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Boundary variational formulations and numerical solution techniques for unilateral contact problems

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Abstract. In this paper the numerical solution of the elastic frictionless contact problem is obtained by means of boundary discretization techniques. Variational formulations in terms of boundary tractions are given in presence of both bilateral and unilateral constraints. The discretization of the boundary functional is examined from the point of view of the theory of approximation and it is proved that the coerciveness (but not the symmetry) of the continuum problem is preserved when standard B.E.Ms are employed. As a consequence, the contact problem can be cast as a L.C.P. having, as coefficient matrix, a generally non symmetric P matrix. A simple, but meaningful example is discussed in some detail.

1 Introduction

In recent years a lot of interest has been devoted to the mathematical formulation of structural problems involving unilateral constraints (Duvaut and Lions 1972; Panagiotopoulos 1985; Del Piero and Maceri 1987; Kikuchi and Oden 1988) and their numerical solutions.

An important class of these problems is the frictionless contact between a deformable body and a rigid support or between two deformable bodies. In these cases the contact area is "a priori" unknown and the unilateral conditions have to be imposed on the relative displacements and the mutual reactions. These problems with "ambiguous" boundary conditions were firstly stated by Signorini (1933) and thoroughly studied by Fichera (1964, 1972). It is worth noting that all of them can be naturally expressed in terms of variational inequalities stating nothing but the principle of virtual or complementary virtual work in its inequality form. As it is well known, these statements are fully equivalent to minimizing on a convex set the potential or the complementary energy respectively. Since the constraints concern boundary variables only, it is quite natural to look for a numerical solution by means of boundary element techniques. As a matter of fact a lot of papers on this subject have recently appeared in the technical literature (see for instance Tralli et al. 1988). However, in most cases, heuristic iterative procedures (trial and error) are employed, without any critical discussion, to achieve a solution of the attained set of algebraic equalities and inequalities.

The first variational formulations defined on the contact area only, rather than on the whole domain, were proposed quite recently, just in the seventies. For instance Kalker and Van Randen (1972) formulated a minimum principle for boundary tractions, whereas Duvaut (1976) suggested a dual variational statement in terms of boundary displacements. In a recent contribution of Bufler (1985) the boundary variational formulations of the frictionless contact problem are re-examined from a mechanical point of view. In that paper the role of the rigid body motions in a contact involving a "stamp" and a foundation (see also Kikuchi and Song 1980) is focused. Finally, a thorough mathematical treatment of variational boundary formulations in terms of Green's function is reported in Kikuchi and Oden (1988, Chap. 8).

Very recently Panagiotopoulos and Lazaridis (1985, 1987) derived the two dual boundary variational statements from the classical minimum potential and complementary energy principles by means of saddle point formulations using appropriate Lagrangian functions. In detail, the extremum formulations of the discretized problem are derived straightforwardly from the corresponding ones of the continuum problem by approximating Green's function or its inverse by means of standard (i.e. based on collocations) direct boundary element procedures. That allows to use quadratic programming tools with a very low number of sign constrained variables, tractions or displacements, in the contact area.

The aim of this paper is to analyse the numerical solution of elastic frictionless contact problems discretized with "standard" direct boundary elements. In Sect. 2 a variational formulation of linear elastic problems with bilateral constraints is given in terms of boundary tractions; their mathematical setting is also discussed. Then variational problems over convex sets, e.g. the Signorini–Fichera problem, are dealt with in Sect. 3. The discretization of the boundary functional is examined in Sect. 4 from the point of view of the theory of approximation and it is shown that the coerciveness of the continuum problem is maintained. Furthermore (Sect. 5) the approximation of Green's function by means of the standard direct boundary element method is studied: the symmetry is generally lost whereas the coerciveness is preserved. In Sect. 6 the B.E. numerical solution of the frictionless contact problem is presented briefly. As a consequence of the previous results the stationarity conditions of the discretized problem can be expressed in the form of a linear complementarity problem having, as coefficient matrix, a non symmetric P matrix. Finally a simple, but meaningful example, is discussed in some detail.

2 "Boundary" variational formulations

For the sake of simplicity let us discuss in the first place a paradigmatic two-dimensional linear problem.

Consider an open, bounded, simply connected domain Ω of the plane $z = 0$ in the Euclidean space $\mathbb{R}^3$ referred to the orthogonal Cartesian reference system $(0, X_i, i = 1, 2, 3)$. The boundary $\Gamma \equiv \partial\Omega$ is assumed to be sufficiently regular. Ω is occupied by a linear elastic body (Fig. 1).

