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Boundary element analysis of dynamic coupled thermoelasticity problems

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Abstract. Some possibility of numerical analysis of coupled dynamic problems of linear elastic heat conductors on classical thermoelasticity theory by using the boundary element method is shown in this paper. The boundary integral equation formulation and its numerical implementation of the two-dimensional problem are developed in the manner by the newly derived fundamental solution for the coupled equations of elliptic type in the transformed space and the numerical inversion of Laplace transformation. The boundary element unsteady solutions of the first and second Danilovskaya problems and the Sternberg and Chakravorty problem in the half-space are demonstrated through comparison with the existing solutions.

1 Introduction

Thermoelasticity deals with the interaction between temperature, stress and elastic deformation due to both mechanical and thermal loadings. The basic system of differential equations governing thermoelastic phenomena is more complicated and coupled one than the Navier equation in elasticity. The system is composed of a wave-type equation of the displacement field and a diffusion-type equation of the temperature field. The first analytical solution in dynamic uncoupled thermoelasticity was obtained by Danilovskaya (1950). Further extensions and developments to that analytical solution was reported by Danilovskaya (1952), Sternberg and Chakravorty (1959) and Boley and Tolins (1963). The class of problem which admits closed-form solution in dynamic thermoelasticity is extremely small. Therefore numerical techniques have to be resorted to for more complex geometries and boundary conditions and more complex coupled field theories. Numerical solutions through the finite element method have been reported by Nickell and Sackman (1968), Odean and Kross (1968), Ting and Chen (1982), Prevost and Tao (1983), Tamma and Railkar (1988).

The boundary element method based on the boundary integral equation formulation of the problem has been recognized as the one of effective approximate procedures for the numerical solutions of various problems in continuum mechanics (Banerjee and Butterfield 1981; Brebbia et al. 1984). Especially, applications of this method to the static or dynamic problems in linear elasticity may be in the satisfactory stage. In spite of the existence of many investigations and applications in this field, some attempts have been made on only formulation to apply the method to dynamic problems in coupled thermoelasticity by Predeleanu (1981), Tanaka and Tanaka (1981), Sládek and Sládek (1983).

This paper concerns with the boundary element application to the dynamic problems in two dimensional linear coupled thermoelasticity. First of all, we apply the Laplace transform to the governing differential equation because of difficulty to construct the time-dependent fundamental solution (Tosaka 1986). And then, we derive the boundary integral equations of displacement and temperature fields in the transformed space and present the coupled fundamental solution in a closed form. The boundary element unsteady solutions of the first and second Danilovskaya problems and the Sternberg and Chakravorty problem in the half-space demonstrate the versatility and accuracy of the proposed method through comparison with the existing solutions.

Throughout this paper, the summation convention on repeated indices is used. A dot ($\dot{}$) is used to denote time differentiation and a coma $(\ )_{,i}$ to denote partial differentiation with respect to x_i $(i = 1, 2)$.

2 Governing equations

We consider the coupled dynamic problems of a linear elastic heat conductor based on the so-called classical thermoelasticity theory. Let Ω be a finite domain with boundary Γ occupied by a homogeneous, isotropic, elastic body. And, let $T = [0, t]$ be the time interval. The basic equations of linearized coupled thermoelasticity can be given as follows (Carlson 1972):

the strain-displacement relation

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{1}$$

the thermal gradient-temperature relation

$$g_i = \theta_{,i}, \tag{2}$$

the equations of motion

$$\tau_{ji,j} + b_i = \rho \ddot{u}_i, \tag{3}$$

the energy equation

$$-q_{i,i} - \gamma\theta_0 \dot{u}_{i,i} + p = c\dot{\theta}, \tag{4}$$

the stress-strain-temperature relation (Duhamel–Neumann law)

$$\tau_{ij} = (\lambda u_{k,k} - \gamma\theta)\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{5}$$

the heat conduction equation (Fourier law)

$$q_i = -k g_i = -k\theta_{,i}, \tag{6}$$

in which $u_i, e_{ij}, \tau_{ij}, q_i, b_i, \rho$ are displacements, strain measures, Cauchy stress tensor, heat flux, body forces, and the density of the elastic body, respectively.

The initial-boundary-value problem of the resulting field equations in terms of displacements u_i and temperature θ can be written as follows:

field equations

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \gamma\theta_{,i} + \rho b_i = \rho \ddot{u}_i, \quad \theta_{,ii} - \frac{1}{\kappa}\dot{\theta} - \eta \dot{u}_{i,i} + \frac{1}{\kappa} Q = 0, \tag{7,8}$$

where

$$\kappa = \frac{k}{\rho c}, \quad \eta = \frac{\gamma}{k}\theta_0, \quad Q = \frac{\kappa}{k} p, \tag{9}$$

initial conditions

$$\begin{aligned} &u_i(x, 0) = {}_0u_i(x) \quad u_{i,i}(x, 0) = {}_0e(x) \\ &\dot{u}_i(x, 0) = {}_0v_i(x) \qquad \theta(x, 0) = {}_0\theta(x), \end{aligned} \tag{10}$$

boundary conditions,

$$\begin{aligned} &u_i = \hat{u}_i && \text{on} \quad \Gamma_u \times T && \theta = \hat{\theta} && \text{on} \quad \Gamma_\theta \times T \\ &\tau_i = \tau_{ij} n_j = \hat{\tau}_i && \text{on} \quad \Gamma_\tau \times T && q = q_i n_i = \hat{q} && \text{on} \quad \Gamma_q \times T. \end{aligned} \tag{11}$$

Here the boundary Γ with the unit normal vector component n_i may be split into the displacement boundary Γ_u, the traction one Γ_τ, the temperature one Γ_θ, and the heat flux one Γ_q.

