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Diodorean Modality in Minkowski Spacetime

Abstract. The Diodorean interpretation of modality reads the operator $\Box$ as "it is now and always will be the case that". In this paper time is modelled by the four-dimensional Minkowskian geometry that forms the basis of Einstein's special theory of relativity, with "event" y coming *after* event x just in case a signal can be sent from x to y at a speed *at most* that of the speed of light (so that y is in the causal future of x).

It is shown that the modal sentences valid in this structure are precisely the theorems of the well-known logic ***S4.2***, and that this system axiomatises the logics of two and three dimensional spacetimes as well.

Requiring signals to travel slower than light makes no difference to what is valid under the Diodorean interpretation. However if the "is now" part is deleted, so that the temporal ordering becomes irreflexive, then there are sentences that distinguish two and three dimensions, and sentences that can be falsified by approaching the future at the speed of light, but not otherwise.

The Stoic logician Diodorus Chronus described the *necessary* as being that which both *is* and *will always be* the case. This temporal interpretation of modality has been exhaustively investigated by the methods of contemporary formal logic within the context of *linear* temporal orderings (cf. Chapter II of [1] for a survey of this work). The present paper is a contribution to the study of modalities in branching time, and is concerned with the most significant of all non-linear time structures, viz. the four-dimensional Minkowskian spacetime that forms the basis of Einstein's theory of special relativity. Since the temporal ordering of spacetime points is directed (indeed any two have a *least* upper bound) it follows, as observed by Arthur Prior in [1, p. 203], that the associated Diodorean modal logic contains the system *S4.2*. We shall prove that it is in fact precisely *S4.2*, and that this holds also for two and three-dimensional spacetime.

The language of propositional modal logic comprises sentences constructed from sentence letters $p, q, r, \ldots$ by Boolean connectives and the modal $\Box$ ('it will always be'). The connective $\Diamond$ ('it will (at some time) be') is defined as $\sim \Box \sim$.

A *time-frame* is a structure $\mathscr{T} = (T, \leqslant)$ comprising a non-empty set T of times (moments, instants, events) on which $\leqslant$ is a *reflexive* and *transitive* ordering. A frame is *directed* if any two elements of it have an upper bound, i.e.

for all $t, s \in T$ there exists a $v \in T$ with $t \leqslant v$ and $s \leqslant v$.

A $\mathscr{T}$-valuation is a function V assigning to each sentence letter p a set $V(p) \subseteq T$ (the set of times at which p is true). The valuation is then extended to all sentences via the obvious definitions for the Boolean connectives, together with

$$t \in V(\Box A) \quad \text{iff} \quad t \leqslant s \quad \text{implies} \quad s \in V(A).$$

Hence $t \in V(\Diamond A)$ iff for some $s \in V(A)$, $t \leqslant s$.
The reflexivity of $\leqslant$ gives $\Box$ the Diodorean '*is* and always will be' interpretation. A sentence A is *valid* in $\mathscr{T}$, $\mathscr{T} \models A$, iff $V(A) = T$ holds for every $\mathscr{T}$-valuation V.

A function $f\colon T \to T'$ is a *p-morphism* from a frame $\mathscr{T} = (T, \leqslant)$ to a frame $\mathscr{T}' = (T', \leqslant')$ if it satisfies

P1: $t \leqslant s$ implies $f(t) \leqslant' f(s)$

P2: $f(t) \leqslant' v$ implies that there exists some $s \in T$ with $t \leqslant s$ and $f(s) = v$.

We write $\mathscr{T} \twoheadrightarrow \mathscr{T}'$ to mean that there is a p-morphism from $\mathscr{T}$ to $\mathscr{T}'$ that is *surjective* (onto).

p-MORPHISM LEMMA. *If $\mathscr{T} \twoheadrightarrow \mathscr{T}'$, then for any sentence A, $\mathscr{T} \models A$ only if $\mathscr{T}' \models A$.*

If $T' \subseteq T$ is *future-closed* under $\leqslant$, i.e. whenever $t \in T'$ and $t \leqslant s$ we have $s \in T'$, then $\mathscr{T}' = (T', \leqslant)$ is called a *subframe* of $\mathscr{T}$. By the transitivity of $\leqslant$, for each t the set $\{s\colon t \leqslant s\}$ is the base of a subframe, called the subframe of $\mathscr{T}$ *generated by* t. In general an element 0 of T is called an *initial point* of $\mathscr{T}$ if $0 \leqslant s$ holds for all $s \in T$. Thus t is an initial point of the subframe generated by t. A frame with an initial point will be simply called a *generated frame*.

SUBFRAME LEMMA. *If $\mathscr{T}'$ is a subframe of $\mathscr{T}$, then for any sentence A, $\mathscr{T} \models A$ only if $\mathscr{T}' \models A$.*

The logic ***S4.2*** may be axiomatised as follows;

AXIOMS: All instances of tautologies, and the schemata

I $\Box(A \supset B) \supset (\Box A \supset \Box B)$
II $\Box A \supset A$
III $\Box A \supset \Box\Box A$
IV $\Diamond\Box A \supset \Box\Diamond A$.

RULES: Modus Ponens, and
Necessitation: From A derive $\Box A$

Axiom I is valid on all frames, as is the rule of Necessitation, regardless of the properties of $\leqslant$. The validity of II depends on reflexivity of $\leqslant$, III requires transitivity, while IV is valid if $\leqslant$ is directed. Thus $\vdash_{S4.2} A$ implies that A is valid on all directed frames. The following strong version of the converse to this statement may be found in [2].

COMPLETENESS THEOREM. *If* $\not\vdash_{S4.2} A$, *then there is a finite generated and directed frame* $\mathcal{T}$ *with not* $\mathcal{T} \models A$.

We have not required that a frame be *partially ordered*, i.e. that $\leqslant$ be antisymmetric (indeed there is no sentence whose validity requires it). Thus the equivalence relation defined on T by

$$t \approx s \quad \text{iff} \quad t \leqslant s \quad \text{and} \quad s \leqslant t$$

will in general be non-trivial. The $\approx$ -equivalence classes are called the *clusters* of $\mathcal{T}$. They are ordered by putting

$$\hat{t} \leqslant \hat{s} \quad \text{iff} \quad t \leqslant s,$$

(where $\hat{t}$ is the cluster containing t etc.), and *this* is an antisymmetric ordering. Thus we may conveniently visualise a frame as a partially-ordered collection of clusters, with the relation $\leqslant$ being universal within each cluster.

