J.F.A.K. van Benthem

Some Kinds of Modal Completeness

Abstract. In the modal literature various notions of "completeness" have been studied for normal modal logics. Four of these are defined here, viz. (plain) *completeness, first-order completeness, canonicity* and possession of the *finite model property* — and their connections are studied. Up to one important exception, all possible inclusion relations are either proved or disproved. Hopefully, this helps to establish some order in the jungle of concepts concerning modal logics. In the course of the exposition, the interesting properties of *first-order definability* and preservation under *ultrafilter extensions* are introduced and studied as well.

1. Introduction

Completeness theorems exist for many well-known modal logics. Not all modal logics admit of such results, however, as was shown by K. Fine (cf. [7]) and S. K. Thomason (cf. [18]). Still, even within the realm of "complete" logics, there exist differences: some are more complete than others, so to speak. In this paper we study a few special kinds of completeness, viz. "completeness cum first-order definability", "first-order-completeness" and "canonicity". Moreover, the "finite model property" will be treated in an appendix. The connections between these concepts will be given, as far as they are known at present. Hopefully, some unity will eventually emerge from research like this. Our main new result is that all first-order complete modal logics are canonical. This extends a result in [8] to the effect that all complete modal logics which are first-order definable are canonical. Cf. also [3], a paper whose notions and results will be used here repeatedly.

This paper is concerned with propositional modal logic, with primitives $\neg$ (negation), $\rightarrow$ (material implication) and $\Box$ (necessity). Other logical constants are defined in the usual manner, viz. $\wedge$ (conjunction), $\vee$ (inclusive disjunction), $\leftrightarrow$ (material equivalence) and $\Diamond$ (possibility). Semantic structures are *frames* $\mathfrak{F} = \langle W, R\rangle$ consisting of a set W (of "worlds") and a binary "alternative relation" R on W. A couple $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$, where $\mathfrak{F}$ is a frame and V a *valuation on* $\mathfrak{F}$ assigning subsets of W to proposition letters, is called a *model*. $\mathfrak{M} \vDash \varphi[w]$ ("φ is *true in* $\mathfrak{M}$ *at* w") is defined in the obvious way, using the well-known Kripke clause for $\Box$. Then, $\mathfrak{F} \vDash \varphi[w]$ ("φ is true in $\mathfrak{F}$ at w") may be defined by the stipulation that $\langle \mathfrak{F}, V \rangle \models \varphi[w]$ for all valuations V on $\mathfrak{F}$. The parameter w may be removed as follows: $\mathfrak{M} \models \varphi$ ("φ is true in $\mathfrak{M}$") if $\mathfrak{M} \models \varphi[w]$ for all $w \in W$; $\mathfrak{F} \models \varphi$ goes similarly. Finally, an intermediate notion turns out to be useful. A *general frame* $\langle \mathfrak{F}, \mathscr{W} \rangle$ consists of a frame $\mathfrak{F}$ together with a set $\mathscr{W}$ of subsets of W which is closed under the set-theoretic operations $-$ (complement with respect to W), $\cap$ (intersection) and m (modal projection with respect to $\mathfrak{F}$: $m(X) =_{\mathrm{def}} \{w \in W \mid \exists v \in X \ Rwv\}$). The truth definition is adapted as follows. $\langle \mathfrak{F}, \mathscr{W} \rangle \models \varphi[w]$ if $\langle \mathfrak{F}, V \rangle \models \varphi[w]$ for all valuations V on $\mathfrak{F}$ which assign values in $\mathscr{W}$ only. (Note that a frame $\mathfrak{F}$ may be identified with the general frame $\langle \mathfrak{F}, P(W) \rangle$, where $P(W)$ is the power set of W.) $\langle \mathfrak{F}, \mathscr{W} \rangle \models \varphi$ is defined in the obvious way. These semantic definitions give rise to the following three notions of modal semantic consequence. For a set Σ of modal formulas and a modal formula φ, $\Sigma \models_{\mathfrak{F}} \varphi$ ($\Sigma \models_{\mathfrak{M}} \varphi$, $\Sigma \models_{\mathfrak{G}\mathfrak{F}} \varphi$) if, for all frames (models, general frames) in which every formula of Σ is true, φ is true as well.

On the syntactic side, there is the *minimal modal logic* ***K*** consisting of a complete propositional basis (with modus ponens as its sole rule of inference) with the modal axiom schema $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and the modal rule of inference "to infer $\Box\varphi$ from φ" ("necessitation"). The best-known general modal completeness theorem is the following:

(1) $\quad \Sigma \vdash_{\boldsymbol{K}} \varphi \quad$ iff $\quad \Sigma \models_{\mathfrak{M}} \varphi, \quad$ for all $\quad \Sigma, \varphi$.

Another popular version of the minimal modal logic, called $\boldsymbol{K_s}$, has single axioms instead of schemata and a rule of substitution. In view of many confirming instances, it seemed a plausible guess that $\boldsymbol{K_s}$ would axiomatize $\models_{\mathfrak{F}}$. But, counter-examples were published in [7] and [18]. Indeed, truth in frames is essentially a second-order notion, and, in [19], the consequence relation of a strong fragment of second-order logic was effectively reduced to $\models_{\mathfrak{F}}$, thus showing this notion to be un-axiomatizable. This led to the formulation of a different general completeness theorem for $\boldsymbol{K_s}$, viz.

(2) $\quad \Sigma \vdash_{\boldsymbol{K_s}} \varphi \quad$ iff $\quad \Sigma \models_{\mathfrak{G}\mathfrak{F}} \varphi, \quad$ for all Σ, φ.

Moreover, Thomason's negative result leads us to treasure whatever stronger modal completeness results we have even more. These form the subject of the next section, in which some important notions of modal completeness are introduced. The main result is proven in a separate section (3): all first-order complete modal logics are canonical. Section 4 consists in a short discussion of the converse implication, which may quite well hold, but which has eluded proof up to now. In Section 5, a kind of appendix to the preceding parts, the familiar concept of the "finite model property" is compared with the notions introduced in previous sections. Finally, Section 6 is devoted to a general, almost "philosophical" discussion of completeness results.

2. Some varieties of modal completeness

A traditional modal completeness theorem is typically like the following. Let ***K4*** be the modal logic obtained from $\boldsymbol{K_S}$ by adding the axiom $\Box p \to \Box\Box p$. Now, for all modal formulas φ,

$\vdash_{K4} \varphi$ iff φ is true in all frames whose alternative relation is transitive.

Thus, there is a correspondence between ***K4*** (or, rather, its characteristic axiom $\Box p \to \Box\Box p$) and transitivity. Several concepts may be isolated from this observation.

2.1. Definition. A set Σ of modal formulas is *complete* ($\Sigma \in \mathbf{C}$) if, for all modal formulas φ,

$$\Sigma \vdash_{K_S} \varphi \quad \text{iff} \quad \Sigma \models_{\mathfrak{F}} \varphi.$$

A modal formula φ is *complete* if $\{\varphi\}$ is.

