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Pretabular Varieties of Modal Algebras

Abstract. We study modal logics in the setting of varieties of modal algebras. Any variety of modal algebras generated by a finite algebra — such a variety is called *tabular* — has only finitely many subvarieties, i.e. is of finite height. The converse does not hold in general. It is shown that the converse does hold in the lattice of varieties of *K4*-algebras. Hence the lower part of this lattice consists of tabular varieties only. We proceed to show that there is a continuum of *pretabular* varieties of *K4*-algebras — those are the non-tabular varieties all of whose proper subvarieties are tabular — in contrast with Maksimova's result that there are only five pretabular varieties of *S4*-algebras.

Much of the literature on modal logics has been engaged in introducing new logics and comparing them with existing ones regarding their strength. Such investigations are really part of the more ambitious attempt to provide a description of the lattice of all modal logics. Though such a description, even of the lattice of normal extensions of $\boldsymbol{K}$, to which we will restrict our attention, seems to be well out of reach yet, a considerable amount of information on the lattice has been obtained by now. The observation that the lattice of normal extensions of $\boldsymbol{K}$ is dually isomorphic to the lattice $\Lambda(\mathbf{M})$ of subvarieties of the variety $\mathbf{M}$ of modal algebras enables us to invoke general results of the algebraic theory of lattices of varieties, in particular the results obtained by B. Jónsson for varieties of algebras whose lattices of congruences are distributive. To mention just a few of the immediate consequences of these general results: the lattice $\Lambda(\mathbf{M})$ is atomic (Makinson [10] showed that there are two atoms), $\Lambda(\mathbf{M})$ is complete, distributive and dually Brouwerian, every $\mathbf{K} \in \Lambda(\mathbf{M})$, $\mathbf{K} \neq \mathbf{M}$, has a cover in $\Lambda(\mathbf{M})$ and all of the varieties corresponding with the familiar modal logics like ***K***, ***T***, ***K4***, ***S4***, ***S4.3***, ***S5*** are join-irreducible. Using somewhat more elaborate methods it is possible to characterize the varieties in $\Lambda(\mathbf{M})$ having $\mathfrak{m}$ covers for any cardinal number $\mathfrak{m}$ ([2]).

It follows from one of Jónsson's results that any congruence distributive variety generated by a finite algebra — such a variety is called a *tabular* variety — has only finitely many subvarieties. Hence tabular varieties are close to the bottom of the lattice. Day [6] proved that for varieties of Heyting algebras a converse holds: any variety of Heyting algebras having only finitely many subvarieties is generated by a finite algebra. This result makes, in principle, a description of the bottom part of the lattice of varieties of Heyting algebras possible. In [5] (see also [4]) we obtained a similar result for the lattice Λ(**MRT**) of subvarieties of the variety **MRT** of interior algebras (dually isomorphic to the lattice of normal extensions of ***S4***). The next step was to investigate the non--tabular varieties of Heyting algebras and interior algebras, and, to begin with, the minimal ones among them (which can easily be shown to exist), called *pretabular* varieties. This was done by Maksimova [11] and [12], and she came to the unexpected conclusion that there are only three pretabular varieties of Heyting algebras and five pretabular varieties of interior algebras (the last result was obtained independently in [7]). The question arises if results of this nature can be extended to larger sublattices of Λ(**M**). Not to the full lattice Λ(**M**), as was shown in [2]. Indeed, the atoms of Λ(**M**) have $2^{\aleph_0}$ covers in Λ(**M**), which gives rise to a multitude of pretabular varieties and destroys every hope to give a description of even the lowest part of Λ(**M**). In the present paper we investigate the lattice Λ(**MT**) of subvarieties of the variety **MT** which corresponds with the modal logic ***K4***, axiomatized by the formula $\Box p \to \Box\Box p$.

After a preliminary section 0 we prove in section 1 that the cover of a tabular variety in Λ(**MT**) is tabular. This extends the results mentioned before and it seems that it cannot be improved essentially[1]. In fact, we give an example of a non-tabular cover of an atom of Λ(**MT**) which satisfies $\Box^2 p \leftrightarrow \Box^3 p$ and hence is only "just" outside of **MT**.

From this it follows that every pretabular variety in Λ(**MT**) is generated by its finite members, as in Λ(**MRT**). However, here the parallel ends. Whereas Λ(**MRT**) contains only five pretabular varieties, it turns out in section 2 that Λ(**MT**) contains a continuum of pretabular varieties.

In the final section we investigate the pretabular varieties in some sublattices of Λ(**MT**). It is shown that the lattice of subvarieties of the variety corresponding to the modal logic axiomatized by Löb's formula contains a countable number of pretabular varieties. The developed methods easily yield the known results for **MRT** and Heyting algebras. Finally we show that the subvariety of **MT** defined by the equation $0^0 \neq 0$ (corresponding with the modal logic D4) contains only finitely many pretabular varieties.

0. Preliminaries

Modal formulas are formed in the usual way from a denumerable set of proposition letters $p, q, r, \ldots$, the classical connectives $\vee$, $\wedge$, $\neg$, $\to$, $\bot$, $\top$ and the unary "modal" operator $\Box$ ("necessarily"). A (*normal*)

[1] The results of section 1 do hold, however, for the lattice of subvarieties of the variety corresponding to the modal logic axiomatized by the slightly weaker law $p \wedge \Box p \to \Box\Box p$.

modal logic is a set of modal formulas containing the classical tautologies and the axiom $\Box(p\to q)\to(\Box p\to\Box q)$ and closed under the rules of modus ponens, substitution and necessitation. A set Γ of modal formulas is said to *axiomatize* a logic L if L is the smallest normal modal logic containing Γ. The formula $\Box p\to\Box\Box p$ axiomatizes the logic ***K4***; by adding $\Box p\to p$ as an axiom we obtain an axiomatization of ***S4***.

A (*Kripke*) *frame* $\mathfrak{F}$ is a pair (W, R) where W is a set — the set of worlds — and R is a binary relation on W — the accessibility relation. We will often write Rwv instead of $(w, v)\in R$. Given a modal formula φ, a frame $\mathfrak{F}$ and a valuation V (that is, a map assigning subsets of W to the proposition letters), satisfaction of φ in $(\mathfrak{F}, V)$ at w, in symbols $(\mathfrak{F}, V)\models\varphi[w]$, is defined by means of the usual inductive truth definition. In particular, $(\mathfrak{F}, V)\models\Box\varphi[w]$ iff for every $v\in W$ such that Rwv $(\mathfrak{F}, V)\models\varphi[v]$. Furthermore, $\mathfrak{F}\models\varphi[w]$ iff $(\mathfrak{F}, V)\models\varphi[w]$ for every valuation V, and $\mathfrak{F}\models\varphi$ iff $\mathfrak{F}\models\varphi[w]$ for every $w\in W$.

If $\mathfrak{F} = (W, R)$ is a frame, $W_1\subseteq W$, then $\mathfrak{F}_1 = \big(W_1, R\cap(W_1\times W_1)\big)$ is called a *generated subframe* if $\forall w\in W_1\forall v\in W[Rwv\Rightarrow v\in W_1]$. The smallest generated subframe of $\mathfrak{F}$ containing a given element $w\in W$ is denoted by $\mathfrak{F}_w = (W_w, R_w)$ and is said to be the frame *generated by* w. If $\mathfrak{F} = \mathfrak{F}_w$ for some $w\in W$ then $\mathfrak{F}$ is called a *generated* frame. If $\mathfrak{F}_i = (W_i, R_i)$, $i = 1, 2$, are frames and $f\colon W_1\to W_2$ is a map then $f\colon\mathfrak{F}_1\to\mathfrak{F}_2$ is said to be an $\mathfrak{F}$*-morphism* if

(i) for all $w, v\in W_1$, if R_1wv then $R_2f(w)f(v)$
(ii) for all $w\in W_1, v\in W_2$, if $R_2f(w)v$ then there is a $v'\in W_1$ such that R_1wv' and $f(v') = v$.

A *modal algebra* is an algebra $\mathfrak{A} = (A, +, \cdot, ', 0, 1, {}^\circ)$ such that $(A, +, \cdot, ', 0, 1)$ is a Boolean algebra, $+$ and $\cdot$ denoting lattice sum and product respectively, $'$ denoting complementation and 0 and 1 the smallest and largest element respectively. The unary operation $^\circ$ satisfies the laws

$$(x\cdot y)^\circ = x^\circ\cdot y^\circ$$

and

$$1^\circ = 1.$$

The variety of modal algebras will be denoted by $\mathbf{M}$ and the lattice of subvarieties of a variety $\mathbf{K}$ by $\Lambda(\mathbf{K})$. The domain of algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{A}_1, \ldots$ will always be denoted by $A, B, A_1, \ldots$ If we want to emphasize that an operation $+$ or a polynomial p is to be evaluated in the algebra $\mathfrak{A}$ we may write $+^{\mathfrak{A}}$ or $+_{\mathfrak{A}}$ and $p^{\mathfrak{A}}$ or $p_{\mathfrak{A}}$. To any modal formula φ we may assign an $\mathbf{M}$-polynomial $\hat{\varphi}$ by replacing the proposition letters by variables, the classical connectives by the corresponding Boolean operations and $\Box$ by $^\circ$. The map $L\to\{\mathfrak{A}\in\mathbf{M}\mid\mathfrak{A}$ satisfies $\hat{\varphi} = 1, \varphi\in L\}$ establishes an anti-

-isomorphism between the lattice of modal logics and the lattice of varieties of modal algebras. The following varieties will be of particular importance:

MT defined by $x^\circ \leqslant x^{\circ\circ}$ $(\sim \boldsymbol{K4})$
MRT defined by $x^\circ \leqslant x$, $x^\circ \leqslant x^{\circ\circ}$ $(\sim \boldsymbol{S4})$
$\mathbf{M}^n$ defined by $x^{\circ^{n-1}} \leqslant x^{\circ^n}$,

where $x^{\circ^0} = x$, $x^{\circ^n} = (x^{\circ^{n-1}})^\circ$, $n = 1, 2, 3, \ldots$ Observe that $\mathbf{MT} = \mathbf{M}^2$. The algebras in **MRT** are also known as *interior algebras*.

Associated with every frame $\mathfrak{F} = (W, R)$ is a modal algebra, denoted by $\mathfrak{F}^+$, and called the *Kripke algebra* of $\mathfrak{F}$: $\mathfrak{F}^+ = (\mathscr{P}(W), \cup, \cap, ', \emptyset, W, l_\mathfrak{F})$, where for $A \subseteq W$ $l_\mathfrak{F}(A) = \{w \in W \mid \forall v \in W[Rwv \Rightarrow v \in A]\}$. For any modal algebra $\mathfrak{A}$ there is a frame $\mathfrak{F}_\mathfrak{A} = (W_\mathfrak{A}, R_\mathfrak{A})$, such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{F}_\mathfrak{A}^+$. The set $W_\mathfrak{A}$ consists of all prime filters of $\mathfrak{A}$, and for $w, v \in W_\mathfrak{A}$ $R_\mathfrak{A} wv$ iff $\forall a \in A[a^\circ \in w \Rightarrow a \in v]$. The map $a \to \bar{a} = \{w \in W_\mathfrak{A} \mid a \in w\}$ is an embedding of modal algebras. The pair $(\mathfrak{F}_\mathfrak{A}, \{\bar{a} \mid a \in A\})$ is called the *general frame representing* $\mathfrak{A}$.

If $\mathbf{K}$ is a class of algebras then $S(\mathbf{K})$ and $H(\mathbf{K})$ are the classes of subalgebras and homomorphic images of algebras in $\mathbf{K}$, respectively; $P(\mathbf{K})$ and $P_U(\mathbf{K})$ are the classes of direct products and ultra products of families of algebras in $\mathbf{K}$. The variety generated by a class $\mathbf{K}$ of algebras, $HSP(\mathbf{K})$, will be denoted by $V(\mathbf{K})$; the class of subdirectly irreducibles of a class $\mathbf{K}$ by $\mathbf{K}_{SI}$. We will often use Jónsson's [8] result that for any class $\mathbf{K}$ of algebras $V(\mathbf{K})_{SI} \subseteq HSP_U(\mathbf{K})$ provided the lattices of congruences of the algebras in $V(\mathbf{K})$ are distributive. In particular, under this condition, $V(\mathbf{K})_{SI} \subseteq HS(\mathbf{K})$ whenever $\mathbf{K}$ is a finite set of finite algebras. A variety is called *tabular* if it is generated by a finite algebra; it is called *pretabular* if it is not tabular but every proper subvariety is tabular. A variety is called *locally finite* if the finitely generated algebras in it are finite.

