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A nonlinear composite shell element with continuous interlaminar shear stresses

F. Gruttmann[1], W. Wagner[1], L. Meyer[1], P. Wriggers[2]

[1] Institut für Baumechanik und Numerische Mechanik, Universität Hannover, Appelstr. 9A, D-30167 Hannover, Germany
[2] Institut für Mechanik, Technische Hochschule Darmstadt, Hochschulstr. 1, D-64289 Darmstadt, Germany

Abstract. A numerical model for layered composite structures based on a geometrical nonlinear shell theory is presented. The kinematic is based on a multi-director theory, thus the in-plane displacements of each layer are described by independent director vectors. Using the isoparametric approach a finite element formulation for quadrilaterals is developed. Continuity of the interlaminar shear stresses is obtained within the nonlinear solution process. Several examples are presented to illustrate the performance of the developed numerical model.

List of symbols

Ω	reference surface
ξ^α	convected coordinates of the shell middle surface
${}^i\zeta$	coordinate in thickness direction
${}^ih$	thickness of layer i
$\mathbf{X}_0$	position vector of the reference surface
${}^i\mathbf{x}_0$	position vector of midsurface of layer i
$\mathbf{t}_k$	orthonormal basis system in the reference configuration
${}^i\mathbf{a}_k$	orthonormal basis system of layer i
$\delta{}^i\mathbf{w}$	axial vector
$\mathbf{R}_0$	orthonormal tensor in the reference configuration
${}^i\mathbf{R}$	orthonormal tensor of layer i
${}^i\boldsymbol{\sigma}$	Cauchy stress tensor
${}^i\mathbf{P}$	First Piola–Kirchhoff stress tensor
${}^i\mathbf{q}$	vector of interlaminar stresses
${}^i\mathbf{n}_\alpha, {}^i\mathbf{m}_\alpha$	vector of stress resultants and stress couple resultants
ν_α	components of the normal vector of boundary Ω_α
${}^iN^{\alpha\beta}, {}^iQ^\alpha, {}^iM^{\alpha\beta}$	stress resultants and stress couple resultants of First Piola–Kirchhoff tensor
${}^i\tilde{N}^{\alpha\beta}, {}^i\tilde{Q}^\alpha, {}^i\tilde{M}^{\alpha\beta}$	stress resultants and stress couple resultants of Second Piola–Kirchhoff tensor
${}^i\varepsilon_{\alpha\beta}, {}^i\kappa_{\alpha\beta}, {}^i\gamma_{\alpha\beta}$	strains of layer i
Λ_K	transformation matrix
$\mathbf{u}_0$	displacement vector of layer 1
${}^i\beta_\alpha$	local rotational degrees of freedom of layer i

1 Introduction

In recent years the application of composite materials in structures has become increasingly popular. The advantages of these materials are high strength and stiffness ratios coupled with a low specific weight. Thus composites are used in highly loaded light weight structures. Often the designed constructions are thin shells which are very sensitive against loss of stability.

The structural response is characterized by the fact that laminated composites typically have very large bending modulus to shear modulus ratios. Due to the varying fiber orientations and the anisotropy of the material each ply tries to behave independently of the other plys. Thus, large edge stresses are necessary to preserve compatibility of deformations. Therefore the precise computation of stress distribution and especially the interlaminar shear stresses is crucial for composite shells. However, the analysis of composite structures is more complicated when compared to metallic structures, because laminated composite shells are characterized by bending-stretching coupling.

Theories for multilayered plates and shells have been subject of much research since the early 1970s, see e.g. Pagano (1970), Epstein and Glockner (1977), Tsai and Hahn (1980), Reddy (1984), Niederstedt (1985). Since there are very many papers on this subject, only a few representative results are mentioned in the following. For detailed surveys we refer to Kapania and Raciti (1989) and Noor et al. (1989).

Shear deformation models are variants of the Reissner–Mindlin theory which were originally proposed for homogeneous isotropic plates. A generalization of the shear-deformable plate theory leads to the classical laminated plate theory where coupling of bending and stretching occurs. Closed form solutions for bending problems of cross-ply and angle-ply laminates have been presented by Whitney and Pagano (1970). These models do not account for continuity of the normal and shear stress components acting on laminar interfaces. The performance of the first-order shear-deformation model is dependent on the factors used for adjusting the transverse shear stiffness, see e.g. Whitney and Pagano (1970). Several approaches have been proposed for calculating shear correction factors for laminates.

Cubic polynomials in thickness direction are used in third-order shear-deformation models to interpolate the displacements. However these models do not fulfill continuity of the stresses across the interfaces. Levinson (1980) presented a higher order shear deformation theory with a displacement field satisfying the condition of zero transverse shear stress on the top and bottom surfaces. A variationally consistent set of governing equations has been derived by Reddy (1984) from the principle of virtual displacements.

In discrete-layer models the transverse variation of the displacement field is represented by piecewise linear functions (multi-director theory). The number of field equations and edge boundary conditions depends on the number of layers. Each layer is considered as a homogeneous shell with constant material properties.

Numerical models for linear composite plate and shells have been considered in e.g. Epstein and Huttelmaier (1983), Toledano and Murakami (1987), Reddy et al. (1989), Ladeur and Batoz (1989), Li and Owen (1989), Tessler and Saether (1991), Lee and Liu (1992). Finite Elements based on mixed variational principles are developed in e.g. Puchta and Reddy (1984), Jing and Liao (1989), Pinski and Jasti (1989), Peseux and Dubigeon (1991). Hierarchical models are discussed in Babuška et al. (1992).

Geometrical nonlinear FE-formulations are presented in e.g. Reddy and Chandrashekhara (1985), Huttelmaier and Epstein (1990), Wagner and Gruttmann (1991), Wagner and Stein (1992). In the paper of Dorninger and Rammerstorfer (1990) material non-linearities in terms of stiffness degradation due to matrix or fibre cracking are taken into account. Yoda and Atluri (1992) perform postbuckling analysis of stiffened laminated composite panels. A finite strip formulation is applied within a 5-parameter higher-order shear deformation theory.

The objective of this paper is to present a geometrical nonlinear formulation of a layered shell with orthogonal anisotropic material behaviour and an associated finite element formulation. In the first part we describe the kinematics of the shell. Each layer has constant material properties thus may be treated as a homogeneous shell. The reference surface is the middle surface of the first layer. We assume constant transverse shear strains for each layer. Based on this assumption the field equations are derived, where the interlaminar stresses are taken into account. Continuity of the interlaminar stresses is automatically obtained within the nonlinear solution process.

The second part is concerned with the finite element formulation for quadrilaterals. For the problems considered here the local rotations are moderate. Within the so-called isoparametric approach we use bilinear shape functions to interpolate the reference surface, the displacement vector and the two rotations for each layer. The displacement vector refers to the reference surface (bottom layer). We present the discrete weak form of equilibrium and derive analytically the tangent stiffness matrix. Numerical examples show the effectivity of the proposed FE-model.

2 Variational formulation of the boundary value problem

In this section the variational formulation of the boundary value problem is presented. We consider the kinematics of a layered shell, derive the static field equations and the associated weak formulation. Furthermore the orthogonal anisotropic material law is formulated.

2.1 Kinematics of the shell

The shell consists of N layers with orthotropic material behaviour. Each layer is treated as a homogeneous shell with constant material properties over the layer thickness ${}^ih$. In the definitions and relations that follow Greek subscripts and superscripts refer to covariant and contravariant surface tensor components, respectively. The summation convention applies to each repeated pair of indices. Commas are used to denote partial differentiation based on the geometry of the reference surface Ω (midsurface of the bottom layer). The position vector $\mathbf{X}_0$ is labeled with convective coordinates ξ^α. An orthonormal basis system $\mathbf{t}_k$ is attached to this surface where $\mathbf{t}_3$ is the normal vector. Accordingly the unit vectors $\mathbf{t}_\alpha$ are given by partial differentiation of the position vector $\mathbf{X}_0$ with respect to the associated coordinates. This leads to a convenient representation of the orthogonal anisotropic material law (see Sect. 2.3).

Furthermore the orthonormal basis system ${}^i\mathbf{a}_k$ describes the deformed cross section of layer i. Due to the transverse shear strains the director vector ${}^i\mathbf{a}_3$ is not normal to the current middle surface of layer i.

The transformations between the different base systems are given by

$$\mathbf{t}_k(\xi^\alpha) = \mathbf{R}_0(\xi^\alpha)\mathbf{e}_k, \quad {}^i\mathbf{a}_k(\xi^\alpha, t) = {}^i\mathbf{R}(\xi^\alpha, t)\mathbf{t}_k(\xi^\alpha) \tag{1}$$

where $\mathbf{R}_0$ and ${}^i\mathbf{R}$ are proper orthogonal tensors. The associated components are specified for moderate rotations in Sect. 3.