It is well known that the mixed boundary value problem obtained by assigning, for instance, the load vector $\boldsymbol{q}$ on $\Gamma q \subset \Gamma$ and by taking into account the essential boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma_0 \subset \Gamma$, $\Gamma \cap \Gamma_q \equiv 0$, has a unique solution. More precisely, such solution can be found by minimizing the potential energy functional

$$J(\boldsymbol{u}) = \tfrac{1}{2} a(\boldsymbol{u}, \boldsymbol{u}) - \int_{\Gamma_q} \boldsymbol{q}\boldsymbol{u} \, \mathrm{d}\gamma \tag{2.1}$$

with

$$a(\boldsymbol{u}, \boldsymbol{u}) = \iint_{\Omega} C_{ijhk} \varepsilon_{ij}(\boldsymbol{u}) \varepsilon_{h_k}(\boldsymbol{u}) \, \mathrm{d}\Omega, \tag{2.2}$$

where C_{ijhk} $(i, j, h, k = 1, 2)$ is Hooke's tensor of elasticity, assumed to be constant all over the body and obeying the classical symmetry and ellipticity conditions

$$C_{ijhk} = C_{jihk} = C_{khji}; \quad C_{ijhk}\varepsilon_{ij}\varepsilon_{hk} \geqq c\varepsilon_{ij}\varepsilon_{hk};$$
$$\forall \varepsilon = \{\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})\, i, j = 1, 2\} \in R^2, \quad c \text{ const} > 0.$$

The minimization of $J(\boldsymbol{u})$ is to be performed over all the admissible displacements, which in the

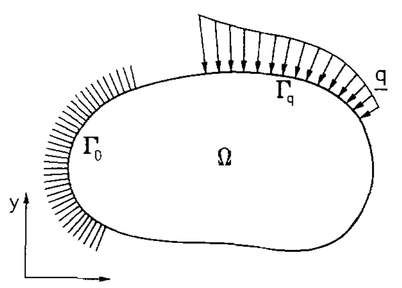


Fig. 1. Mixed boundary value problem for a two-dimensional elastic domain

present case turn out to be

$$\boldsymbol{u} \in U = \{[H^1(\Omega)]^2; \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_\circ\}. \tag{2.3}$$

A boundary variational formulation of the problem we are considered can be immediately found if Green's matrix, related to the boundary conditions, is known. For any vector $\boldsymbol{p} \in [H^{-1/2}(\Gamma)]^2$ we suppose to have determined a matrix $\boldsymbol{G}(\boldsymbol{x}, \xi)$ such that the displacements field induced by $\boldsymbol{p}(\xi)$ can be expressed as

$$\boldsymbol{v}(\boldsymbol{x}) = \int_\Gamma \boldsymbol{G}(\boldsymbol{x}, \xi)\boldsymbol{p}(\xi)\, \mathrm{d}\gamma(\xi), \quad \boldsymbol{x} \in \Omega. \tag{2.4}$$

Clearly, $\boldsymbol{v} \in U$ as defined in (2.3). It is worth noting that (2.4) does not provide the complete solution of the problem, because $\boldsymbol{p}(\xi)$ is known on Γ_q, where it is equal to the prescribed load, vanishes on $\Gamma - (\Gamma_q \cup \Gamma_\circ)$, but it is "a priori" unknown on $\Gamma_\circ$. Actually, along the constrained boundary, a reaction force distribution, depending on the load conditions, arises in order to provide the global equilibrium of the body.

However, we can note that $\boldsymbol{v}(\boldsymbol{x})$, as obtained in (2.4), belongs to the subspace V of U defined as follows:

$$\boldsymbol{v} \in V \equiv \{[H^1(\Omega)]^2; \boldsymbol{v} \neq \boldsymbol{0} \text{ on } \Gamma_\circ; a(\boldsymbol{v}, \boldsymbol{v}') - \langle \boldsymbol{p}, \boldsymbol{v}' \rangle_\Gamma = 0, \quad \forall \boldsymbol{v}' \in U\}, \tag{2.5}$$

where $\langle \boldsymbol{p}, \boldsymbol{v}' \rangle_\Gamma = \int_\Gamma \boldsymbol{p}(\xi)\boldsymbol{v}'(\xi)\, \mathrm{d}\gamma(\xi)$ denotes the inner product on the boundary.

We can conclude that the functional $J(\circ)$ can be equivalently minimized over V instead of U.

In particular, the solution $\boldsymbol{u} \in V \subset U$ of the minimization problem will satisfy the Euler equation for $J(\circ)$, namely:

$$a(\boldsymbol{u}, \boldsymbol{v}') - \int_{\Gamma_q} \boldsymbol{q}(\xi)\boldsymbol{v}'(\xi)\, \mathrm{d}\gamma(\xi) = 0, \quad \forall \boldsymbol{v}' \in U. \tag{2.6}$$

Then we can take both $\boldsymbol{u}$ and the test functions $\boldsymbol{v}'$ as belonging to V; this amounts to defining:

$$\boldsymbol{v}'(\xi) = \int_\Gamma \boldsymbol{G}(\xi, \boldsymbol{x})\boldsymbol{p}'(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x}). \tag{2.7}$$

According to the definition (2.5) of $V \subset U$, we have now

$$a(\boldsymbol{u}, \boldsymbol{v}') = \int_\Gamma \boldsymbol{p}(\xi)\boldsymbol{v}'(\xi)\, \mathrm{d}\gamma(\xi) \tag{2.8}$$

if $\boldsymbol{u} \in V$ and, therefore, the final equation we obtain from (2.6) is

$$\int_{\Gamma_x}\int_{\Gamma_\xi} \boldsymbol{p}(\xi)\boldsymbol{G}(\xi, \boldsymbol{x})\boldsymbol{p}'(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x})\, \mathrm{d}\gamma(\xi) - \int_{\Gamma_x}\int_{\Gamma_{q,\xi}} \boldsymbol{q}(\xi)\boldsymbol{G}(\xi, \boldsymbol{x})\boldsymbol{p}'(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x})\, \mathrm{d}\gamma(\xi) = 0. \tag{2.9}$$