K4, or even $\{\Box p \to \Box\Box p\}$, is complete in this sense. For, one direction is immediate. A routine induction on the length of derivations in $\boldsymbol{K_S}$ shows that

$$\text{if} \quad \Sigma \vdash_{K_S} \varphi, \quad \text{then} \quad \Sigma \models_{\mathfrak{F}} \varphi, \quad \text{for all} \quad \Sigma, \varphi.$$

If, on the other hand, not $\vdash_{K4} \varphi$, then — by the above result — φ fails in some transitive frame. Then, since $\Box p \to \Box\Box p$ is true in any frame whose alternative relation is transitive, φ fails in some frame in which ***K4*** is true, i.e., not ***K4*** $\models_{\mathfrak{F}} \varphi$.

Not only is $\Box p \to \Box\Box p$ implied by $\forall x \forall y (Rxy \to \forall z (Ryz \to Rxz))$ (transitivity), but, conversely, if $\Box p \to \Box\Box p$ is true in a frame, then that frame is transitive. (Cf., e.g., [1].) Thus, $\Box p \to \Box\Box p$ is "first-order definable" in an obvious sense.

2.2. Definition. A set Σ of modal formulas is *first-order definable* ($\Sigma \in M1$) if a set Δ of first-order sentences (in R and $=$) exists such that, for all frames $\mathfrak{F}$, $\mathfrak{F} \models \Sigma$ iff $\mathfrak{F} \models \Delta$. (Here, $\mathfrak{F} \models \Sigma$ if, for all $\varphi \in \Sigma$, $\mathfrak{F} \models \varphi$; etc.) A modal formula φ is *first-order definable* if $\{\varphi\}$ is.

It is not hard to show that, if a modal formula is defined by some set of first-order sentences, then it is defined by a single such sentence already. (Cf. [1].)

Finally, ***K4*** is complete in the following pleasant sense.

2.3. Definition. A set Σ of modal formulas is *first-order complete* ($\Sigma \in C1$) if a set Δ of first-order sentences (in R and $=$) exists such that, for all modal formulas φ,

$$\Sigma \vdash_{K_S} \varphi \text{ iff } \Delta \models_{\mathfrak{F}} \varphi.$$

Between these concepts, there are several known connections; some quite obvious, others less so. Let us first consider the first two.

2.4. Lemma. $C \nsubseteq M1$; $M1 \nsubseteq C$
There are modal formulas outside of $M1 \cup C$.

Proof: "Löb's Formula" $\Box(\Box p \to p) \to \Box p$ is in C (cf. [17]), but it is not in $M1$ (cf. [1]).

The set of modal formulas
$\{\Box p \to p,\ \Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p),\ \Box(p \to \Box p) \to (\Diamond p \to p),\ \Box\Diamond p \to \Diamond\Box p\}$
is in $M1$, but not in C (cf. [4]).

The modal formula $\Box(\Box(p \to \Box p) \to \Box\Box\Box p) \to p$ is neither in $M1$ nor in C (cf. [4]). QED.

Next, the first and third concepts will be compared.

2.5. Lemma. $C1 \subseteq C$; $C \nsubseteq C1$.

Proof: The first assertion is trivial (cf. the above argument showing that ***K4*** is complete). For, if, for all modal formulas φ, $\Sigma \vdash_{K_S} \varphi$ iff $\Delta \models_{\mathfrak{F}} \varphi$, then consider any modal formula φ such that $\Sigma \models_{\mathfrak{F}} \varphi$. It follows from the above equivalence that $\Delta \models_{\mathfrak{F}} \psi$ for each $\psi \in \Sigma$, and hence $\Delta \models_{\mathfrak{F}} \varphi$, and so $\Sigma \vdash_{K_S} \varphi$.

To prove the second assertion, consider the set Σ consisting of Löb's Formula.
It was shown in [17] that, for all modal formulas φ, $\Sigma \vdash_{K_S} \varphi$ iff φ is true in all finite irreflexive trees. Therefore, the modal formulas of the form $\Box^n \bot$ (a contradiction $\bot$ preceded by n occurrences of $\Box$) are not derivable from Σ: $\Box^n \bot$ fails at 0 in a finite strict linear order of length $n+1$. Now, suppose- for the sake of reductio ad absurdum- that $\Sigma \in C1$, i.e., for some set Δ of first-order sentences in R and $=$,

$\Sigma \vdash_{K_S} \varphi$ iff $\Delta \models_{\mathfrak{F}} \varphi$, for all modal formulas φ.

Consider the following set of first-order formulas

$$\Delta \cup \{Rxy_1 \wedge \ldots \wedge Ry_n y_{n+1} \mid n \geqslant 1\}.$$

Each of its finite subsets is satisfiable in some frame $\mathfrak{F}$ for some world w (For, since $\Box^{n+1} \bot$ is not derivable from Σ, it follows that not $\Delta \models_{\mathfrak{F}} \Box^{n+1} \bot$, and hence a frame $\mathfrak{F}$ exists in which Δ is true, whereas $\Box^{n+1} \bot$ fails at some world w.) Then, by the compactness theorem for first-order logic, the above set is simultaneously satisfiable. I.e., a frame $\mathfrak{F}$ exists with a world w such that (i) Δ is true in $\mathfrak{F}$, and (ii) an infinite ascending sequence of worlds $w = w_1 R w_2 R w_3 R \ldots$ exists. But, here is our contradiction. For, setting $V(p) = W - \{w_i \mid i \in N\}$ yields $\langle \mathfrak{F}, V \rangle \models \neg \mathrm{LF}[w]$, and hence Σ is not true in $\mathfrak{F}$ (although Δ is). QED.

As for the second and third concepts, $M1 \nsubseteq C1$, because $M1 \nsubseteq C$. The other connections are as follows.

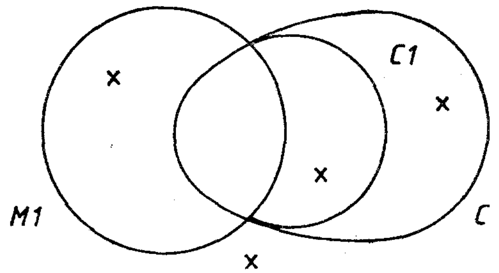
2.6. Lemma. $C1 \nsubseteq M1$; $M1 \cap C \subseteq C1$.

Proof: The second assertion follows trivially from the above definitions. The first assertion, which strengthens the above result that $C \nsubseteq M1$, follows from a proof in [8]. The modal formula $\Diamond\Box(p \vee q) \to \Diamond(\Box p \vee \Box q)$ belongs to $C1$- the relevant first-order property is

$$\forall x\, \forall y \Big(Rxy \to \exists z \Big(Rxz \wedge \forall u \big(Rzu \to \big(Ryu \wedge \forall v (Rzv \to u = v)\big)\big)\Big)\Big) -$$

but not to $M1$.