An element ∞ of T is called *final* in $\mathcal{T}$ if $t \leqslant \infty$ holds for all t in T. All such final points are $\approx$-equivalent and so they form a single cluster. Notice that if $\mathcal{T}$ is directed and *finite* then it must have at least one final point. A unique final point can be adjoined to any frame $\mathcal{T}$ by forming the frame $\mathcal{T}^\infty = (T \cup \{\infty\}, \leqslant)$ where ∞ is some object not a member of T, and the ordering is that of $\mathcal{T}$ extended by

$$\{(s, \infty) \colon s \in T \cup \{\infty\}\}.$$

Notice that $\mathcal{T}^\infty$ is always directed, as the final point serves as upper bound for any two elements.

The key to our characterisation of the logic of spacetime is the structure of the infinite binary-branching frame $\mathcal{B} = (B, \leqslant)$. The members of $\mathcal{B}$ are the finite sequences of the form $x = x_1x_2 \ldots x_n$, where each $x_i \in \{0, 1\}$. Such a sequence is of *length* n, denoted $l(x) = n$. We include the case $n = 0$, so that B contains the empty sequence $x = \varnothing$. The ordering is defined by specifying that for sequences $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$ we have

$$\begin{aligned} x \leqslant y \quad &\text{iff} \quad x \text{ is an initial segment of } y \\ &\text{iff} \quad n \leqslant m \quad \text{and} \quad y = x_1x_2 \ldots x_n y_{n+1} \ldots y_m. \end{aligned}$$

Thus $\mathcal{B}$ is partially-ordered, with $\varnothing$ as initial poind. The successors $\{y\colon x \leqslant y\}$ of x in $\mathcal{B}$ are just the sequences that extend x, and so x has exactly two *immediate* successors, viz. $x0$ and $x1$ (cf. Figure 1). We shall also refer to $l(x)$ as the *level* of x in $\mathscr{B}$.

In what follows we shall use the abbreviations

$$1^r \quad \text{for} \quad \underbrace{11\ldots.1}_{r \text{ times}}$$

$$\text{and} \quad 0^r \quad \text{for} \quad \underbrace{00\ldots.0}_{r \text{ times}}, \quad \text{where } r \geqslant 0.$$

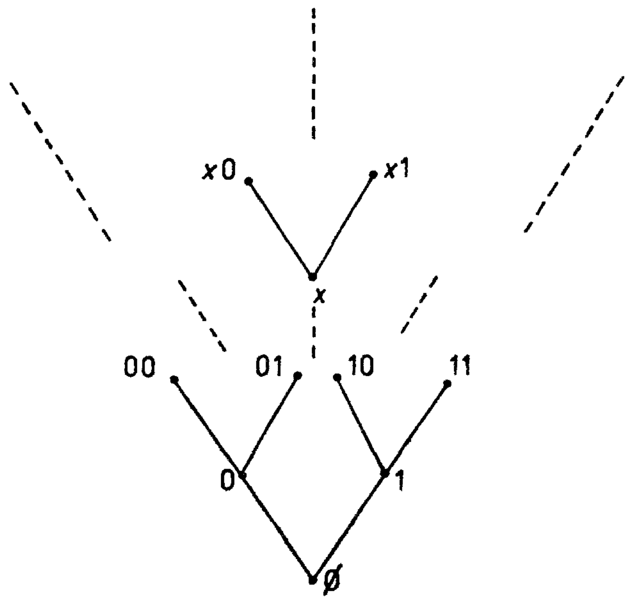

Figure 1

The following result is due originally to Dov Gabbay, and was independently discovered by Johan van Benthem. The construction we use in the proof is that devised by the latter.

THEOREM 1. *If $\mathscr{T}$ is any finite generated frame, then $\mathscr{B} \twoheadrightarrow \mathscr{T}$.*

PROOF: We develop inductively an assignment of members of $\mathscr{T}$ to the 'nodes' of the binary tree $\mathscr{B}$ to obtain the desired p-morphism. *Step One*: Let 0 be an initial point of the generated frame $\mathscr{T}$. Assign 0 to the initial point $\varnothing$ of $\mathscr{B}$.
Inductive Step: Suppose that $x \in B$ has been assigned an element t of T, but that no $\mathscr{B}$-successor of x has received an assignment. Such a point x that is used to initiate an inductive step will be called a *primary node* of $\mathscr{B}$ for the p-morphism being defined.

Now let $t_1, \ldots, t_k$ be all of the $\leqslant$-future points (i.e. $t \leqslant t_i$) of t in $\mathscr{T}$. Take the least j such that $k+1 \leqslant 2^j$. This j is the *bound* of x: $\beta(x) = j$. Notice that $k \geqslant 1$, since at least $t \leqslant t$, and so $j \geqslant 1$.

Suppose $l(x) = n$. Then we assign t to all $\mathscr{B}$-successors of x up to and including level $n+j$ (cf. Figure 2).