If A is a set, $|A|$ will denote its cardinality. The set of natural numbers $1, 2, 3, \ldots$ will be denoted by N. If $(P, \leqslant)$ is a partially ordered set and $x \in P$ then $(x]$ stands for $\{y \in P \mid y \leqslant x\}$, $[x)$ for $\{y \in P \mid y \geqslant x\}$ and for $x, y \in P$ $[x, y] = \{z \in P \mid x \leqslant z \leqslant y\}$. If $x, y \in P$ and $x < y$ such that for all $z \in P$ satisfying $x \leqslant z \leqslant y$ either $x = z$ or $z = y$ we say that y covers x and we write $x \prec y$. If $\mathfrak{A}$ is an algebra and $S \subseteq A$ then $[S]$ (or $[S]^\mathfrak{A}$) is the subalgebra of $\mathfrak{A}$ generated by S. If we want to emphasize that $[S]$ is to be considered an algebra belonging to $\mathbf{K}$ we also write $[S]_\mathbf{K}$. If $\mathfrak{m}$ is a cardinal, the free algebra on $\mathfrak{m}$ generators in the variety $\mathbf{K}$ will be denoted by $\mathfrak{F}_\mathbf{K}(\mathfrak{m})$.

If $\mathfrak{A} \in \mathbf{M}$ and $F \subseteq A$ then F is called an *open filter* if F is a filter such that for any $x \in F$, $x^\circ \in F$. In [4] we showed that the lattice of congruences of a modal algebra is isomorphic to the lattice of its open filters — hence distributive, so we may apply Jónsson's theorem to classes of modal algebras. If F is an open filter in $\mathfrak{A}$ we write $\mathfrak{A}/F$ for the associated quotient algebra. $\mathfrak{A}$ is subdirectly irreducible iff it has a smallest open filter, $\neq \{1\}$. In particular, for any frame $\mathfrak{F}$, $\mathfrak{F}^+$ is subdirectly irreducible iff $\mathfrak{F}$ is a generated frame. If $\mathfrak{A} \in \mathbf{M}$, $u \in A$ such that $u^\circ \geqslant u$, then $[u)$ is an open filter in $\mathfrak{A}$ and $\mathfrak{A}/[u) \subseteq (u]$, where $(u]$ will always be assumed to be endowed with the usual Boolean operations and the operation ${}^{\circ(u]}$ given by $x^{\circ(u]} = x^\circ \cdot u$, for $x \in (u]$. Finally, for any class $\mathbf{K}$ of modal algebras, we have $HS(\mathbf{K}) = SH(\mathbf{K})$ since modal algebras have the congruence extension property. For further references concerning lattice theory and universal algebra, consult [1].

1. Tabular varieties

In this section we want to show that in $\Lambda(\mathbf{MT})$ any cover of a tabular variety is tabular.

1.1 Definition. Let $\mathfrak{F} = (W, R)$ be a frame. An n-tuple $(w_1, \ldots, w_n)$, $w_i \in W$, $i = 1, \ldots, n$, $n \in N$, is called an *R-chain* (*of length* n) if, for $i = 1, \ldots, n-1$, Rw_iw_{i+1} and $\neg Rw_{i+1}w_i$. The height $h(\mathfrak{F})$ of $\mathfrak{F}$ is the supremum of the lengths of R-chains in $\mathfrak{F}$; if $\mathfrak{F} = (\varnothing, \varnothing)$ we put $h(\mathfrak{F}) = \mathbf{0}$. By the height $h(w)$ of an element $w \in W$ we understand $h(\mathfrak{F}_w)$.

The class of frames of height $\leqslant n$ turns out to be modally definable.

1.2 Definition. Let for $n = 0, 1, \ldots$ φ_n be the modal formula defined by the clauses:

(i) $\varphi_0 = \bot$
(ii) if φ_n has been defined then
$\varphi_{n+1} = p_{n+1} \to \Box(\Box\neg p_{n+1} \to \varphi_n)$.

1.3 Lemma. *Let $\mathfrak{F} = (W, R)$ be a frame. If $w_1 \in W$ then $\mathfrak{F} \models \varphi_n[w_1]$ iff for every R-chain $(w_1, \ldots, w_m)$* in $\mathfrak{F}$, $m \leqslant n$.

Proof: For $n = 0$ the assertion holds. Suppose it holds for $n = k$ as well. Let $\mathfrak{F} \models \varphi_{k+1}[w_1]$ and let $(w_1, \ldots, w_m)$ be an R-chain in $\mathfrak{F}$. We claim that $\mathfrak{F} \models \varphi_k[w_2]$. Indeed, let V be any valuation and let V' be the valuation satisfying $V'(p_i) = V(p_i)$, if $i \neq k+1$, and $V'(p_{k+1}) = \{w_1\}$. Since p_{k+1} does not occur in φ_k, $(\mathfrak{F}, V') \models \varphi_k[w_2]$ if and only if $(\mathfrak{F}, V) \models \varphi_k[w_2]$. But, since $\mathfrak{F} \models \varphi_{k+1}[w_1]$, certainly $(\mathfrak{F}, V') \models \varphi_{k+1}[w_1]$. Because $V'(p_{k+1}) = \{w_1\}$ and $\neg Rw_2w_1$, we find that $w_2 \in V'(\Box\neg p_{k+1})$. Since Rw_1w_2 it follows that $(\mathfrak{F}, V') \models \varphi_k[w_2]$, and hence that $(\mathfrak{F}, V) \models \varphi_k[w_2]$. Since V was arbitrary we have shown that $\mathfrak{F} \models \varphi_k[w_2]$ and it follows that $m-1 \leqslant k$, thus $m \leqslant k+1$.

For the converse, assume that $\mathfrak{F} \not\models \varphi_{k+1}[w_1]$. There is a valuation V, such that $(\mathfrak{F}, V) \not\models \varphi_{k+1}[w_1]$; hence there is a $w_2 \in W$, such that Rw_1w_2 and $w_2 \in V(\Box\neg p_{k+1})$ but $(\mathfrak{F}, V) \not\models \varphi_k[w_2]$. Since $w_1 \in V(p_{k+1})$, $\neg Rw_2w_1$, and since $\mathfrak{F} \not\models \varphi_k[w_2]$ there is an R-chain $(w_2, \ldots, w_m)$ such that $m > k+1$. Then $(w_1, \ldots, w_m)$ is an R-chain in $\mathfrak{F}$ such that $m > k+1$.

1.4 Definition. Let $\mathbf{K}(n) = \{\mathfrak{A} \in \mathbf{M} \mid \mathfrak{A} \models \hat{\varphi}_n = 1\}$, $n \in N$.

The Kripke algebras belonging to $\mathbf{K}(n)$ are the algebras $\mathfrak{F}^+$, where $\mathfrak{F}$ is a frame of height $\leqslant n$ in virtue of Lemma 3. Subalgebras of such Kripke algebras also belong to $\mathbf{K}(n)$, and in fact, every algebra in $\mathbf{K}(n)$ is a subalgebra of a Kripke algebra $\mathfrak{F}^+$ with $h(\mathfrak{F}) \leqslant n$:

1.5 Lemma. *Let $\mathfrak{A} \in \mathbf{M}$, and let $(\mathfrak{F}_{\mathfrak{A}}, \mathscr{W}_{\mathfrak{A}})$ be the general frame representing $\mathfrak{A}$. Then $\mathfrak{A} \models \hat{\varphi}_n = 1$ iff $h(\mathfrak{F}_{\mathfrak{A}}) \leqslant n$.*

Proof: The proof is similar to that of lemma 3; we only need to modify the first half of the proof of lemma 3 slightly. Indeed, the thing we have to be careful about is the choice of $V'(p_{k+1})$. If $w_1, w_2 \in W_{\mathfrak{A}}$, such that $R_{\mathfrak{A}} w_1 w_2$, $\neg R_{\mathfrak{A}} w_2 w_1$ then there is an $a \in A$ such that $a^\circ \in w_2$, $a \notin w_1$. Put $V'(p_{k+1}) = \bar{a}' \in \mathscr{W}_{\mathfrak{A}}$, (where $\bar{a} = \{w \in W_{\mathfrak{A}} \mid a \in w\}$).

1.6 Corollary. $\mathbf{K}(n) = \{\mathfrak{A} \in \mathbf{M} \mid \mathfrak{A} \in S(\mathfrak{F}^+), \mathfrak{F}$ *a frame*, $h(\mathfrak{F}) \leqslant n\}$.

1.7 Definition. Let $\mathfrak{A} \in \mathbf{M}$. The height $h(\mathfrak{A})$ of $\mathfrak{A}$ is $\inf\{n \in N \mid \mathfrak{A} \in \mathbf{K}(n)\}$. And if $\mathbf{K} \subseteq \mathbf{M}$ is a variety, the height $h(\mathbf{K})$ of $\mathbf{K}$ is $\inf\{n \in N \mid \mathbf{K} \subseteq \mathbf{K}(n)\}$.

Hence, if $h(\mathfrak{A}) = n$ then there is a frame $\mathfrak{F}$ of height n such that $\mathfrak{A} \in S(\mathfrak{F}^+)$. Let $\mathbf{KT}(n) = \mathbf{K}(n) \cap \mathbf{MT}$. Observe that if $\mathfrak{A} \in \mathbf{KT}(n)$, then the frame $\mathfrak{F}_{\mathfrak{A}}$ of height $\leqslant n$ obtained in lemma 5 is transitive, so $\mathfrak{F}_{\mathfrak{A}}^+ \in \mathbf{KT}(n)$ as well.

1.8 Theorem[2]. *$\mathbf{KT}(n)$ is locally finite, $n \geqslant 0$.*

Proof: The proof is by induction on n.

(i) $\mathbf{KT}(0) = V(\mathbf{1})$. Here $\mathbf{1}$ denotes the one-element algebra.

(ii) Suppose $\mathbf{KT}(n)$ is locally finite and $\mathfrak{A} \in \mathbf{KT}(n+1)_{SI}$ is finitely generated, say by $x_1, \ldots, x_m$. Let $\mathfrak{A} \in S(\mathfrak{F}^+)$, where $\mathfrak{F} = (W, R)$ is a generated transitive frame of height $\leqslant n+1$. Let $W_1 = \{w \in W \mid h(w) \leqslant n\}$, $R_1 = R \cap (W_1 \times W_1)$ and $\mathfrak{F}_1 = (W_1, R_1)$. Then $\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}$ — and we may assume it to be a proper subframe — the height of which is $\leqslant n$. Since the map $x \mapsto x \cap W_1$ constitutes a homomorphism, $S = \{x \cap W_1 \mid x \in A\}$, being a finitely generated algebra in $\mathbf{MT}$ of height $\leqslant n$, is finite. Furthermore, if $x \in A$ then $x^\circ \leqslant W_1$ unless $x = 1$ or $x = W_1$ and $W \setminus W_1$ consists of an irreflexive element, since $x^\circ \leqslant x^{\circ\circ}$. Hence $|A| = |[\{x_1, \ldots, x_m\} \cup S]_{\mathbf{B}}| \leqslant 2^{2^{m+K}}$, where $K = |\mathfrak{F}_{\mathbf{KT}(n)}(m)|$ and $\mathbf{B}$ denotes the variety of Boolean algebras. It follows that there are only finitely many subdirectly irreducible algebras in $\mathbf{KT}(n+1)$ generated by m elements. Every m-generated algebra in $\mathbf{KT}(n+1)$ can be embedded in a finite product of those, and hence will be finite.

Theorem 2.1 of [3] has an obvious generalization to the setting of $\mathbf{M}^n$. We need only a special case:

[2] This result was proved earlier in [14], Ch. II, thm. 6.5.

1.9 Theorem. *Let $\mathfrak{A} \in \mathbf{MT}$ be finitely generated and let $\mathfrak{B}$ be a finite algebra. If $g\colon \mathfrak{A} \to \mathfrak{B}$ is a subjective homomorphism then there is an $a \in A$ such that $a \leqslant a^\circ$ and such that $g \mid (a]\colon (a] \to \mathfrak{B}$ is an isomorphism.*

Proof: Let $\mathfrak{A} = [\{a_1, \ldots, a_m\}]_{\mathrm{M}}$ and $b_i = g(a_i)$, $i = 1, \ldots, m$. Then $\mathfrak{B} = [\{b_1, \ldots, b_m\}]_{\mathrm{M}}$. Let $p_{b_i}(x_1, \ldots, x_m)$ be the projection onto the i-th coordinate, $i = 1, \ldots, m$, and let for $b \in B$, $b \neq b_1, \ldots, b_m$, $p_b(x_1, \ldots, x_m)$ be any $\mathbf{M}$-polynomial such that $p_b(b_1, \ldots, b_m) = b$. Let Ω be the set of formulas of the form $p_b + p_c = p_{b+c}$, $p_b \cdot p_c = p_{b\cdot c}$, $p_{b'} = (p_b)'$, $p_b^\circ = (p_b)^\circ$, $p_0 = 0$, $p_1 = 1$, for $b,c \in B$. Every such formula is $\mathbf{M}$-equivalent to one of the form $r = 1$; let $\Omega' = \{r_i = 1 \mid i = 1, \ldots, k\}$ be the set of equivalents of this form of the formulas in Ω. Then $\mathfrak{B}$ is free on $\{b_1, \ldots \ldots, b_m\}$ with respect to the relations $r_i(b_1, \ldots, b_m) = 1$, $i = 1, \ldots, k$. Let $y = \prod_{i=1}^{k} r_i(a_1, \ldots, a_m)$ and $a = y \cdot y^\circ$. Then $a^\circ = (y \cdot y^\circ)^\circ = y^\circ \cdot y^{\circ\circ} = y^\circ \geqslant a$, thus $[a)$ is an open filter. According to a remark in the preliminaries, $(a] \cong \mathfrak{A}/[a)$, where in $(a]$, $x^{\circ(a]} = x^\circ \cdot a$. Since $g^{-1}(\{1\}) \supseteq [a)$, $g \mid (a]\colon (a] \to \mathfrak{B}$ is a homomorphism, satisfying $g(a_i \cdot a) = b_i$, $i = 1, \ldots, m$. Note that $(a]$ is generated by $a_1 \cdot a, \ldots, a_m \cdot a$. Also $r_j^{(a]}(a_1 \cdot a, \ldots, a_m \cdot a) = r_j^{\mathfrak{A}}(a_1, \ldots \ldots, a_m) \cdot a = a$, so $(a]$ satisfies the relations $r_j(a_1 \cdot a, \ldots, a_m \cdot a) = 1$, $i = 1, \ldots, k$. Hence there is a homomorphism $h\colon \mathfrak{B} \to (a]$ such that $h(b_i) = a_i \cdot a$, $i = 1, \ldots, m$. Therefore $h \circ g \mid (a] = id \mid (a]$, and it follows that g is an isomorphism.