Thus, the situation may be pictured as follows:



A fourth related concept arises not so much from modal completeness *results* as from their *proofs*. Originally, the technique used was that of semantic tableaux (cf. [12]), but, later on in [13], a Henkin type method came into vogue. This uses *Henkin frames* composed of maximally consistent sets and an alternative relation R defined as the set of all couples $\langle \Sigma_1, \Sigma_2 \rangle$ such that, for all modal formulas φ, $\Box\varphi \in \Sigma_1$ only if $\varphi \in \Sigma_2$. A canonical valuation V may be defined on Henkin frames by setting $V(p) = \{\Sigma \in W \mid p \in \Sigma\}$ for each proposition letter p. This yields a *Henkin model*. Finally, a *Henkin general frame* may be defined using the set $\mathscr{W}$ of all sets of the form $\{\Sigma \mid \varphi \in \Sigma\}$, where φ is a modal formula. Any modal logic is true in its corresponding Henkin general frame, but it need not be true in the underlying Henkin frame. Those logics which are (like ***K4***) are called "canonical" in [8]. That paper also contains a semantic characterization of Henkin models, which may be applied to general frames to yield the following concept (cf. [9], or [3]).

2.7. Definition. A general frame $\langle \mathfrak{F}, \mathscr{W} \rangle$ is *descriptive* if

(1) $\forall x \in W \forall y \in W \big(x = y \leftrightarrow \forall P \in \mathscr{W} (Py \to Px)\big)$,

(2) $\forall x \in W \forall y \in W \big(Rxy \leftrightarrow \forall P \in \mathscr{W} (Py \to \exists z \in P\ \ Rxz)\big)$, and

(3) if all finite subsets of a subset of $\mathscr{W}$ have a non-empty intersection, then that whole subset has a non-empty intersection.

Note, at least, that Henkin general frames are descriptive. It may be shown that logics like ***K4*** are preserved in passing from a descriptive general frame in which they are true to the underlying frame. (The proof of this is like the familiar one in the completeness proof:

$$\forall x \forall \varphi \ \Box\varphi \to \Box\Box\varphi \in x,$$
$$\forall x \forall \varphi (\Box\varphi \in x \Rightarrow \Box\Box\varphi \in x),$$
$$\forall x \forall \varphi \big(\Box\varphi \in x \Rightarrow \forall y (Rxy \Rightarrow \Box\varphi \in y)\big),$$
$$\forall x \forall \varphi \Big(\Box\varphi \in x \Rightarrow \forall y \big(Rxy \Rightarrow \forall z (Ryz \Rightarrow \varphi \in z)\big)\Big),$$
$$\forall x \forall y \Big(Rxy \Rightarrow \forall z \big(Ryz \Rightarrow \forall \varphi (\Box\varphi \in x \Rightarrow \varphi \in z)\big)\Big),$$
$$\forall x \forall y \big(Rxy \Rightarrow \forall z (Ryz \Rightarrow Rxz)\big).$$

Thus, R is transitive and hence $\Box p \to \Box\Box p$ is true in the whole frame. Note that only property (2) was used!) Generally, this preservation property gives rise to the following concept (cf. [3]).

2.8. Definition. A set Σ of modal formulas is *canonical* ($\Sigma \in CAN$) if, for all descriptive general frames $\langle \mathfrak{F}, \mathscr{W} \rangle$ such that $\langle \mathfrak{F}, \mathscr{W} \rangle \models \Sigma$, $\mathfrak{F} \models \Sigma$.

Canonical sets have quite interesting properties.

2.9. Lemma. $CAN \subseteq C$; $C \nsubseteq CAN$; $CAN \nsubseteq M1$; $M1 \cap C \subseteq CAN$.

Proof: That $CAN \subseteq C$ is clear from the above considerations. Löb's Formula belongs to C, but not to CAN, however. The above-mentioned formula $\Diamond\Box(p \vee q) \to \Diamond(\Box p \vee \Box q)$ belongs to CAN as well (though not to $M1$). Finally, that $M1 \cap C \subseteq CAN$ follows from a semantic characterization of canonical sets found in [3]; which also contains proofs for the other above assertions. (The fourth assertion is essentially due to [8].) QED.

The concept of "canonicity" is surely the most abstract of the ones introduced up to now. It will become more familiar from the arguments in section 3. The quickest way to get acquainted with it is to think of its role in Henkin (general) frames. E.g., a canonical set Σ of modal formulas will be true in the underlying Henkin frame of any modal logic containing Σ. Thus, it is *generally complete* in the following sense: for all sets $\Sigma' \supseteq \Sigma$ and for all modal formulas φ, $\Sigma' \vdash_{K_s} \varphi$ iff, for all general frames $\langle \mathfrak{F}, \mathscr{W} \rangle$ in which Σ' is true and such that Σ is true in $\mathfrak{F}$, $\langle \mathfrak{F}, \mathscr{W} \rangle \models \varphi$. Now, it was shown in [15] that all first-order complete sets are generally complete in this sense. (Mortimer's argument can be easily adapted to yield this result, that is.)

Here again, the connection between $C1$ and CAN turns up. Already we noted that proofs of canonicity often proceed by finding a first-order condition which is true in the Henkin frame and which implies the relevant set of modal formulas. Such proofs establish, at the same time, that this set is first-order complete. The question then arises as to what is the exact relation between CAN and $C1$. To keep the conceptual complexity of modal completeness theory down to a minimum, it would be most satisfactory to have

$C1 = CAN$,

and this may quite conceivably be the case. But, in this paper, we have only been able to prove one half of this equation:

$C1 \subseteq CAN$.

The proof is in Section 3. Section 4 contains a discussion of the converse inclusion.

3. All first-order complete modal logics are canonical

The following concept is to be found in [10].

3.1. DEFINITION. The *ultrafilter extension* $ue(\mathfrak{F})$ of a frame $\mathfrak{F}$ $(= \langle W, R\rangle)$ is the frame $\langle ue(W), ue(R)\rangle$ with

(1) $ue(W)$ is the set of all ultrafilters on W, and

(2) $ue(R)$ is the set of all couples $\langle U_1, U_2\rangle$ such that, for each set $X \subseteq W$, if $X \in U_2$, then the modal projection $m(X)$ of X is in U_1.

It may be shown that each modal formula which is true in $ue(\mathfrak{F})$ is also true in $\mathfrak{F}$. The converse does not hold, however. E.g., Löb's Formula is true in certain frames, without being true in their ultrafilter extensions. (Cf. [3].) Still, one may define the following concept.

3.2. DEFINITION. A set Σ of modal formulas is *preserved under ultrafilter extensions* ($\Sigma \in PUE$) if, for all frames $\mathfrak{F}$, if $\mathfrak{F} \models \Sigma$, then $ue(\mathfrak{F}) \models \Sigma$.

PUE is an interesting class. E.g., we have the following connections with the notions of Section 2.

3.3. LEMMA. $C \nsubseteq PUE$, $PUE \nsubseteq C$ $\quad$ $C1 \subseteq PUE$, $PUE \nsubseteq C1$
$M1 \subseteq PUE$, $PUE \nsubseteq M1$ $\quad$ $CAN \subseteq PUE$, $PUE \nsubseteq CAN$.

PROOF: Löb's Formula is in C, but not in PUE. That $PUE \nsubseteq C$ follows from the fact that $M1 \subseteq PUE$ and $M1 \nsubseteq C$ (lemma 2.4).

That $M1 \subseteq PUE$ is shown in [3]. That $PUE \nsubseteq M1$ follows from the fact that $CAN \subseteq PUE$ and $CAN \nsubseteq M1$ (Lemma 2.9).