The kinematic of the shell is based on the assumption of layerwise constant transverse shear strains. Thus the position vector of layer i follows from

$${}^i\mathbf{x}(\xi^\alpha, {}^i\zeta) = {}^i\mathbf{x}_0(\xi^\alpha) + {}^i\zeta\,{}^i\mathbf{a}_3(\xi^\alpha) \qquad -\frac{{}^ih}{2} \leqq {}^i\zeta \leqq +\frac{{}^ih}{2} \tag{2}$$

with ${}^i\zeta$ the coordinate in thickness direction. Since ${}^i\mathbf{a}_3$ is a unit vector we assume inextensibility in thickness direction. This assumption is valid for thin layers and is also justified by the restriction to small strains. Further equations concerning the kinematics of nonlinear shells especially with finite rotations are given in e.g. Gruttmann et al. (1989), Wriggers and Gruttmann (1990).

2.2 Field equations and weak formulation

It is common practice in the analysis of thin shells to assume that external forces are applied to the midsurface of the shell configuration. Here external loading is acting in normal direction to the reference configuration. Since the shell is inextensible in thickness direction there are no effects of surface loading within the variational formulation.

Hence the stress boundary conditions of the shell problem can be written as

$${}^0\mathbf{q} = \mathbf{0} \quad {}^N\mathbf{q} = \hat{\mathbf{q}} = {}^N\tau^3\mathbf{t}_3. \tag{3}$$

The set of static field equations are derived using the local balance equations of the three-dimensional theory, accordingly

$$\operatorname{Div}\mathbf{P} + \rho\mathbf{b} = \mathbf{0}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{4}$$

Here $\boldsymbol{\sigma}$ and $\mathbf{P}$ denote the symmetric Cauchy stress tensor and the First Piola–Kirchhoff stress tensor, respectively. The body forces $\rho\mathbf{b}$ are neglected in the following equations.

The stress resultants and interlaminar stresses of layer i are introduced according to Fig. 1

$$ {}^i\mathbf{n}^\alpha := \int_{-{}^ih/2}^{{}^ih/2} \mathbf{P}\mathbf{e}_\alpha d\zeta \quad {}^i\mathbf{m}^\alpha := \int_{-{}^ih/2}^{{}^ih/2} ({}^i\mathbf{x} - {}^i\mathbf{x}_0) \times \mathbf{P}\mathbf{e}_\alpha d\zeta. \tag{5} $$

Inserting kinematical assumption (2) into $(5)_2$ one obtains

$$ {}^i\mathbf{m}^\alpha = {}^i\mathbf{a}_3 \times {}^i\tilde{\mathbf{m}}^\alpha \quad {}^i\tilde{\mathbf{m}}^\alpha = \int_{-{}^ih/2}^{{}^ih/2} \zeta \mathbf{P}\mathbf{e}_\alpha d\zeta \tag{6} $$

hence ${}^iM^{\alpha 3} = {}^i\mathbf{m}^\alpha \cdot {}^i\mathbf{a}_3 \equiv 0$ is incorporated.

Integrating (4) in thickness direction and inserting (5) yields the set of static field equations

$$ {}^i\mathbf{n}^\alpha|_\alpha + {}^i\mathbf{q} - {}^{i-1}\mathbf{q} = \mathbf{0}, \quad {}^i\mathbf{m}^\alpha|_\alpha + {}^i\mathbf{x}_{0,\alpha} \times {}^i\mathbf{n}^\alpha + \frac{{}^ih}{2} {}^i\mathbf{a}_3 \times ({}^i\mathbf{q} + {}^{i-1}\mathbf{q}) = \mathbf{0}, \quad \text{in } \Omega \tag{7} $$

and boundary conditions

$$ v_\alpha {}^i\mathbf{n}^\alpha = \mathbf{0} \quad v_\alpha {}^i\mathbf{m}^\alpha = \mathbf{0} \quad \text{on } \partial\Omega_\sigma, \quad {}^i\mathbf{u} = {}^i\hat{\mathbf{u}} \quad {}^i\mathbf{a}_3 = {}^i\hat{\mathbf{a}}_3 \quad \text{on } \partial\Omega_u. \tag{8} $$

The associated weak formulation is obtained by weighting the field equations with test functions $\delta^i\mathbf{x}_0$ and $\delta^i\mathbf{w}$. Summation over the layers yields

$$ \begin{aligned} &\int_{(\Omega)} \sum_{i=1}^N [({}^i\mathbf{n}^\alpha|_\alpha + ({}^i\mathbf{q} - {}^{i-1}\mathbf{q}))\cdot\delta^i\mathbf{x}_0] d\Omega \\ &+ \int_{(\Omega)} \sum_{i=1}^N \left[\left({}^i\mathbf{m}^\alpha|_\alpha + {}^i\mathbf{x}_{0,\alpha} \times {}^i\mathbf{n}^\alpha + \frac{{}^ih}{2} {}^i\mathbf{a}_3 \times ({}^i\mathbf{q} + {}^{i-1}\mathbf{q}) \right)\cdot\delta^i\mathbf{w} \right] d\Omega \\ &+ \int_{\partial\Omega_\sigma} \sum_{i=1}^N [(v_\alpha {}^i\mathbf{n}^\alpha)\cdot\delta^i\mathbf{x}_0 + (v_\alpha {}^i\mathbf{m}^\alpha)\cdot\delta^i\mathbf{w}] d\partial\Omega = 0 \end{aligned} \tag{9} $$

where $\delta^i\mathbf{w}$ follows from the variation of the basis system ${}^i\mathbf{a}_k$ in the current configuration

$$ \delta^i\mathbf{a}_k = \delta^i\mathbf{w} \times {}^i\mathbf{a}_k. \tag{10} $$

In the following the contribution of the interlaminar stresses within the variational formulation is examined, accordingly with ${}^0\mathbf{q} = \mathbf{0}$

$$ \begin{aligned} &\int_{(\Omega)} \sum_{i=1}^N \left[({}^i\mathbf{q} - {}^{i-1}\mathbf{q})\cdot\delta^i\mathbf{x}_0 + \frac{{}^ih}{2} {}^i\mathbf{a}_3 \times ({}^i\mathbf{q} + {}^{i-1}\mathbf{q})\cdot\delta^i\mathbf{w} \right] d\Omega \\ &= \int_{(\Omega)} \sum_{i=1}^N \left[({}^i\mathbf{q} - {}^{i-1}\mathbf{q})\cdot\left(\delta^i\mathbf{x}_0 + \frac{{}^ih}{2}\delta^i\mathbf{a}_3 \right) + {}^{i-1}\mathbf{q}\cdot{}^ih\delta^i\mathbf{a}_3 \right] d\Omega \\ &= \int_{(\Omega)} \left[\sum_{i=1}^{N-1} {}^i\mathbf{q}\cdot(\delta^i\hat{\mathbf{x}} - \delta^{i+1}\hat{\mathbf{x}}) + {}^i\mathbf{q}\cdot{}^{i+1}h\delta^{i+1}\mathbf{a}_3 \right] d\Omega + \int_{(\Omega)} [{}^N\mathbf{q}\cdot\delta^N\hat{\mathbf{x}}] d\Omega \\ &= \int_{(\Omega)} \left[\sum_{i=1}^{N-1} ({}^i\mathbf{q} - {}^i\mathbf{q})\cdot{}^{i+1}h\delta^{i+1}\mathbf{a}_3 \right] d\Omega + \int_{(\Omega)} [{}^N\mathbf{q}\cdot\delta^N\hat{\mathbf{x}}] d\Omega \\ &= \int_{(\Omega)} \hat{\mathbf{q}}\cdot\delta\mathbf{u} d\Omega = \delta\Pi_{\text{ext}}. \end{aligned} \tag{11} $$

where $\delta^i\hat{\mathbf{x}} = \delta^i\mathbf{x}_0 + \frac{{}^ih}{2}\delta^i\mathbf{a}_3 = \delta^{i+1}\mathbf{x}_0 - \frac{{}^{i+1}h}{2}\delta^{i+1}\mathbf{a}_3$. Thus continuity of the interlaminar stresses across the interfaces is automatically attained within the nonlinear solution process.