Now we are ready to prove the main lemma.

1.10 Lemma. *Let $\mathfrak{A} \in \mathbf{MT}$ be finitely generated. If $h(\mathfrak{A}) = \infty$ then for every $n \in N$ there is a $\mathfrak{B}_n \in H(\mathfrak{A})$ such that $h(\mathfrak{B}_n) = n$.*

Proof: Let $\mathfrak{F} = (W, R)$ be the canonical frame $\mathfrak{F}_{\mathfrak{A}}$; we may think of $\mathfrak{A}$ as a subalgebra of $\mathfrak{F}^+$. Since $\mathfrak{A} \in \mathbf{MT}$, R is transitive. Since $h(\mathfrak{A}) = \infty$, $h(\mathfrak{F}) = \infty$. First we show that if there is a $w \in W$ such that $h(w) = n$ then there is an algebra $\mathfrak{B}_n \in H(\mathfrak{A})$ such that $h(\mathfrak{B}_n) = n$. The map $f\colon \mathfrak{A} \to \mathfrak{F}_w^+$ defined by $x \mapsto x \cap W_w$ is a homomorphism since $\mathfrak{F}_w$ is a generated subframe of $\mathfrak{F}$. Because $h(\mathfrak{F}_w) = n$, $h(f[\mathfrak{A}]) \leqslant n$, and as $f[\mathfrak{A}]$ is finitely generated, it is finite in virtue of theorem 8. By theorem 9 there is an $a \in A$ such that $a \leqslant a^0$ and such that $f \mid (a]\colon (a] \to f[\mathfrak{A}]$ is an isomorphism. Since $f(a) = 1$, $a \geqslant W_w$. We claim that if $v \in a$, then $\{v\} \in (a] \subseteq A$. The properties of $\mathfrak{F}_{\mathfrak{A}}$ guarantee that $\{v\} = \Pi\{a \cdot x \mid v \in x \in A\}$. But the set $\{a \cdot x \mid v \in x \in A\}$ is contained in $(a]$ and therefore finite; hence $\{v\} \in (a]$. Since $f \mid (a]$ is an isomorphism and $\{v\} \neq 0$ it follows that for every $v \in a$ $0 \neq f(\{v\}) = \{v\} \cap W_w$, whence, in fact, $a = W_w$. Thus $(a] = \mathfrak{F}_w^+ \in H(\mathfrak{A})$. Since $h(\mathfrak{F}_w) = n$, we have $h(\mathfrak{F}_w^+) \leqslant n$; however, it is easily seen (and it also follows from Lemma 5) that $h(\mathfrak{F}_w^+) < n$ is impossible. Hence $h(\mathfrak{F}_w^+) = n$, and by putting $\mathfrak{B}_n = \mathfrak{F}_r^+$ we obtain the sought-for algebra.

In order to complete the proof of the lemma we have to show that there are elements $w \in W$ of arbitrary finite height. Suppose $n \in N$ is the smallest natural number such that there is no $w \in W$ satisfying $h(w) = n$. Then there are neither any elements $w \in W$ satisfying $h(w) \in N$, $h(w) > n$. Let $W_{n-1} = \{w \in W \mid h(w) \leqslant n-1\}$. Then $\mathfrak{F}_{n-1} = (W_{n-1}, R \cap (W_{n-1} \times \times W_{n-1}))$ is a generated subframe of $\mathfrak{F}$ (possibly empty), satisfying $W_{n-1} \neq W$ since $h(\mathfrak{F}) = \infty$, $h(\mathfrak{F}_{n-1}) = n-1$. The map $f\colon \mathfrak{A} \to \mathfrak{F}^{+}_{n-1}$ defined by $x \to x \cap W_{n-1}$ is a homomorphism, and $h(f[\mathfrak{A}]) \leqslant n-1$. Since $f[\mathfrak{A}]$ is finitely generated, $f[\mathfrak{A}]$ is finite, and as in the first part of this proof, it follows that $W_{n-1} = a_{n-1} \in A$ and that $h((a_{n-1}]) = h(f[\mathfrak{A}]) = n-1$. Observe that $a_{n-1} \leqslant a^{\circ}_{n-1}$. If $a_{n-1} < a^{\circ}_{n-1}$ then let $w \in a^{\circ}_{n-1} \setminus a_{n-1}$ and we see that $h(w) = n$, a contradiction. Thus $a_{n-1} = a^{\circ}_{n-1}$. Consider $\mathscr{G} = \{F \subseteq A \mid F$ is an open filter, $F \subsetneq [a_{n-1})\}$. Then $\mathscr{G} \neq \varnothing$, since $\{1\} \in \mathscr{G}$. Also, by Zorn's lemma, $\mathscr{G}$ has a maximal element, say G_0. We claim that $\mathfrak{A}/G_0$ has height n. Indeed, let $g\colon \mathfrak{A} \to \mathfrak{A}/G_0$ be the canonical homomorphism. Suppose that $g(a_{n-1}) < g(v) < 1$, $v \in A$. We may then assume that $a_{n-1} < v$, $v \notin G_0$. Therefore the open filter generated by $G_0 \cup \{v\}$ equals $[a_{n-1})$, which implies the existence of an $u \in G_0$ such that $u \cdot v \cdot v^{\circ} \leqslant a_{n-1}$, whence $u^{\circ} \cdot v^{\circ} \leqslant a^{\circ}_{n-1} = a_{n-1}$. It follows that $g(v)^{\circ} = g(v^{\circ}) = g(v^{\circ}) \cdot g(u^{\circ}) = g(v^{\circ} \cdot u^{\circ}) \leqslant g(a_{n-1})$, whence $g(v)^{\circ} = g(a_{n-1})$ (*). Now, let $\mathfrak{F}^1 = (W^1, R^1)$ be the generated subframe of $\mathfrak{F}$ corresponding with $\mathfrak{A}/G_0$ — i.e., let $W^1 = \{w \in W \mid w \supseteq G_0\}$, $R^1 = R \cap W^1 \times W^1$. (Recall that $\mathfrak{F}$ is the canonical frame of $\mathfrak{A}$). Then we may think of A/G_0 as $\{x \cap W^1 \mid x \in A\}$. If $w_1, w_2 \in W^1 \setminus a_{n-1}$, $w_1 \neq w_2$, then Rw_1w_2. For if $\neg Rw_1w_2$ then there is an $a \in A$ such that $a^{\circ} \in w_1$, $a \notin w_2$. Let $v = (a + a_{n-1}) \cap W^1$. Then we have an element in $\mathfrak{A}/G_0$ satisfying $a_{n-1} < v < 1$. For since $a^{\circ}_{n-1} = a_{n-1}$ and a° *non* $\leqslant a_{n-1}$, we infer that $a \cap W^1$ *non* $\leqslant a_{n-1}$, thus $a_{n-1} < v$, and because $w_2 \notin v$, $v < 1$. However, $v^{\circ} \geqslant (a^{\circ} + a_{n-1}) \cap W_1 > a_{n-1}$, contradicting (*). Thus $W^1 \setminus a_{n-1}$ is a cluster. Since $W^1 > a_{n-1}$ there is a $w \in W^1 \setminus a_{n-1}$. Then $h(w) \leqslant n$, and because $w \notin a_{n-1}$, $h(w) = n$ — contradictory to our assumption that there are no elements of height n in $\mathfrak{F}$.

In this lemma, the assumption of transitivity (i.e., $\mathfrak{A} \in \mathbf{MT}$) is essential. For example, let $\mathfrak{F} = (\{1, 2, 3\}, \{(1, 2), (2, 3), (3, 1)\})$. Then $h(\mathfrak{F}^{+}) = \infty$, $\mathfrak{F}^{+}$ is finite but $H(\mathfrak{F}^{+})$ contains no algebras of finite height except **1**.

The next result is an extension of [4], 7.2 to the setting of **MT**.

1.11 Lemma. *Let* $\mathbf{K} \subseteq \mathbf{MT}$ *be a locally finite variety.* $\mathbf{K}$ *contains an infinite subdirectly irreducible iff* $\mathbf{K}$ *contains infinitely many finite subdirectly irreducibles.*

Proof: Let $\mathfrak{A} \in \mathbf{K}_{SI}$ be given, $|A|$ infinite. First we show that any finite subalgebra of $\mathfrak{A}$ is subdirectly irreducible. Let $\mathfrak{B} \in S(\mathfrak{A})$ be such that B is finite. Let $a \in A$, $a < 1$ such that $[a)$ is the smallest open filter, $\neq \{1\}$, in $\mathfrak{A}$. Let $b = \sum\{x \cdot x^{\circ} \mid x \in B, x < 1\}$. The element b is well defined since B is finite and $b \in B$. Since for all $x \in B$ such that $x < 1$, $x \cdot x^{\circ} \leqslant a$, $b \leqslant a$ as well, and hence $[b \cdot b^{\circ})$ is the smallest open filter in $\mathfrak{B}$, $\neq \{1\}$. Thus $\mathfrak{B}$ is subdirectly irreducible.

Now we define an infinite sequence of finite subalgebras of $\mathfrak{A}$ as follows. Let $\mathfrak{A}_0 = [\varnothing]_{\mathbf{M}} \in S(\mathfrak{A})$, then $\mathfrak{A}_0$ is finite since **K** is locally finite. If $\mathfrak{A}_n \in S(\mathfrak{A})$ has been defined such that $|A_n|$ is finite and $\mathfrak{A}_n$ is subdirectly irreducible, choose $x_{n+1} \in A \setminus A_n$ and let $\mathfrak{A}_{n+1} = [A_n \cup \{x_{n+1}\}]_{\mathbf{M}} \subseteq \mathfrak{A}$. Then $\mathfrak{A}_{n+1}$ is finite and subdirectly irreducible, by the remarks in the first paragraph of the proof.

For the converse we only need to note that the property of being subdirectly irreducible in **MT** is first order expressible. Hence, if **K** contains infinitely many finite subdirectly irreducibles any non-principal ultraproduct of these will provide an infinite subdirectly irreducible in **K**.

Observe that this lemma holds in fact for any variety $\mathbf{K} \subseteq \mathbf{M}^n$, $n \in N$.

Now we are ready to prove the result we are aiming at:

1.12 THEOREM. *Let* $\mathbf{K} \in \Lambda(\mathbf{MT})$. *Then* **K** *is tabular iff* $|\Lambda(\mathbf{K})|$ *is finite.*

PROOF. $\Rightarrow$ Since $\mathbf{K}_{SI}$ is finite, by Jónsson's theorem. $\Leftarrow$ By Lemma 10 (and lemma 5) **K** cannot contain finitely generated algebras of infinite height or of arbitrary finite height. Hence $\mathbf{K} \subseteq \mathbf{KT}(n)$, for some $n \in N$, and therefore **K** is locally finite. By Lemma 11, **K** cannot contain an infinite subdirectly irreducible, since otherwise it would possess infinitely many finite ones and would therefore have infinitely many subvarieties. Thus **K** contains only finite subdirectly irreducibles, and, again, only finitely many ones. Hence **K** is generated by a finite algebra.

1.13 COROLLARY. *Let* $\mathbf{K}_1, \mathbf{K}_2 \in \Lambda(\mathbf{MT})$ *be such that* $\mathbf{K}_1$ *is tabular and* $\mathbf{K}_1 \prec \mathbf{K}_2$. *Then* $\mathbf{K}_2$ *is tabular as well.*

PROOF: By theorem 12, $\Lambda(\mathbf{K}_1)$ is finite. Since $\Lambda(\mathbf{MT})$ is distributive, $\Lambda(\mathbf{K}_2)$ is finite, too. Hence $\mathbf{K}_2$ is tabular.