That $C1 \subseteq PUE$ will be proven below (Lemma 3.6). That $PUE \nsubseteq C1$ follows from $PUE \nsubseteq C$, $C1 \subseteq C$.

That $CAN \subseteq PUE$ is shown in the above-mentioned paper. That $PUE \nsubseteq CAN$ follows from $PUE \nsubseteq C$, $CAN \subseteq C$. QED.

We have no example of a modal formula in *PUE*, but outside of *M1* and *CAN*. It is a plausible conjecture that such formulas exist, however. (E.g., a modal formula which is not complete, but which is true in the same frames as the above-mentioned formula $\Diamond\Box(p\vee q)\rightarrow\Diamond(\Box p\vee\Box q)$ would be one.)

The following result is, again, in the above-mentioned paper:

3.4. Lemma. $CAN = C\cap PUE$.

For any set Σ of modal formulas, the *modal completion* $C(\Sigma)$ of Σ, defined as

$\{\varphi \mid$ for all frames $\mathfrak{F}$, if $\mathfrak{F}\models\Sigma$, then $\mathfrak{F}\models\varphi\}$, is a complete set of modal formulas which is true in exactly the same frames as Σ. The following result may, then, be deduced from Lemma 3.4.

3.5. Corollary. *For any set Σ of modal formulas, $\Sigma\in PUE$ if and only if $C(\Sigma)\in CAN$.*

Now, we turn to *C1*. Since $C1\subseteq C$, it suffices- by Lemma 3.4- to show that $C1\subseteq PUE$, in order to prove that $C1\subseteq CAN$. Here, then, is the crucial result.

3.6. Theorem. $C1\subseteq PUE$.

Proof: Let Σ be a set in *C1*. Say, Δ is a set of first-order sentences such that $\Sigma\vdash_{K_s}\varphi$ iff $\Delta\models_{\mathfrak{F}}\varphi$, for all modal formulas φ. Now, let $\mathfrak{F}$ be any frame in which Σ is true. It is to be shown that Σ is true in $ue(\mathfrak{F})$. To see this, consider any ultrafilter $\mathscr{U}$ in $ue(W)$: it is to be shown that $ue(\mathfrak{F})\models\Sigma[\mathscr{U}]$. By well-known modal results (cf. [17]), it suffices to look at the subframe $TC(ue(\mathfrak{F}),\mathscr{U})$ of $ue(\mathfrak{F})$ which is *generated by* $\mathscr{U}$ (taking R-successors, and R-successors of R-successors, etc.). Now, take unary predicate constants **X** corresponding to the subsets X of $W(\mathfrak{F}=\langle W,R\rangle)$. Let the set Δ' consist of Δ together with all formulas of the forms

(i) $\mathbf{X}u$ $(X\in\mathscr{U})$, where u is some fixed individual variable,

(ii) $\forall y\,(R^n uy\rightarrow(\mathbf{W}-\mathbf{X}y\leftrightarrow\neg\mathbf{X}y))$ $(n\geqslant 0)$,

where the notational convention is as follows:

- "R^0xy" stands for "$x=y$",
- "$R^{n+1}xy$" stands for "$\exists z_{n+1}(R^n xz_{n+1}\wedge Rz_{n+1}y)$",

(iii) $\forall y\,(R^n uy\rightarrow(\mathbf{X}\cap\mathbf{Z}y\leftrightarrow(\mathbf{X}y\wedge\mathbf{Z}y)))$, and

(iv) $\forall y\,(R^n uy\rightarrow(\mathbf{m}(\mathbf{X})y\leftrightarrow\exists z(Ryz\wedge\mathbf{X}z)))$.

Claim. *Each finite subset of Δ' is satisfiable.*

Proof: Suppose otherwise. Then finitely many formulas $\alpha_1,\ldots,\alpha_k$ of the above four forms exist such that $\Delta\models\neg(\alpha_1\wedge\ldots\wedge\alpha_k)$. Say, the unary predicate constants $\mathbf{X}_1,\ldots,\mathbf{X}_n$ occur in $\alpha_1,\ldots,\alpha_k$. Consider the proposition letters $p_1,\ldots,p_n$. Replace $\alpha_1,\ldots,\alpha_k$ by modal formulas as follows:

- if α_i is of form (i), take the appropriate p_j,

— if α_i is of form (ii), take the appropriate $\Box^n(p_j \leftrightarrow \neg p_k)$, where "$\Box^n$" denotes prefixing of n copies of "$\Box$",
— if α_i is of form (iii), take the appropriate $\Box^n(p_j \leftrightarrow (p_k \wedge p_l))$, and, finally,
— if α_i is of form (iv), take the appropriate $\Box^n(p_j \leftrightarrow \Diamond p_k)$. This yields modal formulas $\alpha_1^m, \ldots, \alpha_k^m$, for which it is easy to see that $\Delta \models_{\mathfrak{F}} \neg(\alpha_1^m \wedge \ldots \wedge \alpha_k^m)$ (by the fact that $\Delta \models \neg(\alpha_1 \wedge \ldots \wedge \alpha_k)$). But, then, by the above assumption, $\Sigma \vdash_{K_s} \neg(\alpha_1^m \wedge \ldots \wedge \alpha_k^m)$. Since $\mathfrak{F} \models \Sigma$, it follows that $\mathfrak{F} \models \neg\neg(\alpha_1^m \wedge \ldots \wedge \alpha_k^m)$. In particular, for the valuation V on $\mathfrak{F}$ defined by $V(p_i) = X_i$ $(1 \leqslant i \leqslant n)$ and arbitrary elsewhere, it holds that $\langle \mathfrak{F}, V \rangle \models \neg\neg(\alpha_1^m \wedge \ldots \wedge \alpha_k^m)$. Here is our intended contradiction. For, the formulas of form (i) among $\alpha_1, \ldots, \alpha_k$ involve finitely many sets in $\mathcal{U}$ (which have a non-empty intersection, $\mathcal{U}$ being an ultrafilter), and hence a world w exists satisfying all of them. Moreover, all formulas of the forms (ii), (iii), (iv) are true under the intended interpretation (by virtue of their form). It follows that-taking w for u- $\alpha_1, \ldots, \alpha_k$ are satisfiable under the intended interpretation, and hence that $\langle \mathfrak{F}, V \rangle \models \alpha_i^m[w]$ $(1 \leqslant i \leqslant k)$. QED.

From the claim, it follows, by the compactness theorem for first-order logic, that Δ' is satisfiable, say $\langle \mathfrak{F}_1, \mathbf{X}_1 \rangle_{X \subseteq W} \models \Delta'[w_1]$. Moreover, by standard model-theoretic arguments (cf. [5]), this model has an $\aleph_0$-*saturated* elementary extension $\langle \mathfrak{F}_2, \mathbf{X}_2 \rangle_{X \subseteq W}$, in which Δ' is still satisfied at w_1. Now, consider the generated subframe $TC(\langle \mathfrak{F}_2, \mathbf{X}_2 \rangle_{X \subseteq W}, w_1)$ $(= \mathfrak{F}_2(w_1))$. The latter structure is vital: it will be mapped onto $TC(ue(F), \mathcal{U})$ by the following function f.