The components of the stress resultants are introduced with respect to the basis system $\{{}^i\mathbf{x}_{0,\alpha}, {}^i\mathbf{a}_3\}$

$$ {}^i\mathbf{n}^\alpha = {}^iN^{\alpha\beta}\,{}^i\mathbf{x}_{0,\beta} + {}^iQ^\alpha\,{}^i\mathbf{a}_3, \quad {}^i\tilde{\mathbf{m}}^\alpha = {}^i\tilde{M}^{\alpha\beta}\,{}^i\mathbf{x}_{0,\beta} + {}^i\tilde{M}^{\alpha 3}\,{}^i\mathbf{a}_3. \tag{12} $$

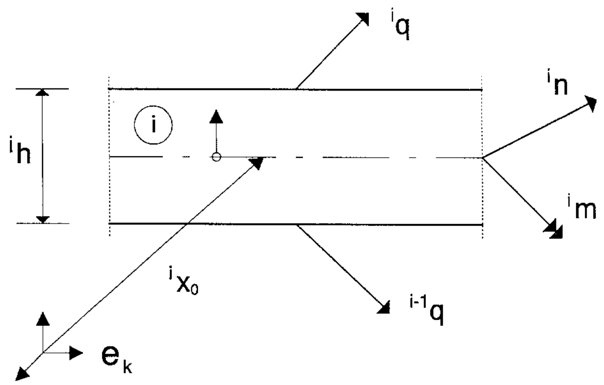


Fig. 1. Stress resultants and interlaminar stresses of layer i

Furthermore we define strains of layer i

$$ {}^i\varepsilon_{\alpha\beta} := \tfrac{1}{2}({}^i\mathbf{x}_{0,\alpha}\cdot{}^i\mathbf{x}_{0,\beta} - {}^i\mathbf{X}_{0,\alpha}\cdot{}^i\mathbf{X}_{0,\beta}), \quad {}^i\kappa_{\alpha\beta} := {}^i\mathbf{x}_{0,\alpha}\cdot{}^i\mathbf{a}_{3,\beta} - {}^i\mathbf{X}_{0,\alpha}\cdot\mathbf{t}_{3,\beta}, \quad {}^i\gamma_{\alpha} := {}^i\mathbf{x}_{0,\alpha}\cdot{}^i\mathbf{a}_{3} - {}^i\mathbf{X}_{0,\alpha}\cdot\mathbf{t}_{3}. \tag{13} $$

These strain measures are the Green–Lagrangian strains $E_{ij} = \frac{1}{2}(\mathbf{x}_{,i}\cdot\mathbf{x}_{,j} - \mathbf{X}_{,i}\cdot\mathbf{X}_{,j})$ of the three-dimensional theory considering kinematic assumption (2).

Application of the divergence theorem to (9) and inserting (11–13) leads after some algebraic manipulations to

$$ \int_{({}^i\Omega)} \sum_{i=1}^{N} [{}^i\tilde{N}^{\alpha\beta}\delta{}^i\varepsilon_{\alpha\beta} + {}^i\tilde{M}^{\alpha\beta}\delta{}^i\kappa_{\alpha\beta} + {}^i\tilde{Q}^{\alpha}\delta{}^i\gamma_{\alpha}]\,d\Omega - \delta\Pi_{\text{ext}} = 0. \tag{14} $$

The virtual work of the external forces is deduced from

$$ \delta\Pi_{\text{ext}} = \int_{(\Omega)} \hat{\mathbf{q}}\cdot\delta\mathbf{u}\,d\Omega = \int_{(\Omega)} {}^N\tau^3\,\delta w\,d\Omega, \quad \delta w = \mathbf{t}_3\cdot\delta\mathbf{u}. \tag{15} $$

In (14) symmetric membrane forces ${}^i\tilde{N}^{\alpha\beta}$ and shear forces ${}^i\tilde{Q}^{\alpha}$ are given by the relation

$$ {}^i\tilde{N}^{\alpha\beta} = {}^iN^{\alpha\beta} - {}^ib^{\alpha i}_{\gamma}\tilde{M}^{\beta\gamma}, \quad {}^i\tilde{Q}^{\alpha} = {}^iQ^{\alpha} - {}^ib^{3i}_{\gamma}\tilde{M}^{\alpha\gamma} \tag{16} $$

where the curvatures follow from the derivative of the director vector

$$ {}^i\mathbf{a}_{3,\alpha} = {}^ib^{\beta i}_{\alpha}\mathbf{x}_{0,\beta} + {}^ib^{3i}_{\alpha}\mathbf{a}_3. \tag{17} $$

As shown in e.g. Simo et al. (1989) the stress couple resultants ${}^i\tilde{M}^{\alpha 3}$ do not contribute to the virtual work.

In Appendix A an alternative representation of the weak formulation is derived by inserting the equilibrium Eq. $(7)_1$ into the equilibrium of stress couple resultants $(7)_2$.

2.3 Orthogonal anisotropic material law

Besides the kinematical relations and the equilibrium equations we have to formulate a constitutive law to determine the deformations of the shell. In a pure mechanical theory we neglect thermal influences on the deformation process and thus are lead to a hyperelastic, isothermal constitutive law.

Each layer is considered as a homogeneous shell with constant material properties over the layer thickness. In contrast to a first order shear deformation theory no coupling occurs between ${}^i\tilde{N}^{\alpha\beta}$ and ${}^i\tilde{M}^{\alpha\beta}$. It is assumed in the following that the skew-symmetric part of the stress couple resultants ${}^i\tilde{M}^{\alpha\beta}$ is zero, i.e. only the symmetric part of the curvatures ${}^i\kappa_{\alpha\beta}$ is used. Thus the stress resultants as defined in the previous section are written in matrix notation and expressed as

$$ {}^i\mathbf{S} = {}^i\mathbf{D}\,{}^i\mathbf{E} \tag{18} $$

with

$${}^i\mathbf{S} = [{}^i\tilde{N}^{11}, {}^i\tilde{N}^{22}, {}^i\tilde{N}^{12}, {}^i\tilde{M}^{11}, {}^i\tilde{M}^{22}, {}^i\tilde{M}^{12}, {}^i\tilde{Q}^1, {}^i\tilde{Q}^2]^T, \quad {}^i\mathbf{E} = [{}^i\varepsilon_{11}, {}^i\varepsilon_{22}, 2{}^i\varepsilon_{12}, {}^i\kappa_{11}, {}^i\kappa_{22}, 2{}^i\kappa_{(12)}, {}^i\gamma_1, {}^i\gamma_2]^T$$

$$i\mathbf{D} = \begin{bmatrix} {}^i\mathbf{D}^m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^i\mathbf{D}^b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}^i\mathbf{D}^s \end{bmatrix}, \quad {}^i\mathbf{D}^m = {}^ih\,{}^i\mathbf{C} \quad {}^i\mathbf{D}^b = \frac{{}^ih^3}{12}\,{}^i\mathbf{C} \quad {}^i\mathbf{D}^s = {}^ih\,{}^i\mathbf{C}_S. \tag{19}$$

The material matrix ${}^i\mathbf{C}$ refers to the orthonormal coordinate system $\mathbf{t}_\alpha$ and is deduced from the local matrix $\mathbf{C}_L$ by a simple transformation

$${}^i\mathbf{C} = {}^i\mathbf{T}^T \mathbf{C}_L\,{}^i\mathbf{T}$$

$${}^i\mathbf{T} = \begin{bmatrix} \cos^2{}^i\varphi & \sin^2{}^i\varphi & \sin{}^i\varphi\cos{}^i\varphi \\ \sin^2{}^i\varphi & \cos^2{}^i\varphi & -\sin{}^i\varphi\cos{}^i\varphi \\ -2\sin{}^i\varphi\cos{}^i\varphi & 2\sin{}^i\varphi\cos{}^i\varphi & \cos^2{}^i\varphi - \sin^2{}^i\varphi \end{bmatrix} \tag{20}$$

$$\mathbf{C}_L = \begin{bmatrix} \hat{\mathbf{C}}_L & 0 \\ 0 & G_{12} \end{bmatrix} \quad \hat{\mathbf{C}}_L = \frac{1}{1 - \nu^2(E_2/E_1)} \begin{bmatrix} E_1 & \nu E_2 \\ \nu E_2 & E_2 \end{bmatrix}.$$

Here, ${}^i\varphi$ is the angle between fiber direction and basis vector $\mathbf{t}_1$. Transformation of the shear stiffness yields

$${}^i\mathbf{C}_S = {}^i\mathbf{T}_S^T \mathbf{C}_{SL}\,{}^i\mathbf{T}_S \quad {}^i\mathbf{T}_S = \begin{bmatrix} \cos{}^i\varphi & \sin{}^i\varphi \\ -\sin{}^i\varphi & \cos{}^i\varphi \end{bmatrix} \quad \mathbf{C}_{SL} = \begin{bmatrix} G_{13} & 0 \\ 0 & G_{23} \end{bmatrix}. \tag{21}$$

Using (18) plane stress condition is incorporated. The material law is postulated in terms of Second Piola–Kirchhoff stresses and Green Lagrangian strains. We restrict ourselves to small strains thus transformations between different stress tensors are neglectable and we can use material parameters of the linear theory.

3 Finite element formulation

In this section the finite element formulation of quadrilaterals is developed. The initial geometry is approximated using standard bilinear shape functions. Since shear deformations are considered in the theory presented only C^0-continuity for the displacements and rotations is needed at the element boundaries.