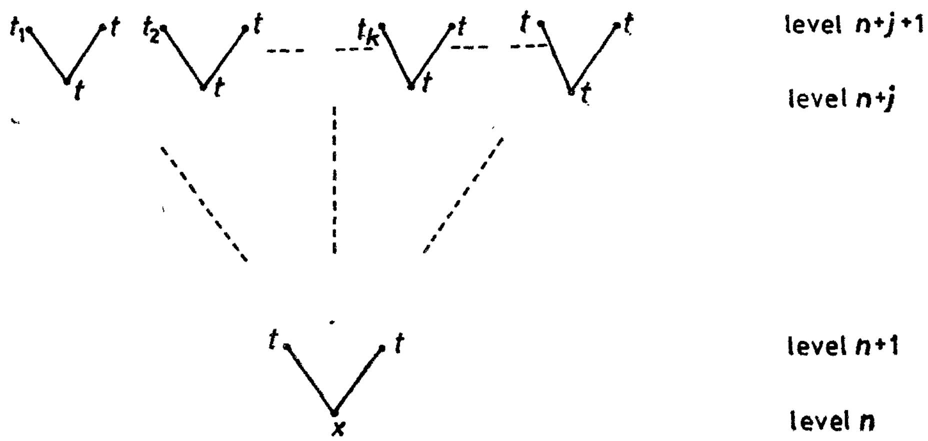

Figure 2

Now let $y_1, \ldots, y_k$ be any k of the 2^j successors of x at level $n+j$. Assign t_1 to one of the immediate successors of y_1, and t to the other. Assign t_2 to one of the immediate successors of y_2, and t to the other. Continue this process up to y_k, thereby giving assignments to $2k$ of the 2^{j+1} successors of x at level $n+j+1$. Let all the other nodes at this level be assigned t (there are such nodes, as $2k < 2^{j+1}$).

The nodes at levels $n+1$ through $n+j$ are designated as *intermediate* nodes for the construction, while the nodes at level $n+j+1$ are new *primary* nodes. The inductive step is then repeated for each of the latter, and so on. Since $j \geqslant 0$, the immediate successors of x at level $n+1$ must receive an assignment (in fact the same one as x). Hence by induction, every member of B gets an assignment, and a function f: $B \to T$ may be defined by letting $f(x)$ be the member of T assigned to x. Since $\{t: 0 \leqslant t\} = T$, every member of T will be assigned at least once already after the first inductive step, and so f is onto. To prove clause P1 of the p-morphism definition, observe that if $x \leqslant y$, and $f(x) = t$ say, then y will be assigned a future point of t in $\mathscr{T}$, hence $f(x) \leqslant f(y)$ (a rigorous argument would proceed by induction on the level of y above x, and use the transitivity of $\leqslant$).

For P2, suppose that $f(x') \leqslant s$, where $f(x') = t$. If x' is primary at level n, such as the x in Figure 2, then there is a point y at level $n+j+1$ that is assigned s, hence $x' \leqslant y$ and $f(y) = s$. If however x' is intermediate, then since all points at level $n+j$ have at least one successor at $n+j+1$ that is assigned t, there will be some such primary node z with $x' \leqslant z$ and $f(z) = t$. Then by the argument of the previous sentence, there will be a y with $z \leqslant y$, and hence $x' \leqslant y$, such that $f(y) = s$. This completes the proof. ■

We note in passing that the modal logic ***S4*** has as basis the axioms for ***S4.2*** without the schema IV. It is well known that any non-theorem for ***S4*** is falsifiable on a finite generated (reflexive and transitive) frame, and hence by Theorem 1 and the p-Morphism Lemma is falsifiable on $\mathscr{B}$. Thus for any sentence A we have

$$\vdash_{S4} A \quad \text{iff} \quad \mathscr{B} \models A,$$

so that $\mathscr{B}$ is a characteristic frame for ***S4***.

It is apparent that the proof of Theorem 1 as given requires only that $k \leqslant 2^j$. The reason for the stronger constraint is that we have to refine the construction to ensure that f satisfies some combinatorial conditions that will allow us to define a p-morphism on spacetime. In the proof of Theorem 1 the chosen nodes $y_1, \ldots, y_k$ at level $n+j$ will be called *special* intermediate points. The other intermediate points are *ordinary*. Then since there are $2^j \geqslant k+1$ points above x at level $n+j$;

(a) f can be defined so that for primary x the intermediate node $x0^j$ is ordinary (where $j = \beta(x)$).

We then give the definition of f in the inductive step related to Figure 2 quite explicitly as follows:

if z is an intermediate point,

(b) let $f(z) = t = f(x)$,

and if z is at level $n+j$, then

(c) if z is ordinary, let $f(z0) = f(z1) = t = f(x)$,

while

(d) if $z = y_i$ is special, let $f(z0) = t_i$ and $f(z1) = t$.

Thus the only case in which an intermediate node has a different assignment to one of its immediate successors is when the node is a special point z, and the successor is the primary point $z0$. Moreover in the case of a primary point x, the successors $x0$ and $x1$ at level $n+1$ are intermediate, as $\beta(x) \geqslant 1$, and so (Figure 2) have the same f-value as x. Altogether then we have that for *any* point z in B,

(e) $f(z) = f(z1)$:

and

(f) if z is not a special intermediate point, then $f(z) = f(z0)$.

From (e) we deduce that

(g) for all $z \in B$ and all r, $f(z) = f(z1^r)$.

Next we consider nodes of the form $x0^r$, for primary x. If $r \leqslant j = \beta(x)$, $x0^r$ is intermediate and so has the same f-value as x by (b). But by (a), $x0^j$ is not special, so by (f), $f(x0^{j+1}) = f(x0^j) = f(x)$. Since $x0^{j+1}$ is primary, the argument may be repeated up to the next level of primary points, and so by induction,

(h) if x is primary, then $f(x) = f(x0^r)$, for all r.

LEMMA 2. *For any* $x \in B$,

(i) *if* x *is special, then* $f(x) = f(x10^r)$, *all* r

and

(ii) *otherwise* $f(x) = f(x01^r)$, *all* r.

PROOF: For (i), if x is special then $f(x) = f(x1)$ by (d), and since $x1$ is primary, $f(x1) = f(x10^r)$ by (h).

If however x is not special, then $f(x) = f(x0)$ by (f), and then $f(x0)$ $f(x01^r)$ by (g). ∎

Our next step is to produce a characteristic frame for **S4.2** by placing an infinite final cluster at the top of $\mathscr{B}$. Let

$$\Omega = \{\infty_0, \infty_1, \ldots, \infty_n, \ldots\}$$

be an infinite set of objects disjoint from B. Define a frame

$$\mathscr{B}^\Omega = (B \cup \Omega, \leqslant)$$

by taking the ordering $\leqslant$ to be that of $\mathscr{B}$ extended by

$$\{(s, \infty_n)\colon s \in B \cup \Omega \text{ and } n \in N\}$$

where $N = \{0, 1, 2, \ldots\}$. Then $\mathscr{B}^\Omega$ has Ω as its set of final points, with $\infty_n \approx \infty_m$ for all n and m.