On the other hand, it is quite evident that Eq. (2.9) can be viewed as the Euler equation of the boundary functional

$$J^*(\boldsymbol{p}) = \frac{1}{2}\int_{\Gamma_x}\int_{\Gamma_\xi} \boldsymbol{p}(\xi)\boldsymbol{G}(\xi, \boldsymbol{x})\boldsymbol{p}'(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x})\, \mathrm{d}\gamma(\xi) - \int_{\Gamma_x}\int_{\Gamma_{q,\xi}} \boldsymbol{q}(\xi)\boldsymbol{G}(\xi, \boldsymbol{x})\boldsymbol{p}'(\boldsymbol{x})\, \mathrm{d}\gamma(\boldsymbol{x})\, \mathrm{d}\gamma(\xi). \tag{2.10}$$

The present development is similar to the discussion contained in a recent book of Kikuchi and Oden (Kikuchi and Oden 1988, pp. 216–220), where the reader can find also a simple proof of the ellipticity of the operator described by means of Green's function. For our successive results, however, it will be useful to have the explicit expression (2.10) of the boundary functional.

3 Boundary problems over convex sets

The variational formulation presented in the previous section exhibits some interesting features, even though for the classical problem considered before it is completely equivalent to any "direct" formulation in terms of virtual forces (Kalker and Van Randen 1972; Bufler 1985; Kikuchi and Oden 1988).

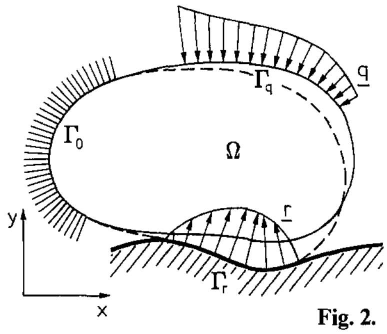


Fig. 2. Two-dimensional elastic body on a rigid frictionless support: the "obstacle" problem

First of all, the coerciveness of the bilinear form involved in the definition of $J^*(\circ)$ allows a straightforward extension of the classical error analysis for potential problems, as we will discuss with some detail in the next session. Secondly, a variational formulation, that involves only boundary forces, seems the most favourable framework for studying a whole class of problems, where some constraint is imposed on the domain of the functional. A classical example is the Signorini–Fichera problem, also known as "obstacle problem", illustrated in Fig. 2, but we will see that the remarks we are presenting are far more general.

The "obstacle problem" can be approached by considering again the potential energy functional (2.1) and by modifying the set of the admissible displacements as follows:

$$\boldsymbol{u} \in K \equiv \{[H^1(\Omega)]^2;\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_\circ;\ \boldsymbol{u}\cdot\boldsymbol{n} \leqq \Delta(x) \text{ on } \Gamma_r\} \tag{3.1}$$

where $\boldsymbol{n}$ denotes the outer normal on $\Gamma \equiv \partial\Omega$ and $\Delta(x)$ the gap measured along the outer normal between the body, before the deformation, and the obstacle surface. We note that K is a convex subset of U as defined in (2.3); as usual, it is convenient to take into account the constraint

$$\boldsymbol{u}\cdot\boldsymbol{n} - \Delta(x) \leqq 0, \quad \text{on } \Gamma_r \tag{3.2}$$

by means of a Kuhn–Tucker multiplier λ. Thus we are led to consider the following problem: find a stationary point of

$$J_\Delta(\boldsymbol{u};\lambda) = \tfrac{1}{2}a(\boldsymbol{u},\boldsymbol{u}) - \int_{\Gamma_q} \boldsymbol{q}\cdot\boldsymbol{u}\,\mathrm{d}\gamma - \int_{\Gamma_r} \lambda(\boldsymbol{u}\cdot\boldsymbol{n} - \Delta(x))\,\mathrm{d}\gamma \tag{3.3}$$

with $\boldsymbol{u} \in U$ and $\lambda \in H^{-1/2}(\Gamma)$; $\langle \lambda, 1 \rangle_{\Gamma'} \leqq 0, \forall \Gamma' \subset \Gamma_r$.

In the present formulation, we have implicity supposed that the contact surface Γ_r has empty intersection with both Γ_q and $\Gamma_\circ$. The assumption $\Gamma_r \cap \Gamma_\circ = \emptyset$ obeys to the mechanics of the problem itself; on the contrary, it is easy to verify that $\Gamma_r \cap \Gamma_q = \emptyset$ is far from essential and it has been introduced only for the sake of simplicity.