3.1 Approximation of geometry and displacements

In the paper of Rammerstorfer (1992) besides the two rotations of the shell middle surface $2(N-1)$ additional rotational degrees of freedom are used at each node. This leads to a rather complicated representation of shell strains for layer N. Therefore in this formulation the director vector of each layer is described with independent degrees of freedom.

Let us assume that the shell surface is approximated by a finite element discretization of the form

$$\Omega^h = \bigcup_{e=1}^{n_{\text{elm}}} \Omega_e, \tag{22}$$

where n_{elm} is the number of finite elements in the discretization, and Ω_e denotes a typical element.

Within an element Ω_e, the position vector $\mathbf{X}_0$ of the middle surface of the first layer is interpolated by

$$\mathbf{X}_0 = \sum_{K=1}^{4} N_K(\xi,\eta)\mathbf{X}_{0K}. \tag{23}$$

Here the functions $N_K(\xi,\eta)$ are the standard bilinear element shape functions. Then the local cartesian basis system $\mathbf{t}_i$ and associated coordinates s_α are deduced from (23). The normal vector $\mathbf{t}_3$ is obtained by the cross product of the tangential vectors $\mathbf{X}_{0,\xi}, \mathbf{X}_{0,\eta}$, whereas $\mathbf{t}_1$ and $\mathbf{t}_2$ can be chosen as any arbitrary set of orthonormal vectors normal to the undeformed director vector. Then application of the jacobian matrix $[J_{\alpha\beta}]$ with components $J_{\alpha\beta} = \mathbf{X}_{0,\alpha}\cdot\mathbf{t}_\beta$ yields the derivatives of the shape functions N_K with respect to the coordinates s_α.

Furthermore the displacement vector of the bottom layer $\mathbf{u}_0$, the incremental rotation vector $\mathrm{d}^i\mathbf{a}_3$ and the initial director vector $\mathbf{t}_3$ are interpolated

$$\mathbf{u}_0 = \sum_{K=1}^{4} N_K(\xi,\eta)\mathbf{u}_{0K}, \quad \mathrm{d}^i\mathbf{a}_3 = \sum_{K=1}^{4} N_K(\xi,\eta)\mathrm{d}^i\mathbf{a}_{3K}, \quad \mathbf{t}_3 = \frac{\sum_{K=1}^{4} N_K(\xi,\eta)\mathbf{t}_{3K}}{\|\sum_{K=1}^{4} N_K(\xi,\eta)\mathbf{t}_{3K}\|}. \tag{24}$$

Here $\mathrm{d}^i\mathbf{a}_{3K}$ is given by transformation of the local rotational degrees of freedom ${}^i\beta_{\alpha K}$ to the cartesian coordinate system $\mathbf{e}_i$

$$\mathrm{d}^i\mathbf{a}_{3K} = \boldsymbol{\Lambda}_K{}^i\boldsymbol{\beta}_K, \quad \boldsymbol{\Lambda}_K = [-\mathbf{t}_{2K}, \mathbf{t}_{1K}], \quad {}^i\boldsymbol{\beta}_K = \begin{bmatrix} {}^i\beta_{1K} \\ {}^i\beta_{2K} \end{bmatrix}. \tag{25}$$

Now we are able to express the director vector of the current configuration and the position vector of layer i in initial and current configuration, respectively

$${}^i\mathbf{a}_3 = \mathbf{t}_3 + \mathrm{d}^i\mathbf{a}_3, \quad {}^i\mathbf{X}_0 = \mathbf{X}_0 + \left(\frac{{}^1h}{2} + \sum_{j=2}^{i-1} {}^jh + \frac{{}^ih}{2}\right)\mathbf{t}_3, \quad {}^i\mathbf{x}_0 = \mathbf{X}_0 + \mathbf{u}_0 + \frac{{}^1h}{2}{}^1\mathbf{a}_3 + \sum_{j=2}^{i-1} {}^jh\,{}^j\mathbf{a}_3 + \frac{{}^ih}{2}{}^i\mathbf{a}_3. \tag{26}$$

Note that ${}^i\mathbf{X}_{0,\alpha}$ is in general not a unit vector

$$\|{}^i\mathbf{X}_{0,\alpha}\| = \begin{cases} 1 & i=1 \\ >1 & i=2,N. \end{cases} \tag{27}$$

Thus expression $\|{}^i\mathbf{X}_{0,\alpha}\|$ has to be considered when computing the strains.

Finally the variations of ${}^i\mathbf{x}_{0,\alpha}$, ${}^i\mathbf{a}_3$ and ${}^i\mathbf{a}_{3,\alpha}$ can be expressed by inserting the FE-interpolations (24)

$$\delta^i\mathbf{x}_{0,\alpha} = \sum_{K=1}^{4} N_{K,\alpha}\left[\delta\mathbf{u}_K^0 + \boldsymbol{\Lambda}_K\left(\frac{{}^1h}{2}\delta^1\boldsymbol{\beta}_K + \sum_{j=1}^{i-1} {}^jh\delta^j\boldsymbol{\beta}_K + \frac{{}^ih}{2}\delta^i\boldsymbol{\beta}_K\right)\right]$$

$$\delta^i\mathbf{a}_3 = \sum_{K=1}^{4} N_K\boldsymbol{\Lambda}_K\delta^i\boldsymbol{\beta}_K, \quad \delta^i\mathbf{a}_{3,\alpha} = \sum_{K=1}^{4} N_{K,\alpha}\boldsymbol{\Lambda}_K\delta^i\boldsymbol{\beta}_K. \tag{28}$$

Since the tangential vectors $\mathbf{t}_{\alpha K}$ at the nodes have to be unique we are restricted to differentiable surfaces Ω. This nodal basis system is computed within the mesh generation in such a way that special boundary conditions of the calculated problem can be accommodated. Using transformation (25) we are restricted to moderate rotations. An extension to finite rotations is basically possible, however it is not necessary for the class of problems considered here. Concerning the problem of finite rotations for laminated shells we refer to Wagner and Gruttmann (1992).

3.2 *Discretized variational formulation and linearization*

Inserting the preliminary interpolations of the last section into the virtual work expression we now derive the FE-equations in matrix notation. In order to compute the virtual work we need to compute the variations of the strains

$$\delta^i\varepsilon_{\alpha\beta} = \tfrac{1}{2}(\delta^i\mathbf{x}_{0,\alpha}\cdot{}^i\mathbf{x}_{0,\beta} + {}^i\mathbf{x}_{0,\alpha}\cdot\delta^i\mathbf{x}_{0,\beta}), \quad \delta^i\kappa_{\alpha\beta} = \delta^i\mathbf{x}_{0,\alpha}\cdot{}^i\mathbf{a}_{3,\beta} + {}^i\mathbf{x}_{0,\alpha}\cdot\delta^i\mathbf{a}_{3,\beta}$$
$$\delta^i\gamma_\alpha = \delta^i\mathbf{x}_{0,\alpha}\cdot{}^i\mathbf{a}_3 + {}^i\mathbf{x}_{0,\alpha}\cdot\delta^i\mathbf{a}_3. \tag{29}$$

Introducing the displacement vector of node K

$$\mathbf{v}_K = [\mathbf{u}_K^0, {}^1\boldsymbol{\beta}_K \cdots {}^j\boldsymbol{\beta}_K \cdots {}^i\boldsymbol{\beta}_K \cdots {}^N\boldsymbol{\beta}_K]^T \tag{30}$$

the virtual strains can be expressed as

$$\delta^i\mathbf{E} = [\delta^i\varepsilon_{11}, \delta^i\varepsilon_{22}, 2\delta^i\varepsilon_{12}, \delta^i\kappa_{11}, \delta^i\kappa_{22}, 2\delta^i\kappa_{(12)}, \delta^i\gamma_1, \delta^i\gamma_2]^T$$

$$\delta^i\mathbf{E} = \sum_{K=1}^{4} {}^i\mathbf{B}_K^T\delta\mathbf{v}_K \quad {}^i\mathbf{B}_K = \begin{bmatrix} {}^i\mathbf{B}_K^{mu} & {}^i\mathbf{B}_K^{m\beta} \\ {}^i\mathbf{B}_K^{bu} & {}^i\mathbf{B}_K^{b\beta} \\ {}^i\mathbf{B}_K^{su} & {}^i\mathbf{B}_K^{s\beta} \end{bmatrix}. \tag{31}$$