For any w in the domain of $\mathfrak{F}_2(w_1)$, $f(w) =_{\text{def}} \{X \subseteq W \mid w \in \mathbf{X}_2\}$.

Claim. *f is a function from $\mathfrak{F}_2(w_1)$ into $ue(\mathfrak{F})$ such that*
(1) $f(w_1) = \mathcal{U}$,
(2) *if R_2wv, then $ue(R)f(w)f(v)$,* and
(3) *if $ue(R)f(w)\mathcal{U}'$, then a world v exists in the domain of $\mathfrak{F}_2(w_1)$ such that R_2wv and $f(v) = \mathcal{U}'$.*

Proof: To show that f assigns values in $ue(W)$, it is to be checked that, for any w, $f(w)$ is an ultrafilter on W. Here are the relevant conditions:
— if $X \in f(w)$, then $w \in \mathbf{X}_2$ and hence $w \notin \mathbf{W} - \mathbf{X}_2$ (and so $W - X \notin f(w)$). For, since w is in the domain of $TC(\langle \mathfrak{F}_2, \mathbf{X}_2 \rangle_{X \subseteq W}, w_1)$, a natural number n exists such that $R_2^n w_1 w$ (by the definition of generated subframes). Now, the formula $\forall y\,(R^n uy \to (\mathbf{W} - \mathbf{X}y \leftrightarrow \neg \mathbf{X}y))$
belongs to Δ', whence it is satisfied in our structure:
which yields the required conclusion.
— if $X \notin f(w)$, then $w \notin \mathbf{X}_2$ and, arguing like above, it follows that $x \in \mathbf{W} - \mathbf{X}_2$ and so $W - X \in f(w)$.
— if $X, Z \in f(w)$, then $w \in \mathbf{X}_2$ and $w \in \mathbf{Z}_2$, and, again arguing like above,

it follows that $w \in \mathbf{X} \cap Z_2$ and so $X \cap Z \in f(w)$. (The reasons for the choice of our formulas of the forms (ii), (iii) and (iv) will have become clear by now.) Finally,

— if $X \in f(w)$ and $X \subseteq Z$, then $w \in \mathbf{X}_2$ and $X \cap Z = X$; i.e., again like above, $w \in \mathbf{X}_2$ and $w \in Z_2$, and so $Z \in f(w)$.

It follows that $f(w)$ is an ultrafilter on W.

That $f(w_1) = \mathscr{U}$ follows from the fact that all formulas of form (i) are true at w_1.

Now, suppose that R_2wv; with, say, $R_2^n w_1 w$ (and hence $R_2^{n+1} w_1 v$). To show that $ue(R)f(w)f(v)$, it suffices to prove that, for any set $X \in f(v)$, $m(X) \in f(w)$ (by the definition of $ue(R)$; cf. 3.1). So, suppose that $X \in f(v)$, i.e., $v \in \mathbf{X}_2$. Because the formula

$$\forall y (R^n uy \rightarrow (\mathbf{m}(\mathbf{X}) y \leftrightarrow \exists z (Ryz \wedge \mathbf{X}z))$$

belongs to Δ' and, therefore, holds in $\mathfrak{F}_2(w_1)$, it follows that $w \in \mathbf{m}(\mathbf{X}_2)$, and hence $m(X) \in f(w)$.

Finally, suppose that $ue(R)f(w)\mathscr{U}'$. A world v is to be found in the domain of $\mathfrak{F}_2(w_1)$ such that R_2wv and $f(v) = \mathscr{U}'$. To discover that world, consider the following set of formulas, $\{\mathbf{X}x \mid X \in \mathscr{U}'\} \cup \{Rux\}$.

Each of its finite subsets is satisfiable in $\langle \mathfrak{F}_2, \mathbf{X}_2 \rangle_{X \subseteq W}$. For, consider any $X_1, \ldots, X_k \in \mathscr{U}' \cdot X = X_1 \cap \ldots \cap X_k \in \mathscr{U}'$ and hence $m(X) \in f(w)$ (by the definition of $ue(R)$). Then $w \in \mathbf{m}(\mathbf{X})_2$ (by the definition of f). Like above (using a formula of form (iv)), an R_2-successor v of w may be found such that $v \in \mathbf{X}_2$. Note that v belongs to the domain of $F_2(w_1)$, because w does. Next, using formulas of the form (iii), it follows that $v \in \mathbf{X}_{i_2}$ $(1 \leqslant i \leqslant k)$. Now, because the above set is finitely satisfiable, and because $\langle \mathfrak{F}_2, \mathbf{X}_2 \rangle_{X \subseteq W}$ is $\aleph_0$-saturated, this set is simultaneously satisfiable in that structure, say at a world v. Clearly, this world is the required one. QED.

The time has come for the final argument completing the proof of Theorem 3.6. Recall that, in any frame in which Δ is true, Σ is true. Now, because $\Delta \subseteq \Delta'$ and Δ' is satisfied in $\langle \mathfrak{F}_2, \mathbf{X}_2 \rangle_{X \subseteq W}$ (at w_1), it follows that $\mathfrak{F}_2 \models \Delta$ and hence that $\mathfrak{F}_2 \models \Sigma$. Therefore, $TC(\mathfrak{F}_2, w_1) \models \Sigma$, for — by well-known modal results — truth of modal formulas is preserved under generated subframes (cf. [17]). Now, the function f defined above is easily seen to be a *p-morphism* (in the sense of that same work) from $TC(\mathfrak{F}_2, w_1)$ onto $TC(ue(\mathfrak{F}), \mathscr{U})$; and hence — again by a well-known modal result — $TC(ue(\mathfrak{F}), \mathscr{U}) \models \Sigma$. QED.

The following consequence has been announced already.

3.7. Corollary. $C1 \subseteq CAN$.

Yet another corollary is the result that $M1 \cap C \subseteq CAN$ (cf. Lemma 2.9), because $M1 \cap C \subseteq C1$ (cf. Lemma 2.6). But, our result is stronger, because $C1 \nsubseteq M1 \cap C$ (cf. Lemma 2.6).

It remains to be remarked that the above proof is a more complicated version of one in [3] showing that for any frame $\mathfrak{F}$, $ue(\mathfrak{F})$ is a p-morphic image of some frame elementarily equivalent to $\mathfrak{F}$.

4. Are all canonical modal logics first-order complete?

It was remarked above that proofs of canonicity often involve the construction of a first-order property with respect to which the relevant set of modal formulas is complete. This is true for our example, but also for Fine's more complicated formula $\Diamond\Box(p\vee q)\to\Diamond(\Box p\vee\Box q)$. In fact, it is the method of proof of H. Sahlqvist's general completeness theorem in [16]. Now, such observations suggest a method of proof for the inclusion $CAN\subseteq C1$:
Suppose that $\Sigma\in CAN$ and show that it is complete with respect to the first-order theory of its own Henkin frame! To this end, it suffices to show that Σ is true in any frame which is elementarily equivalent to its Henkin frame. Unfortunately, this has turned out to be easier said than done. And so the question in the title remains open.