1.14 COROLLARY. *Let* $\mathbf{K} \in \Lambda(\mathbf{MT})$ *be tabular. Then* **K** *has finitely many covers in* $\Lambda(\mathbf{MT})$.

PROOF: Let $\mathbf{K}_{SI} = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$. The $\mathfrak{A}_i$, $i = 1, \ldots, n$ are finite. Let $n = \max\{h(\mathfrak{A}_i) \mid i = 1, \ldots, n\}$, and let $k = \min\{m \mid \mathfrak{A}_i$ is m-generated, $i = 1, \ldots, n\}$. We claim that whenever $\mathbf{K} \prec \mathbf{K}_1$ then $\mathbf{K}_{1SI}$ consists of (finite) $(k+1)$-generated algebras of height $\leqslant n+1$. It will follow that whenever $\mathbf{K} \prec \mathbf{K}_1$ then $\mathbf{K}_1 = V(\mathbf{K}_{1SI})$ where $\mathbf{K}_{1SI} \subseteq H\big(\mathfrak{F}_{\mathbf{KT}(n+1)}(k+1)\big)$. Since $\mathfrak{F}_{\mathbf{KT}(n+1)}(k+1)$ is finite this shows that **K** has only finitely many covers.

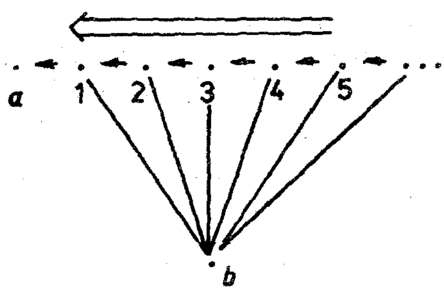
Firstly, suppose that $\mathbf{K} \prec \mathbf{K}_1$ and that $\mathfrak{B} \in \mathbf{K}_{1SI}$ is such that $h(\mathfrak{B}) = h > n+1$, $\mathfrak{B}$ finite. Then $\mathfrak{B} \cong \mathfrak{F}^+$ for some frame $\mathfrak{F} = (W, R)$ of height h. Let $w \in W$ be such that $h(w) = n+1$. Then $\mathfrak{F}_w^+$ is a subdirectly irreducible algebra of height $n+1$ and $\mathfrak{F}_w^+ \in H(\mathfrak{B})$. Let $\mathbf{K}_2 = \mathbf{K} + V(\mathfrak{F}_w^+)$, then $\mathbf{K} \leqslant \mathbf{K}_2 \leqslant V(\mathfrak{B})$, and since $h(\mathbf{K}) < h(\mathbf{K}_2) < h(\mathbf{K}_1))$, $K < K_2 < \mathbf{K}_1$, a contradiction.

Next suppose that $\mathbf{K} \prec \mathbf{K}_1$ and that $\mathfrak{B} \in \mathbf{K}_{1SI}$ is such that $\mathfrak{B}$ is not $(k+1)$-generated. Choose r minimal k such that $\mathfrak{B} = [\{b_1, \ldots, b_r\}]_{\mathbf{M}}$, and let $\mathfrak{B}_1 = [\{b_1, \ldots, b_{r-1}\}]_{\mathbf{M}}$. Then $\mathfrak{B}_1$ is a subdirectly irreducible algebra in virtue of the first half of the proof of lemma 11. Since $\mathfrak{B}_1$ is not k-generated — otherwise $\mathfrak{B}$ would be $(k+1)$-generated — we have $\mathbf{K} < \mathbf{K} + V(\mathfrak{B}_1) \leqslant \mathbf{K}_1$. However, since $|B| > |B_1|$, $\mathfrak{B} \notin V(\mathfrak{B}_1)$, $\mathfrak{B} \notin \mathbf{K}$, thus $\mathbf{K} + V(\mathfrak{B}_1) < \mathbf{K}_1$. We thus arrived at a contradiction.

We mentioned already that in $\Lambda(\mathbf{M})$ a cover of a tabular variety need not be tabular. This phenomenon occurs already at a low level. Let $\mathbf{2}$ and $\mathbf{2}^+$ denote the modal algebras $\{0, 1\}$ with an operator $^\circ$ satisfying $0^\circ = 0$, $1^\circ = 1$ and $0^\circ = 1^\circ = 1$, respectively. It is well-known (see, for example, [4]) that $V(\mathbf{2})$ and $V(\mathbf{2}^+)$ are the atoms of $\Lambda(\mathbf{M})$. We shall now give an example of a non-tabular cover of $V(\mathbf{2}^+)$, which belongs to $\mathbf{M}^3$.

Let $\mathfrak{F} = (W, R)$ where $W = N \cup \{a, b\}$, $a, b \notin N$, $a \neq b$, and where

$$Rwv \quad \text{iff} \quad \begin{cases} w \neq a, b, \ v = a \\ \{w, v\} = \{b, n\} \ \text{for some } n \in N \\ w > v, \ w, v \in N \end{cases}$$



Let $\mathfrak{A} = [\varnothing]_{\mathbf{M}} \in S(\mathfrak{F}^+)$. Note that A consists of the finite and cofinite subsets of W. Indeed, $0^\circ = \{a\} \in A$, $\{a\}'^\circ = \{a, b\} \in A$, $\{a, b\}^\circ = \{a, 1\} \in A$ and if $\{a, 1, \ldots, k\} \in A$, $k \geqslant 1$ then $\{a, 1, \ldots, k+1\} = \{a, b, 1, \ldots, k\}^\circ \in A$ Thus A contains all atoms and therefore all finite and cofinite sets. Furthermore, if $x \in A$ and x is finite then so is x°, and if x is cofinite we have the following cases:

(i) $x = 1$. Then $x^\circ = 1$.
(ii) $x = \{b\}'$ or $x = \{a\}'$. Then $x^\circ = \{a, b\}$.
(iii) $x \leqslant \{n\}'$, $n \in N$. Then $x^\circ \subseteq \{a\} \cup [1, n]$.

It follows that the finite and cofinite subsets of W form a subalgebra of $\mathfrak{F}^+$, which equals $\mathfrak{A}$. Since $\mathfrak{A}$ is 0-generated, it is the free object on 0 generators in the variety $V(\mathfrak{A})$. Now let $\mathfrak{B} \in V(\mathfrak{A})_{SI}$, and let $\mathfrak{B}_0 = [\varnothing]_{\mathbf{M}} \in S(\mathfrak{B})$. Then $B_0 \in H(\mathfrak{F}_{V(\mathfrak{A})}(0)) = H(\mathfrak{A})$. Let $f\colon \mathfrak{A} \to \mathfrak{B}_0$ be an onto homomorphism. If f is not $1-1$, then there is an $x \in A$, $x \neq 1$ such that $f(x) = 1$. By the remarks above, $x^\circ \subseteq \{a, b\} \cup [1, n]$ for some $n \in N$, hence $x^{\circ\circ\circ} = \{a\}$. It follows that $\mathfrak{B}_0 \cong \mathbf{2}^+$, i.e. the modal algebra $\{0, 1\}$, satisfying $0^\circ = 1^\circ = 1$.

Therefore $\mathfrak{B}$ satisfies $0^\circ = 1$. However, the only subdirectly irreducible modal algebra satisfying $0^\circ = 1$ is $\mathbf{2}^+$, so $\mathfrak{B} \cong \mathbf{2}^+$. On the other hand, if f is 1—1, then $\mathfrak{A} \in S(\mathfrak{B}_0) \subseteq S(\mathfrak{B})$. Hence, if $\mathbf{K}$ is a variety such that $V(\mathbf{2}^+) \subsetneq \mathbf{K} \subseteq V(\mathfrak{A})$ then there is a $\mathfrak{B} \in \mathbf{K}_{SI} \subseteq V(\mathfrak{A})_{SI}$ such that $\mathfrak{B} \not\cong \mathbf{2}^+$, which implies that $\mathfrak{A} \in S(\mathfrak{B}) \subseteq \mathbf{K}$, whence $\mathbf{K} = V(\mathfrak{A})$. Since $\mathfrak{A} \notin V(\mathbf{2}^+)$, $V(\mathbf{2}^+) < V(\mathfrak{A})$, so $V(\mathfrak{A})$ covers $V(\mathbf{2}^+)$. Because $V(\mathfrak{A})_{FSI} = \{\mathbf{2}^+\}$, as we have seen, $V(\mathfrak{A})$ is nontabular. And since $\forall x \in A$, $x \neq 1$, $x^{\circ\circ} = \{a\} = x^{\circ\circ\circ}$, and obviously also $1^{\circ\circ} = 1^{\circ\circ\circ}$, $\mathfrak{A}$ satisfies the equation $x^{\circ^2} = x^{\circ^3}$. Thus $V(\mathfrak{A}) \subseteq \mathbf{M}^3$.

A slight modification of this example provides a non-tabular cover in $\mathbf{M}^3$ of $V(\mathbf{2})$. Indeed, let $\mathfrak{F}' = (W, R')$, where $R' = R \cup \{(a, a)\}$, and let $\mathfrak{A}' = [\{a\}]_{\mathbf{M}} \in S(\mathfrak{F}'^+)$. Then $V(\mathbf{2}) \prec V(\mathfrak{A})$, as one can show using a somewhat more elaborate argument.

For an example of a family of non-tabular covers of $V(\mathbf{2})$ of the cardinality of the continuum, belonging to $\mathbf{MR}^3$, we refer to [2]. The simpler example we gave here can be used to obtain a continuum of covers of $V(\mathbf{2}^+)$ (and in a similar way of $V(\mathbf{2})$) in $\Lambda(\mathbf{M}^4)$. We indicate the procedure briefly. Let for $M \subseteq N$ $\mathfrak{F}_M = (W, R_M)$, where $R_M = R \setminus \{(b, n) \mid n \in M\}$, and let $\mathfrak{A}_M = [\emptyset]_{\mathbf{M}} \in S(\mathfrak{F}_M^+)$. Then $\mathfrak{A}_M$ satisfies $x^{\circ^3} = x^{\circ^4}$, hence $\mathfrak{A}_M \in \mathbf{M}^4$. By an argument similar to the one used above we show that $V(\mathbf{2}^+) \prec V(\mathfrak{A}_M)$. Finally, since for $M, M' \subseteq N$, $M \neq M'$, clearly $\mathfrak{A}_M \not\cong \mathfrak{A}_{M'}$, it follows that $V(\mathfrak{A}_M) \neq V(\mathfrak{A}_{M'})$, because $\mathfrak{A}_M = \mathfrak{F}_{V(\mathfrak{A}_M)}(0)$.

2. Pretabular varieties in $\Lambda(\mathbf{MT})$

A variety $\mathbf{K} \subseteq \mathbf{M}$ is called *pretabular* if $\mathbf{K}$ is non-tabular but every proper subvariety is tabular. Using the fact that each tabular variety is finitely axiomatizable (see, for example, [4] or [13]), a straight forward application of Zorn's lemma shows that every non-tabular variety contains a pretabular one. The continuously many covers of $V(\mathbf{2})$ and $V(\mathbf{2}^+)$ are examples of pretabular varieties. They are not generated by their finite members. In $\mathbf{MT}(= \mathbf{M}^2)$ the situation is nicer, however.

2.1 THEOREM. *Let* $\mathbf{K} \in \Lambda(\mathbf{MT})$. *If* $\mathbf{K}$ *is pretabular, then* $\mathbf{K}$ *is generated by its finite members.*

PROOF: If $\mathbf{K} \subseteq \mathbf{KT}(n)$ for some $n \in N$ then $\mathbf{K}$ is locally finite, and hence certainly generated by its finite members. If $\mathbf{K} \nsubseteq \mathbf{KT}(n)$, $n \in N$, then $\mathbf{K}$ contains finitely generated algebras of height n, for every $n \in N$, in virtue of Lemma 1.10. Let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be a sequence of finitely generated algebras in $\mathbf{K}$ such that $h(\mathfrak{A}_n) = n$, $n = 1, 2, \ldots$. The $\mathfrak{A}_n$ are finite, again by the local finiteness of the $\mathbf{KT}(n)$, and $\mathbf{K}_1 = V(\{\mathfrak{A}_n \mid n = 1, 2, \ldots\})$ is non-tabular, since if it were tabular, it would be contained in some $\mathbf{KT}(n)$, contradicting the fact that $h(\mathfrak{A}_{n+1}) = n+1$, whence $\mathfrak{A}_{n+1} \notin \mathbf{KT}(n)$. Hence, by pretabularity of $\mathbf{K}$, $\mathbf{K} = \mathbf{K}_1$ and $\mathbf{K}$ is generated by its finite members.

