOF. Let $\mathfrak{M} = \langle U, Ill, Leg, V \rangle$ be any model. Let us write *Non* for the set of elements of U that belong to neither *Ill* nor to *Leg* (and thus are both non-illegal and non-legal). In other words, $Non = (U - Ill) \cap$

$\cap (U - Leg)$. Define $\mathfrak{M}^* = \langle U^*, Ill^*, Leg^*, V^* \rangle$ by

$$\begin{aligned} U^* &= \{\langle u, x\rangle : u \in U \ \&\ x \in \{0, 1\}\}, \\ Ill^* &= \{\langle u, x\rangle \in U^* : u \in Ill \ \&\ x \in \{0, 1\}\} \cup \{\langle u, 0\rangle : u \in Non\}, \\ Leg^* &= \{\langle u, x\rangle \in U^* : u \in Leg \ \&\ x \in \{0, 1\}\} \cup \{\langle u, 1\rangle : u \in Non\}, \\ V^*(\pi) &= \{\langle u, x\rangle \in U^* : u \in V(\pi) \ \&\ x \in \{0, 1\}\}, \text{ for each event letter } \pi. \end{aligned}$$

Note that, for all $a \in \Theta$, the following three conditions are equivalent:

(i) $u \in V(a)$,
(ii) $\langle u, 0\rangle \in V^*(a)$,
(iii) $\langle u, 1\rangle \in V^*(a)$.

This is readily proved by induction on a: the basis is provided by the definition of V^*, and the inductive step is trivial.

It is clear that $\mathfrak{M}^*$ is a deontic action model and that $Ill^* \cup Leg^* = U^*$. But we must show that $\mathfrak{M}$ and $\mathfrak{M}^*$ are equivalent. This we do by induction on A. The only interesting case occurs in the basic step, viz., when A is of the form $Forb\, a$ or $Perm\, a$. These subcases are similar, and so it will be enough to treat just the former here. First assume that $\mathfrak{M} \models Forb\, a$. Then $V(a) \subseteq Ill$. Take any $\langle u, x\rangle \in V^*(a)$. Then $u \in V(a)$, hence $u \in Ill$, and so $\langle u, x\rangle \in Ill^*$. This argument shows that $V^*(a) \subseteq Ill^*$. Therefore, $\mathfrak{M}^* \models Forb\, a$. Next assume that $\mathfrak{M} \not\models Forb\, a$. Then there exists some $u \in V(a)$ such that $u \notin Ill$. Consequently, $\langle u, 1\rangle \in V^*(a)$, and, by the definition of Ill^*, $\langle u, 1\rangle \notin Ill^*$. Therefore, $\mathfrak{M}^* \not\models Forb\, a$. ∎

What all this shows is that we are justified in maintaining that the Closure Principle is satisfied at the level of outcomes — it makes no difference to the present logic to think that every possible outcome is either positively illegal or positively legal.

At the level of events the situation is different. If one would try to express the Closure Principle — "everything is either forbidden or permitted" — in formulas, then the first schema that comes to mind is probably

(C0) $\quad Forb\, a \vee Perm\, a.$

However, it is certainly not the case that all instances of this schema are valid. In particular, the instance $Forb\, \mathbf{1} \vee Perm\, \mathbf{1}$ is verified only by models $\langle U, Ill, Leg, V\rangle$ such that either $U = Ill$ or $U = Leg$, and in such models there is no need for deontic logic. Thus (C0) is not an acceptable schema.

A weaker and therefore more plausible candidate is this:

(C1) $\quad a \neq \mathbf{1} \rightarrow (Forb\, a \vee Perm\, a).$

Indeed, one may even consider strengthening (C1) into an equivalence:

(C1′) $\quad a \neq \mathbf{1} \leftrightarrow (Forb\, a \vee Perm\, a).$

The parallel between (C1′) and (D3) is obvious, and the resulting axiom system would be nicely symmetric. However, also this effort is mistaken, as the following algebraic argument shows. Let $\mathfrak{A} = \langle A, \cap, \cup, -, 0, 1, F, P\rangle$ be any deontic action algebra validating (C1). Then it readily follows that

$$(*) \qquad A - \{1\} \subseteq F \cup P.$$

Suppose now that F is a proper ideal and that there are elements $a, b \in F$, both distinct from 0. Since F is proper, $\bar{b} \notin F$, and $b \neq 0$ implies $\bar{b} \neq 1$; therefore, by (*), $\bar{b} \in P$. Furthermore, $a \cap \bar{b} \in F$ (since $a \in F$) and $a \cap \bar{b} \in P$ (since $\bar{b} \in P$), and so $a \cap \bar{b} \in F \cap P$. But $\mathfrak{A}$ is a deontic action algebra, so evidently $a \cap \bar{b} = 0$. Consequently $a \leqslant b$. By a similar argument, $b \leqslant a$. Therefore $a = b$.