Now we can repeat step-by-step the deduction of Sect. 2; once introduced the space V of the admissible displacements, which can be expressed by means of Green's matrix $\boldsymbol{G}(x,\xi)$, we can assume also the test functions $\boldsymbol{v}'$ as belonging to the same V. Thus the Euler equation in terms of boundary forces, corresponding to (2.9), is obtained:

$$\begin{aligned} &\int_{\Gamma_x}\int_{\Gamma_\xi} \boldsymbol{p}(\xi)\boldsymbol{G}(\xi,x)\boldsymbol{p}'(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(\xi) - \int_{\Gamma_x}\int_{\Gamma_{q,\xi}} \boldsymbol{q}(\xi)\boldsymbol{G}(\xi,x)\boldsymbol{p}'(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(\xi) \\ &\quad - \int_{\Gamma_x}\int_{\Gamma_{r,\xi}} \lambda(\xi)\boldsymbol{n}(\xi)\boldsymbol{G}(\xi,x)\boldsymbol{p}'(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(\xi) + \int_{\Gamma_{r,x}} \lambda(x)\Delta(x)\,\mathrm{d}\gamma(x) \geqq 0, \quad \forall \boldsymbol{p}' \in [H^{-1/2}(\Gamma)]^2. \end{aligned} \tag{3.4}$$

Finally, we can observe that (3.4) is the minimum condition for the functional

$$J^*_\Delta(\boldsymbol{p};\lambda) = \frac{1}{2}\int_{\Gamma_x}\int_{\Gamma_\xi} \boldsymbol{p}(\xi)\boldsymbol{G}(\xi,x)\boldsymbol{p}(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(\xi) + \int_{\Gamma_{r,x}} \lambda(x)\Delta(x)\,\mathrm{d}\gamma(x) \tag{3.5}$$

over the convex set

$$\boldsymbol{p} \in K \equiv \{[H^{-1/2}(\Gamma)]^2;\ \boldsymbol{p}(x) = \boldsymbol{q}(x), \text{ on } \Gamma_q \text{ and } \boldsymbol{p}(x) = \lambda(x)\boldsymbol{n}(x);\ \langle \lambda, 1 \rangle_{\Gamma'} \leqq 0, \forall \Gamma' \subset \Gamma_r\}. \tag{3.6}$$

In (3.4), the conditions on Γ_q and Γ_r are explicitly imposed in the weak sense, which is correct, since $H^{-1/2}(\Gamma)$ is a distributional space, and $\boldsymbol{p}(\boldsymbol{x})$ in principle does not need to have well-defined point values. Let us remark that the well known extremum principles with respect to the tractions of the contact area (Kalker and Van Randen 1972; Bufler 1985; Panagiotopoulos 1985; Panagiotopoulos and Lazaridis 1987) are completely equivalent to (3.5), differing from it in the additional term

$$\frac{1}{2}\int_{\Gamma_{q,x}}\int_{\Gamma_{q,\xi}} \boldsymbol{q}(\boldsymbol{x})\boldsymbol{G}(\boldsymbol{x},\xi)\boldsymbol{q}'(\xi)\,\mathrm{d}\gamma(\boldsymbol{x})\,\mathrm{d}\gamma(\xi)$$

which is not subject to variations [see Eq. (3.6)]. Moreover, the present result differs from the one of the previous section in what we have here an additional unknown, the contact surface Γ_r, which cannot be identified "a priori". Yet, this is not a trouble, since $\lambda(\boldsymbol{x})$ is required to be non-positive and vanishing values are allowed. Thus, in principle, we can take $\Gamma_r = \Gamma\backslash\Gamma_0$, and in solution $\lambda(\boldsymbol{x})$ will vanish outside the contact area.

The relevant point in the present formulation, however, resides in what (3.4), or equivalently functional (3.6), has been obtained by using Green's matrix for the elastic problem without obstacle. In mechanical terms, this amounts to considering the solution of the complete problem as the superposition of two partial solutions, the one corresponding to the body constrained at Γ_0 and loaded with $\boldsymbol{q}(\boldsymbol{x})$ along Γ_q, the other to the body constrained at Γ_0 and loaded with $\lambda(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$ along Γ_r. The basic hypothesis here is that the reaction of the contact surface is completely frictionless: this allows in practice to use mathematical programming tools (i.e. linear complementary or, equivalently, quadratic programming), in addition to iterative procedures (trial and error), to solve the problem numerically [see for instance Cannarozzi (1980) for a discussion of this point]. The more general case, where the contact reaction involves also a tangent component due to friction, $\boldsymbol{r}(\boldsymbol{x}) = \lambda(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) + \mu(\boldsymbol{x})\boldsymbol{t}(\boldsymbol{x})$, presents some additional difficulties. First of all, an existence result for the solution can be proved only under the rather restrictive assumption that the total work of the friction forces be "small" with respect to the global deformation energy (Nečas et al. 1980). Besides, the coefficient $\mu(\boldsymbol{x})$ of the tangent component of the reaction force does not need to be sign constrained, and this involves considerable difficulties in providing a complementary setting of the problem (Alliney 1988).

4 Some remarks from the theory of approximation

In abstract terms, our basic problem: find $\boldsymbol{p}\in[H^{-1/2}(\Gamma)]^2$ such that

$$\int_{\Gamma_x}\int_{\Gamma_\xi} \boldsymbol{p}(\xi)\boldsymbol{G}(\xi,\boldsymbol{x})\boldsymbol{p}'(\xi)\,\mathrm{d}\gamma(\boldsymbol{x})\,\mathrm{d}\gamma(\xi) - \int_{\Gamma_x}\int_{\Gamma_{q,\xi}} \boldsymbol{q}(\xi)\boldsymbol{G}(\xi,\boldsymbol{x})\boldsymbol{p}'(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x})\,\mathrm{d}\gamma(\xi) = 0 \quad \forall \boldsymbol{p}'\in[H^{-1/2}(\Gamma)]^2 \tag{4.1}$$

belongs to a wide class of elliptic problems, for which sharp theoretical results are available. In this section, we will draw freely from earlier studies (namely, Le Roux 1977; Nedelec and Planchard 1973), and we will focus our attention on the practical implication of the theory, when dealing with numerical approximate models.