THEOREM 3. *If* $\mathscr{T}$ *is finite directed and generated, then* $\mathscr{B}^\Omega \twoheadrightarrow \mathscr{T}$.

PROOF: By Theorem 1 there is a p-morphism $f\colon \mathscr{B} \twoheadrightarrow \mathscr{T}$. We lift this map to $B \cup \Omega$. Since $\mathscr{T}$ is directed it has final points, and these form a (non-empty) cluster, C say. We extend f by mapping Ω *onto* C in any surjective manner. Since the relevant frame orderings are universal within C and Ω, and each of these clusters consists of final points, it is readily seen that such an extension of f yields the desired surjective p-morphism. ∎

Applying the Completeness Theorem given above for **S4.2** to Theorem 3, we deduce

COROLLARY 4. *For any sentence* A,

$$\vdash_{S4.2} A \quad \textit{iff} \quad \mathscr{B}^\Omega \models A.$$

We turn now to the structure of spacetime itself. If $x = (x_1, \ldots, x_n)$ is an n-tuple of real numbers, let

$$\mu(x) = x_1^2 + x_2^2 + \ldots + x_{n-1}^2 - x_n^2.$$

Then by *n-dimensional spacetime*, for $n \geqslant 2$, we mean the frame

$$\boldsymbol{T}^n = (\boldsymbol{R}^n, \leqslant)$$

where $\boldsymbol{R}^n$ is the set of all real n-tuples, and for x and y in $\boldsymbol{R}^n$ we have

$$\begin{aligned} x \leqslant y \quad &\text{iff} \quad \mu(y-x) \leqslant 0 \quad \text{and } x_n \leqslant y_n \\ &\text{iff} \quad \sum_{i=1}^{n-1} (y_i - x_i)^2 \leqslant (y_n - x_n)^2 \quad \text{and} \quad x_n \leqslant y_n. \end{aligned}$$

Then $\boldsymbol{T}^n$ is a partially-ordered frame, which is directed. As an upper bound of x and y we have, for example, $z = (x_1, \ldots, x_{n-1}, z_n)$, where

$$z^n = \sum_{i=1}^{n-1} (x_i - y_i)^2 + |x_n| + |y_n|.$$

THEOREM 5. $\boldsymbol{T}^{n+1} \twoheadrightarrow \boldsymbol{T}^n$.

PROOF: Let $f\colon (x_1, \ldots, x_{n+1}) \mapsto (x_2, \ldots, x_{n+1})$ be the (surjective) projection map. Then if $x \leqslant y$ in $\boldsymbol{T}^{n+1}$, we have

$$\sum_{i=1}^{n} (y_i - x_i)^2 \leqslant (y_{n+1} - x_{n+1})^2 \quad \text{and} \quad x^{n+1} \leqslant y^{n+1}.$$

But then as $(y_1 - x_1)^2 \geqslant 0$,

$$\sum_{i=2}^{n} (y_i - x_i)^2 \leqslant \sum_{i=1}^{n} (y_i - x_i)^2 \leqslant (y_{n+1} - x_{n+1})^2$$

and so $f(x) \leqslant f(y)$ in $\boldsymbol{T}^n$, establishing P1 for f.

For P2, if $f(x) \leqslant y$ in $\boldsymbol{T}^n$, where $x = (x_1, \ldots, x_{n+1})$ and $y = (y_2, \ldots, y_{n+1})$, let $z = (x_1, y_2, \ldots, y_{n+1}) \in \boldsymbol{R}_{n+1}$. Then $z_1 - x_1 = 0$, so

$$\begin{aligned} \sum_{i=1}^{n} (z_i - x_i)^2 &= \sum_{i=2}^{n} (z_i - x_i)^2 \\ &= \sum_{i=2}^{n} (y_i - x_i)^2 \leqslant (y_{n+1} - x_{n+1}) = (z_{n+1} - x_{n+1}). \end{aligned}$$

Thus $x \leqslant z$, and by definition $f(z) = y$. Therefore f is the desired p-morphism. ∎

Minkowski spacetime is $\boldsymbol{T}^4$. The intended interpretation of $x \leqslant y$ is that a signal can be sent from 'event' x to 'event' y at a speed at most that of the speed of light, and so y is in the 'causal' future of x (assuming a choice of coordinates that gives the speed of light as one unit of distance per unit of time).

The frame T^2 is depicted in Figure 3. For each point $t = (x, y)$ in the plane the future consists of all points on or above the upwardly directed rays of slopes $+1$ and -1 emanating from t.

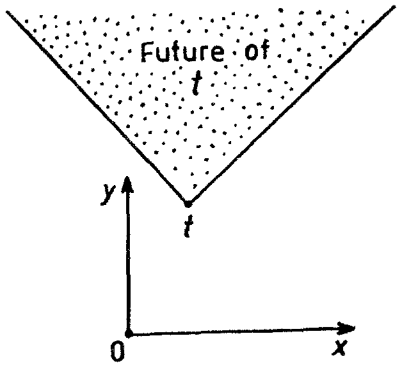

Figure 3

By performing the isometry of rotating the plane clockwise through 45^0 about the origin 0, the picture becomes that of Figure 4,

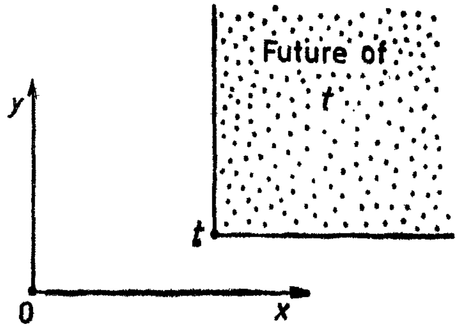

Figure 4

in which the future points of t are precisely those above and to the right of t. The rotation is a bijective p-morphism (isomorphism of frames) and so from now on we will identify T^2 with the structure of Figure 4. This is done largely to make the costructions to follow more tractable, but notice that it reveals T^2 as the direct product of the real linear frame $(\boldsymbol{R}, \leqslant)$ with itself, as we now have

$$(*) \qquad (x_1, y_1) \leqslant (x_2, y_2) \quad \text{iff} \quad x_1 \leqslant x_2 \quad \text{and} \quad y_1 \leqslant y_2.$$