Thus far the positive results concerning tabular and pretabular varieties holding for **MRT** are valid in **MT** as well. Though it would not seem unreasonable to hope for an analogue of Maksimova's result [11], that there is only a finite number of pretabular varieties in **MRT**, the apparent similarity between **MRT** and **MT** turns out to be deceptive: in fact, **MT** contains $2^{\aleph_0}$ pretabular varieties, as we will show now. Our point of departure is the interior algebra $B(\mathfrak{F}_{\mathbf{H}}(1))$, i.e., an algebra in **MRT** whose lattice of open elements is the free algebra on one generator in the variety **H** of Heyting algebras, and which is generated by its open elements as a Boolean algebra (cf. [3]). By modifying the frame $\mathfrak{F} = (W, R)$, used to represent $B(\mathfrak{F}_{\mathbf{H}}(1))$, — in fact, by making the elements of suitable subsets of W irreflexive — we obtain 0-generated infinite algebras which turn out to generate pretabular varieties.

We need some preparation.

2.2 Definition. If p is an **M**-polynomial then p^* is an **M**-polynomial defined by induction according to the following clauses:

(i) If p is a variable x_i, then $p^* = p = x_i$.

(ii) Suppose q^* has been defined for all polynomials q of length $\leqslant n$. Then, if p has length $n+1$, we define:

a) if $p = q+r$ then $p^* = q^*+r^*$

b) if $p = q\cdot r$ then $p^* = q^*\cdot r^*$

c) if $p = q'$ then $p^* = (q^*)'$

d) if $p = q^\circ$ then $p^* = q^*\cdot q^{*\circ}$.

2.3 Lemma. *Let $\mathfrak{F} = (W, R)$ be a frame, $\mathfrak{A} = \mathfrak{F}^+$. Let $\mathfrak{F}^r = (W, R^r)$ where $R^r = R\cup\{(w, w) \mid w \in W\}$ and let $\mathfrak{A}^r = (\mathfrak{F}^r)^+$. Then, for any $a_1, \ldots, a_n \in A$ and any n-ary **M**-polynomial p,*

$$p^{*\mathfrak{A}}(a_1, \ldots, a_n) = p^{\mathfrak{A}^r}(a_1, \ldots, a_n).$$

Proof: By induction on the length of p. If p is a variable then the statement obviously holds. Next assume that the assertion holds for all polynomials of length $\leqslant n$, and let p have length $n+1$. The only interesting case is $p = q^\circ$, for some polynomial q. Then

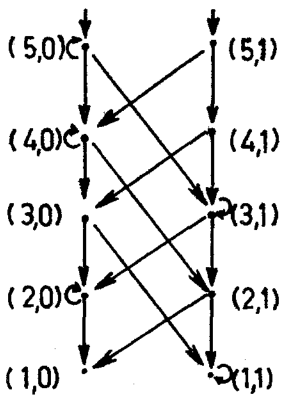
$$\begin{aligned} p^{\mathfrak{A}^r}(a_1, \ldots, a_n) &= (q^{\mathfrak{A}^r}(a_1, \ldots, a_n))^\circ \\ &= l_{\mathfrak{F}^r}(q^{*\mathfrak{A}}(a_1, \ldots, a_n)) \\ &= \{w \in W \mid \forall v \in W[R^r wv \Rightarrow v \in q^{*\mathfrak{A}}(a_1, \ldots, a_n)]\} \\ &= \{w \in W \mid \forall v \in W[[Rwv \vee w = v] \Rightarrow v \in q^{*\mathfrak{A}}(a_1, \ldots, a_n)]\} \\ &= l_{\mathfrak{F}}(q^{*\mathfrak{A}}(a_1, \ldots, a_n))\cap q^{*\mathfrak{A}}(a_1, \ldots, a_n) \\ &= (q^{*\mathfrak{A}})^\circ(a_1, \ldots, a_n)\cap q^{*\mathfrak{A}}(a_1, \ldots, a_n) \\ &= p^{*\mathfrak{A}}(a_1, \ldots, a_n). \end{aligned}$$

2.4 Definition. Let $W = N\times\{0, 1\}$. For $M \subseteq N$ let R_M be the transitive binary relation on W given by:

$$R_M(m,i)(n,j) \quad \text{iff} \quad \begin{cases} i = j,\ m > n \\ i = 1,\ j = 0,\ m > n \\ i = 0,\ j = 1,\ m > n+1 \\ i = j = 0,\ m = n,\ m \notin M \\ i = j = 1,\ m = n,\ m \in M \end{cases}$$

Let $\mathfrak{F}_M = (W, R_M)$, and let $\mathfrak{A}_M = [\varnothing]_{\mathbf{M}} \in S(\mathfrak{F}_M^+)$.

The bottom of $\mathfrak{F}_M$, with $M \cap [1,5] = \{1,3\}$, looks like:



Observe that the subalgebra of $(\mathfrak{F}_M^r)^+$ of finite and cofinite subsets of W is isomorphic to $B(\mathfrak{F}_{\mathbf{H}}(1))$. The set $\{(1,0)\}$ corresponds with the free generator of $\mathfrak{F}_{\mathbf{H}}(1)$ (which also generates $B(\mathfrak{F}_{\mathbf{H}}(1))$), and hence there are for every $n \in N$ unary $\mathbf{M}$-polynomials p_{2n-1} and p_{2n} such that, $p_{2n-1}(\{(1,0)\}) = \{(n,0)\}$ and $p_{2n}(\{(1,0)\}) = \{(n,1)\}$ in $(\mathfrak{F}_M^r)^+$. By lemma 3 for any $M \subseteq N$, $(p_{2n-1}^*)\mathfrak{F}_M^+(\{(1,0)\}) = \{(n,0)\}$, $(p_{2n}^*)\mathfrak{F}_M^+(\{(1,0)\}) = \{(n,1)\}$. Now, whenever $1 \in M \subseteq N$, we have $0^\circ = \{(1,0)\}$ in $\mathfrak{F}_M^+$, so if $1 \in M$ then $\mathfrak{A}_M = [\varnothing]_{\mathbf{M}} \in S(\mathfrak{F}_M^+)$ will contain all atoms of $\mathfrak{F}_M^+$, and therefore the finite and cofinite subsets of W.

2.5 LEMMA. *If $1 \in M \subseteq N$ then $V(\mathfrak{A}_M)$ is a pretabular variety.*

PROOF: By the remarks preceding the lemma $\mathfrak{A}_M$ contains all finite and cofinite subsets of W. Let $\mathbf{K}$ be a variety such that $\mathbf{K} \subseteq V(\mathfrak{A})$. First we show that if $\mathbf{K}$ contains an infinite subdirectly irreducible then $\mathfrak{A}_M \in \mathbf{K}$ and hence $\mathbf{K} = V(\mathfrak{A}_M)$. Let $\mathfrak{A} \in \mathbf{K}_{SI}$ be an infinite algebra. By Jónsson's theorem $\mathfrak{A} \in HSP_U(\mathfrak{A}_M)$. So let $\mathfrak{A}_1 \in P_U(\mathfrak{A}_M)$, $\mathfrak{A}_2 \in S(\mathfrak{A}_1)$, $h\colon \mathfrak{A}_2 \to \mathfrak{A}$ an onto homomorphism. Note that since $\mathfrak{A}_M \cong \mathfrak{F}_{V(\mathfrak{A}_M)}(0)$, $[\varnothing]_{\mathbf{M}}^{\mathfrak{A}_1} = [\varnothing]_{\mathbf{M}}^{\mathfrak{A}_2} \cong \mathfrak{A}_M$. If $[\varnothing]_{\mathbf{M}}^{\mathfrak{A}} \cong \mathfrak{A}_M$ then there is a $u \in [\varnothing]_{\mathbf{M}}^{\mathfrak{A}_2} \subseteq A_2$ such that $u^\circ \neq 1$, $h(u^\circ) = 1$, and hence $\mathfrak{A} \in H((u^\circ]^{\mathfrak{A}_2})$. Let $u^\circ = q_{\mathfrak{A}_2}(0)$ for some unary polynomial q. Since the algebra $(q_{\mathfrak{A}_M}(0)]^{\mathfrak{A}_M}$ is finite, $(q_{\mathfrak{A}_M}(0)]^{\mathfrak{A}_M} \cong (q_{\mathfrak{A}_1}(0)]^{\mathfrak{A}_1}$, by the properties of ultra products. Since $(q_{\mathfrak{A}_1}(0)]^{\mathfrak{A}_1}$ is 0-generated, $(q_{\mathfrak{A}_1}(0)]^{\mathfrak{A}_1} = (u^\circ]^{\mathfrak{A}_2}$, and hence $(u^\circ]^{\mathfrak{A}_2}$ is finite. But that would imply that $\mathfrak{A}$ is finite as well, contradictory to our assumption. It follows that $[\varnothing]_{\mathbf{M}}^{\mathfrak{A}} \cong \mathfrak{A}_M$, whence $\mathfrak{A}_M \in S(\mathfrak{A}) \subseteq \mathbf{K}$.

Next suppose $\mathfrak{A} \in \mathbf{K}_{FSI}$. By the argument in the previous paragraph we see that $\mathfrak{A} \in H\big((u^\circ]^{\mathfrak{A}_M}\big)$ for some $u^\circ \in A_M$, and hence $\mathfrak{A} \cong (u]^{\mathfrak{A}_M}$ for some $u \in A_M$. More precisely, the finite subdirectly irreducible algebras in $V(\mathfrak{A}_M)$ are the algebras $(\mathfrak{F}_M)^+_{(n,i)}$, $n \in N$, $i \in \{0, 1\}$. Here $(\mathfrak{F}_M)_{(n,i)}$ denotes, as usual, the subframe of $\mathfrak{F}_M$ generated by the element (n, i). Now suppose $\mathbf{K}_{FSI}$ is an infinite set. Since $(\mathfrak{F}_M)^+_{(n,i)} \in H\big((\mathfrak{F}_M)^+_{(n,i)}\big)$ whenever $n+1 < m$ it follows that $\mathbf{K}_{FSI} = V(\mathfrak{A}_M)_{FSI}$. But clearly $\mathfrak{A}_M \in SP\big(\big\{(\mathfrak{F}_M)^+_{(n,i)} \mid n \in N,\ i \in \{0,1\}\big\}\big)$, and therefore $\mathbf{K} \supseteq V(\mathbf{K}_{FSI}) \supseteq V(\mathfrak{A}_M)$. We have thus shown that every proper subvariety of $V(\mathfrak{A}_M)$ contains no infinite subdirectly irreducibles and only finitely many finite ones, and is therefore tabular.

2.6 THEOREM. *$\Lambda(\mathbf{MT})$ contains $2^{\aleph_0}$ pretabular varieties.*

PROOF: Let $M, M' \subseteq N$, such that $1 \in M$, $1 \in M'$, and suppose that $\mathfrak{A}_M \cong \mathfrak{A}_{M'}$. Let $f\colon \mathfrak{A}_M \to \mathfrak{A}_{M'}$ be an isomorphism. Since $f(0) = 0$, $f(\{(n, i)\}) = f\big(p^{*\mathfrak{A}_M}_{2n+(i-1)}(0^\circ)\big) = p^{*\mathfrak{A}_{M'}}_{2n+(i-1)}(0^\circ) = \{(n, i)\}$. Hence $l_{\mathfrak{F}_M}(\{(n, i)\}') \subseteq \{(n, i)\}'$ iff $l_{\mathfrak{F}_{M'}}(\{(n, i)\}') \leqslant \{(n, i)\}'$, from which it follows that (n, i) is reflexive in $\mathfrak{F}_M$ iff (n, i) is reflexive in $\mathfrak{F}_{M'}$. Thus $M = M'$. Hence, if $M, M' \subseteq N$, $1 \in M$, $1 \in M'$, $M \neq M'$, then $\mathfrak{A}_M \neq \mathfrak{A}_{M'}$. But since $\mathfrak{A}_M$ and $\mathfrak{A}_{M'}$ are 0-generated, $\mathfrak{F}_{V(\mathfrak{A}_M)}(0) \cong \mathfrak{A}_M$ and $\mathfrak{F}_{V(\mathfrak{A}_{M'})}(0) \cong \mathfrak{A}_{M'}$, it follows then that $V(\mathfrak{A}_M) \neq V(\mathfrak{A}_{M'})$. Hence, by lemma 5, there are $2^{\aleph_0}$ pretabular varieties in $\Lambda(\mathbf{MT})$.

This refutes the claim in [13] that **MRT** and **MT** behave similarly, in this respect.

3. Pretabular varieties in some sublattices of $\Lambda(\mathbf{MT})$

Though the result of the previous section indicates that it will be difficult to describe all pretabular varieties belonging to $\Lambda(\mathbf{MT})$, some more information can be obtained if we restrict ourselves to pretabular varieties having certain desirable properties. We need some preparation. First we show that every finite algebra of given height contains at least one element of a certain set of finite algebras of that height as a subalgebra.