In this place, we introduce the Laplace transformation procedure to solve the above initial-boundary-value problem. By means of an application of the Laplace transform to the initial-boundary-value problem of the coupled thermoelasticity, we can rewrite the problem into the boundary-value problem of elliptic type.

Applying formally the Laplace transform defined by

$$\bar{u}_i(x,\omega)=\int_0^\infty \exp(-\omega t)u_i(x,t)\,\mathrm{d}t \tag{12}$$

to the coupled system of Eqs. (7) and (8) and taking into consideration of the initial conditions (10), we can get the following elliptic type differential equations in the transformed space:

$$\mu\bar{u}_{i,jj}+(\lambda+\mu)\bar{u}_{j,ij}-\rho\omega^2\bar{u}_i-\gamma\bar{\theta}_{,i}=-G_i \tag{13}$$

$$\bar{\theta}_{,ii}-\frac{\omega}{\kappa}\bar{\theta}-\eta\omega\bar{u}_{i,i}=-H \tag{14}$$

where the pseudo body force G_i and the pseudo heat supply H are defined by, respectively,

$$G_i=\bar{b}_i+\rho\omega_0u_i+\rho_0v_i,\quad H=\frac{1}{\kappa}\bar{Q}+\frac{1}{\kappa_0}\theta+\eta_0e. \tag{15}$$

Consequently, the initial-boundary-value problem reduces to the boundary-value problem of the set of (13) and (14) with the boundary conditions in the transformed space.

For the sake of derivation of the boundary integral formulation of the boundary-value problem, it is convenient to rewrite the above system of (13) and (14) into the following matrix form:

$$\mathrm{L}_{ij}\bar{U}_j=\bar{B}_i\quad(i,j=1,2,3) \tag{16}$$

where each quantities for two dimensional case are given as

$$[\mathrm{L}_{ij}]=\begin{bmatrix}\mu\Delta+(\lambda+\mu)D_1^2-\rho\omega^2 & (\lambda+\mu)D_1D_2 & -\gamma D_1\\ (\lambda+\mu)D_1D_2 & \mu\Delta+(\lambda+\mu)D_2^2-\rho\omega^2 & -\gamma D_2\\ -\eta\omega D_1 & -\eta\omega D_2 & \Delta-\dfrac{\omega}{\kappa}\end{bmatrix} \tag{17}$$

$$\bar{U}_j=\{\bar{u}_1\quad\bar{u}_2\quad\bar{\theta}\}^{\mathrm{T}},\quad \bar{B}_i=\{-G_1-G_2-H\}^{\mathrm{T}}. \tag{18, 19}$$

Here we use the notations $D_i=\partial/\partial x_i$ $(i=1,2)$ and Δ, the Laplacian, as the abbreviation of differentiation.

3 Formulation of integral equations

In order to derive the integral equation formulation of the coupled differential Eqs. (16), let us start with the following weighted residual expression of the system (16) for the weighting tensor V^*_{ik}

$$\int_\Omega(\mathrm{L}_{ij}\bar{U}_j-\bar{B}_i)V^*_{ik}\,\mathrm{d}\Omega=0. \tag{20}$$

After integration by parts and some manipulations, we can obtain the following boundary integral equation set in terms of the transformed unknown functions:

$$\begin{aligned}c_{kj}\bar{U}_k(y;\omega)=&\int_\Gamma\{\bar{\tau}_\alpha(x;\omega)V^*_{\alpha j}(x,y;\omega)-\bar{u}_\alpha(x;\omega)\Sigma^*_{\alpha j}(x,y;\omega)\}\mathrm{d}\Gamma(x)\\&+\int_\Gamma\{\bar{\theta}_{,n}(x;\omega)V^*_{3j}(x,y;\omega)-\bar{\theta}(x;\omega)V^*_{3j,n}(x,y;\omega)\}\mathrm{d}\Gamma(x)\\&-\int_\Omega\bar{B}_i(x;\omega)V^*_{ij}(x,y;\omega)\mathrm{d}\Omega(x),\quad(i,j,k=1,2,3,\ \alpha=1,2),\end{aligned} \tag{21}$$

where c_{ij} is the shape coefficient matrix to be determined with properties on both the location of a boundary point x and the local geometry at the source point y, and the vector $\bar{\tau}_\alpha$ and the corresponding quantity $\Sigma^*_{\alpha j}$ are defined by, respectively,

$$\bar{\tau}_\alpha = \bar{\tau}_{\alpha\beta} n_\beta = \{(\lambda \bar{u}_{k,k} - \gamma\bar{\theta})\delta_{\alpha\beta} + \mu(\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha})\} n_\beta, \tag{22}$$

$$\Sigma^*_{\alpha j} = \{(\lambda V^*_{kj,k} + \eta\omega V^*_{3j})\delta_{\alpha\beta} + \mu(V^*_{\alpha j,\beta} + V^*_{\beta j,\alpha})\} n_\beta. \tag{23}$$

Here the weighting tensor V^*_{jk} must be determined as the solution which satisfies the differential equation

$$L_{ij} V^*_{jk} = -\delta_{ik}\delta(x-y). \tag{24}$$

From the above it is evident that the weighting tensor is the fundamental solution tensor for the adjoint operator L_{ij} of L_{ij} given by

$$[L_{ij}] = \begin{bmatrix} \mu\Delta + (\lambda+\mu)D_1^2 - \rho\omega^2 & (\lambda+\mu)D_1D_2 & \eta\omega D_1 \\ (\lambda+\mu)D_1D_2 & \mu\Delta + (\lambda+\mu)D_2^2 - \rho\omega^2 & \eta\omega D_2 \\ \gamma D_1 & \gamma D_2 & \Delta - \dfrac{\omega}{\kappa} \end{bmatrix}. \tag{25}$$