Now let $T_0^2 = \{t\colon\ 0 \leqslant t\}$ be the 'first quadrant' of the plane, consisting of all points with non-negative coordinates. A *future-open box* in T_0^2 is a subset of the form $[a, b) \times [c, d)$ (cf. Figure 5)

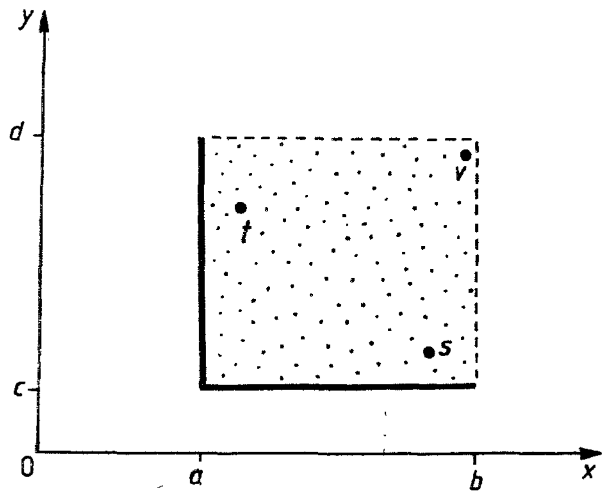

Figure 5

Notice that any two members t, s of a future-open box have an upper bound v within the box, and that v may be chosen to lie on the diagonal line joining (a, c) to (b, d).

THEOREM 6. *Any future-open box is temporally isomorphic to* $\boldsymbol{T}_0^2$.

PROOF: It is a fact of classical analysis that there is a bijection $f\colon [a, b) \mapsto [0, \infty) = \{e\colon 0 \leqslant e\}$ that preserves order, i.e. has $x \leqslant y$ iff $f(x) \leqslant f(y)$. Figure 6 displays one method of geometrically constructing f.

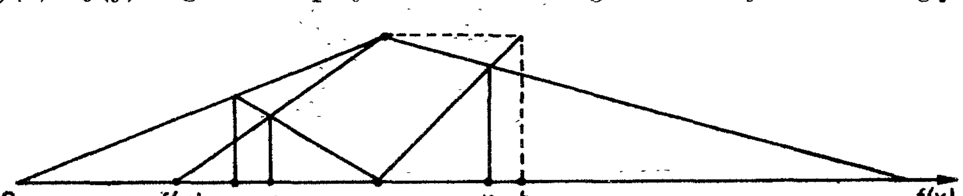

Figure 6

Likewise, there is an order-isomorphism $g\colon [c, d) \to [0, \infty)$. Then the map $(x, y) \mapsto (f(x), g(y))$ gives a bijection between $[a, b) \times [c, d)$ and $\boldsymbol{T}_0^2$ that preserves the temporal ordering defined on each by $(*)$. ∎

COROLLARY 7. *Any two future-open boxes are temporally isomorphic.*

From now on we focus on the structure of the *unit box* $I = [0, 1) \times \times [0, 1)$.

THEOREM 8. $I \twoheadrightarrow \Omega$

PROOF: Here Ω is considered as a frame in its own right, consisting of an infinite set of points all related to each other by $\leqslant$. The idea of the proof is that each ∞_n is made to correspond to a subset A_n of I that is *cofinal* with I, i.e.

$$\text{for each } t \in I \text{ there is some } s \in A_n \text{ with } t \leqslant s.$$

We can do this by making rational cofinal assignments up the diagonal of I to $\infty_1, \infty_2, \ldots$, and mapping everything else to ∞_0 (Figure 7).

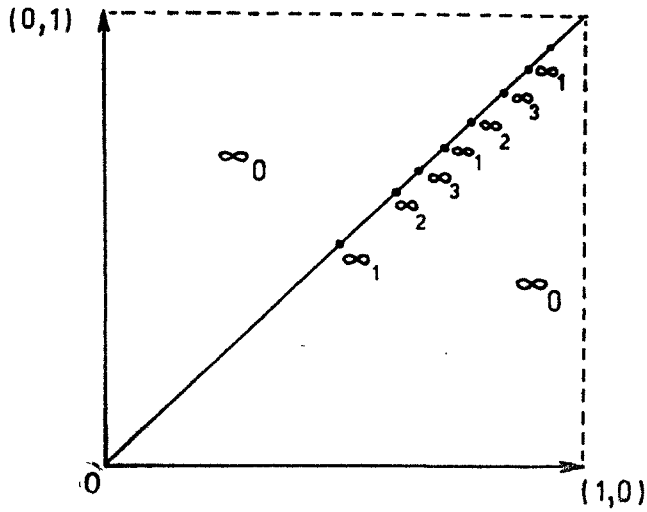

Figure 7

Thus we map $(\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4}), \ldots$ to ∞_1; $(\frac{2}{3}, \frac{2}{3}), (\frac{8}{9}, \frac{8}{9}), \ldots$ to ∞_2; $(\frac{4}{5}, \frac{4}{5}), (\frac{24}{25}, \frac{24}{25}), \ldots$ to ∞_3; and so on. Formally, let $\pi_1, \pi_2, \ldots, \pi_n, \ldots$ be a listing without repetition of the prime numbers in order of increasing magnitude, starting with $\pi_1 = 2$. Then if $(x, y) \in I$,

(i) if $x = y = 1 - \dfrac{1}{(\pi_n)^k}$ for some $k \geqslant 1$, put $f(x, y) = \infty_n$

and

(ii) otherwise put $f(x, y) = \infty_0$.

