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Independent Propositional Modal Logics*

Abstract. We show that the join of two classical [respectively, regular, normal] modal logics employing distinct modal operators is a conservative extension of each of them.

A *propositional modal language* $\mathscr{L}$ has a countably infinite set of propositional variables and a set $C(\mathscr{L})$ of connectives comprising the Boolean connectives and a set $N(\mathscr{L})$, at most countable, of unary "necessity" connectives $\Box$; $F(\mathscr{L})$ is the set of formulas of $\mathscr{L}$. A *classical modal logic* [1] is a set $\boldsymbol{S}$ of formulas of a propositional modal language $\mathscr{L}_S$, containing all the Boolean tautologies and closed under Substitution, Detachment, and RE (if $\Box \in N(\mathscr{L}_S)$ and $\alpha \equiv \beta \in \boldsymbol{S}$ then $\Box\alpha \equiv \Box\beta \in \boldsymbol{S}$). Two such logics $\boldsymbol{S}$ and $\boldsymbol{T}$ are *independent* if $N(\mathscr{L}_S) \cap N(\mathscr{L}_T) = \emptyset$, and their *join* $\boldsymbol{S}\oplus\boldsymbol{T}$ is the smallest such logic containing their set-theoretic union $\boldsymbol{S}\cup\boldsymbol{T}$. To say that $\boldsymbol{S}\oplus\boldsymbol{T}$ is a *conservative extension* of $\boldsymbol{S}$ is to say that $(\boldsymbol{S}\oplus\boldsymbol{T}) \cap F(\mathscr{L}_S) = \boldsymbol{S}$.

A *regular modal logic* is a classical modal logic containing $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ for each $\Box \in N(\mathscr{L}_S)$ and closed under RR (if $\Box \in N(\mathscr{L}_S)$ and $\alpha \to \beta \in \boldsymbol{S}$ then $\Box\alpha \to \Box\beta \in \boldsymbol{S}$); a *normal modal logic* is a regular modal logic which is closed under RN (if $\Box \in N(\mathscr{L}_S)$ and $\alpha \in \boldsymbol{S}$ then $\Box\alpha \in \boldsymbol{S}$) [1]. It is easy to see, however, that a classical modal logic is regular if and only if it contains $\Box(p \wedge q) \equiv (\Box p \wedge \Box q)$, and normal if and only if it contains $\Box(p \wedge q) \equiv (\Box p \wedge \Box q)$ and $\Box(p \vee -p)$. Thus the "regular join" [respectively, "normal join"] of two regular [respectively, normal] modal logics $\boldsymbol{S}$ and $\boldsymbol{T}$, that is, the smallest regular [respectively, normal] modal logic containing $\boldsymbol{S}\cup\boldsymbol{T}$, is the same as their "classical join" $\boldsymbol{S}\oplus\boldsymbol{T}$. So it is not necessary to treat regular and normal logics separately from classical ones.

THEOREM. *If $\boldsymbol{S}$ and $\boldsymbol{T}$ are independent classical modal logics and $\boldsymbol{T}$ is consistent, then $\boldsymbol{S}\oplus\boldsymbol{T}$ is a conservative extension of $\boldsymbol{S}$.*

PROOF: Let $\mathfrak{A}_S = \langle A_S, F_S \rangle$ be the Lindenbaum-Tarski algebra of $\boldsymbol{S}$, that is, $A_S = \{/\alpha/ \mid \alpha \in F(\mathscr{L}_S)\}$ where $/\alpha/ = /\beta/ \leftrightarrow (\alpha \equiv \beta) \in \boldsymbol{S}$, and $F_S = \langle *_S \mid * \in C(\mathscr{L}_S) \rangle$ where $*_S(/\alpha_1/, \ldots, /\alpha_n/) = /*\alpha_1 \ldots \alpha_n/$ if $*$ is n-ary.

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Each $a \in F(\mathscr{L}_S)$ determines the polynomial f_a over $\mathfrak{A}_S$, and $a \in S$ if and only if f_a is identically 1 in $\mathfrak{A}_S$. The reduct $\mathfrak{A}_S^\circ = \langle A_S, -_S, \vee_S \rangle$ is a countably infinite (unless S is inconsistent, in which case the theorem holds trivially) atomless (if $/a/ \neq 0$ and p does not occur in a then $0 < /p \wedge a/ < /a/$) Boolean algebra.

Similarly, the reduct $\mathfrak{A}_T^\circ$ of the Lindenbaum-Tarski algebra $\mathfrak{A}_T = \langle A_T, F_T \rangle$ of T is a countably infinite (since T is consistent) atomless Boolean algebra. All countably infinite atomless Boolean algebras are isomorphic [2, p. 28]; let φ be an isomorphism from $\mathfrak{A}_T^\circ$ onto $\mathfrak{A}_S^\circ$. Let $\mathfrak{A} = \langle A_S, F_{S,T} \rangle$ be the expansion of $\mathfrak{A}_S$ such that φ is an isomorphism $\mathfrak{A}_T$ onto the reduct of $\mathfrak{A}$ to the language of $\mathfrak{A}_T$, that is $F_{S,T} = \langle *_{S,T} \mid * \in C(L_{S\oplus T}) \rangle$, $*_{S,T} = *_S$ if $* \in C(\mathscr{L}_S)$, and $*_{S,T}(a_1, \ldots, a_n) = \varphi\big(*_T(\varphi^{-1}a_1, \ldots, \varphi^{-1}a_n)\big)$ if $* \in C(\mathscr{L}_T)$.

Let $\Delta = \{a \in F(\mathscr{L}_{S\oplus T}) \mid f_a \text{ is identically } 1 \text{ in } \mathfrak{A}\}$. Then Δ is a classical modal logic, $S \cup T \subseteq \Delta$, and $\Delta \cap F(\mathscr{L}_S) \subseteq S$, from which it follows that $S \oplus T$ is a conservative extension of S.

References

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