This argument shows that, under the adopted hypothesis, F is either improper or contains at most one element in addition to the null element. By the same token, P too is either improper or contains at most one element other than the null element. This obviously makes it impossible to accept (C1) or (C1′).

If the Closure Principle is to be rendered in the object language, then the following schema may be the best candidate:

$$\text{(C2)} \qquad Forb\,\pi \vee Perm\,\pi,$$

π ranging over the set of event letters. We shall now state a theorem which offers some semantic support for this view and why the system that results when (C2) is added to the axiom system of Section 3 deserves to be called *the basic closed deontic logic of urn model action* (B.C.D.). In order to arrive at a semantics which seems to articulate the intuitions behind the Closure Principle, we must change the preceding modellings in some ways. Restricting ourselves to model theory, we require that in a model $\langle U, Ill, Leg, V\rangle$

(i) $Ill \cup Leg = U$,
(ii) for every π, $V(\pi) \subseteq Ill$ or $V(\pi) \subseteq Leg$.

This semantics — which is no longer of the usual frame type or second-order type (see Thomason [5]) — is adequate for B.C.D.:

THEOREM 5.3. *Let Γ be any set of formulas and A any formula. Then the following conditions are equivalent:*
(i) *Γ implies A in B.C.D.*
(ii) *Γ model theoretically implies A.*

Given the completeness proof for B.O.D., a proof is easily adapted from the proof of the lemma at the beginning of this section.

Notice that, unlike B.O.D., B.C.D. is not closed under substitution of arbitrary terms for event letters.

6. Other operators

In our logic, as in ordinary deontic logic, it is possible to define other operators in terms of the primitive ones. Thus let us introduce *P-based obligation* and *P-based prohibition*:

$$Obl_P\,\alpha =_{df} \neg Perm\,\bar{\alpha},$$
$$Forb_P\,\alpha =_{df} \neg Perm\,\alpha,$$

and *F-based obligation* and *F-based prohibition*:

$$Obl_F\,\alpha =_{df} Forb\,\bar{\alpha},$$
$$Perm_F\,\alpha =_{df} \neg Forb\,\alpha.$$

If $\mathfrak{M} = \langle U, Ill, Leg, V\rangle$ is a deontic action model, then the semantic conditions of the new operators work out as follows (conditions for closed frames within brackets):

$$\mathfrak{M} \models Obl_P\,\alpha \quad \text{iff} \quad (U - Leg) \nsubseteq V(\alpha) \qquad [\text{iff} \quad Ill \nsubseteq V(\alpha)];$$
$$\mathfrak{M} \models Forb_P\,\alpha \quad \text{iff} \quad (U - Leg) \cap V(\alpha) \neq \varnothing \qquad [\text{iff} \quad Ill \cap V(\alpha) \neq \varnothing];$$
$$\mathfrak{M} \models Obl_F\,\alpha \quad \text{iff} \quad (U - Ill) \subseteq V(\alpha) \qquad [\text{iff} \quad Leg \subseteq V(\alpha)];$$
$$\mathfrak{M} \models Perm_F\,\alpha \quad \text{iff} \quad (U - Ill) \cap V(\alpha) \neq \varnothing \qquad [\text{iff} \quad Leg \cap V(\alpha) \neq \varnothing].$$

It follows that the following schemata are valid in B.O.D.:

$$Obl_F(\alpha \frown \beta) \leftrightarrow (Obl_F\,\alpha \wedge Obl_F\beta),$$
$$Perm_F(\alpha \smile \beta) \leftrightarrow (Perm_F\,\alpha \vee Perm_F\beta).$$

Thus the F-based operators behave in important ways as the well known operators of ordinary deontic logic. This is not the case with the P-based ones, for $Obl_P(\alpha \frown \beta) \leftrightarrow (Obl_P\,\alpha \wedge Obl_P\beta)$ and $Perm(\alpha \smile \beta) \leftrightarrow (Perm\,\alpha \vee Perm\,\beta)$ are both non-valid; in their place we have the valid $Obl_P(\alpha \frown \beta) \leftrightarrow (Obl_P\,\alpha \vee Obl_P\beta)$ and (D2). In other words, *Perm* is a strong permission concept, expressing a kind of free choice permission, and $Perm_F$ is a weak one: α is permitted in the strong sense if and only if *every* α-outcome is legal, while α is permitted in the weak sense if and only if *some* α-outcome is legal.