That Pl holds for f is immediate, as Ω is a cluster. But the cofinality of the ∞_n-assignments along the diagonal, together with the fact that each point t in I has $\leqslant$-successors on the diagonal, gaurantees that $A_n = f^{-1}(\infty_n)$ is cofinal with I. This cofinality ensures that f satisfies P2. ■

THEOREM 9. *If $\mathscr{T}$ is finite, generated, and directed, then $I \twoheadrightarrow \mathscr{T}$.*

PROOF: By Theorem 3 there exists a p-morphism $f\colon \mathscr{B}^{\Omega} \twoheadrightarrow \mathscr{T}$. We define a map $g\colon I \to B \cup \Omega$ which will compose with f to give the desired result. This is done by assigning each point in $B \cup \{\infty\}$ a future-open box contained in I, through a series of *temporary* and then *permanent* labellings. *Step One*: Temporarily assign the initial point $\emptyset$ of $\mathscr{B}$ to I.

Inductive Step: Suppose $x \in B$ has been temporarily assigned a box within I. Divide this box into four equal future-open boxes (Figure 8). *Permanently* assign the lower left-hand box to x and the upper right-hand one to ∞. *Temporarily* assign the upper left-hand box to $x0$, and the lower right-hand one to $x1$.

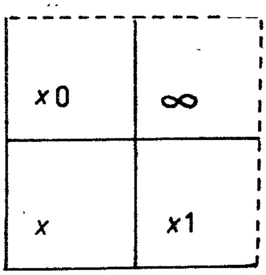

Figure 8

When all members of B have inductively received *permanent* assignments, the picture is as in Figure 9.

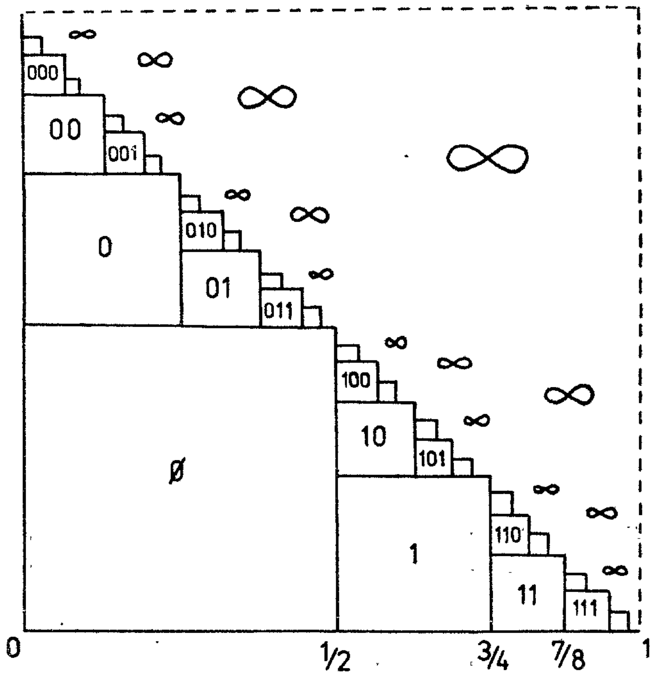

Figure 9

It is apparent that

(**) if $z \leqslant y$ in $\mathscr{B}$, then the box permanently assigned y lies inside the one temporarily assigned z.

LEMMA 10. *If $t \in I$ belongs to the box assigned $x \in B$, then there is some $z \in B$ with $f(x) = f(z)$, and such that the box assigned z lies entirely inside the I-future of t.*

PROOF: As indicated in Figure 10,

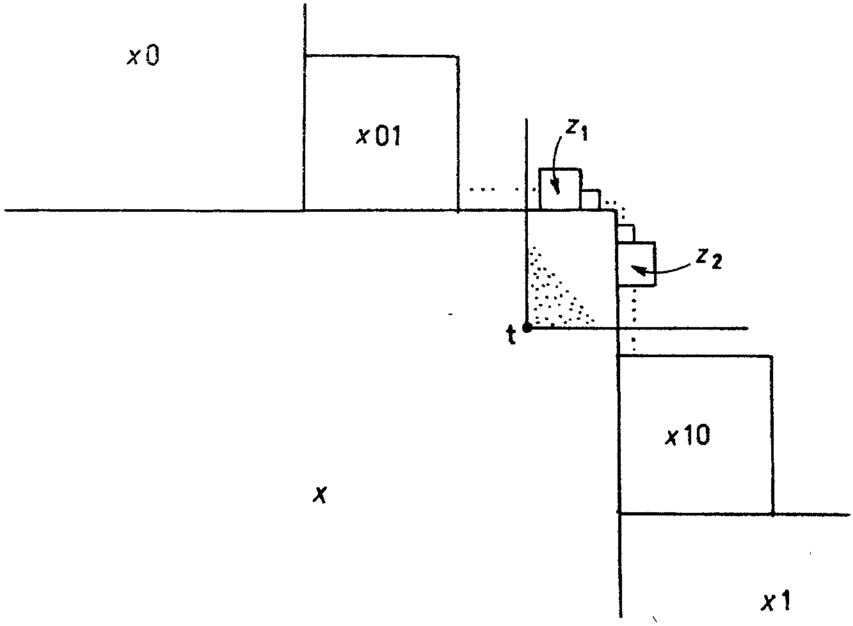

Figure 10

by taking r large enough we can ensure that the boxes assigned $z_1 = x01^r$ and $z_2 = x10^r$ both lie inside the future of t. Then by Lemma 2, if x is a special point for the construction of f as in Theorem 1, we may take $z = z_2$ to fulfill Lemma 10, while if x is not special, $z = z_1$ meets our requirements.

To continue with the proof of Theorem 9, we define a map $g\colon I \to B \cup \Omega$ as follows:

(i) the members of the future-open box permanently assigned $x \in B$ in Figure 9 are all mapped to x by g.
(ii) each box assigned ∞ in Figure 9 is mapped p-morphically onto Ω by g. This is done by the method of Theorem 8, noting Corollary 7.

Next a surjective map $h\colon I \to \mathscr{T}$ is defined by putting $h(t) = f(g(t))$, for all $t \in I$. To show that h satisfies P1, suppose $t \leqslant s$ in I. Then if $h(s)$ is final in $\mathscr{T}$, immediately $h(t) \leqslant h(s)$. If $h(s) = f(g(s))$ is not final, then (cf. proof of Theorem 3) $g(s) \notin \Omega$ and so $g(s) \in B$. But since $t \leqslant s$, the permanent B-assignment to s will be a sequence extending the one assigned to t (Figure 8), i.e. $g(t) \leqslant g(s)$. But then $f(g(t)) \leqslant f(g(s))$, as f satisfies P1.