For the sake of simplicity, we re-write (3.1) as

$$a^*(\boldsymbol{p},\boldsymbol{p}') = b^*(\boldsymbol{q},\boldsymbol{p}'), \quad \forall \boldsymbol{p}'\in[H^{-1/2}(\Gamma)]^2. \tag{4.2}$$

First of all, we can note that the bilinear form $a^*(\circ,\circ):[H^{-1/2}(\Gamma)]^2 \times [H^{-1/2}(\Gamma)]^2 \to \mathbb{R}$ is continuous and coercive; furthermore, $a^*(\boldsymbol{p},\boldsymbol{p})$ represents one norm over $[H^{-1/2}(\Gamma)]^2$, which is equivalent to the usual one (Le Roux 1977, Theorem 1.1). Classical results also state that the solution $\boldsymbol{p}$ of (3.2) is the trace on the boundary of a suitable stress tensor defined inside the domain. In order to obtain a discrete version of problem (3.2), we have to approximate both the ambient space $[H^{-1/2}(\Gamma)]^2$ and the curved boundary Γ. We will denote with Γ_h the approximation of the boundary by means of straight segments and with P_h^2 a finite dimensional subspace of $[H^{-1/2}(\Gamma_h)]^2$.

The new problem is then: find $\boldsymbol{p}_h\in P_h^2$ such that

$$a_h^*(\boldsymbol{p}_h,\boldsymbol{p}_h') = b_h^*(\boldsymbol{q}_h,\boldsymbol{p}_h'), \quad \forall \boldsymbol{p}_h'\in P_h^2, \tag{4.3}$$

where $\boldsymbol{q}_h$ is an approximation to the prescribed load $\boldsymbol{q} \in [H^{-1/2}(\Gamma)]^2$ and $a_h^*(\circ,\circ), b_h^*(\circ,\circ)$ are the bilinear forms corresponding to (3.2), now defined on the approximate boundary Γ_h.

The error we introduce by searching for $\boldsymbol{p}_h$ which satisfies (3.3) has two different sources: (i) the approximation of $[H^{-1/2}(\Gamma)]^2$ with one finite dimensional subspace P_h^2; (ii) the approximation of the true boundary Γ with a piecewise linear curve Γ_h. The discussion that follows is based on the further assumption that the approximating subspace P_h^2 is in practice the space of vectors, having components, which are constants over any straight segment of Γ_h. Thus we have

$$P_h^2(\Gamma_h) \subset [L^2(\Gamma_h)]^2 \tag{4.4}$$

and

$$[L^2(\Gamma_h)]^2 \equiv [H^0(\Gamma_h)]^2 \subset [H^{-1/2}(\Gamma_h)]^2. \tag{4.5}$$

Under such hypotheses, we claim that the bilinear form $a_h^*(\circ,\circ): P_h^2 \times P_h^2 \to \mathbb{R}$ is coercive for sufficiently small values of the parameter h. Let $T: P_h^2(\Gamma_h) \to [H^{-1/2}(\Gamma)]^2$ denote the operator, which maps a vector defined on the approximated boundary Γ_h into a corresponding vector defined on the exact boundary Γ; such operator can be explicitly built up, as illustrated in Le Roux (1977). From now onwards, we will assume that the boundary Γ is at least of class $\mathscr{C}^{m+2}$.

For any $\boldsymbol{p}_h \in P_h^2(\Gamma_h)$, we have the following inequalities (Le Roux 1977):

$$K_1 \| T\boldsymbol{p}_h \|_0 \leqq \| \boldsymbol{p}_h \|_{0,h} \leqq K_2 \| T\boldsymbol{p}_h \|_0; \quad K_1, K_2 > 0, \tag{4.6}$$

where $\| \circ \|_{0,h}$ denotes the norm of $L^2(\Gamma_h)$ and $\| \circ \|_0$ the norm of $L^2(\Gamma)$. Furthermore, we have the following:

Proposition 1. *For any $\boldsymbol{p}_h \in P_h^2(\Gamma_h)$ the following inequality holds*

$$\| T\boldsymbol{p}_h \|_{-1/2} \geqq c_1 h^{1/2} \| T\boldsymbol{p}_h \|_0. \tag{4.7}$$

Proposition 2. *For any $\boldsymbol{p}_h \in P_h^2(\Gamma_h)$, the following inequality holds*

$$| a^*(T\boldsymbol{p}_h, T\boldsymbol{p}_h) - a_h^*(\boldsymbol{p}_h, \boldsymbol{p}_h) | \leqq c_2 h^{m+1} \| \boldsymbol{p}_h \|_{0,h}^2. \tag{4.8}$$

The proof of these propositions is but a straightforward extension to the vector case of the result of Le Roux (1977) and it will not be reported here.