Moreover, from the derived integral Eq. (21) and the stress–strain-temperature relation, the stress tensor at some point y in the domain Ω can be expressed with the following integral equation:

$$\begin{aligned} \bar{\tau}_{\alpha\beta}(y;\omega) = & \int_\Gamma \{\bar{\tau}_\gamma(x;\omega) S_{\alpha\beta\gamma}(x,y;\omega) - \bar{u}_\gamma(x;\omega) D_{\alpha\beta\gamma}(x,y;\omega)\} \mathrm{d}\Gamma(x) \\ & + \int_\Gamma \{\bar{\theta}_{,n}(x;\omega) S_{\alpha\beta 3}(x,y;\omega) - \bar{\theta}(x;\omega) D_{\alpha\beta 3}(x,y;\omega)\} \mathrm{d}\Gamma(x) \\ & - \int_\Omega \bar{B}_k(x;\omega) S_{\alpha\beta k}(x,y;\omega) \mathrm{d}\Omega(x), \quad (\alpha,\beta,\gamma = 1,2,\ k = 1,2,3), \end{aligned} \tag{26}$$

where we introduce the quantities

$$S_{\alpha\beta k} = (\lambda V^*_{km,m} - \gamma V^*_{k3})\delta_{\alpha\beta} + \mu(V^*_{k\alpha,\beta} + V^*_{k\beta,\alpha}) \tag{27}$$

$$D_{\alpha\beta\gamma} = (\lambda \Sigma^*_{\gamma m,m} - \gamma \Sigma^*_{\gamma 3})\delta_{\alpha\beta} + \mu(\Sigma^*_{\gamma\alpha,\beta} + \Sigma^*_{\gamma\beta,\alpha}) \tag{28}$$

$$D_{\alpha\beta 3} = \{\lambda(V^*_{3m,n})_{,m} - \gamma V^*_{33,n}\}\delta_{\alpha\beta} + \mu\{(V^*_{3\alpha,n})_{,\beta} + (V^*_{3\beta,n})_{,\alpha}\}. \tag{29}$$

4 Fundamental solutions

Let us consider how to construct the fundamental solution tensor defined by (24). We follow the methodology developed by Tosaka (1985) which was originated from Hörmander (1946).

We may put the fundamental solutions tensor V^*_{ij} into the following potential expression:

$$V^*_{ij} = \mu_{ij}\phi^*, \tag{30}$$

where ϕ^* is the scalar function and μ_{ij} is the transposed cofactor operator of L_{ij} such that

$$\begin{aligned} \mu_{\alpha\beta} &= (\lambda+2\mu)\left[\left\{\Delta^2 - \left(\frac{\omega^2}{c_1^2} + (1+\delta)\frac{\omega}{\kappa}\right)\Delta + \frac{\omega^2}{c_1^2}\frac{\omega}{\kappa}\right\}\delta_{\alpha\beta} - \left\{\frac{\lambda+\mu}{\lambda+2\mu}\left(\Delta - \frac{\omega}{\kappa}\right) - \delta\right\}\Delta D_\alpha D_\beta\right] \\ \mu_{\alpha 3} &= -\eta\omega\mu\left(\Delta - \frac{\omega^2}{c_2^2}\right), \quad \mu_{3\beta} = -\mu\gamma\left(\Delta - \frac{\omega^2}{c_2^2}\right), \quad \mu_{33} = \mu(\lambda+2\mu)\left(\Delta - \frac{\omega^2}{c_1^2}\right)\left(\Delta - \frac{\omega^2}{c_2^2}\right), \end{aligned} \tag{31}$$

where c_1 and c_2 are velocities of longitudinal and transverse waves, respectively, and δ denotes thermomechanical coupling parameter given by

$$\delta = \frac{\eta\gamma\kappa}{\lambda + 2\mu}. \tag{32}$$

Substitution of expression (30) into Eq. (24) yields

$$L\phi^* = -\delta(x-y) \tag{33}$$

where L is the determinant of L_{ij}. From the adjoint operator (25) we can get immediately

$$L = \mu(\lambda+2\mu)(\Delta^3 - \beta_1\Delta^2 + \beta_2\Delta - \beta_3) \tag{34}$$

with

$$\beta_1 = \frac{\omega^2}{c_1^2} + \frac{\omega^2}{c_2^2} + (1+\delta)\frac{\omega}{\kappa}, \quad \beta_2 = \frac{\omega^2}{c_1^2}\left(\frac{\omega^2}{c_2^2} + \frac{\omega}{\kappa}\right) + \frac{\omega^2}{c_2^2}\frac{\omega}{\kappa}(1+\delta), \quad \beta_3 = \frac{\omega^2}{c_1^2}\frac{\omega^2}{c_2^2}\frac{\omega}{\kappa}. \tag{35}$$

After all, it is found that the scalar function ϕ^* is the fundamental solution for the differential operator L. We can rewrite the above expression (34) into the following factorized form (Kupradze 1979):

$$L = \mu(\lambda+2\mu)(\Delta - h_1^2)(\Delta - h_2^2)(\Delta - h_3^2) \tag{36}$$

in which the coefficients h_1^2, h_2^2 and h_3^2 must be determined as these which satisfy

$$h_1^2 + h_2^2 = \frac{\omega^2}{c_1^2} + (1+\delta)\frac{\omega}{\kappa}, \quad h_1^2 h_2^2 = \frac{\omega^2}{c_1^2}\frac{\omega}{\kappa}, \quad h_3^2 = \frac{\omega^2}{c_2^2}. \tag{37}$$

In this place, if we introduce the assumption $1+\delta \doteqdot 1$, then the above coefficients can be determined with the following closed form (Tosaka 1986; Suh and Tosaka 1987):

$$h_1^2 = \frac{\omega}{\kappa}, \quad h_2^2 = \frac{\omega^2}{c_1^2}, \quad h_3^2 = \frac{\omega^2}{c_2^2}. \tag{38}$$

However this assumption is applicable to only the restricted case in which the thermomechanical coupling is very week.