For P2, suppose $h(t) = f(g(t)) \leqslant v$ in $\mathscr{T}$. If $g(t)$ is a member of B, then by Lemma 10 there exists some $z \in B$ that is assigned a box entirely inside the future of t and that has $f(z) = f(g(t)) \leqslant v$. Since f satisfies P2, there is some $y \in B$ with $z \leqslant y$ and $f(y) = v$. But then (cf. (**)) the box assigned y also lies inside the future of t, and so if we choose an element s from this box, so that $g(s) = y$, we have $t \leqslant s$ and $h(s) = f(y) = v$. On the other hand, if $g(t) \in \Omega$, then $h(t)$ is final in $\mathscr{T}$, and therefore so is v, hence $v = f(\infty_n)$ for some n. Let s' be some point in the region assigned ∞, i.e. $g(s') \in \Omega$, that has $t \leqslant s'$. Then using part (ii) of the definition of g we obtain some s with $s' \leqslant s$ and $g(s) = \infty_n$. Then $t \leqslant s$ and $h(s) = f(\infty_n) = v$ as required.

This completes the proof of Theorem 9. ∎

THEOREM 11. *For any sentence* A,

$$\vdash_{S4.2} A \quad \text{iff} \quad \boldsymbol{T}^n \models A \quad \text{iff} \quad I \models A$$

PROOF: If $\vdash_{S4.2} A$, then A is valid on all directed frames and thus in particular on $\boldsymbol{T}^n$. But if $\boldsymbol{T}^n \models A$, application $(n-2$ times$)$ of the p-Morphism Lemma to Theorem 5 gives $\boldsymbol{T}^2 \models A$. The Subframe Lemma then gives $\boldsymbol{T}_0^2 \models A$, which in turn by Theorem 6 yields $I \models A$. To complete the cycle of implications, observe by Theorem 9 that if $I \models A$ then A is valid on all finite generated and directed frames, and so by the Completeness Theorem given earlier, A is a theorem of ***S4.2***. ∎

Slower-than-light Signals

In $\boldsymbol{T}^n$, define

$$x \prec y \text{ iff } \mu(y-x) < 0 \text{ and } x_n < y_n.$$

Then $x \prec y$ holds just in case a signal can be sent from x to y at less than the speed of light. The reflexive relation

$$xRy \quad \text{iff} \quad x = y \text{ or } x \prec y$$

yields the same logic as before — we leave it to the reader to analyse the above proof to verify that the valid sentences on $(\boldsymbol{T}^n, R)$ are precisely the ***S4.2*** theorems.

The End of Time

Amongst the possible future fates of our universe is that expansion will eventually give way to contraction and collapse to a singularity. In this event, any future-oriented path in spacetime will come to an end (the singularity). Formally, this corresponds to the frame condition

$$(\dagger) \qquad \forall x \exists y (x \leqslant y \;\&\; \forall w (y \leqslant w \supset y = w)).$$

In a directed partially-ordered frame there can be only one y as in (†), namely a unique final point, for if y has no successors then an upper bound for y and any other point can only be y itself.

The logic ***K2*** extends the system ***S4.2*** by the additional axiom schema

$$\Box\Diamond A \supset \Diamond\Box A,$$

which is valid on frames satisfying (†). Conversely the work of Segerberg [3] may be used to show:

If A is not a ***K2***-theorem, then A can be falsified on a finite generated directed frame whose final cluster has only one member.

Thus ***K2*** is characterised by the finite generated directed frames with a *unique* final point. Any such frame $\mathscr{T}$ is a p-morphic image of I^∞, as may be deduced from $I \twoheadrightarrow \mathscr{T}$. Indeed any p-morphism $\mathscr{T}_1 \twoheadrightarrow \mathscr{T}$ can be lifted to $\mathscr{T}_1^\infty \twoheadrightarrow \mathscr{T}$ by mapping ∞ to the unique final point of $\mathscr{T}$. We leave it to the reader to use that observation to verify, for any sentence A that

$$\vdash_{K2} A \quad \text{iff} \quad \boldsymbol{T}^{n^\infty} \models A \quad \text{iff} \quad I^\infty \models A \quad \text{iff} \quad \mathscr{B}^\infty \models A.$$

Irreflexive Time

Tense logic, as a branch of modal logic, is generally taken to be concerned with *irreflexive* orderings, so that a point is not considered to be in its own future. In spacetime there are two natural *strict* orderings, viz. the relation

$$x \prec y \quad \text{iff} \quad \mu(y-x) < 0 \quad \text{and} \quad x_n < y_n$$

defined earlier, and

$$x\alpha y \quad \text{iff} \quad x \neq y \quad \text{and} \quad x \leqslant y.$$

(α is the relation 'after' axiomatised by Robb in [4]).

The logic of these two orderings can be distinguished in terms of the validity of modal sentences. There may be two propositions A and B that are true in the future at two points that can only be reached by travelling (in opposite directions) at the speed of light (cf. Figure 11).

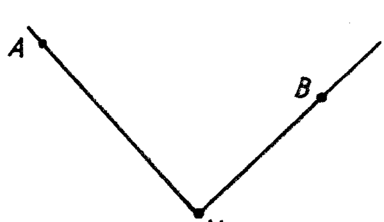

Figure 11

In this situation, $\Diamond A \vee \Diamond B$ will be true now, but never again, and hence the sentence

$$\Diamond A \wedge \Diamond B \supset \Diamond(\Diamond A \wedge \Diamond B)$$

is not valid when α is the temporal ordering. It is however valid under $\prec$, since a slower-than-light journey can always be made to go faster, so we could wait some time and then travel at a greater speed to A and B Figure 12).