Now we are able to prove the main result, as anticipated before:

Theorem. *For sufficiently small h and for any $\boldsymbol{p}_h \in P_h^2(\Gamma_h)$ as previously defined (piecewise constant functions on Γ_h), the bilinear form $a_h^*(\circ,\circ)$ is coercive with respect to the norm of $L^2(\Gamma_h)$.*

Proof. From (4.8), we obtain immediately

$$a_h^*(\boldsymbol{p}_h, \boldsymbol{p}_h) \geqq a^*(T\boldsymbol{p}_h, T\boldsymbol{p}_h) - c_2 h^{m+1} \| \boldsymbol{p}_h \|_{0,h}^2. \tag{4.9}$$

Since $a^*(\circ,\circ)$ is coercive and induces an equivalent norm on $[H^{-1/2}(\Gamma)]^2$, we have also

$$a^*(T\boldsymbol{p}_h, T\boldsymbol{p}_h) \geqq c_3^2 \| T\boldsymbol{p}_h \|_{-1/2}^2. \tag{4.10}$$

Using now (4.7), we obtain the further inequality.

$$a^*(T\boldsymbol{p}_h, T\boldsymbol{p}_h) \geqq c_3^2 c_1^2 h \| T\boldsymbol{p}_h \|_0^2. \tag{4.11}$$

Recalling the bounds (4.6), related to the operator T, we have finally:

$$a^*(T\boldsymbol{p}_h, T\boldsymbol{p}_h) \geqq \left(\frac{c_3 c_1}{K_2} \right)^2 h \| \boldsymbol{p}_h \|_0^2. \tag{4.12}$$

After substitution of (4.12) into (4.9) we obtain

$$a_h^*(\boldsymbol{p}_h, \boldsymbol{p}_h) \geqq h(\alpha - c_2 h^m) \| \boldsymbol{p}_h \|_{0,h}^2 \tag{4.13}$$

with $\alpha = (c_3 c_1 / K_2)^2$. Equation (4.13) proves the theorem; notice that the present result is somewhat stronger than the simple coerciveness w.r.t. to the norm of $[H^{-1/2}(\Gamma_h)]^2$.

5 On the numerical solution by direct B.E.Ms

The results reported in the previous section constitute the ground for the numerical solution of the contact problem. Namely the choice of the approximating subspace $P_h^2(\Gamma_h)$ as defined in (4.4), by using piecewise constant shape functions, allows a dramatic reduction of the computational effort. It is a matter of routine to verify that this amounts to using a zero-order integration formula over any interval of Γ_h or, from a different perspective, to resorting to a suitable collocation method in the evaluation of the line integrals.

Unfortunately, the esplicit form of Green's operator (2.4) is known only in special cases. We remind, however, that in discrete problems the inverse of the "stiffness matrix" plays the same role as Green's function of our approximation scheme, as remarked by Lazaridis and Panagiotopoulos (1987a, b). This suggests that Green's operator of our formulation could be given an approximate representation exactly in that way. Whenever standard direct B.E.Ms are employed, however, some comments are due. Namely, it is well-known that the stiffness matrices provided by direct B.E. techniques are not, as a rule, symmetric. A part from special cases (e.g. when coupling F.E. and B.E. solutions—see Brebbia et al. 1984, Chap. 13; and Banerjee and Butterfield 1981, Chap. 14), where certain symmetrization procedures are currently used, non-symmetric matrices still provide valid approximate solutions. Of course, classical symmetry properties, as the Maxwell–Betti theorem, hold only asymptotically for the discretized problem.

In our case, the approximation outlined above amounts to an imperfect evaluation of the bilinear form (4.1)—of course over Γ_h instead of Γ. The integrals like

$$\int_{\Gamma_{h,\xi}} G(\xi, x)p'(x)\,\mathrm{d}\gamma(x) \tag{5.1}$$

will be evaluated by using the corresponding values of the inverse stiffness matrix over any sub-interval of the boundary: A simple argument, based on the integral average theorem, shows that this introduces into our evaluation an error term, which is proportional to the characteristic length h of the sub-intervals of the boundary; indeed, for $p'(x)=p'_h\in P_h^2$ the constant terms can be brought out of the integration operator: what remains is just one integral average of Green's function. As a matter of fact, we are using a zero-order interpolation method; in any case, a complete discussion, together with error analysis, for higher order interpolation/collocation methods can be found e.g. in Arnold and Wendland (1983) and Saranen and Wendland (1985). Taking into account the second line integration [see (4.1)], we obtain

$$\int_{\Gamma_{h,x}} \int_{\Gamma_{h,\xi}} p_h(\xi)G(\xi, x)p'(x)\,\mathrm{d}\gamma(x)\,\mathrm{d}\gamma(\xi) = a_h^*(p_h, p'_h) + h^2 e(p_h, p'_h) = \tilde{a}_h^*(p_h, p'_h), \tag{5.2}$$

where the term $h^2(e(p_h, p'_h)$ accounts for the error introduced into the evaluation of (5.1), after the second integration. Of course, the bilinear form $e(\circ,\circ)$—in practice unknown—cannot be assumed to be symmetric: to be precise, this is just the non-symmetric error contribution which affects the discrete models based on "direct" B.E. formulations. As we noted before, this error contribution is proportional to h^2 and, therefore, such term has been pointed out in Eq. (5.2).