3.1 DEFINITION. Let $\mathfrak{F} = (W, R)$ be a frame. An equivalence relation $\Theta \subseteq W \times W$ is called a congruence relation (on $\mathfrak{F}$) if for all $w, w', v \in W$

if Rwv and $(w, w') \in \Theta$ then there is a $v' \in W$
such that $Rw'v'$ and $(v, v') \in \Theta$.

Furthermore, if Θ is a congruence on $\mathfrak{F}$, then $\mathfrak{F}/\Theta$ will denote the frame $(W/\Theta, R/\Theta)$ where $W/\Theta = \{\overline{w} \mid W \in W\}$, where $\overline{w}$ (or $\overline{w}^\Theta$) stands for

$\{v \in W \mid (w, v) \in \Theta\}$, and $R/\Theta\ \bar{w}\bar{v}$ iff there are $w' \in \bar{w}$, $v' \in \bar{v}$ such that $Rw'v'$. The map $W \to W/\Theta$ defined by $w \to \bar{w}$ will be denoted by ν_Θ.

3.2 PROPOSITION. *The map ν_Θ is an $\mathfrak{F}$-morphism from $\mathfrak{F}$ to $\mathfrak{F}/\Theta$ whenever Θ is a congruence on $\mathfrak{F}$.*

PROOF: By the definition of R/Θ, ν_Θ preserves the relation. In order to check the second property of $\mathfrak{F}$-morphisms, let $R/\Theta\ \bar{w}\bar{v}$. There are $w_1, v_1 \in W$ such that $w_1 \in \bar{w}$, $v_1 \in \bar{v}$ and Rw_1v_1. But then, since Θ is a congruence, there is a $v' \in \bar{v}$ such that Rwv' and $(v_1, v') \in \Theta$. Hence $\nu_\Theta(v') = \bar{v}' = \bar{v}$, as was to be shown.

3.3 COROLLARY. *Let $\mathfrak{F}$ be a frame and let Θ be a congruence on $\mathfrak{F}$. Then $(\mathfrak{F}/\Theta)^+ \in S(\mathfrak{F}^+)$.*

We recall the definition of a cluster.

3.4 DEFINITION. Let $\mathfrak{F} = (W, R)$ be a frame. A set $C \subseteq W$ is called a *cluster* if

(i) for all $w, v \in C$, Rwv.

(ii) C is maximal with respect to this property.

In particular, every cluster consists of reflexive elements. And if R is transitive then it is easily seen that different clusters are disjoint.

3.5 DEFINITION. Let $\mathfrak{F} = (W, R)$ be a transitive frame. Then γ (or, more explicitly, $\gamma_\mathfrak{F}$) will denote the relation $\{(w, v) \mid w, v \in W,\ w = v$ or w and v belong to the same cluster$\}$.

3.6 PROPOSITION. *Let $\mathfrak{F} = (W, R)$ be a transitive frame. Then γ is a congruence relation on $\mathfrak{F}$ and $h(\mathfrak{F}) = h(\mathfrak{F}/\gamma)$.*

PROOF: First observe that γ is an equivalence relation. Let $w, w', v \in W$ such that Rwv and $(w, w') \in \gamma$. If $w = w'$ then $Rw'v$ and if $w \neq w'$ then $Rw'w$, Rwv, whence, by transitivity, $Rw'v$. In order to verify the last statement, we need only observe that the image under ν_γ of an R-chain is an R-chain.

If $\mathfrak{F} = (W, R)$ is a transitive frame then $\mathfrak{F}/\gamma$ is transitive as well, and does not contain any clusters except possibly one-element clusters, and hence no circuits. Thus any sequence $(w_1, \ldots, w_n)$, $w_i \in W$, $i = 1, \ldots, n$ satisfying Rw_iw_{i+1}, $w_i \neq w_{i+1}$, is an R-chain in $\mathfrak{F}$.

As before, let, for $m \in N$, $W_m = \{w \in W \mid h(w) \leqslant m\} \subseteq W$ and $\mathfrak{F}_m = (W_m, R \cap (W_m \times W_m))$.

3.7 DEFINITION. Let $\mathfrak{F} = (W, R)$ be a transitive frame such that $W = \bigcup_{m=1}^{\infty} W_m$, containing no clusters having more than one element. λ or, more explicitly, $\lambda_\mathfrak{F}$ will denote the binary relation defined by induction on the height as follows:

(i) $\lambda_1 \subseteq W_1 \times W_1$ is the relation defined by

$$(w, v) \in \lambda_1 \quad \text{iff} \quad \begin{cases} Rww \text{ and } Rvv \\ \quad \text{or} \\ \neg Rww \text{ and } \neg Rvv, \end{cases}$$

for $w, v \in W$, such that $h(w) = h(v) = 1$.

(ii) Next suppose that $\lambda_m \subseteq W_m \times W_m$ has been defined. Let $\lambda_{m+1} = \lambda_m \cup \{(w, v) \mid h(w) = h(v) = m+1, (w, v) \text{ satisfies } (*)\}$, where (*) denotes the property:

$$(*) \quad \begin{cases} (Rww \text{ and } Rvv) \text{ or } (\neg Rww \text{ and } \neg Rvv) \\ \qquad\qquad \text{and} \\ \{\bar{u}^{\lambda_m} \mid Rwu, h(u) \leqslant m\} = \{\bar{u}^{\lambda_m} \mid Rvu, h(u) \leqslant m\}. \end{cases}$$

Now let $\lambda = \bigcup_{m=1}^{\infty} \lambda$.

3.8 Proposition. *Let $\mathfrak{F}$ be a transitive frame such that $W = \bigcup_{m=1}^{\infty} W_m$, containing no clusters having more than one element. The relation λ is a congruence relation on $\mathfrak{F}$ and $\mathfrak{F}/\lambda$ contains finitely many elements of height n, for every $n \in N$.*

Proof: First we show that the λ_m, $m = 1, 2, \ldots$ are congruences on $\mathfrak{F}_m$, $m = 1, 2, \ldots$, with the additional property that if $(w, v) \in \lambda_m$ then $h(w) = h(v) \leqslant m$.

(i) Clearly λ_1 is a congruence on $\mathfrak{F}_1$. Indeed, it is an equivalence relation, and if $w, w', v \in W_1$ such that Rwv, and $(w, w') \in \lambda_1$, then, since $h(w) = 1$, $w = v$, hence Rww and thus $Rw'w'$, whence we can take v' to be w'. Furthermore, if $(w, v) \in \lambda_1$, then $h(w) = h(v) = 1$.

(ii) Suppose that λ_m is a congruence on $\mathfrak{F}_m$ such that if $(w, v) \in \lambda_m$ then $h(w) = h(v) \leqslant m$. Firstly, using the fact that λ_m is an equivalence relation on W_m, we see that λ_{m+1} is an equivalence on W_{m+1}. Let w, w', $v \in W_{m+1}$ be such that Rwv and $(w, w') \in \lambda_{m+1}$. If $h(w) \leqslant m$ then $w \in W_m$ and $(w, w') \in \lambda_m$, and since Rwv, $h(v) \leqslant m$, so $v \in W_m$ as well. Because by assumption λ_m is a congruence, there is a $v' \in W_m$ such that $Rw'v'$ and $(v, v') \in \lambda_m$. Next suppose that $h(w) = m+1$. Then $h(w') = m+1$ and (w, w') satisfies (*). Suppose $h(v) = m+1$. Then $w = v$ since by our remark preceding definition 3.7 our assumptions on $\mathfrak{F}$ guarantee that if Rwv, $w \neq v$, $h(v) = m+1$ then $h(w) \geqslant m+2$. Hence Rww, so $Rw'w'$ as well and we may choose for v' the element w' itself. Now suppose $h(v) \leqslant m$. Then $\bar{v}^{\lambda_m} \in \{\bar{u}^{\lambda_m} \mid Rwu, h(u) \leqslant m\} = \{\bar{u}^{\lambda_m} \mid Rw'u, h(u) \leqslant m\}$, so there is a u, such that $Rw'u$, and $\bar{u}^{\lambda_m} = \bar{v}^{\lambda_m}$. But then $(v, u) \in \lambda_m \subseteq \lambda_{m+1}$, so we may choose v' to be the element u. Again, we see that $(w, v) \in \lambda_{m+1}$ implies $h(w) = h(v) \leqslant m+1$.

We verify that $\lambda = \bigcup_{m=1}^{\infty} \lambda_m$ is a congruence relation on $\mathfrak{F}$. If $w \in W$, then $w \in W_m$ and hence $(w, w) \in \lambda_m \subseteq \lambda$. If $(w, v) \in \lambda$ then $(w, v) \in \lambda_m$ for some m, and hence $(v, w) \in \lambda_m \subseteq \lambda$. Furthermore, if $(w, v) \in \lambda$, $(v, u) \in \lambda$, then there is an $m \in N$ such that both $(w, v) \in \lambda_m$ and $(v, u) \in \lambda_m$. Hence $(w, u) \in \lambda_m \subseteq \lambda$. Thus λ is an equivalence relation on W. Now let $w, w', v \in W$ such that Rwv and $(w, w') \in \lambda$. But then, since there is an $m \in N$ such that $w, w', v \in W_m$, $(w, w') \in \lambda_m$. Since λ_m is a congruence there is a $v' \in W_m$ such that $Rw'v'$ and $(v, v') \in \lambda_m \subseteq \lambda$.

The second assertion of the proposition is easily proved by induction. First note that since $(w, v) \in \lambda$ implies $h(w) = h(v)$, for every $w \in W$, $h(w) = h(\overline{w}^{\lambda})$. Now, there are at most two equivalence classes consisting of elements of height 1, giving rise to two elements of height 1 in W/λ: a reflexive element and an irreflexive one. Now assume that W/λ contains finitely many elements of height $\leqslant m$. Let $w, v \in W$ have height $m+1$, and suppose w, v are both reflexive. Note that if $(w, v) \notin \lambda$ then $\{\overline{u} \mid Rwu, h(u) \leqslant m\} \neq \{\overline{u} \mid Rvu, h(u) \leqslant m\}$. Hence the map $\overline{w}^{\lambda} \to \{\overline{u} \mid Rwu, h(u) \leqslant m\}$ is one-to-one from the set of reflexive elements of height $m+1$ in $\mathfrak{F}/\lambda$ to the set of subsets of the finite set of elements of $\mathfrak{F}/\lambda$ of height $\leqslant m$. Hence there are only finitely many reflexive elements of height $m+1$ in $\mathfrak{F}/\lambda$, and in the same way one proves that there are only finitely many irreflexive elements of height $m+1$ in $\mathfrak{F}/\lambda$. Since any element is either reflexive or irreflexive this proves the claim.

It follows from 3.6 and 3.8 that for every $n \in N$ there are finitely many frames, which we will denote by $\mathfrak{F}_i^n = (W_i^n, R_i^n)$, $1 \leqslant i \leqslant m_n$, such that for any transitive frame $\mathfrak{F}$ of height n there is an i, $1 \leqslant i \leqslant m_n$ such that $(\mathfrak{F}/\gamma)/\lambda \cong \mathfrak{F}_i^n$. This observation enables us to give another characterization of the varieties $\mathbf{KT}(n)$, $n \in N$, which played an important role in section 1. Recall that a finite subdirectly irreducible algebra $\mathfrak{B}$ belonging to a variety $\mathbf{K}$ is called *splitting in* $\mathbf{K}$ if and only if the class $\{\mathfrak{A} \in \mathbf{K} \mid \mathfrak{B} \notin SH(\mathfrak{A})\}$ is a variety, which then is denoted by $\mathbf{K}/\mathfrak{B}$. If $\mathfrak{B}$ is splitting in $\mathbf{K}$ then for any subvariety $\mathbf{K}_1$ of $\mathbf{K}$, $V(\mathfrak{B}) \subseteq \mathbf{K}_1$ or $\mathbf{K}_1 \subseteq \mathbf{K}/\mathfrak{B}$ (but not both); hence the lattice of subvarieties of $\mathbf{K}$ is the disjoint union of the intervals $[V(\mathfrak{B}))$ and $(\mathbf{K}/\mathfrak{B}]$. In [13] it was shown that every finite subdirectly irreducible in $\mathbf{MT}$ is splitting. For a further discussion of splitting varieties we refer to [2] and [5].

Let for $n \in N$ $\mathbf{S}^n = \{\mathfrak{B}_1^n, \ldots, \mathfrak{B}_{k_n}^n\}$ denote the set of finite subdirectly irreducible algebras of height n belonging to $H(\{(\mathfrak{F}_i^n)^+ \mid 1 \leqslant i \leqslant m_n\})$. Observe that since $\mathfrak{F}_i^n$, $1 \leqslant i \leqslant m_n$ are finite, $\mathbf{S}^n$ is a finite set of finite algebras.

3.9 Theorem. *For $n \in N$, $\mathbf{KT}(n) = \bigcap\{\mathbf{MT}/\mathfrak{B} \mid \mathfrak{B} \in \mathbf{S}^{n+1}\}$.*

Proof: (i) Since by 1.6 $\mathbf{KT}(n)$ consists of algebras of height $\leqslant n$, $\mathbf{S}^{n+1} \cap \mathbf{KT}(n) = \varnothing$, whence $\mathbf{KT}(n) \subseteq \bigcap\{\mathbf{MT}/\mathfrak{B} \mid \mathfrak{B} \in \mathbf{S}^{n+1}\}$.