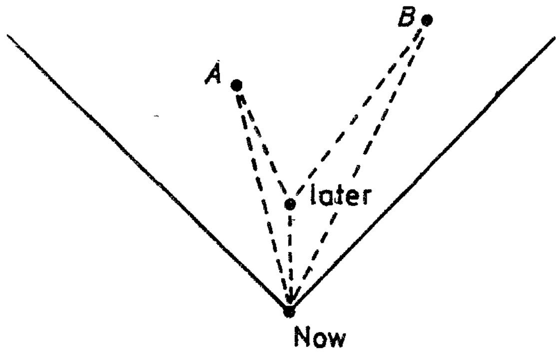

Figure 12

These observations apply to T^n for all $n \geqslant 2$. However by pushing the idea a little further we can produce a sentence whose truth is dimension--dependent. For, in *three*-dimensional spacetime we can find at least three points that can only be reached by travelling in different directions at the speed of light. In T^3, the future of t is represented by the upper half of a right circular cone centered on t (Figure 13).

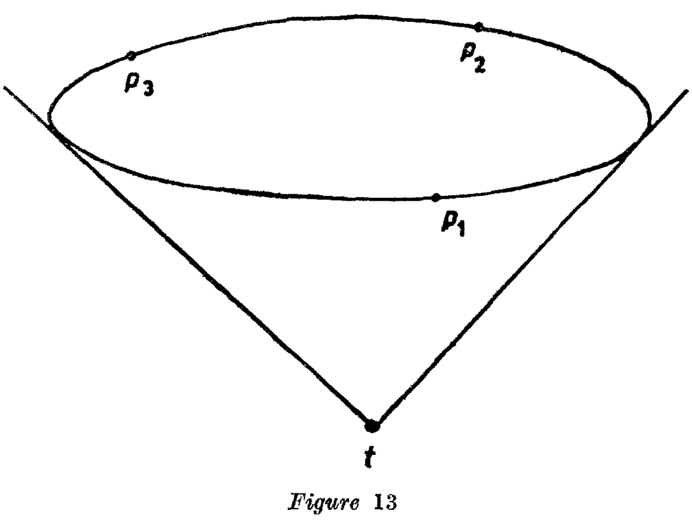

Figure 13

Thus in (T^3, α), and indeed in (T^n, α) for $n \geqslant 3$, we can falsify the following sentence (here i and j range over $\{1, 2, 3\}$);

$$\big(\bigwedge_i \Diamond p_i\big) \wedge \big(\bigwedge_{i\neq j} \Box(p_i \supset \sim \Diamond p_j)\big) \supset \bigvee_{i\neq j} \big(\Diamond(\Diamond p_i \wedge \Diamond p_j)\big).$$

However this sentence is valid under $\prec$ for all $n \geqslant 2$, and is valid under α in T^2.

PROBLEMS:
1. Axiomatise the logics corresponding to α and to $\prec$ in the various dimensions.
2. Analyse the logic of *discrete* spacetime (i.e. when $\boldsymbol{R}$ is replaced by Z).

Acknowledgement. I am very much indebted to Johan van Benthem for a stimulating and fruitful dialogue, without which I doubt that I would ever have completed this jigsaw puzzle.

References

[1] A. PRIOR, ***Past, Present and Future***, Oxford 1967.
[2] K. SEGERBERG, ***An Essay in Classical Modal Logic***, Filosofiska studier 13, Uppsala Universitet, 1971.
[3] K. SEGERBERG, *Decidability of* ***S4.1***, ***Theoria*** 34 (1968), pp. 7-20.
[4] A. A. ROBB, ***A Theory of Time and Space***, Cambridge 1914.

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(EDITOR'S FOOTNOTE.) It has come to the editor's attention that there is substantial overlap between Rob Goldblatt's paper and work done by Valentin Shehtman, Moscow. The results claimed by Shehtman can be described as follows.

Let $\boldsymbol{R}$ be the set of reals. Define a relation R on $\boldsymbol{R}^2$ by the condition

$$\langle x, y\rangle R \langle x', y'\rangle \quad \text{iff} \quad x \leqslant x' \quad \text{and} \quad y \leqslant y'.$$

Any subset $X \subseteq \boldsymbol{R}^2$ determines a modal logic $L(X)$, viz., the normal logic determined by the frame $\langle X, R \upharpoonright X\rangle$. Shehtman's work is summarized by the following five theorems.

THEOREM 1. *If X is an open polygon, then $L(X)$ is* ***S4*** *or* ***S4.2***.

THEOREM 2. *If X is a closed polygon, then $L(X)$ is* ***S4.1*** *or* ***S4.1.2***.

THEOREM 3. $L(\boldsymbol{R}^2) = $ ***S4.2***.

THEOREM 4. *If X is any open bounded domain with a smooth boundary, then* $L(X) = $ ***S4***.

THEOREM 5. *If X is any compact domain with a smooth boundary, then $L(X) = \mathbf{S4.1}$.*

(Here ***S4.1*** and ***S4.2*** are the normal extensions of ***S4*** by $\Box\Diamond p \to \Diamond\Box p$ and $\Diamond\Box p \to \Box\Diamond p$, respectively, while ***S4.1.2*** is the smallest normal logic extending both ***S4.1*** and ***S4.2***).

Shehtman's results are stated without proof in the collection of abstracts of the Soviet Conference on Mathematical Logic in Kishinjev, 1976. Evidently, Goldblatt's main result coincides with Theorem 3; in a private communication to the editor, Shehtman remarks that Goldblatt's proof is "almost the same" as his.

Shehtman's work was carried out during 1973-76. The suggestion to study problems of the present kind he attributes to A. G. Dragalin. He also remarks that Goldblatt's Theorem 1, which is due to J. F. A. K. van Benthem, was first proved by Dragalin as early as 1973 (see his book Математический интуиционизм:: Введение в теорию доказательств, Moscow 1979, p. 131).

It goes without saying that the question of priority does not detract from the beauty of Goldblatt's masterly presentation. Still, it is a bit saddening that work of the highest quality should be carried out in East and West by logicians who are sometimes not aware of each other's existence, let alone each other's work. The overlap between Goldblatt's and Shehtman's work should give reason to pause: in the editor's opinion it affords convincing argument that we need to improve the international network of scholarly contact and exchange.

K.S.