Here the matrices

$${}^i\mathbf{B}_K^{mu} = \begin{bmatrix} N_{K,1}\,{}^i\mathbf{x}_{0,1}^T \\ N_{K,2}\,{}^i\mathbf{x}_{0,2}^T \\ N_{K,1}\,{}^i\mathbf{x}_{0,2}^T + N_{K,2}\,{}^i\mathbf{x}_{0,1}^T \end{bmatrix}, \quad {}^i\mathbf{B}_K^{bu} = \begin{bmatrix} N_{K,1}\,{}^i\mathbf{a}_{3,1}^T \\ N_{K,2}\,{}^i\mathbf{a}_{3,2}^T \\ N_{K,1}\,{}^i\mathbf{a}_{3,2}^T + N_{K,2}\,{}^i\mathbf{a}_{3,1}^T \end{bmatrix}, \quad {}^i\mathbf{B}_K^{su} = \begin{bmatrix} N_{K,1}\,{}^i\mathbf{a}_3^T \\ N_{K,2}\,{}^i\mathbf{a}_3^T \end{bmatrix} \tag{32}$$

refer to $\mathbf{u}_K^0$, whereas the matrices

$${}^i\mathbf{B}_K^{m\beta} = {}^i\mathbf{H}^1\,{}^i\mathbf{B}_K^{mu}\boldsymbol{\Lambda}_K, \quad {}^i\mathbf{B}_K^{b\beta} = ({}^i\mathbf{H}^1\,{}^i\mathbf{B}_K^{bu} + {}^i\mathbf{H}^2\,{}^i\mathbf{B}_K^{mu})\boldsymbol{\Lambda}_K, \quad {}^i\mathbf{B}_K^{s\beta} = ({}^i\mathbf{H}^1\,{}^i\mathbf{B}_K^{su} + {}^i\mathbf{H}^2\,{}^i\mathbf{B}_K^{ss})\boldsymbol{\Lambda}_K$$
$${}^i\mathbf{B}_K^{ss} = \begin{bmatrix} N_K\,{}^i\mathbf{x}_{0,1}^T \\ N_K\,{}^i\mathbf{x}_{0,2}^T \end{bmatrix}, \quad {}^i\mathbf{H}^1 = \left[\frac{{}^1h}{2} \cdots {}^jh \cdots \frac{{}^ih}{2} \cdots 0\right], \quad {}^i\mathbf{H}^2 = [0 \cdots 0 \cdots 1 \cdots 0] \tag{33}$$

refer to the rotational degrees of freedom $\{{}^1\boldsymbol{\beta}_K \cdots {}^j\boldsymbol{\beta}_K \cdots {}^i\boldsymbol{\beta}_K \cdots {}^N\boldsymbol{\beta}_K\}$. The finite element approximation of the virtual work (14) yields

$$\delta\mathbf{v}^T\mathbf{g}^h(\mathbf{v}, \delta\mathbf{v}) = \bigcup_{e=1}^{n_{elm}} \sum_{K=1}^{4} \delta\mathbf{v}_K^T \int_{(\Omega_e)} \left(\sum_{i=1}^{N} {}^i\mathbf{B}_K^T\,{}^i\mathbf{S} - N_K\hat{\mathbf{q}}\right) d\Omega_e = 0. \tag{34}$$

The solution of the nonlinear algebraic Eq. (34) is obtained by Newton's method. For this purpose we have to construct the tangent stiffness of $\mathbf{g}^h$ by computing the directional derivative. This leads to the following incremental solution scheme

$$D\mathbf{g}^h(\mathbf{v}_k)\cdot\Delta\mathbf{v}_{k+1} = -\mathbf{g}^h(\mathbf{v}_k), \quad \mathbf{v}_{k+1} = \mathbf{v}_k + \Delta\mathbf{v}_{k+1}. \tag{35}$$

The tangential stiffness matrix is derived by employing a consistent linearization procedure to the weak form. The external loads are not displacement dependent, thus

$$D\mathbf{g}^h\cdot\Delta\mathbf{v} = \int_{(\Omega)} \sum_{i=1}^{N} [\Delta^i\tilde{N}^{\alpha\beta}\delta^i\varepsilon_{\alpha\beta} + \Delta^i\tilde{M}^{\alpha\beta}\delta^i\kappa_{\alpha\beta} + \Delta^i\tilde{Q}^\alpha\delta^i\gamma_\alpha + {}^i\tilde{N}^{\alpha\beta}\Delta\delta^i\varepsilon_{\alpha\beta} + {}^i\tilde{M}^{\alpha\beta}\Delta\delta^i\kappa_{\alpha\beta} + {}^i\tilde{Q}^\alpha\Delta\delta^i\gamma_\alpha]\, d\Omega \tag{36}$$

with

$$\Delta\delta^i\varepsilon_{\alpha\beta} = \tfrac{1}{2}(\delta^i\mathbf{x}_{0,\alpha}\cdot\Delta^i\mathbf{x}_{0,\beta} + \Delta^i\mathbf{x}_{0,\alpha}\cdot\delta^i\mathbf{x}_{0,\beta}), \quad \Delta\delta^i\kappa_{\alpha\beta} = \delta^i\mathbf{x}_{0,\alpha}\cdot\Delta^i\mathbf{a}_{3,\beta} + \Delta^i\mathbf{x}_{0,\alpha}\cdot\delta^i\mathbf{a}_{3,\beta},$$
$$\Delta\delta^i\gamma_\alpha = \delta^i\mathbf{x}_{0,\alpha}\cdot\Delta^i\mathbf{a}_3 + \Delta^i\mathbf{x}_{0,\alpha}\cdot\delta^i\mathbf{a}_3. \tag{37}$$

Hence the general form of the tangent stiffness for one finite element consists of a material and geometrical part

$$\delta\mathbf{v}^T D\mathbf{g}^h(\mathbf{v}, \delta\mathbf{v})\cdot\Delta\mathbf{v} = \bigcup_{e=1}^{n_{elm}} \sum_{K=1}^{4} \sum_{L=1}^{4} \delta\mathbf{v}_K^T\mathbf{K}_{TKL}\Delta\mathbf{v}_L, \quad \mathbf{K}_{TKL} = \int_{(\Omega_e)} \sum_{i=1}^{N} ({}^i\mathbf{B}_K^T\,{}^i\mathbf{D}\,{}^i\mathbf{B}_L + {}^i\mathbf{G}_{KL})\, d\Omega_e. \tag{38}$$

The geometrical part is derived by inserting the element shape functions into (37) and is expressed as

$$
{}^i\mathbf{G}_{KL} = \begin{bmatrix} {}^i\mathbf{G}^{uu}_{KL} & {}^i\mathbf{G}^{u\beta}_{KL} \\ {}^i\mathbf{G}^{\beta u}_{KL} & {}^i\mathbf{G}^{\beta\beta}_{KL} \end{bmatrix}, \quad {}^i\mathbf{G}^{uu}_{KL} = {}^i\hat{N}\mathbf{1}
$$

$$
{}^i\mathbf{G}^{u\beta}_{KL} = \left[{}^i\hat{N}\frac{{}^1h}{2} \quad {}^i\hat{N}\,{}^jh \quad {}^i\hat{A}^1 \quad 0 \right] \mathbf{\Lambda}_L, \quad {}^i\mathbf{G}^{\beta u}_{KL} = \left\{ \left[{}^i\hat{N}\frac{{}^1h}{2} \quad {}^i\hat{N}\,{}^jh \quad {}^i\hat{A}^2 \quad 0 \right] \mathbf{\Lambda}_K \right\}^T \tag{39}
$$

$$
{}^i\mathbf{G}^{\beta\beta}_{KL} = \begin{bmatrix} {}^i\hat{N}\frac{{}^1h^2}{4} & {}^i\hat{N}\frac{{}^1h}{2}{}^jh & {}^i\hat{A}^1\frac{{}^1h}{2} & 0 \\ {}^i\hat{N}\,{}^jh\frac{{}^1h}{2} & {}^i\hat{N}\,{}^jh^2 & {}^i\hat{A}^1\,{}^jh & 0 \\ {}^i\hat{A}^2\frac{{}^1h}{2} & {}^i\hat{A}^2\,{}^jh & {}^i\hat{A}^3\frac{{}^ih}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{\Lambda}^T_K \mathbf{\Lambda}_L
$$

where

$$
{}^i\hat{N} = \tfrac{1}{2}{}^i\tilde{N}^{\alpha\beta}(N_{K,\alpha}N_{L,\beta} + N_{K,\beta}N_{L,\alpha}), \quad {}^i\hat{M} = {}^i\tilde{M}^{\alpha\beta}N_{K,\beta}N_{L,\alpha}, \quad {}^i\hat{Q}^1 = {}^i\tilde{Q}^{\alpha}N_{K,\alpha}N_L, \quad {}^i\hat{Q}^2 = {}^i\tilde{Q}^{\alpha}N_{L,\alpha}N_K,
$$

$$
{}^i\hat{A}^1 = {}^i\hat{N}\frac{{}^ih}{2} + {}^i\hat{M} + {}^i\hat{Q}^1, \quad {}^i\hat{A}^2 = {}^i\hat{N}\frac{{}^ih}{2} + {}^i\hat{M} + {}^i\hat{Q}^2, \quad {}^i\hat{A}^3 = {}^i\hat{N}\frac{{}^ih}{2} + 2{}^i\hat{M} + {}^i\hat{Q}^1 + {}^i\hat{Q}^2. \tag{40}
$$

Associated with shear elastic shell formulations are locking phenomena. Since our main goal is the formulation of layered shell elements we use selective-reduced integration technique to overcome locking. Thus the shear-terms in residual and tangential stiffness are obtained by one-point integration.