From the relations (37) the coefficients h_1^2 and h_2^2 can be determined as the roots of the following quadratic equation of unknown h^2.

$$(h^2)^2 + \left[\frac{\omega^2}{c_1^2} + (1+\delta)\frac{\omega}{\kappa}\right]h^2 + \frac{\omega^2}{c_1^2}\frac{\omega}{\kappa} = 0. \tag{39}$$

Making use of the determined roots h_1^2 and h_2^2, the fundamental solution of the operator (36) can be given as

$$\phi^*(x, y; \omega) = \frac{1}{2\pi\mu(\lambda+2\mu)}\sum_{k=1}^{3} W_k K_0(h_k r), \tag{40}$$

where $K_0(r)$ is the modified Bessel function of the second kind of order zero with the argument $r = \|x-y\|$ and

$$W_1 = \frac{-1}{(h_1^2-h_2^2)(h_3^2-h_1^2)}, \quad W_2 = \frac{-1}{(h_1^2-h_2^2)(h_2^2-h_3^2)}, \quad W_3 = \frac{-1}{(h_2^2-h_3^2)(h_3^2-h_1^2)}. \tag{41}$$

After all, we can determine each component of fundamental solution tensor from (30) and (31), respectively, by using of the derived fundamental solution (40) as follows:

$$\begin{aligned} V_{\alpha\beta}^* &= \frac{1}{2\pi\rho c_2^2}\sum_{k=1}^{3} W_k[\Psi_k(r)\delta_{\alpha\beta} - \chi_k^*(r) r_{,\alpha} r_{,\beta}] \\ V_{\alpha 3}^* &= \frac{\eta\omega}{2\pi\rho c_1^2}\sum_{k=1}^{3} W_k \xi_k^*(r) r_{,\alpha}, \quad V_{3\beta}^* = \frac{\gamma}{2\pi\rho c_1^2}\sum_{k=1}^{3} W_k \xi_k^*(r) r_{,\beta}, \quad V_{33}^* = \frac{1}{2\pi}\sum_{k=1}^{3} W_k \zeta_k^*(r). \end{aligned} \tag{42}$$

And, we can also obtain the quantities Σ^*_{ij},

$$\Sigma^*_{\alpha\beta}=\frac{1}{2\pi}\sum_{k=1}^{3}W_k\left[\left(\Psi^*_{k,r}-\chi^*_k\frac{1}{r}\right)\left(r_{,\alpha}n_\beta+\frac{\partial r}{\partial n}\delta_{\alpha\beta}\right)-2\chi^*_{k,r}r_{,\alpha}r_{,\beta}\frac{\partial r}{\partial n}-\chi^*_k\frac{2}{r}\left(r_{,\beta}n_\alpha-2r_{,\alpha}r_{,\beta}\frac{\partial r}{\partial n}\right)\right.$$
$$\left.+\left(\frac{c_1^2}{c_2^2}-2\right)\left(\Psi^*_{k,r}-\chi^*_{k,r}-\chi^*_k\frac{1}{r}\right)r_{,\beta}n_\alpha+\frac{\delta}{\kappa}\omega\xi^*_k r_{,\beta}n_\alpha\right]$$
$$\Sigma^*_{\alpha 3}=\frac{\eta\omega}{2\pi}\sum_{k=1}^{3}W_k\left[2\frac{c_2^2}{c_1^2}\left\{\left(\xi^*_{k,r}-\xi^*_k\frac{1}{r}\right)r_{,\alpha}\frac{\partial r}{\partial n}+\xi^*_k\frac{1}{r}n_\alpha\right\}+\left(1-2\frac{c_2^2}{c_1^2}\right)\left(\xi^*_{k,r}+\xi^*_k\frac{1}{r}\right)n_\alpha+\xi^*_k n_\alpha\right] \tag{43}$$
$$\Sigma^*_{3\beta}=-\frac{\gamma}{2\pi\rho c_1^2}\sum_{k=1}^{3}W_k\left[\left(\xi^*_{k,r}-\xi^*_k\frac{1}{r}\right)r_{,\beta}\frac{\partial r}{\partial n}+\xi^*_k\frac{1}{r}n_\beta\right],\quad \Sigma^*_{33}=\frac{1}{2\pi}\sum_{k=1}^{3}W_k\zeta^*_{k,r}\frac{\partial r}{\partial n}.$$

Here we introduce the following functions:

$$\Psi^*_k=\left\{\left(h_k^2-\frac{\omega}{\kappa}\right)\left(h_k^2-\frac{\omega^2}{c_1^2}\right)-\frac{1}{\kappa}\delta\omega h_k^2\right\}K_0(rh_k)+\left\{\frac{1}{2(1-\nu)}\left(h_k^2-\frac{\omega}{\kappa}\right)-\frac{1}{\kappa}\delta\omega\right\}h_k\frac{1}{r}K_1(rh_k)$$
$$\chi^*_k=h_k^2\left\{\frac{1}{2(1-\nu)}\left(h_k^2-\frac{\omega}{\kappa}\right)-\frac{1}{\kappa}\delta\omega\right\}K_2(rh_k),\quad \xi^*_k=h_k\left(h_k^2-\frac{\omega^2}{c_2^2}\right)K_1(rh_k),$$
$$\eta^*_k=\left(h_k^2-\frac{\omega^2}{c_1^2}\right)\left(h_k^2-\frac{\omega^2}{c_2^2}\right)K_0(rh_k). \tag{44}$$