Now, inequality (4.13) has to be verified for $\tilde{a}_h^*(\circ,\circ)$ approximating the "true" bilinear form $a_h^*(\circ,\circ)$. Therefore, we have to verify that

$$\tilde{a}_h^*(p_h, p_h) = a_h^*(p_h, p_h) + h^2 e(p_h, p_h) \geqq h(\alpha - c_2 h^m)\|p_h\|_{0,h}^2. \tag{5.3}$$

Here, it is sufficient to note that for h sufficiently small we have

$$a_h^*(p_h, p_h) \geqq h(\alpha - c_2 h^m)\|p_h\|_{0,h}^2 - h^2 e(p_h, p_h) \tag{5.4}$$

even though the sign of $e(p_h, p_h)$ is undetermined. It follows that:

$$\tilde{a}_h^*(p_h, p_h) \geqq h(\alpha - c_2 h^m)\|p_h\|_{0,h}^2 \tag{5.5}$$

for any $p_h \in P_h^2(\Gamma_h)$.

Finally let us remark that, on the basis of the quoted papers of M. N. Le Roux (1974, 1977), inequalities analogous to (4.13) and (4.19) hold also for more refined boundary element models, for instance for the linear one.

6 A numerical example

In Sect. 3 variational formulations of the Signorini–Fichera problem have been presented either as a variational inequality (3.4), or as a minimum principle.

In this context the use of direct boundary techniques provides, as previously discussed, a discretized bilinear form $\tilde{a}_h^*(\boldsymbol{p}_h, \boldsymbol{p}_h)$, whose coefficient matrix $\hat{\boldsymbol{Z}}$ is non-symmetric. Consequently, the problem at hand cannot be directly formulated as a minimum problem and standard quadratic programming tools cannot be employed.

Nevertheless inequality (5.5), in virtue of a well known result attributed to Fiedler and Ptàk, guarantees that $\hat{\boldsymbol{Z}}$ is a P matrix, that is all its principal minors have strictly positive determinants. It is even obvious to remark that any principal submatrix of $\hat{\boldsymbol{Z}}$ is a P matrix.

A more detailed discussion of direct boundary element solutions of contact problems is reported in Tralli, Alessandri and Alliney (1988); however for the sake of completeness some of the main points are mentioned in the following.

(i) Variational inequality (3.4) (rather than the related minimum principle) is reduced to a variational inequality defined only on the contact surface Γ_r in terms of the contact pressures distribution $\boldsymbol{p}_r(\boldsymbol{x}) = \lambda(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$. This can be obtained by taking into account explicitly the constraint $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{q}(\boldsymbol{x})$ on Γ_q; moreover the essential boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on Γ_0 is directly imposed during the evaluation of the approximate influence coefficients. The variational inequality so obtained states nothing but the principle of virtual forces (Bufler 1985).

(ii) By discretizing the problem with constant B.E., as discussed in the previous Sections, the following linear complementarity problem (L.C.P.) (6.1) is straightforwardly obtained:

$$\bar{\boldsymbol{W}} = \boldsymbol{Z}\boldsymbol{P} + (\boldsymbol{W}_q + \boldsymbol{\Delta}) \geqq \boldsymbol{0}; \quad \boldsymbol{P} \geqq \boldsymbol{0}; \quad \bar{\boldsymbol{W}}^T \boldsymbol{P} = 0, \tag{6.1a}$$

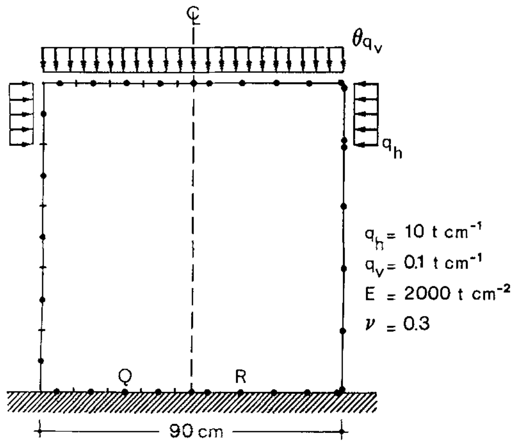
where $\boldsymbol{P}$ is the vector of "nodal" contact forces and $\boldsymbol{W}_q$ and $\boldsymbol{\Delta}$ are, respectively, the vectors of the mean normal displacements, induced by the assigned external load distribution $\boldsymbol{q}$, and the gaps.

The L.C.P. coefficient matrix $\boldsymbol{Z}$ ($Np \times Np$ if Np denotes the number of elements approximating Γ_r) is a principal submatrix of $\hat{\boldsymbol{Z}}$ from which it can be obtained by deleting rows and columns; therefore, it turns out to be a P matrix itself. As well known this property represents the necessary and sufficient condition to get a unique solution of L.P.C. (6.1) for any vector of assigned data $(\boldsymbol{W}_q + \boldsymbol{\Delta})$; the analogy with the corresponding continuum problem $(\Gamma_0 = \phi)$ is evident. Moreover Lemke's algorithm guarantees that the solution of the discretized problem so far discussed exists and can be achieved in a finite number of steps.