(ii) Let $\mathfrak{A} \in \bigcap\{\mathbf{MT}/\mathfrak{B} \mid \mathfrak{B} \in \mathbf{S}^{n+1}\}$ be finitely generated and assume that $\mathfrak{A} \notin \mathbf{KT}(n)$. Then $h(\mathfrak{A}) > n$, and by 1.10 there is a $\mathfrak{B} \in H(\mathfrak{A})$ such that $h(\mathfrak{B}) = n+1$. In virtue of 1.8, $\mathfrak{B}$ is finite, and hence so is $\mathfrak{F}_{\mathfrak{B}}$. Since $\mathfrak{B} \in \mathbf{MT}$, $\mathfrak{F}_{\mathfrak{B}}$ is transitive. By Lemmas 6 and 8, $(\mathfrak{F}_{\mathfrak{B}}/\gamma)/\lambda \cong \mathfrak{F}_i^{n+1}$, for some i, $1 \leqslant i \leqslant m_{n+1}$, whence there exists a $\mathfrak{B}_i^{n+1} \in \mathbf{S}^{n+1}$ such that $\mathfrak{B}_i^{n+1} \in HS(\mathfrak{B})$. But then $\mathfrak{B}_i^{n+1} \in HSH(\mathfrak{A}) \subseteq V(\mathfrak{A})$, contradicting our assumption that $\mathfrak{A} \in \bigcap\{\mathbf{MT}/\mathfrak{B} \mid \mathfrak{B} \in \mathbf{S}^{n+1}\}$.

Observe that if $\mathfrak{F}$ is a reflexive and transitive finite frame of height n then $(\mathfrak{F}/\gamma)/\lambda$ is the linearly ordered frame K_n on n elements. It follows that for $n \in N$ $\mathbf{KT}(n) \cap \mathbf{MR} = \mathbf{MRT}/(K_n)^+$; these are just the varieties we used in [4] to establish the result that a cover in the lattice $\Lambda(\mathbf{MRT})$ of a tabular variety is tabular.

Now we want to use lemmas 6 and 8 to obtain some more information concerning pretabular varieties. The first result deals with pretabular varieties of finite height.

3.10 THEOREM. *For $n \in N$, $\mathbf{KT}(n)$ contains only finitely many pretabular varieties.*

PROOF: Let $\mathbf{K} \subseteq \mathbf{KT}(n)$ be a pretabular variety. Since $\mathbf{KT}(n)$ is locally finite, $\mathbf{K}$ is generated by its finite members. Let $\mathbf{K} = V(\{\mathfrak{A}_n \mid n = 1, 2, \ldots\})$, where every $\mathfrak{A}_n$ is finite and subdirectly irreducible. Since $\mathbf{K} \subseteq \mathbf{KT}(n)$, $h(\mathfrak{A}_i) \leqslant n$ for $i = 1, 2, \ldots$. Because $\mathbf{K}$ is pretabular $\mathbf{K}$ is generated by any infinite subset of $\{\mathfrak{A}_i \mid i = 1, 2, \ldots\}$, whence we may as well assume that $h(\mathfrak{A}_i) = n$, $i = 1, 2, \ldots$. Let $\mathfrak{A}_i = \mathfrak{F}_i^+$, $i = 1, 2, \ldots$, where $\mathfrak{F}_i = (W_i, R_i)$. First suppose that, up to isomorphism, there are only finitely many algebras $\mathfrak{F}_i/\gamma_i$, $i = 1, 2, \ldots$ (we write γ_i for $\gamma_{\mathfrak{F}_i}$). Then we may assume, in virtue of the pretabularity, that there is an $\mathfrak{F} = (W, R)$ such that for all $i \in N$ $\mathfrak{F}_i/\gamma_i \cong \mathfrak{F}$; let $\mathfrak{F}/\lambda = \mathfrak{F}_j^n$, where $1 \leqslant j \leqslant m_n$. Since W is finite, there is a $w_\circ \in W$ such that $|\nu_{\gamma_i}^{-1}(\{w_\circ\})|$, $i = 1, 2, \ldots$ is an unbounded sequence, and since $h(\mathbf{K}) = n$, we may assume that $h(w_\circ) = n$. Define a relation π_i on $\mathfrak{F}_i$ as follows: for $w, v \in W_i$,

$$(w, v) \in \pi_i \quad \text{iff} \quad \begin{cases} \nu_\lambda \circ \nu_{\gamma_i}(w) = \nu_\lambda \circ \nu_{\gamma_i}(v) \text{ if } h(w) = h(v) < n \\ w = v \text{ if } h(w) = h(v) = n \end{cases}$$

Clearly, π_i is an equivalence relation. In order to verify that π_i is a congruence relation, let $w, w', v \in W_i$, such that $(w, w') \in \pi_i$, Rwv. If $h(w) = n$, then apparently $w = w'$ so we may choose $v' = v$. If $h(w) < n$ then we use the fact that γ_i, λ are congruences. Since $\mathfrak{F}_i^+$ is subdirectly irreducible, $\mathfrak{F}_i$ is generated, and hence so is $\mathfrak{F}_i/\pi_i$. In fact, $\mathfrak{F}_i/\pi_i$ is isomorphic to the frame $\big((W_j^n \setminus \{\nu_\lambda(w_\circ)\}) \cup \nu_{\gamma_i}^{-1}(\{w_\circ\}), R\big)$, where

$$Rwv \quad \text{iff} \quad \begin{cases} R_j^n wv \text{ and } h(w) < n \\ \quad \text{or} \\ h(w) = n \end{cases}$$

(the W_i are assumed to be disjoint from W_j^n.) Hence $\mathfrak{F}_i/\pi_i$ is obtained from $\mathfrak{F}_j^n$ by replacing the root — which is a reflexive element in this case — by a cluster. Since the $(\mathfrak{F}_i/\pi_i)^+$ are subdirectly irreducible, and since there are infinitely many non isomorphic algebras among them, **K** is generated by the set $\{(\mathfrak{F}_i/\pi_i)^+ \mid i = 1, 2, \ldots\}$. Since there are only finitely many frames $\mathfrak{F}_j^n$ to choose from, this gives rise to only a finite number of pretabular varieties of height n of this type.

Next suppose there are infinitely many non-isomorphic algebras $\mathfrak{F}_i/\gamma_i$. Then we may as well assume that the $\mathfrak{F}_i$ themselves have no clusters containing more than one element. We also may assume, again by pretabularity of **K**, that there is a j, $1 \leqslant j \leqslant m_n$, such that $\mathfrak{F}_i/\lambda_i = \mathfrak{F}_j^n$, for all $i = 1, 2, \ldots$. Again, there is a $w \in W_j^n$ such that $|\nu_{\lambda_i}^{-1}(\{w\})|$, $i = 1, 2, \ldots$ is an unbounded sequence. Because the $\mathfrak{F}_i$ are generated frames, any such w will satisfy $h(w) < n$. Let $w_\circ$ be an element of maximal height k with this property. We claim that $k = n-1$. For if $k < n-1$, then there is for every $l \in N$ a $w_{i_l} \in W_{i_l}$ such that $h(w_{i_l}) \leqslant n-1$ and w_{i_l} has at least l successors in $\nu_{\lambda_{i_l}}^{-1}(\{w_0\})$; then, however, would the $\mathfrak{F}_{w_{i_l}}$, $l = 1, 2, \ldots$ be an infinite sequence of generated frames, and hence **K** would be generated by the $\mathfrak{F}_{w_{i_l}}^+$, $l = 1, 2, \ldots$. But since $h(\mathfrak{F}_{w_{i_l}}) \leqslant n-1$, this would contradict our assumption that $h(\mathbf{K}) = n$. We thus showed that $h(w_\circ) = k = n-1$. Now define π_i on $\mathfrak{F}_i$ as follows:

for $w, v \in W_i$ let

$$(w, v) \in \pi_i \quad \text{iff} \quad \begin{cases} \nu_{\lambda_i}(w) = \nu_{\lambda_i}(v) \neq w_\circ \\ \text{or} \\ w = v. \end{cases}$$

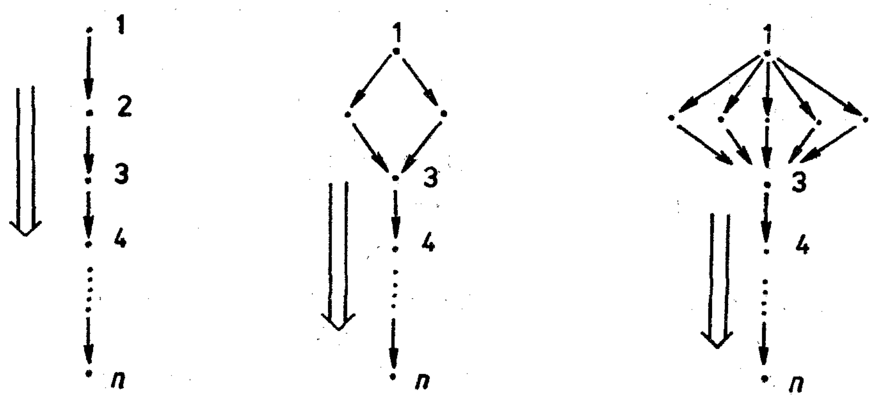
Clearly, π_i is an equivalence relation. To see that π_i is a congruence, let $w, w', v \in W_i$ such that $(w, w') \in \pi_i$ and Rwv. If $h(w) = n$ then $w = w'$ — since the $\mathfrak{F}_i$ are generated frames — and we may choose $v' = v$. Similarly if $h(w) = n-1$ and $w = w'$, which takes care of the case $\nu_{\lambda_i}(w) = w_\circ$. If $h(w) < n$ and $\nu_{\lambda_i}(w) \neq w_\circ$ then $\nu_{\lambda_i}(v) \neq w_\circ$. Since $(w, w') \in \lambda_i$ and λ_i is a congruence there is a $v' \in W_i$ such that $R_i w' v'$ and $(v, v') \in \lambda_i$. But because $\nu_{\lambda_i}(v) = \nu_{\lambda_i}(v') \neq w_\circ$, $(v, v') \in \pi_i$. Note that $\mathfrak{F}_i/\pi_i$ is a generated frame, and that there are infinitely many pairwise non isomorphic algebras among the $(\mathfrak{F}_i/\pi_i)^+$, $i = 1, 2, \ldots$. Therefore **K** is generated by $\{(\mathfrak{F}_i/\pi_i)^+ \mid i = 1, 2, \ldots\}$. In fact, $\mathfrak{F}_i/\pi_i$ is isomorphic to the frame $\big((W_j^n \setminus \{w_\circ\}) \cup \nu_{\lambda_i}^{-1}(\{w_\circ\}), R\big)$, where

$$Rwv \quad \text{iff} \quad \begin{cases} R_j^n wv, \ w, v \in W_j^n, w, v \neq w_\circ \\ R_j^n w w_\circ, \ w \in W_j^n, w \neq w_\circ, v \in \nu_{\lambda_i}^{-1}(\{w_\circ\}) \\ R_j^n w_\circ w, \ v \in W_j^n, v \neq w_\circ, w \in \nu_{\lambda_i}^{-1}(\{w_\circ\}) \\ R_j^n w_\circ w_\circ, \ w, v \in \nu_{\lambda_i}^{-1}(\{w_\circ\}), w = v. \end{cases}$$

(the W_i are assumed to be disjoint from W_j^n.)

Since there are only finitely many $\mathfrak{F}_j^{n}$' s, and every $\mathfrak{F}_j^n$ has only finitely many elements of height $n-1$, this gives rise to finitely many pretabular varieties of height n of this kind as well.

In order to summarize the results of the proof of 3.10 we introduce some more notation. Let, for a generated frame $\mathfrak{F}_i^n$, $1 \leqslant i \leqslant m_n$, $\mathfrak{F}_i^n(C_m)$ be the frame obtained from $\mathfrak{F}_i^n$ by replacing the root — i.e. the element of height n — by a cluster containing m elements, as in the first half of the proof of 3.10. Let, for a generated frame $\mathfrak{F}_i^n$, $1 \leqslant i \leqslant m_n$, $\mathfrak{F}_i^n(D_m^w)$ be the frame obtained from $\mathfrak{F}_i^n$ by replacing the element w of height $n-1$ by m "copies", like in the second half of the proof of 3.10. Then it follows from the proof of 3.10 that any pretabular variety $\mathbf{K}$ of height n is either $V(\{\mathfrak{F}_i^n(C_m)^+ \mid m = 1, 2, \ldots\})$, for some i, $1 \leqslant i \leqslant m_n$, or $V(\{\mathfrak{F}_i^n(D_m^w)^+ \mid m = 1, 2, \ldots\}$, for some i, $1 \leqslant i \leqslant m_n$ and some $w \in W_i^n$ of height $n-1$. Although, conversely, not every set of algebras of this form gives rise to a pretabular variety, it is not difficult to see that there are countably many pretabular varieties of finite height in **MT**. Indeed, let $L_n = (\{1, \ldots, n\}, <)$. Then $h(L_n) = n$, L_n is a generated frame and since $S(L_n^+) = \{L_n^+\}$, $L_n \in \{\mathfrak{F}_i^n \mid 1 \leqslant i \leqslant m_n\}$. Let $\mathbf{K}_n = V(\{L_n(D_m^2) \mid m = 1, 2, \ldots\}$. The frames $L_n(D_m^2)$ look like:



$L_n = L_n(D_1^2)$ $\qquad$ $L_n(D_2^2)$ $\qquad$ $L_n(D_5^2)$

For every $n \in N$, $\mathbf{K}_n$ is pretabular, and if $n \neq m$ then $\mathbf{K}_n \neq \mathbf{K}_m$.