5 Numerical solution procedure

We can apply the standard boundary element procedure in order to solve numerically the system of boundary integral Eqs. (21). The boundary $\Gamma(\boldsymbol{x})$ is divided into Ne elements and the unknown functions over each element are discretized with the following expression:

$$\bar{u}_i=\boldsymbol{\Phi}^{\mathrm{T}}\bar{\boldsymbol{u}}_i^N\qquad \bar{\tau}_i=\boldsymbol{\Phi}^{\mathrm{T}}\bar{\boldsymbol{\tau}}_i^N$$
$$\bar{\theta}=\boldsymbol{\Phi}^{\mathrm{T}}\bar{\boldsymbol{\theta}}^N\qquad \bar{\theta}_{,n}=\boldsymbol{\Phi}^{\mathrm{T}}\bar{\boldsymbol{\theta}}^N_{,n} \tag{45}$$

where $\boldsymbol{\Phi}^{\mathrm{T}}$ are the interpolation functions and $\bar{\boldsymbol{u}}_i^N, \bar{\boldsymbol{\tau}}_i^N, \bar{\boldsymbol{\theta}}^N, \bar{\boldsymbol{\theta}}^N_{,n}$ are the nodal values of displacements, tractions, temperature, and temperature gradient, respectively, at nodal point $\boldsymbol{x}_N$ on the boundary element. If the pseudo body force and the pseudo heat supply are assumed to be zero, the discretized form of the boundary integral Eqs. (21) can be expressed as

$$c_{kj}\bar{U}_k(y;\omega)=\sum_{e=1}^{Ne}\bar{\tau}^N_\alpha\int_{\Gamma_e}V^*_{\alpha j}(\boldsymbol{x}(\xi),y;\omega)\boldsymbol{\Phi}^{\mathrm{T}}(\xi)\mathrm{d}\Gamma(\xi)-\sum_{e=1}^{Ne}\bar{u}^N_\alpha\int_{\Gamma_e}\Sigma^*_{\alpha j}(\boldsymbol{x}(\xi),y;\omega)\boldsymbol{\Phi}^{\mathrm{T}}(\xi)\mathrm{d}\Gamma(\xi),$$
$$+\sum_{e=1}^{Ne}\bar{\theta}^N_{,n}\int_{\Gamma_e}V^*_{3j}(\boldsymbol{x}(\xi),y;\omega)\boldsymbol{\Phi}^{\mathrm{T}}(\xi)\mathrm{d}\Gamma(\xi)-\sum_{e=1}^{Ne}\bar{\theta}^N\int_{\Gamma_e}V^*_{3j,n}(\boldsymbol{x}(\xi),y;\omega)\boldsymbol{\Phi}^{\mathrm{T}}(\xi)\mathrm{d}\Gamma(\xi) \tag{46}$$

where $\xi=\{\xi_i\}$ denotes the intrinsic coordinates on Γ_e. For all boundary nodes, we can express the discretized form (46) as the following matrix system of equation

$$\mathbf{H}_1\bar{\boldsymbol{u}}+\mathbf{H}_2\bar{\boldsymbol{\theta}}=\mathbf{G}_1\bar{\boldsymbol{\tau}}+\mathbf{G}_2\bar{\boldsymbol{\theta}}_{,n}. \tag{47}$$

Reordering the system (47) with respect to the unknown, the final system of equation to be solved can be expressed as follows:

$$\mathbf{A}\bar{\boldsymbol{X}}=\boldsymbol{B} \tag{48}$$

where $\mathbf{A}$ is the influence matrix, $\boldsymbol{B}$ is the contribution vector of the known boundary conditions and $\boldsymbol{X}$ is the vector of unknowns in $\bar{u}_i^N, \bar{\theta}^N, \bar{\tau}_i^N, \bar{\theta}^N_{,n}$. The above system (48) can be solved with the proper solver and we can get all the transformed unknowns on the boundary.

The above procedure gives us only the numerical solution in the Laplace transformed space. Then, it is necessary to invert the transform to restore the original variables in the time domain. It is well known that there exist a number of numerical inversion methods (Papoulis 1957; Durbin 1974; Narayanan and Beskos 1982) corresponding to different methods of approximation of the Bromwich integral,

$$f(t)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\exp(\omega t)\bar{f}(\omega)\mathrm{d}\omega. \tag{49}$$

In this paper, we use the method of Hosono (1979) which has recently attracted attention as an efficient and accurate method (Suh and Tosaka 1988a). In the Hosono's method a function defined by (49) is expressed approximately as the following series:

$$f(t)=\frac{\exp(\sigma)}{t}\left[\sum_{n=1}^{N-1}\bar{f}_n+\frac{1}{2^{M+1}}\sum_{m=0}^{M}A_{M,m}\bar{f}_{N+m}\right] \tag{50}$$

with the coefficients,

$$A_{M,M}=1,\quad A_{M,m-1}=A_{M,m}+\frac{(M+1)!}{m!(M-m+1)!}. \tag{51}$$

Here the function $\bar{f}_n$ in the transformed space at the nth sample point ω_n to be defined below, is given by

$$\bar{f}_n=(-1)^n\,\mathrm{Im}\{\bar{f}(\omega_n)\} \tag{52}$$

with

$$\omega_n=\frac{1}{t}\left\{\sigma+i\left(n-\frac{1}{2}\right)\pi\right\} \tag{53}$$

where $\mathrm{Im}\{\bar{f}\}$ denotes the imaginary part of the complex function $\bar{f}$.

According to the solution algorithm based on the numerical inversion formula (50) due to Hosono as shown in Fig. 1, we can get the transient solution of a particular time directly with only $20\sim30$ sample points ω_n in the transformed space.