4 Numerical examples

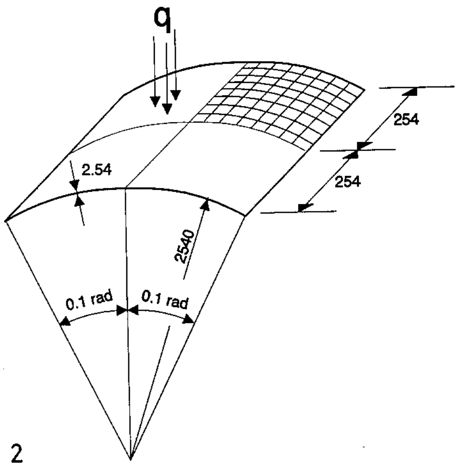
In this section we present several numerical examples which demonstrate effectiveness of the finite element formulation presented above. The obtained results are compared to known solutions in the literature. The elemental scheme was implemented using an enhanced version of the program FEAP documented in Zienkiewicz, Taylor (1988).

4.1 Bending of a clamped cylindrical panel

The first problem is concerned with the bending of a cross-ply [0/90] clamped cylindrical panel under uniform load. This example has been previously investigated in Reddy, Chandrashekhara (1985). The geometrical data are given in Fig. 2. Material data of the cross-ply laminate are

$$
E_1 = 25 \times 10^6\,\text{psi}, \quad E_2 = 10^6\,\text{psi}, \quad G_{12} = G_{13} = 0.5 \times 10^6\,\text{psi},
$$
$$
G_{23} = 0.2 \times 10^6\,\text{psi}, \quad \nu = 0.25. \tag{41}
$$

We discretize one quarter of the shell with 8 × 8 4-node shell elements. The load deflection curve is computed and depicted in Fig. 3. The results agree with a solution obtained with a first order shear deformation theory.



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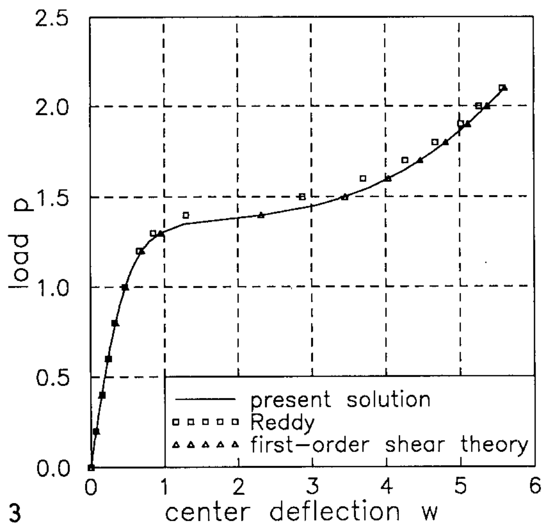

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Figs. 2 and 3. **2** Bending of a clamped cylindrical shell under a uniform loading. **3** Load deflection curve cylindrical shell

4.2 Plate strip subjected to uniform loading

The next example has been investigated by Babuška et al. (1992) using hierarchical models. We consider a plate strip consisting of three plies which are symmetrically arranged with respect to the middle plane (see Fig. 4). The fiber direction is parallel to x in the two outer layers and parallel to y in the central layer. Constant loading is acting in z-direction. The length to hight ratio is $l/h = 10$. Material data are those of Eq. (41). The plate strip is assumed to be in a state of plane strain with respect to the x–z plane. Considering symmetry at $x = l/2$ only one half of the plate strip is modeled using a 15×1 mesh of shell elements. The normalized displacements $\bar{u}_x = 3E_2 u_x(0, z)/(qh)$, normal stresses $\sigma_x(x = l/2)$ and transverse shear stresses $\sigma_{xz}(x = 0)$ are depicted in Figs. 5–7. There is good agreement between our results and the reference solution. Especially the shape of the transverse shear stresses across the thickness could be essentially improved compared to a solution obtained with a first-order shear-deformation theory.

4.3 Rectangular plate under uniform uniaxial extension

This well-known problem, presented in Fig. 8, is examined as next example, see e.g. Ladeur (1992). The laminates considered in this study consist of four identical plies symmetrically stacked in (90/0/0/90) stacking sequences. The width to height ratio is $b/h = 40$, where h represents one layer thickness and b half the width of the plate. The elastic constants with respect to principal material

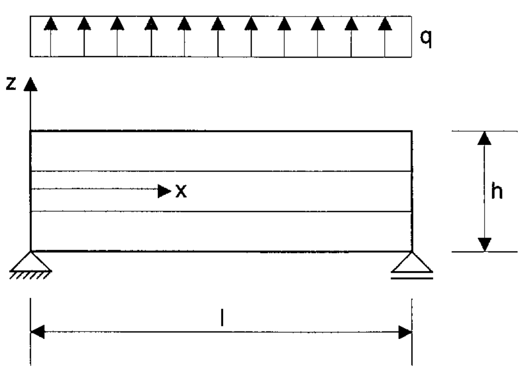
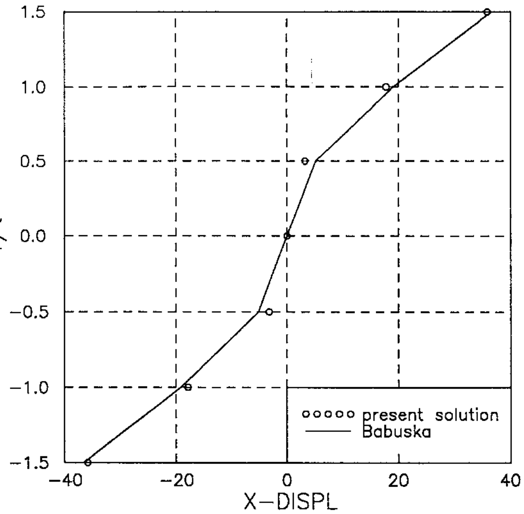


Fig. 4. Plate strip subjected to uniform loading



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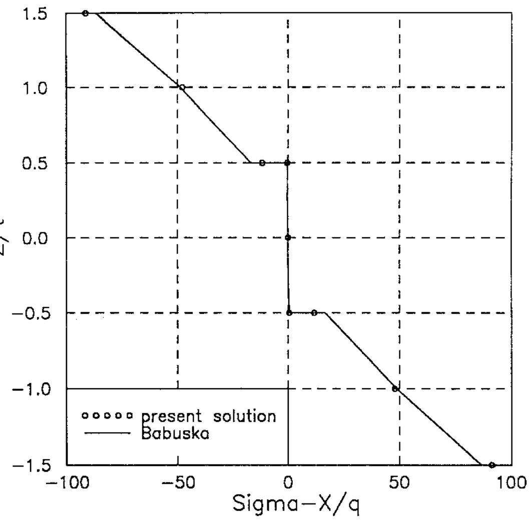

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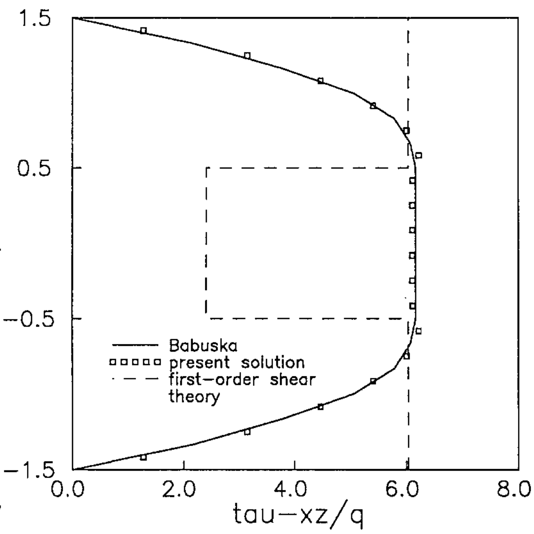

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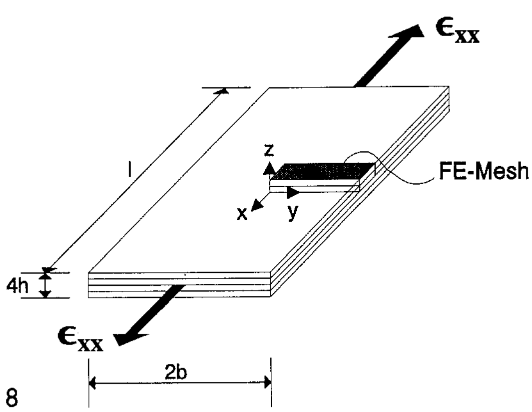
Figs. 5–7. **5** Normalized displacements $\bar{u}_x$. **6** Stresses $\sigma_x(x = l/2, z)$. **7** Shear stresses $\sigma_{xz}(x = 0, z)$