A modal formula which has attracted considerable attention in the literature ([14], [15]) is Löb's formula: $\Box(\Box p \to p) \to \Box p$. Let L be the normal extension of K axiomatized by Löb's formula. The logic L is complete — in fact, it has the finite model property — and $\Box p \to \Box\Box p$ is one of its theorems. Furthermore, for any frame $\mathfrak{F} = (W, R)$, $\mathfrak{F} \models L$ iff R transitive and the converse of R is a wellfounded relation. In particular, if $\mathfrak{F} \models L$ then R is irreflexive. Let **MITA** be the variety of modal algebras corresponding with Löb's logic, i.e. the subvariety of **M** (which, because $\Box p \to \Box\Box p \in L$, turns out to be a subvariety of **MT**) defined by the equation $(x^{\circ\prime} + x)^{\circ\prime} + x^\circ = 1$.

3.11 THEOREM. **MITA** *contains countably many pretabular varieties.*

PROOF: The varieties $\mathbf{K}_n$, given in the preceding paragraph, belong to **MITA**; hence **MITA** contains at least countably many pretabular varieties. In virtue of 3.10, **MITA** contains only countably many pretabular varieties of finite height. Now let $\mathbf{K} \subseteq \mathbf{MITA}$ be a pretabular variety of infinite height. By 2.1, $\mathbf{K}$ is generated by its finite members. Let $\mathfrak{F} = (W, R)$ be a finite frame, such that $\mathfrak{F}^+ \in \mathbf{K}$. Then $\mathfrak{F}$ contains no circuits and is irreflexive, hence $\mathfrak{F}/\lambda = L_n\big(= (\{1, \ldots, n\}, <)\big)$, for some $n \in N$. Since $\mathbf{K}$ is of infinite height $\mathbf{K}$ contains infinitely many L_n^+'s, thus $\mathbf{K} = V(\{L_n^+ \mid n = 1, 2, \ldots\})$. Therefore **MITA** contains only one pretabular variety of infinite height.

Another extreme is the case where the relations of the frames involved are all reflexive, i.e. where we restrict ourselves to subvarieties of the variety **MRT** of interior algebras, defined by the law $x^\circ \cdot x = x^\circ$, in addition to the axioms of **MT**. The pretabular subvarieties of **MRT** were first determined by Maksimova [11] (whose proof is based on an announcement by Kuznetsov [9]), later by Esakia and Meskhi [7] (whose proof, however, contains an essential gap) and by Rautenberg [13].

Let, as before, $K_n = (\{1, \ldots, n\}, \leqslant)$.

3.12 THEOREM. *The pretabular varieties of interior algebras are:*

(i) $V(\{K_1(C_m)^+ \mid m = 1, 2, \ldots\})$
(ii) $V(\{K_2(C_m)^+ \mid m = 1, 2, \ldots\})$
(iii) $V(\{K_2(D_m^2)^+ \mid m = 1, 2, \ldots\})$
(iv) $V(\{K_3(D_m^2)^+ \mid m = 1, 2, \ldots\})$
(v) $V(\{K_n^+ \mid n = 1, 2, \ldots\})$.

PROOF: It is a matter of easy verification to show that these five varieties are pretabular. Now let $\mathbf{K}$ be pretabular, $\mathbf{K} \subseteq \mathbf{MRT}$. By 2.1, $\mathbf{K}$ is generated by its finite members. Let $\mathfrak{F} = (W, R)$ be a finite frame such that $\mathfrak{F}^+ \in \mathbf{K}$. Since every element of w is reflexive, $(\mathfrak{F}/\gamma)/\lambda \cong K_n$, for some $n \in N$. If $\mathbf{K}$ contains infinitely many algebras $\mathbf{K}_n^+$, then $\mathbf{K} = V(\{\mathbf{K}_n^+ \mid n = 1, 2, \ldots\})$. If not, then $\mathbf{K}$ is of finite height, say of height n, and by the remark following 3.10, $\mathbf{K} = V\big(\{K_n(C_m^2)^+ \mid m = 1, 2, \ldots\}\big)$ or $\mathbf{K} = V(\{K_n(D_m^2)^+ \mid m = 1, 2, \ldots\})$. However, since $K_2(C_m)^+ \in S\big(K_n(C_m)^+\big)$ whenever $n \geqslant 2$, and likewise $K_2(D_m^2)^+ \in S\big(K_n(D_m^2)^+\big)$ whenever $n \geqslant 3$ it follows that $\mathbf{K}$ is of height $\leqslant 3$, and is one of the varieties listed in (i)-(iv).

By a result in [5], the lattice of varieties of Heyting algebras is isomorphic to the lattice of subvarieties of the variety **MRTA**, corresponding with the modal logic axiomatized by the formula $\Box\big(\Box(p \to \Box p) \to p\big) \to p$ (in [14] referred to as Grzegorczyk's formula). The only pretabular varieties contained in **MRTA** are the ones listed as (iii), (iv) and (v). Hence there are three pretabular varieties of Heyting algebras (cf. [12]).

As a last example we want to determine the pretabular varieties in **MT** satisfying the equation $0^\circ = 0$, that is the pretabular subvarieties of $\mathbf{MT}/\mathbf{2}^+$. That the ten varieties mentioned in the theorem are pretabular was observed by V. Meskhi; we shall prove now that these are all. Let for $A \subseteq \{1, \ldots, n\}$, $M_n^A = (\{1, \ldots, n\}, < \cup \{(i, i) \mid i \in A\})$.

3.13 THEOREM. *The pretabular subvarieties of* **MT** *satisfying* $0^\circ = 0$ *are*:

(i)-(v) of 3.12
(vi) $V(\{M_2^{\{2\}}(D_m^2)^+ \mid m = 1, 2, \ldots\})$
(vii) $V(\{M_3^{\{1,3\}}(D_m^2)^+ \mid m = 1, 2, \ldots\})$
(viii) $V(\{M_3^{\{2,3\}}(D_m^2)^+ \mid m = 1, 2, \ldots\})$
(ix) $V(\{M_3^{\{3\}}(D_m^2)^+ \mid m = 1, 2, \ldots\})$
(x) $V(\{M_n^{\{n\}+} \mid n = 1, 2, \ldots\})$.

PROOF: It is not difficult to verify that (i)-(x) give rise to different pretabular varieties. Now let $\mathbf{K} \subseteq \mathbf{MT}$ be pretabular and assume that $0^\circ = 0$ in **K**. By 2.1, **K** is generated by its finite members. Let $\mathfrak{F} = (W, R)$ be a finite frame such that $\mathfrak{F}^+ \in \mathbf{K}$. Then $\mathfrak{F}_1 = \mathfrak{F}/\gamma/\lambda$ is a frame which has one (reflexive) element of height 1. Let $\mathfrak{F}_1 = (W_1, R_1)$, and let $w_1, w_2, \ldots, w_n \in W_1$ be such that w_i is reflexive for $i = 1, \ldots n$, Rw_iw_{i+1}, $w_i \neq w_{i+1}$, $i = 1, \ldots, n-1$, and $h(w_n) = 1$. Define $f\colon \mathfrak{F}_1 \to K_n$ by $f(w) = \min\{j \mid j \in \{1, \ldots, n\}, Rww_j\}$. Since for all $w \in W_1$ Rww_n, f is well defined and since the w_i are reflexive and $\neg Rw_iw_j$ if $j < i$ f is onto. Furthermore, f is an $\mathfrak{F}$-morphism. For if $w, v \in W$ such that R_1wv then $\min\{j \mid Rww_j\} \leqslant \min\{j \mid Rvw_j\}$, by transitivity of R, whence $f(w) \leqslant f(v)$. And if $R_1f(w)j$, then $k = f(w) \leqslant j$, whence Rww_k, Rw_kw_j, so Rww_j. But since $f(w_j) = j$, it follows then that f is an $\mathfrak{F}$-morphism. We infer that if **K** contains Kripke algebras of finite frames with reflexive R-chains of arbitrary length, then $\mathbf{K} = V(\{K_n^+ \mid n = 1, 2, \ldots\})$. If **K** is not of finite height but not of this form — i.e., for every frame $\mathfrak{F}$ such that $\mathfrak{F}^+ \in \mathbf{K}$, every reflexive R-chain is of length $\leqslant m$ for some fixed m — then there is for every $n \in N$ a finite frame $\mathfrak{F} = (W, R)$, containing no clusters with more than one element, such that $\mathfrak{F}^+ \in \mathbf{K}$ and $h(\mathfrak{F}) - r = n$, where $r = \max\{h(w) \mid w \text{ reflexive}, w \in W\}$, as an easy argument shows. Let Θ be the equivalence relation on $\mathfrak{F}$ defined by

$$(w, v) \in \Theta \quad \text{iff} \quad \begin{cases} h(w),\ h(v) \leqslant r \\ \quad\text{or} \\ h(w) = h(v) \end{cases}$$

Clearly, Θ is an equivalence relation. And if $w, w', v \in W$ such that $(w, w') \in \Theta$ and Rwv, then if $h(v) \leqslant r$ take for v' the element of $\mathfrak{F}$ of height 1. If $h(v) > r$, then $h(w) > h(v) > r$, so $h(w) = h(w')$. Hence there is a $v' \in W$ such that $Rw'v'$ and $h(v') = h(v)$, i.e., $(v', v) \in \Theta$. Thus Θ is a congruence relation. Clearly $\mathfrak{F}/\Theta \cong M_n^{\{n\}}$, thus $\mathbf{K} = V(\{M_n^{\{n\}+} \mid n = 1, 2, \ldots\})$. It follows that (v) and (x) are the only pretabular varieties of infinite height in **MT** which satisfy $0^\circ = 0$.

The case that **K** is of finite height remains. Suppose that **K** has height n. In 3.10 we have seen that $\mathbf{K} = V(\{\mathfrak{F}_i^n(C_m)^+ \mid m = 1, 2, \ldots\})$, or $\mathbf{K} = V(\{\mathfrak{F}_i^n(D_n^w)^+ \mid m = 1, 2, \ldots\})$ for some i, $1 \leqslant i \leqslant m_n$, where $h(w) = n-1$. On the frames $\mathfrak{F}_i^n(C_m)$ we define a congruence relation Θ_i^1 as follows:

$$(w, v) \in \Theta_i^1 \text{ iff } \ h(w),\ h(v) \leqslant n-1 \quad \text{or} \quad w = v$$

and on the frames $\mathfrak{F}_i^n(D_m^{w_\circ})$ a congruence Θ_i^2 as follows:

$$(w, v) \in \Theta_i^2 \text{ iff } w, v \in W_i^n,\ w, v \neq w_\circ,\ h(w),\ h(v) \leqslant n-1 \text{ or } w = v.$$

Θ_i^1 and Θ_i^2 are congruences, essentially because $\mathfrak{F}_i^n$ has only one element of height 1 which is reflexive, $\mathfrak{F}_i^{n+}$ satisfying the equation $0^\circ = 0$. It follows that $\mathfrak{F}_i^n(C_m)/\Theta_i^1$ is isomorphic to either $K_1(C_m)$ or $K_2(C_m)$ and $\mathfrak{F}_i^n(D_m^{w_0})/\Theta_i^2$ to $K_2(D_m^2)$, $M_2^{\{2\}}(D_m^2)$, $K_3(D_m^2)$, $M_3^{\{1,3\}}(D_m^2)$, $M_3^{\{2,3\}}(D_m^2)$ or $M_3^{\{3\}}(D_m^2)$. Hence **K** is equal to one of the varieties listed under (ii), (iii), (iv), (vi), (vii), (viii) and (ix).

Apparently, it is the fact that $0^\circ > 0$ in general in **MT** which is responsible for the presence of so many pretabular varieties. Note that the varieties $V(\mathfrak{A}_m)$ constructed in section 2 satisfy $0^{\circ\circ} = 0^\circ$ whenever $2 \notin M$ — hence, there are $2^{\aleph_0}$ pretabular varieties in **MT** satisfying $0^{\circ\circ} = 0^\circ$.

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