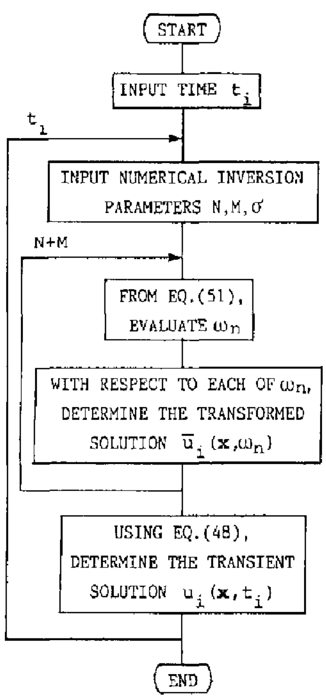


Fig. 1. Solution algorithm

6 Numerical examples

In the following, a number of well-known examples are presented in order to demonstrate the versatility and accuracy of our formulation and its numerical implementation developed in the previous section. For the numerical examples three half-space problems are selected as follows:

a) the first Danilovskaya problem,
b) the second Danilovskaya problem,
c) the Sternberg and Chakravorty problem.

The above three problems are essentially one-dimensional and therefore it may be needed to apply the one-dimensional boundary element method. For one-dimensional problems, we can develop the boundary element method quite easily and actually we can get almost same results as the ones which are obtained in two-dimensional problems and presented later in this paper. The geometry of the problems in the half-space is depicted in Fig. 2.

It is convenient to introduce the following dimensionless variables and we refer to the results on these quantities:

$$\tilde{x}_i = \frac{1}{a} x_i, \quad \tilde{\theta} = \frac{\theta}{T}, \quad \tilde{t} = \frac{\kappa}{a^2} t, \quad \tilde{\tau}_{ij} = \frac{1}{\gamma T} \tau_{ij}, \quad \tilde{u}_i = \frac{\lambda + 2\mu}{a \gamma T} u_i \tag{54}$$

where

$$a = \frac{\kappa}{c_1}. \tag{55}$$

Figure 3 shows the boundary element representation of the domain of problem in which a half-space is modeled by two-dimensional boundary elements constrained to undergo axial displacements only at upper and lower parts. In our calculations, we adopt 58 linear elements and we perform the numerical integration with 8 points Gaussian quadrature on each element. For simplicity, initial conditions, body force, and heat supply are assumed to be zero.

The infinite elastic body in numerical examples is assumed to possess the following material properties corresponding to the carbon steel:

$$\begin{aligned} &\mu = 8.4\,10^6\,\mathrm{kg/cm^2}, && k = 0.105\,\mathrm{cal/cm \cdot s \cdot K}, \\ &\nu = 0.25, && \rho = 7.84\,10^{-3}\,\mathrm{kg/cm^3}, \\ &\alpha = 0.107\,10^{-4}/\mathrm{K} && c = 1.17\,10^2\,\mathrm{cal/kg \cdot K}. \end{aligned} \tag{56}$$

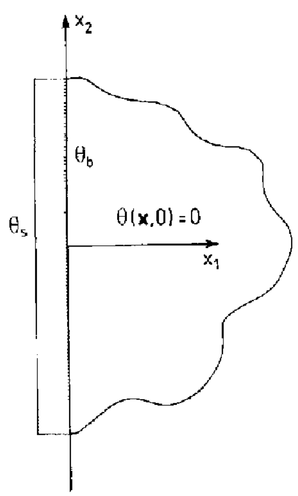


Fig. 2. Problem geometry

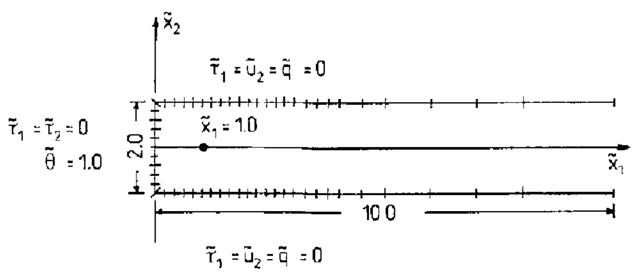


Fig. 3. Boundary element discretization

6.1 *The first Danilovskaya problem*

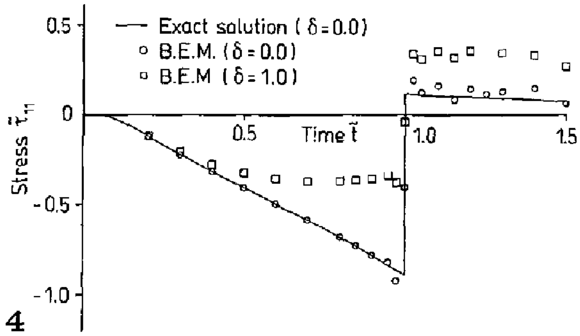
The problem proposed by Danilvoskaya (1950) concerned a linear elastic half-space subjected to a uniform sudden temperature change (i.e., step surface heating) on its boundary plane in which the traction was assumed to be free.

The boundary conditions in this problem are given as

$$\begin{cases} \tilde{\tau}_i(\tilde{x},\tilde{t}) = 0 & \text{on} \quad \tilde{x}_1 = 0 \\ \tilde{\theta}(x,\tilde{t}) = H(\tilde{t}) & \text{on} \quad \tilde{x}_1 = 0 \end{cases} \tag{57}$$

where $H(\tilde{t})$ denotes the Heaviside step function. Danilovskaya obtained the first analytical solution to this problem in the sense of dynamic uncoupled thermoelasticity. Analytical investigations into the fully coupled theory was made by many authors (Boley and Tolins 1963; Dunn 1966) and approximate solutions have recently been obtained by using the finite element method in several papers (Ting and Chen 1982; Prevost and Tao 1983).