5. The finite model property

To show that one must be thankful for what little connections exist between the above notions, here is another important concept, which behaves even less socially.

If a modal formula fails in the Henkin model of ***K4***, then the so-called "filtration method" (cf. [17]) may be applied to that model, changing it into a *finite* transitive model in which the modal formula in question fails. Thus, for all modal formulas φ, $\vdash_{K4}\varphi$ iff φ is true in all finite transitive frames. More generally, this inspires the following concept.

5.1. Definition. A set Σ of modal formulas has the *finite model property* ($\Sigma\in FMP$) if, for all modal formulas φ such that not $\Sigma\vdash_{K_s}\varphi$, a finite model exists in which Σ is true, whereas φ is not.

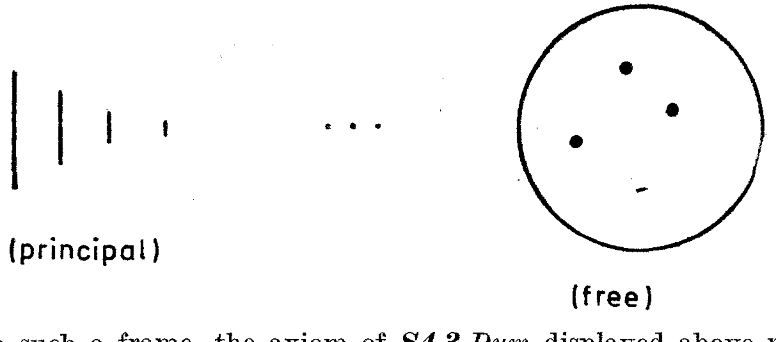
It was shown in [11] that, if $\Sigma\in FMP$, then, for all modal formulas φ such that not $\Sigma\vdash_{K_s}\varphi$, a finite *frame* exists in which Σ is true, whereas φ is not. So, if $\Sigma\in FMP$, then it is *complete* in the sense of Section 2. Moreover, if Σ is recursive and $\Sigma\in FMP$, then $\{\varphi\mid\Sigma\vdash_{K_s}\varphi\}$ is *recursive.*

This concept has turned out to be a happy choice: leading to important results, like Bull's Theorem: "All normal extensions of the modal logic ***S4.3*** have the finite model property" (cf. [6]). Compare such a result with the unfortunate behaviour of our previous notions in this respect. E.g., the modal logic ***S4.3*** ***Dum*** axiomatized by ***S4.3*** together with "Dummett's Axiom"

$$\Box(\Box(p\to\Box p)\to p)\to(\Diamond\Box p\to p),$$

has *FMP*, but it does not belong to *CAN*. (And hence it does not belong to *M1* or *C1* to by previous results.) Still, ***S4.3*** possessed all these properties as well.

That ***S4.3 Dum*** is outside of *CAN* may be shown by noting that $\langle N, \leqslant\rangle \models$ ***S4.3 Dum***, where N is the set of natural numbers. Now, if ***S4.3 Dum*** $\in CAN$, then it would be preserved under ultrafilter extensions (cf. Lemma 3.3). Thus, it suffices to show that ***S4.3 Dum*** is not true in $ue(\langle N, \leqslant\rangle)$. A short look at this frame is required now. Each natural number n corresponds to the principal ultrafilter $\{X \subseteq N \mid n \in X\}$, and indeed these ultrafilters form an isomorphic copy of $\langle N, \leqslant\rangle$ which is a subframe of $ue(\langle N, \leqslant\rangle)$. (Cf. [1].) It remains to determine the position of the *free* ultrafilters on N. Now, let $\mathscr{U}$ be any free ultrafilter on N, and let $\mathscr{V}$ be just any ultrafilter on N. For any set $X \in \mathscr{U}$, X is infinite ($\mathscr{U}$ being free, it does not contain finite sets) and hence $m(X)$ equals N, whence $m(X) \in \mathscr{V}$. Thus, by Definition 3.1, $ue(\leqslant)\mathscr{V}\mathscr{U}$. It follows that $ue(\langle N, \leqslant\rangle)$ consists of a copy of $\langle N, \leqslant\rangle$ succeeded by $2^{\aleph_0}$ (the number of free ultrafilters on N) points on which the relation is universal. Finally, no free ultrafilter stands in the relation $ue(\leqslant)$ to any principal one. For, if $ue(\leqslant)\mathscr{U}\{X \subseteq N \mid n \in X\}$, then- since $\{n\}$ belongs to the latter ultrafilter- $m(\{n\}) = \{k \mid k \leqslant n\} \in \mathscr{U}$, and hence $\mathscr{U}$ contains a finite set. In other words, $ue(\langle N, \leqslant\rangle)$ looks as follows:



But, on such a frame, the axiom of ***S4.3 Dum*** displayed above may be falsified as follows. Define a valuation V on it by setting $V(p) =$ the set of all free ultrafilters together with the set of all principal ultrafilters generated by *odd* natural numbers; and consider the first point in the copy of $\langle N, \leqslant\rangle$: $\Box\big(\Box(p \to \Box p) \to p\big)$ is true at it and so is $\Diamond\Box p$, but p is not. This concludes the proof that ***S4.3 Dum*** is not canonical. Thus, *FMP* has certain advantages which previous notions like *CAN* or *C1* lack.

As it happens, *FMP* is quite unrelated to other notions of completeness:

5.2. Lemma. $FMP \subseteq C$, $C \nsubseteq FMP$ $\quad FMP \nsubseteq C1$, $C1 \nsubseteq FMP$

$FMP \nsubseteq M1$, $M1 \nsubseteq FMP$ $\quad FMP \nsubseteq CAN$, $CAN \nsubseteq FMP$.

PROOF: It suffices to show that (i) $M1 \cap C \nsubseteq FMP$, and (ii) $FMP \nsubseteq CAN$: the other assertions follow from these by previous results.

To prove (i), consider the set $\Sigma = \{\Box p \to p, (\Box p \wedge \Diamond \Diamond q) \to \Diamond(\Box\Box p \wedge \Diamond\Diamond\Diamond q)\}$. Σ is true in the "recession frame" (cf. [4]) $\langle N, R\rangle$ with $R = \{\langle m, n\rangle \mid n \geqslant m-1\}$. Note that the modal formula $\Box p \to \Box\Box p$ is not true in this frame; whence not $\Sigma \vdash_{K_s} \Box p \to \Box\Box p$. But, in any *finite* frame in which Σ is true, $\Box p \to \Box\Box p$ is true as well. (Just note that the modal formula

$$(\Box p \wedge \neg\Box\Box p) \to \Diamond(\Box\Box p \wedge \neg\Box\Box\Box p)$$

is provable in Σ, and apply the argument in [14].) It follows that $\Sigma \notin FMP$. Moreover, $\Sigma \in M1 \cap C$: this follows from the general completeness theorem for modal formulas of this form in [16]. (By the way, the relevant first--order equivalents are

$\forall x Rxx$ for $\Box p \to p$, and

$$\forall x \forall y \Big(Rxy \to \forall z \Big(Ryz \to \exists u \Big(Rxu \wedge \forall s \big(Rus \to \forall t (Rst \to Rxt)\big) \wedge$$
$$\wedge \exists z_1 \big(Ruz_1 \wedge \exists z_2 (Rz_1 z_2 \wedge Rz_2 z)\big)\Big)\Big)\Big)$$ for the second axiom.)