In fact, B.O.D. has many of the virtues sought by von Wright in [6] without suffering from the drawbacks pointed out in Ruzsa [3]. If it is regarded as a defect that in B.O.D. the impossible is permitted as well as forbidden, it is easy to introduce very similar operators without this feature, for example,

$$Perm_1\,\alpha =_{df} Perm\,\alpha \wedge \alpha \neq \mathbf{0},$$
$$Forb_1\,\alpha =_{df} Forb\,\alpha \wedge \alpha \neq \mathbf{0},$$
$$Perm_2\,\alpha =_{df} Perm\,\alpha \wedge \alpha \neq \mathbf{0} \wedge \alpha \neq \mathbf{1},$$
$$Forb_2\,\alpha =_{df} Forb\,\alpha \wedge \alpha \neq \mathbf{0} \wedge \alpha \neq \mathbf{1}.$$

Of these operators, $Perm_1$ and $Perm_2$, respectively, $Forb_1$ and $Forb_2$ would reduce to one another if the following natural condition is placed on deontic action frames $\langle U, Ill, Leg\rangle$: that Ill and Leg be proper subsets of U. The corresponding logic, call it B.O.D.′, is of course axiomatized by adding to the above axiomatization of B.O.D. the further axioms

(D4) $\neg Perm\,\mathbf{1}$,
(D5) $\neg Forb\,\mathbf{1}$.

An operator closely related to $Perm_F$ is $Perm^*$, with the following truth-conditions in a model $\mathfrak{F} = \langle U, Ill, Leg, V\rangle$:

$$\mathfrak{M} \models Perm^* a \quad \text{iff} \quad Leg \cap V(a) \neq \varnothing .$$

Perhaps it is a defect of B.O.D. that this operator cannot be defined in it:

THEOREM 6.1. *$Perm^*$ is not definable in B.O.D. or B.O.D.′.*

PROOF. Suppose by contradiction that $Perm^*$ is definable in B.O.D.′. This means that we may regard $Perm^*$ as an abbreviatory device and that, for each a, $Perm^* a$ is a well-formed formula satisfying the truth-condition above. Consequently, by Theorem 5.1, $Perm_F a \leftrightarrow Perm^* a$ is valid in every closed frame and therefore provable in B.O.D. and *a fortiori* in B.O.D.′. However, take any deontic action model $\langle U, Ill, Leg, V\rangle$ such that $Ill \cup Leg \neq U$. Let π be any event letter and specify that $V(\pi) = U - Leg$. Now we have a contradiction, for $Perm_F \pi$ is true in the resulting model, since $(U - Ill) \cap V(\pi) \neq \varnothing$, yet $Perm^* \pi$ is false, since $Leg \cap V(\pi) = \varnothing$. ∎

7. An action-game

The several preceding concepts of permission should not be regarded as competing attempts to define of *the* concept of permission but rather as concepts with different fields of application. The action-games introduced in [4] may be used to illustrate this point. Games can be tailored to model any of the operators mentioned in this paper, but we will concentrate on $Perm_1$ and $Perm^*$.

Consider an arrangement where in addition to the Umpire there are two players, Player 1 and Player 2, and where the marbles come in two colours, white and red. The game is played as follows. First Player 1 delineates a set of what is to be called "permitted" sets of marbles. Next Player 2 chooses a set of marbles. As in [4] he may not be able to choose every possible set — some may literally speaking be beyond his reach —but what is new is that he will now restrict his choice to among the permitted sets. Finally the Umpire picks a marble from the set chosen by Player 2, if this set is non-empty; if the set is empty, the Umpire will pick any red marble. At the end of the game, the marble picked by the Umpire is given to Player 1. Player 2 also receives some return which depends in a complicated manner on what set he chooses.

Like the games in [4] this is not a very entertaining game, yet it is worth Player 1's while to think about it. He knows that the white marbles are actually made of silver, while the red ones are just clay, and so he is anxious to do what he can to ensure that a white marble is picked by the Umpire. This is not his only concern, though, for Player 2 is his friend, and whatever preferences Player 2 may have, Player 1 would want for him to have as many permitted sets to choose among as possible (as long as this does not endanger his own silver marble).

What should Player 1 do? Evidently much will depend on the Umpire. We consider two cases. (1) *The Umpire is friendly*. This means that the Umpire understands and sympathizes with the desire of Player 1 to acquire a white marble (or white marbles, if the game is played several times), and so he will pick a white marble whenever he is able to under the rules of the game. In this situation it would obviously be rational for Player 1 to resort to the concept of permission expressed by $Perm^*$ and designate as permitted every set that contains at least one white marble. (2) *The Umpire is not friendly*. In fact, let us suppose that he is hostile and will pick a red marble whenever he is able to do so. In this situation it would not be in Player 1's interest to permit any set containing even one red marble, nor, of course, the empty set. What he should do is to resort to the concept of permission expressed by $Perm_1$ and permit exactly the non-empty sets consisting of only white marbles.

The given action-game is artificial, as they all are, but it clearly brings out the difference between $Perm^*$ and $Perm_1$: when in command use $Perm^*$ if you completely trust those depending on your permission, $Perm_1$ if you completely distrust them.

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