We wish to show the results for time history on a particular location $\tilde{x} = 1.0$ in the following figures. This point $\tilde{x} = 1.0$ means the dimensionless location of the thermally induced elastic wave front at the dimensionless time $\tilde{t} = 1.0$. Figures 4–6 show the dimensionless stress $\tilde{\tau}_{11}$, temperature $\tilde{\theta}$ and displacement $\tilde{u}_1$ time histories at $\tilde{x}_1 = 1.0$ in the uncoupled case $\delta = 0.0$ and the coupled case $\delta = 1.0$ through comparison with the exact solutions in the uncoupled case. The present results in the uncoupled case shown in these figures are very good agreement with the exact solutions, although we used a coarse discretization. Especially, in stress, the numerical results put on the exact solution very well, nevertheless there exists the stress-discontinuity at $\tilde{t} = 1.0$. The oscillations take place after passage the stress wave front and tend to converge to the exact solution with increasing time. This oscillations are due to the introduction of the numerical inversion method by Hosono which has a feature to occur the so-called Gibbs phenomenon at the discontinuity, and the oscillations can be reduced to some degree by taking a large number of inversion parameter N in Eq. (50).



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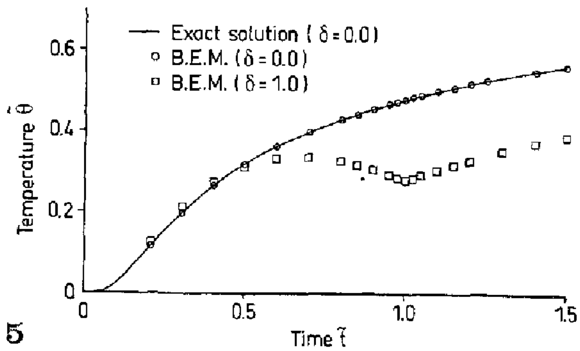

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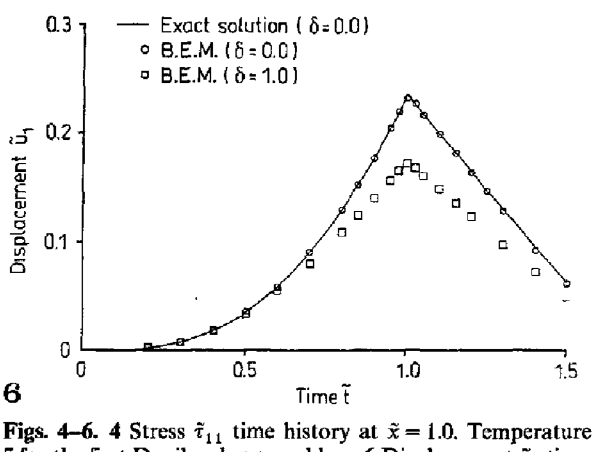

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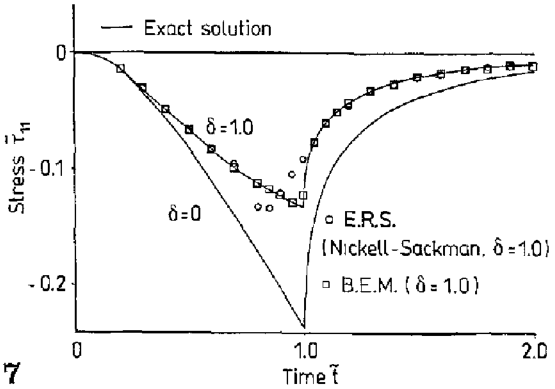
Figs. 4–6. 4 Stress $\tilde{\tau}_{11}$ time history at $\tilde{x} = 1.0$. Temperature $\tilde{\theta}$ time history at $\tilde{x} = 1.0$ for the first Danilovskaya problem. **5** for the first Danilovskaya problem. **6** Displacement $\tilde{u}_1$ time history at $\tilde{x} = 1.0$ for the first Danilovskaya problem



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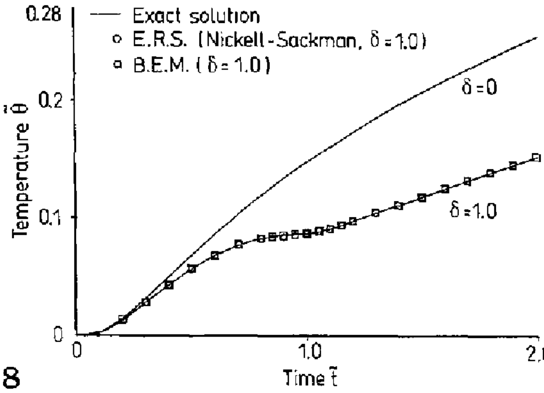

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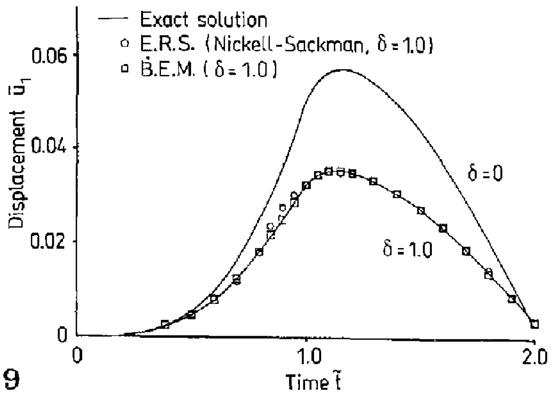

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Figs. 7–9. 7 Stress $\tilde{\tau}_{11}$ time history at $\tilde{x} = 1.0$ for the second Danilovskaya problem with $\delta = 1.0$ and $H = 0.5$. **8** Temperature $\tilde{\theta}$ time history at $\tilde{x} = 1.0$ for the second Danilovskaya problem with $\delta = 1.0$ and $H = 0.5$. **9** Displacement $\tilde{u}_1$ time history at $\tilde{x} = 1.0$ for the second Danilovskaya problem with $\delta = 1.0$ and $H = 0.5$