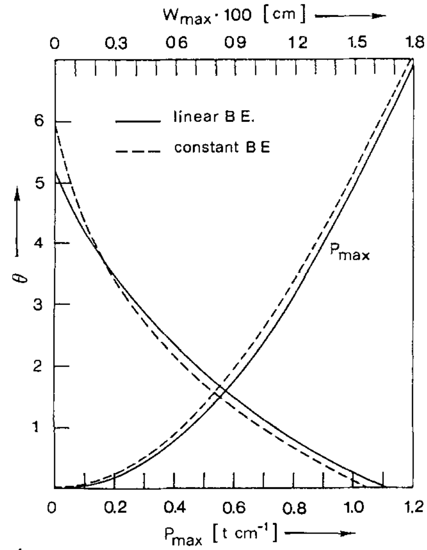
It is worth noting that the boundary constraint $\boldsymbol{u} = \boldsymbol{0}$, assumed so far in Γ_0, is not strictly necessary and, therefore, problems with unconstrained "stamps" can be solved exactly as illustrated before. Namely, the $\boldsymbol{Z}$ matrix coefficients and vector $\boldsymbol{W}_q$ can be evaluated for any isostatic stamp, where auxiliary constraints are arbitrarily introduced for eliminating rigid body motions. Obviously these constraints have to be placed at the nodes with zero tractions prescribed (i.e. nodes not belonging to Γ_q). The explicit enforcement of the three (6 in 3-D problems) global equilibrium equations between the given loads and the contact tractions guarantees that the constraint reactions are equal to zero. By adding these equations to relations (6.1a–c) we obtain a non standard L.C.P.; for such a problem it is always possible to find a complementary solution by Lemke's algorithm, if the convex set defined by the equilibrium equations and (5.1b) is not empty. However, possible rigid-body displacements remain undetermined. From the discussion of the last section we can now conclude that the main properties of the continuum problems (e.g. existence and uniqueness of the solution) are preserved also for the discrete B.E. problem. As a final remark we remind that iterative methods have been recently proposed in the technical literature for the solution of linear complementarity problems with non symmetric matrices. As an example, we treat a simple, unilateral, plane strain, contact problem. The obstacle is assumed to be rigid and frictionless.

In the following we shall describe our experiments with regular boundary discretization, even though a better performance could be obtained using a self-adapting mesh refinement technique (see for instance Rencis and Mullen 1986).

In Fig. 3 the discretization with constant boundary elements is shown. The same mesh has



3



4

Figs. 3 and 4. 3 Elastic punch on a rigid foundation. Constant and linear B.E. discretization; **4** $\vartheta - P_{max}$, W_{max} diagrams for constant and linear B.E. discretization

been used also in the analysis with linear B.E s; obviously the nodes are located at the extremes of each element. However, in order to avoid the second integration, which nevertheless appears absolutely trivial, it is possible to evaluate the constraint on the normal displacement $\boldsymbol{u}\cdot\boldsymbol{n} \leqq \Delta(\boldsymbol{x})$ only at each node (Collocation method). The problem has been studied for increasing values of the parameter ϑ. The contact pressure at the centre point and the displacement at the edge of the base are depicted in Fig. 4. These analyses have been carried out by imposing explicitly the global equilibrium, as previously discussed, and by employing Lemke's algorithm as reported in Ravindran (1972). L.C.Ps have been solved with 9 sign-constrained nodal contact pressures P for the constant B.E. discretization and 11 for the linear B.E. discretization (an additional node has been located at the centre point of the base).

In Table 1 the maximum values of displacements and contact pressures at the base are reported for $\vartheta = 3$. In many problems, even for unconstrained bodies, it is possible to define "a priori" some actual contact nodes with prescribed zero displacements; therefore, in this case it is not necessary to make the equilibrium equations explicit. For the problem at the hand we chose nodes Q and R as actual contact nodes.

For every type of element we report the results obtained either in the case where the equilibrium is explicitly imposed (columns I, V) or in the case where some contact nodes are already known (columns II, VI). Column III reports the results obtained by symmetrizing the $\boldsymbol{Z}$ matrix. As a matter of fact, in the case of equal length elements the $\boldsymbol{Z}$ matrix turns out to be almost symmetric (the measure of the unsymmetry of the $\boldsymbol{Z}$ matrix, computed by means of the following expression (Bauer–Roy scaling)

$$\max \left| \frac{Z_{ij} - Z_{ji}}{\sqrt{|Z_{ii}|}\sqrt{|Z_{jj}|}} \right|$$

Table 1. Maximum values of displacements and contact pressures at the rigid foundation

	Constant elements				Linear elements		
$\vartheta = 3$	I	II	III	IV	V	VI	VII
P_{max} (t. cm.$^{-1}$)	0.760971	0.748998	0.749032	0.748998	0.782141	0.785598	0.785598
W_{max} (cm.)	0.003694	0.003911	0.003921	0.003911	0.003935	0.003795	0.003795

amounts approximately to 1% for constant B.Es and to 8% for linear B.Es, if collocation procedures are applied; all computations were performed in double precision with an Olivetti OC3250 Computer).

Finally in columns IV and VII we report the bilateral solution obtained by supposing the contact area "a priori" known. Columns I and V appear not to be in so good agreement with columns IV and VII; that depends on the fact that B.E.M. results do not satisfy exactly the equilibrium but in an asymptotic way (Brebbia et al. 1984). For instance the B.E. solutions reported in column IV(II) or VII(VI) exhibit an error of 1.57% or 0.44% respectively in computing the equilibrium in the vertical direction.

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