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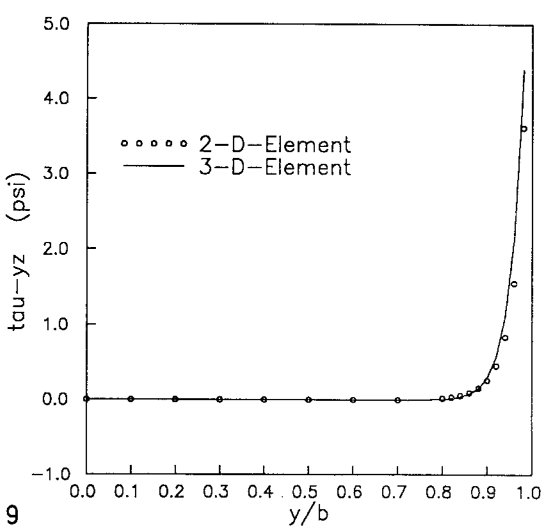

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Figs. 8 and 9. **8** Rectangular plate under uniform uniaxial extension. **9** Distribution of stresses τ_{yz} along the line $z = h/2$

axes are:

$$E_1 = 20 \times 10^6\,\text{psi}, \quad E_2 = 2.1 \times 10^6\,\text{psi}, \quad G_{12} = G_{13} = G_{23} = 0.85 \times 10^6\,\text{psi},$$
$$G_{23} = 0.2 \times 10^6\,\text{psi}, \quad v = 0.25. \tag{42}$$

The stress state is constant in x-direction with $\varepsilon_{xx} = 10^{-4}$, thus one element is sufficient in this direction. Considering symmetry only a quarter of the system is modeled by 50 elements in y-direction. In thickness direction each layer is subdivided into two.

The shear stresses τ_{yz} are computed along the line $z = h/2$ and depicted in Fig. 9. Furthermore a finite element solution obtained with a 3d-element mesh ($x \times y \times z = 1 \times 50 \times 4$) is given. The results show that large edge stresses are necessary to preserve compatibility of deformations. These stresses are mainly responsible for delamination. The developed FE-formulation is able to represent the steep ascent of the shear stresses along the free edges.

5 Concluding remarks

In this paper a geometrical nonlinear theory of laminated shells is outlined. The kinematic is based on the assumption of constant transverse shear strains for each layer. The underlying field equations and the associated weak formulation are derived. Continuity of the interlaminar shear stresses is automatically attained within the nonlinear solution process. The finite element formulation for quadrilaterals is based on the so-called isoparametric approach. Initial geometry, displacement vector and rotations of each layer are interpolated using bilinear functions. Exact linearization of the variational equations leads to tangential stiffness matrices hence to asymptotic quadratic convergence behaviour. Several numerical examples are presented. Especially the shape of the transverse shear stresses across the thickness has been improved compared to a first order shear deformation theory. The computed results are in good agreement with solutions in the literature. An extension of the presented model is possible by introducing stresses in thickness direction.

Appendix A

An alternative expression of the weak formulation can be derived by eliminating the interlaminar stresses within the variational formulation. With continuous interlaminar stresses across the laminar interfaces and summation of the equilibrium Eq. $(7)_1$ over the layers $1 \leqq j \leqq i$ one obtains

$$\sum_{j=1}^{i} [{}^j\mathbf{n}^\alpha|_\alpha + ({}^j\mathbf{q} - {}^{j-1}\mathbf{q})] = \sum_{j=1}^{i} {}^j\mathbf{n}^\alpha|_\alpha + {}^i\mathbf{q} = \mathbf{0}. \tag{43}$$

Inserting (43) into the second equilibrium Eq. $(7)_2$ leads to

$${}^i\mathbf{m}^\alpha|_\alpha + {}^i\mathbf{x}_{0,\alpha} \times {}^i\mathbf{n}^\alpha - \frac{{}^ih}{2}\,{}^i\mathbf{a}_3 \times \left({}^i\mathbf{n}^\alpha|_\alpha + 2\sum_{j=1}^{i-1} {}^j\mathbf{n}^\alpha|_\alpha \right) = \mathbf{0}. \tag{44}$$

Using the definition

$$\Delta{}^i\tilde{\mathbf{m}}^\alpha = \frac{{}^ih}{2}\left({}^i\mathbf{n}^\alpha + 2\sum_{j=1}^{i-1} {}^j\mathbf{n}^\alpha \right) \tag{45}$$

and the derivative of (6)

$${}^i\mathbf{m}^\alpha|_\alpha = {}^i\mathbf{a}_{3,\alpha} \times {}^i\tilde{\mathbf{m}}^\alpha + {}^i\mathbf{a}_3 \times {}^i\tilde{\mathbf{m}}^\alpha|_\alpha \tag{46}$$

one obtains

$${}^i\mathbf{a}_3 \times ({}^i\tilde{\mathbf{m}}^\alpha|_\alpha - \Delta{}^i\tilde{\mathbf{m}}^\alpha|_\alpha) + ({}^i\mathbf{x}_{0,\alpha} \times {}^i\mathbf{n}^\alpha + {}^i\mathbf{a}_{3,\alpha} \times {}^i\tilde{\mathbf{m}}^\alpha) = \mathbf{0}. \tag{47}$$

Weighting (47) with test functions $\delta^i\mathbf{w}$, integration and application of the divergence theorem yields the weak formulation

$$\int_{(\Omega)} \sum_{i=1}^{N} [({}^i\tilde{\mathbf{m}}^\alpha - \Delta^i\tilde{\mathbf{m}}^\alpha)\cdot\delta^i\mathbf{a}_{3,\alpha} - ({}^i\mathbf{x}_{0,\alpha} \times {}^i\mathbf{n}^\alpha + {}^i\mathbf{a}_{3,\alpha} \times {}^i\tilde{\mathbf{m}}^\alpha)\cdot\delta^i\mathbf{w}]\,d\Omega, \quad - \int_{(\Omega)} \hat{\mathbf{q}}\cdot\delta\mathbf{u}\,d\Omega = 0. \tag{48}$$

Inserting the decomposition of the stress resultants (12) with respect to basis system $\{{}^i\mathbf{x}_{0,\alpha}, {}^i\mathbf{a}_3\}$ yields

$$\begin{aligned}
&\int_{(\Omega)} \sum_{i=1}^{N} [({}^i\tilde{M}^{\alpha\beta} - \Delta^i\tilde{M}^{\alpha\beta}){}^i\mathbf{x}_{0,\beta} + ({}^i\tilde{M}^{\alpha 3} - \Delta^i\tilde{M}^{\alpha 3}){}^i\mathbf{a}_3\cdot\delta^i\mathbf{a}_{3,\alpha} \\
&\quad - [{}^i\mathbf{x}_{0,\alpha} \times ({}^iN^{\alpha\beta\,i}\mathbf{x}_{0,\beta} + {}^iQ^{\alpha\,i}\mathbf{a}_3) + {}^i\mathbf{a}_{3,\alpha} \times ({}^i\tilde{M}^{\alpha\beta\,i}\mathbf{x}_{0,\beta} + {}^i\tilde{M}^{\alpha 3\,i}\mathbf{a}_3)\cdot\delta^i\mathbf{w}]\,d\Omega \\
&\quad - \int_{(\Omega)} \hat{\mathbf{q}}\cdot\delta\mathbf{u}\,d\Omega = 0.
\end{aligned} \tag{49}$$