6.2 The second Danilovskaya problem

The first Danilovskaya problem was extended to account for boundary layer thermal conductance along the boundary plane by Danilovskaya (1952). We refer this problem as the second Danilovskaya problem in which the boundary condition (57) is modified with

$$\tilde{q}(x, \tilde{t}) = m\{1 - \tilde{\theta}(\tilde{x}, \tilde{t})\} \quad \text{on} \quad \tilde{x}_1 = 0 \tag{58}$$

where we introduce the following coefficient instead of the boundary-layer thermal conductance h,

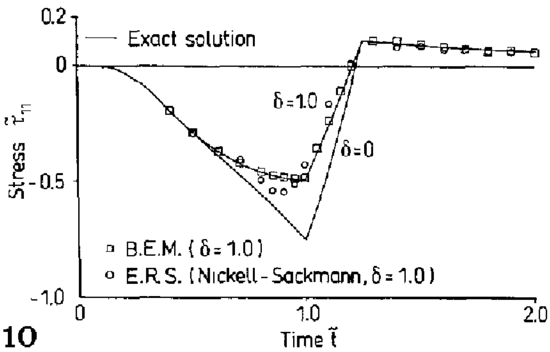
$$m = \frac{\rho\kappa h}{(\lambda + 2\mu)k}. \tag{59}$$

The coupled results obtained by the present method for the stress $\tilde{\tau}_{11}$, temperature $\tilde{\theta}$, displacement $\tilde{u}_1$ at $\tilde{x}_1 = 1.0$ are shown in Figs. 7–9 in which we adopt $\delta = 1.0$ and $m = 0.5$. These results are compared with the exact solutions and extended Ritz solutions by Nickell and Sackman (1968) and excellent agreement is also observed.

6.3 The Sternberg and Chakravorthy problem

As the third problem, let us consider the Sternberg and Chakravorty problem (1959) in which a more realistic ramp-type heating of the boundary plane than the previous first problem subjected to a thermal shock was discussed. The boundary conditions in this problem become

$$\tilde{\theta}(\tilde{x}_i, \tilde{t}) = \begin{cases} \tilde{\theta}/\theta, & 0 \leqq \tilde{\theta} \leqq \theta \\ 1, & \tilde{\theta} \geqq \theta \end{cases}, \quad x_1 = 0, \tag{60}$$



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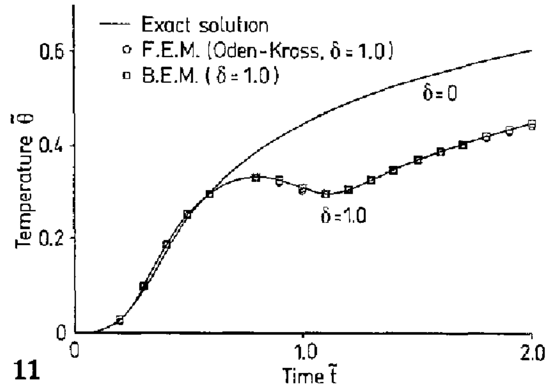

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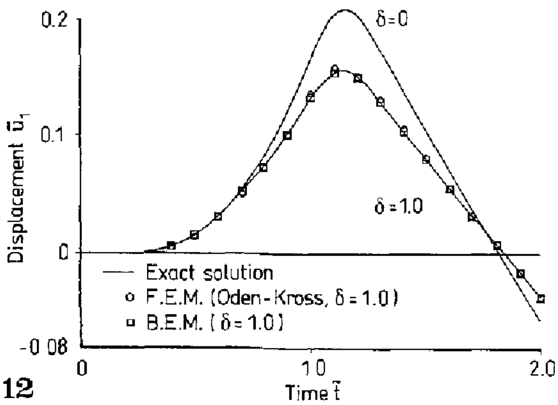

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Figs. 10–12. **10** Stress $\bar{\tau}_{11}$ time history at $\bar{x}=1.0$ for the Sternberg and Chakravorthy problem with $\delta=1.0$ and $t_0=0.25$. **11** Temperature $\bar{\theta}$ time history at $\bar{x}=1.0$ for the Sternberg and Chakravorthy problem with $\delta=1.0$ and $t_0=0.25$. **12** Displacement $\bar{u}_1$ time history at $\bar{x}=1.0$ for the Sternberg and Chakravorthy problem with $\delta=1.0$ and $t_0=0.25$

where

$$\theta=\frac{(\lambda+2\mu)^2 t_0}{\rho^2 \kappa}. \tag{61}$$

Figures 10–12 depict the stress $\tilde{\tau}_{11}$, temperature $\tilde{\theta}$, and displacement $\tilde{u}_1$ time histories for the ramp heating time $t_0=0.25$ in the coupled case $\delta=1.0$. These results are compared with the exact solutions by Nickell and Sackman (1968) and the finite element solutions by Oden and Kross (1968). The present results shown in these figures show excellent agreement with the exact solutions.

7 Conclusion

In this paper, we have developed the boundary element method for obtaining approximate solutions to initial boundary-value problems in the classical dynamic theory of two dimensional coupled thermoelasticity. The coupled fundamental solution in the Laplace transformed space was newly constructed and presented in a closed form. According to the solution algorithm introduced in this paper, we can obtain the transient solution at a particular time directly with fewer time consuming. As numerical examples, the first and second Danilovskaya problems and the Sternberg and Chakravorthy problem have been chosen. Through the comparisons of our results of test problems with other existing results we conclude that the efficiency and applicability of the boundary element method is demonstrated for predicting the dynamic response of coupled or uncoupled thermally induced stress waves, especially the traveling discontinuity phenomena in the normal stress. The present method and solution procedure can be easily developed to three-dimensional problems and to the generalized thermoelastic problems with the finite temperature wave speed. Applications and numerical implementations in above two problems have been reported by our papers (Tosaka and Suh 1987; Suh and Tosaka 1988b; Suh and Tosaka 1989).

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