Next, to prove (ii), consider **S4.3 Dum**. QED.

6. Discussion

All modal logics which had been studied up to 1974 (the year when Fine and Thomason presented the first examples of incompleteness) have turned out to possess one or more of the completeness properties treated in this paper. Indeed, all examples of incomplete modal logics which have been found up to now are, to a certain extent, artificial. (There is no analogon here to Gödel's famous Incompleteness Theorem which showed a natural, existing theory — viz. Peano Arithmetic — to be incomplete.) Given this situation, it is only natural to ask why this should be so. This may not be a scientific question, but let us see if it leads to one.

Now, modal completeness results may be approached from two perspectives. In the first, one lays down a logic (hopefully, a codification of some philosophical insights) and asks if it is complete with respect to Kripke semantics (in the sense of Definition 2.1). In the second perspective, one starts from a class $\mathscr{K}$ of frames and asks for (a recursive axiomatization of) its modal theory $Th_{mod}(\mathscr{K}) =_{\text{def}} \{\varphi \mid \text{for all } \mathfrak{F} \in \mathscr{K}, \mathfrak{F} \models \varphi\}$. Again, up to the present, such questions have always received a satisfactory answer.

As it happens, it is hard to say anything enlightening about the success of the first perspective. The reason cannot be found in the relative *simplicity* of modal logics in the literature, for there are simple incomplete modal logics too (cf. [4]). Still, there might be a common characteristic of complete logics which has been overlooked all the time. Here, then, is an interesting question:

"Is the set of complete modal formulas (viewed as a subset of the set of all modal formulas) recursive?".[1] Another causal factor may consist in an intuitive feeling for *strength*. Any modal logic Σ is contained in a complete logic $C(\Sigma)$ (cf. section 3) which is true in exactly the same frames as Σ itself. Why formulate, e.g., an axiom φ, when the stronger $C(\{\varphi\})$ lies at hand? Yet, here as well, even the first questions remain to be answered, like

"Is $C(\{\varphi\})$ finitely axiomatizable for each modal formula φ?"

Turning to the second perspective, let us note first that classes of frames are always given by means of some condition. Hence the natural questions will assume forms like:

"When does a first-order (second-order) definable class of frames possess a recursive modal theory?" One reason for success in this area is clear. If our class $\mathscr{K}$ is definable by means of a recursive (or even a recursively enumerable) set Δ of first-order sentences, then its modal theory is recursively axiomatizable. For, then,

$\varphi \in Th_{mod}(\mathscr{K})$ iff $\Delta \vDash \varphi$ iff $\Delta \vDash \varphi$, where φ is the standard transcription of a modal formula φ into a first-order formula (cf. [1]), iff $\Delta \vdash \varphi$, by the completeness theorem for first-order logic. And the latter notion is recursively enumerable, by the familiar Gödel type encodings.

Still, in practice, the modal axiomatizations one encounters are not only recursive, but even *finite*. Could the reason for this phenomenon be the following:

"The modal theories of classes of frames defined by single first-order sentences are finitely axiomatizable"? Unfortunately, the answer is negative: which shows what a difficult area of research has been touched upon in this final discussion.

We conclude with the presentation of a counter-example to the last conjecture. Consider the first-order sentence $\alpha = \forall x \exists y (Rxy \wedge \exists ! zRyz)$ ("each world has a successor with exactly one successor"). Consider, in addition, the following set Σ of modal formulas:

$\{(\Box\Diamond p_1 \wedge \ldots \wedge \Box\Diamond p_n) \rightarrow \Diamond(\Box p_1 \wedge \ldots \wedge \Box p_n) \mid n \geqslant 1\}$.

Call these formulas σ_n, $n \geqslant 1$.)

6.1. Lemma. *Σ is complete with respect to α*; i.e., *for each modal formula φ, $\Sigma \vdash_{K_s} \varphi$ iff $\alpha \vDash_{\mathfrak{F}} \varphi$.*

Proof: The direction from left to right follows from the fact that $\alpha \vDash_{\mathfrak{F}} \Sigma$, which may be checked by a routine argument. For the converse direction, suppose that not $\Sigma \vdash_{K_s} \varphi$. Then φ fails in the Henkin frame of Σ. But, this Henkin frame satisfies α, as may be seen from the following calculation.

(1) $\forall x \forall \varphi (\Box\Diamond\varphi \in x \Rightarrow \Diamond\Box\varphi \in x)$.

[1] **Added in print**: This question was answered negatively by S. K. Thomason in a forth-coming paper called "Undecidability of the Completeness Problem of Modal Logic" (April, 1978).

Consider $\Gamma = \{\alpha \mid \Box\alpha \in x\} \cup \{\Box\varphi \mid \Box\Diamond\varphi \in x\}$. This set is consistent. For, otherwise, for certain $\alpha_1, \ldots, \alpha_k$; $\varphi_1, \ldots, \varphi_m$, we would have $\Sigma \vdash_{K_s} (\alpha_1 \wedge \ldots \wedge \alpha_k) \to \neg(\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_m)$, and hence $\Sigma \vdash_{K_s} \Box(\alpha_1 \wedge \ldots \wedge \alpha_k) \to \Box\neg(\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_m)$. Since $\Box(\alpha_1 \wedge \ldots \wedge \alpha_k) \in x$, it follows that $\Box\neg(\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_m) \in x(*)$. On the other hand, $\Box\Diamond\varphi_1, \ldots, \Box\Diamond\varphi_m \in x$, whence $\Box(\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_m) \in x$ and, therefore, $\Diamond(\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_m) \in x$: a contradiction with $(*)$ and the consistency of x. Now, any maximally consistent extension of Γ produces a world y as described in the following line.

(2) $\forall x \exists y (Rxy \wedge \forall\varphi(\Box\Diamond\varphi \in x \Rightarrow \Box\varphi \in y)$. Then

(3) $\forall x \exists y \big(Rxy \wedge \forall z \big(Ryz \Rightarrow \forall\varphi(\Box\Diamond\varphi \in x \Rightarrow \varphi \in z)\big)\big)$.

An argument like above, now considering the set $\{\alpha \mid \Box\alpha \in x\} \cup \{\neg\Diamond\varphi \mid \neg\varphi \in z\}$, yields a world u as described in the next line.

(4) $\forall x \exists y \Big(Rxy \wedge \forall z \Big(Ryz \Rightarrow \exists u \big(Rxu \vee \forall\varphi(\Diamond\varphi \in u \Rightarrow \varphi \in z)\big)\Big)\Big)$. Then

(5) $\forall x \exists y \Big(Rxy \wedge \forall z \Big(Ryz \Rightarrow \exists u \Big(Rxu \vee \forall v \big(Ruv \Rightarrow (\varphi \in v \Rightarrow \varphi \in z)\big)\Big)\Big)\Big)$,

and, finally,

(6) $\forall x \exists y \Big(Rxy \wedge \forall z \Big(Ryz \Rightarrow \exists u \big(Rxu \wedge \forall v (Ruv \Rightarrow v = z)\big)\Big)\Big)$.