Then introducing the curvatures ${}^i\mathbf{a}_{3,\alpha} = {}^ib_\alpha^{\beta\,i}\mathbf{x}_{0,\beta} + {}^ib_\alpha^{3\,i}\mathbf{a}_3$ (17) and considering the identities

$$\begin{aligned}
&[{}^i\mathbf{x}_{0,\alpha} \times {}^iN^{\alpha\beta\,i}x_{0,\beta} + {}^ib_\alpha^{\gamma\,i}\mathbf{x}_{0,\gamma} \times {}^i\tilde{M}^{\alpha\beta\,i}\mathbf{x}_{0,\beta}]\cdot\delta^i\mathbf{w} = ({}^iN^{\alpha\beta} - {}^ib_\gamma^{\alpha\,i}\tilde{M}^{\beta\gamma})({}^i\mathbf{x}_{0,\alpha} \times {}^i\mathbf{x}_{0,\beta})\cdot\delta^i\mathbf{w} \\
&\quad = {}^i\tilde{N}^{\alpha\beta}e_{\alpha\beta}({}^i\mathbf{x}_{0,1} \times {}^i\mathbf{x}_{0,2})\cdot\delta^i\mathbf{w} \equiv 0, \\
&{}^i\tilde{M}^{\alpha 3}[{}^ia_3\cdot\delta^i\mathbf{a}_{3,\alpha} - ({}^i\mathbf{a}_{3,\alpha} \times {}^i\mathbf{a}_3)\cdot\delta^i\mathbf{w}] \\
&\quad = {}^i\tilde{M}^{\alpha 3}[{}^ia_3\cdot\delta^i\mathbf{a}_{3,\alpha} + \delta^ia_3\cdot{}^i\mathbf{a}_{3,\alpha}] = {}^i\tilde{M}^{\alpha 3}\delta(\mathbf{a}_3\cdot\mathbf{a}_{3,\alpha}) \equiv 0, \\
&{}^i\tilde{M}^{\alpha 3}\,{}^ib^{\alpha 3}({}^ia_3 \times {}^i\mathbf{a}_3)\cdot\delta^i\mathbf{w} \equiv 0.
\end{aligned} \tag{50}$$

the virtual work expression becomes

$$\int_{(\Omega)} \sum_{i=1}^{N} [{}^i\hat{\mathbf{m}}^\alpha\cdot\delta^i\mathbf{a}_{3,\alpha} + {}^i\hat{\mathbf{n}}^\alpha\cdot\delta^i\mathbf{a}_3]\,d\Omega - \int_{(\Omega)} \hat{\mathbf{q}}\cdot\delta\mathbf{u}\,d\Omega = 0 \tag{51}$$

where

$${}^i\hat{\mathbf{m}}^\alpha = ({}^i\tilde{M}^{\alpha\beta} - \Delta^i\tilde{M}^{\alpha\beta}){}^i\mathbf{x}_{0,\beta} - \Delta^i\tilde{M}^{\alpha 3\,i}\mathbf{a}_3, \quad {}^i\hat{\mathbf{n}}^\alpha = {}^i\tilde{M}^{\alpha\beta\,i}b_\alpha^{3\,i}\mathbf{x}_{0,\beta} + {}^iQ^{\alpha\,i}\mathbf{a}_3. \tag{52}$$

Using the constraint $\|{}^i\mathbf{a}_3\| = 1$ the curvature ${}^ib_\alpha^3$ can be expressed as ${}^ib_\alpha^3 = -{}^ib_\alpha^\beta({}^i\mathbf{x}_{0,\beta}\cdot{}^i\mathbf{a}_3) \ll 1$, thus can be neglected. The associated finite element formulation is obtained by inserting the interpolation functions according to Sect. 3.

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References

Babuška, I.; Szabó, B. A.; Actis, R. L. (1992): Hierarchic models for laminated composites. Int. J. Num. Meth. Engng. 33, 503–535

Dorninger, K.; Rammerstorfer, F. (1990): A layered composite shell element for elastic and thermoelastic stress and stability analysis at large deformations. Int. J. Num. Meth. Engng. 30, 833–858

Epstein, M.; Glockner, P. G. (1977): Nonlinear analysis of Multilayered shells. Int. J. Solids Struct. 13, 1081–1089

Epstein, M.; Huttelmaier, H. P. (1983): A finite element formulation for multilayered and thick plates. Comp. and Struct. 5, 645–650

Gruttmann, F.; Stein, E.; Wriggers, P. (1989): Theory and numerics of thin elastic shells with finite roations. Ingenieur Archiv. 59, 54–67

Huttelmaier, H. P.; Epstein, M. (1990): A large displacement finite element for multilayered plates. Finite Elements in Analysis and Design 6, 189–196

Jing, H. S.; Liao, M. L. (1989): Partial hybrid stress element for the analysis of thick laminated composite plates. Int. J. Num. Meth. Engng. 28, 2813–2827

Kapania, R. K.; Raciti, S. (1989): Recent advances in analysis of laminated beams and plates, part I: Shear effects and buckling. AIAAJ. 27, 923–934

Ladeur, P. (1992): Presentation of a linear or nonlinear analysis method for multilayered composite plats and shells with edge effects influence. In: Hirsch, C. H.; Zienkiewicz, O. C.; Onate, E. (eds.): Numerical methods in engineering '92, 637–644. London: Elsevier

Ladeur, P.; Batoz, J. L. (1989): Composite plate analysis using a new discrete shear triangular finite element. Int. J. Num. Meth. Engng. 27, 343–359

Lee, C.; Liu, D. (1992): An interlaminar stress continuity theory for laminated composite analysis. Comp. and Struct. 42, 69–78

Levinson, M. (1980): An accurate, simple theory of the statics and dynamics of elastic plates. Mechanics research Communications 7, 343–350.

Li, Z. H.; Owen, D. R. J. (1989): Elastic-plastic analysis of laminated anisotropic shells by a refined finite element laminated model. Comp. and Struct. 32, No. 5, 361–382

Niederstedt, G. (1985): Leichtbau mit kohlefaserversträkten Kunststoffen. Kontakt und Studium Band 167. Sindelfingen: Expert

Noor, A. K.; Burton, W. S.; Peters, J. M. (1989): Assessment of computational models for multilayered composite cylinders. In: Noor, A. K. et al. (eds.): Analytical and Computational models of shells. Vol. 3, ASME 3, 419–441. New York

Pagano, N. J. (1970): Exact solutions for rectangular bidirectional composites and sandwich plates. J. Comp. Mat. 4, 20–34

Peseux, B.; Dubigeon, S. (1991): Equivalent homogeneous finite element for composite materials via Reissner principle. Part II: Finite element for shells. Int. J. Num. Meth. 31, 1497–1509

Pinsky, P.; Jasti, R. V. (1989): A mixed finite element for laminated composite plates based on the use of bubble functions. Eng. Comp. 6, 316–330

Puchta, N. S.; Reddy, J. N. (1984): A mixed shear flexible element for the analysis of laminated plates. Comp. Meth. Appl. Mech. Engng. 44, 213–227

Rammerstorfer, F. G. (1992): Lectures on composite and sandwich shells. In: Rammerstorfer, F. G. (ed.): Nonlinear analysis of shells by finite elements. CISM courses and lectures, No. 328, pp 131–194. Wien, New York: Springer

Reddy, J. N. (1984): A simple higher theory for laminated composite plates. J. Appl. Mech. ASME 51, 745–752

Reddy, J. N.; Chandrashekhara, K. (1985): Nonlinear analysis of laminated shells including transverse shear strains. AIAA Journal 23, 440–441

Reddy, J. N.; Barbero, E. J.; Teply, J. L. (1989): A plate bending element based on a generalized laminate plate theory. Int. J. Num. Meth. Engng. 28, 2275–2292

Simo, J. C.; Fox, D. D.; Rifai, M. S. (1989): On a stress resultant geometrically exact shell model, part I: Formulation and optimal parametrization. Comp. Meth. Appl. Mech. Engng. 72, 267–304

Tessler, A.; Saether, E. (1991): A computationally viable higher-order theory for laminated composite plates. Int. J. Num. Meth. Engng. 31, 1069–1086

Toledano, A.; Murakami, H. (1987): A high-order laminated plate theory with improved in-plane responses. Int. J. Solids Struct. 23, 111–131

Tsai, S. W.; Hahn, H. T. (1980): Introduction to composite materials. Technomic Publishing, Library of Congress, Westport Lancaster, USA

Wagner, W.; Gruttmann, F. (1991): On the stability behaviour of composite shells. In: Ladevèze, P.; Zienkiewicz O. C. (eds.): Proceeding of the European Conference on New Advances in Computational Structural Mechanics, Giens (France) 2–5.4.1991, pp. 539–546

Wagner, W.; Stein, E. (1992): A new finite element formulation for cylindrical shells of composite material. Comp. Eng. (in publication)

Wagner, W.; Gruttmann, F. (1992): A simple finite rotation formulation for composite shell elements. Eng. Comput. (in publication)

Whitney, J. M.; Pagano, N. J. (1970): Shear deformation in heterogeneous anisotropic plates. J. Appl. Mech. ASME 37, 1031–1036

Wriggers, P.; Gruttmann, F. (1990): Large deformations of thin shells: Theory and finite-element-diskretization. In: Noor, A.; Belytschko, T.; Simo, J. C. (eds.): Analytical and computational models of shells. Vol. 3, ASME CED, 135–159

Yoda, T.; Atluri, S. N. (1992): Postbuckling analysis of stiffened laminated composite panels using a higher-order shear deformation theory. Comput. Mech. 9, 390–404

Zienkiewicz, O. C.; Taylor, R. L. (1988): The finite element method. 4th edition, Vol. 1. London: McGraw Hill

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