It is easily seen that (6) implies α. QED.

Lemma 6.1 also expresses the fact that Σ axiomatizes the modal theory of $\{\mathfrak{F} \mid \mathfrak{F} \models \alpha\}$. Now, if this theory were finitely axiomatizable, then, obviously, Σ would be axiomatized by a finite subset of itself. In particular, for a sufficiently large natural number n, the formulas $\sigma_1, \ldots, \sigma_n$ would imply σ_{n+1}. But, this cannot happen:

6.2. Lemma. *For no natural number n, $\{\sigma_1, \ldots, \sigma_n\} \vdash_{K_s} \sigma_{n+1}$.*

Proof: For any natural number n, consider the frame $\mathfrak{F}_n = \langle W_n, R_n \rangle$, with

– $W_n = \{1, \ldots, n+2\} \cup \{\{i, j\} \mid i \neq j;\ 1 \leqslant i, j \leqslant n+2\} \cup \{0\}$, and
– $R_n = \{\langle 0, \{i, j\}\rangle \mid i \neq j;\ 1 \leqslant i, j \leqslant n+2\} \cup$
$\cup \{\langle \{i, j\}, k\rangle \mid i \neq j;\ 1 \leqslant i, j \leqslant n+2;\ k \in \{i, j\}\} \cup$
$\cup \{\langle k, k\rangle \mid 1 \leqslant k \leqslant n+2\}$.

We claim that

(1) $\mathfrak{F}_n \models \sigma_i$ *for* $i = 1, \ldots, n$.
(2) *not* $\mathfrak{F}_n \models \sigma_{n+1}$.

Proof of (1): All formulas σ_i are true at all worlds different from 0: this is easy to check, using the fact that the worlds $1, \ldots, n+2$ each have exactly one R-successor. Next, consider the world 0. If V is any valuation on $\mathfrak{F}_n$ such that $\langle \mathfrak{F}_n, V\rangle \models \Box\Diamond p_1 \wedge \ldots \wedge \Box\Diamond p_i[0]$, then each two-element subset of $\{1, \ldots, n+2\}$ and each set $V(p_j)$ $(1 \leqslant j \leqslant i)$ have a non-empty intersection. It follows that p_j can fail at at most one member of $\{1, \ldots, n+2\}$ $(1 \leqslant j \leqslant i)$. Since $i \leqslant n$, this leaves at least two members k, l of $\{1, \ldots, n+2\}$ at which $p_1, \ldots, p_i$ are all true, and hence $\Box p_1 \wedge \ldots \wedge \Box p_i$ is true at $\{k, l\}$, and, therefore, $\Diamond(\Box p_1 \wedge \ldots \wedge \Box p_i)$ is true at 0. QED.

PROOF OF (2): Set $V(p_i) = \{1, \ldots, n+2) - \{i\}$, for $i = 1, \ldots, n+1$. It follows that all of $p_1, \ldots, p_{n+1}$ are true at $n+2$; but this happens at no other world in $\{1, \ldots, n+2\}$! Clearly, by this choice of a valuation, $\langle \mathfrak{F}_n, V\rangle \models \Box\Diamond p_i[0]$ for $i = 1, \ldots, n+1$. But, by the previous observation, $\Box p_1 \wedge \ldots \wedge \Box p_{n+1}$ is true at no R-successor of 0. QED.

It follows from (1) and (2) that not $\{\sigma_1, \ldots, \sigma_n\} \models_{\mathfrak{F}} \sigma_{n+1}$, and hence that not $\{\sigma_1, \ldots, \sigma_n\} \vdash_{K_8} \sigma_{n+1}$. QED.

It was noted above already that lemma 6.2 has the following consequence.

6.3. COROLLARY. *The modal theory of* $\forall x \exists y(Rxy \wedge \exists!\, zRyz)$ *is no finitely axiomatizable.*

References

[1] J. F. A. K. VAN BENTHEM, *Modal Correspondence Theory*, dissertation, Department of Mathematics, University of Amsterdam, 1976.

[2] —, ***Modal Logic as Second-Order Logic***, report 77-04, 1977, Department of Mathematics, University of Amsterdam; expanded version to appear in Studia Logica Selected Topics Library 1980.

[3] —, *Canonical modal logics and ultrafilter extensions*, ***The Journal of Symbolic Logic*** 44 (1979), pp. 1-8.

[4] —, *Two simple incomplete modal logics*, ***Theoria*** 44 (1978), pp. 25-37.

[5] C. C. CHANG and H. J. KEISLER, ***Model Theory***, North-Holland, Amsterdam, 1973.

[6] K. FINE, *The logics containing S4.3*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik*** 17 (1971), pp. 371-376.

[7] —, *An incomplete logic containing S4*, ***Theoria*** 40 (1974), pp. 23-29.

[8] —, *Some connections between elementary and modal logic*, in: ***Proceedings of the Third Scandinavian Logic Symposium***, Uppsala 1973, North-Holland, Amsterdam 1975, pp. 15-31.

[9] R. I. GOLDBLATT, *Metamathematics of modal logic*, ***Reports on Mathematical Logic*** 6 (1976), pp. 41-77, and 7, pp. 21-52.

[10] R. I. GOLDBLATT and S. K. THOMASON, *Axiomatic classes in propositional modal logic*, in: ***Algebra and Logic, Springer Lecture Notes in Mathematics*** 450, Berlin (1974), pp. 163-173.

[11] R. HARROP, *On the existence of finite models and decision procedures for propositional calculi*, ***Proceedings of the Cambridge Philosophical Society*** 54 (1958), pp. 1-13.

[12] S.A. KRIPKE, *A completensesz theorem in modal logic*, ***The Journal of Symbolic Logic*** 24 (1959), pp. 1-14.

[13] E. J. LEMMON and D. SCOTT, ***An Introduction to Modal Logic***, Blackwell, Oxford 1966, 1977, (Edited by K. Segerberg.)
[14] D. C. MAKINSON, *A normal calculus between T and S4 without the finite model property*, ***The Journal of Symbolic Logic*** 34 (1969), pp. 35-38.
[15] M. MORTIMER, *Some results in modal model theory*, ***The Journal of Symbolic Logic*** 39 (1974), pp. 496-508.
[16] H. SAHLQVIST, *Completeness and correspondence in the First and Second Order semantics for modal logic*, in: ***Proceedings of the Third Scandinavian Logic Symposium***, Uppsala 1973, North-Holland, Amsterdam 1975, pp. 110-143.
[17] K. SEGERBERG, ***An Essay in Classical Modal Logic***, Department of Philosophy, University of Uppsala, 1971.
[18] S. K. THOMASON, *An incompleteness theorem in modal logic*, ***Theoria*** 40 (1974), pp. 30-34.
[19] —, *Reduction of second-order logic to modal logic*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik*** 21 (1975), pp. 107-114.

INSTITUTE OF PHILOSOPHY
GRONINGEN, HOLLAND

Received September 18, 1978.