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A class of finite element methods based on orthonormal, compactly supported wavelets

J. Ko, A. J. Kurdila, M. S. Pilant

Abstract This paper develops a class of finite elements for compactly supported, shift-invariant functions that satisfy a dyadic refinement equation. Commonly referred to as wavelets, these basis functions have been shown to be remarkably well-suited for integral operator compression, but somewhat more difficult to employ for the representation of arbitrary boundary conditions in the solution of partial differential equations. The current paper extends recent results for treating periodized partial differential equations on unbounded domains in R^n, and enables the solution of Neumann and Dirichlet variational boundary value problems on a class of bounded domains. Tensor product, wavelet-based finite elements are constructed. The construction of the wavelet-based finite elements is achieved by employing the solution of an algebraic eigenvalue problem derived from the dyadic refinement equation characterizing the wavelet, from normalization conditions arising from moment equations satisfied by the wavelet, and from dyadic refinement relations satisfied by the elemental domain. The resulting finite elements can be viewed as generalizations of the connection coefficients employed in the wavelet expansion of periodic differential operators. While the construction carried out in this paper considers only the orthonormal wavelet system derived by Daubechies, the technique is equally applicable for the generation of tensor product elements derived from Coifman wavelets, or any other orthonormal compactly supported wavelet system with polynomial reproducing properties.

1 Introduction

The rapidly emerging field of wavelet and multiresolution analysis[1,2] has been developed primarily by researchers in signal processing in the context of multirate filtering techniques, and by approximation theorists. Not only has the emergence of multiresolution analysis affected the understanding of rather abstract approximation theoretic concepts (for example, the smoothness of functions in Sobolev-Besov spaces in DeVore, Jawerth and Popov (1992)), but it has also had an enormous practical impact on lossy filtering and compression techniques in visualization. It is well-known that independent companies have formed over the past five years to design hardware that embodies aspects of multiresolution analysis, and computer and workstation vendors are incorporating wavelet analysis in both software and hardware in forthcoming platforms.

The potential for wavelet and multiresolution analysis in computational mechanics has been noted in Jaffard and Laurencot (1992), but far less progress has been made in this field. Jaffard and Laurencot (1992) has proven that wavelet Galerkin methods can, in principle, yield matrix representations whose iterative solution is characterized by a condition number that is independent of dimension. This remarkable fact should be carefully contrasted to classical h-based finite element methods which generate matrix representations whose condition numbers grow like $O(N^p)$ where N is the number of unknowns and p is the polynomial order of the elment. Thus, the wavelet-Galerkin methods have computational complexity characterizations that compare favorably with hierarchical finite element methods. This class of finite element techniques can yield matrix representations whose condition number grows like $O(N \log N)$, as in Yserentant (1986, 1990). Still it is important to realize that the results in Jaffard and Laurencot (1992), for instance, are analytic in nature, and have not been realized in practice. Dahmen, Prossdorf and Schneider (1992, 1993a, 1993b), and Dahmen and Kunoth (1992) have presented detailed analyses of the condition numbers of partial differential equations with periodic boundary conditions, while Rieder (1993) has investigated the convergence of multigrid methods for wavelet formulation of periodic problems. Likewise, Heurtaux, Planchon and Wickerhauser (1994) employs wavelet Galerkin methods to investigate energy exchange between scales in a two dimensional Burger's equation subject to periodic boundary conditions. Dahlke and Kunoth (1993) derives biorthogonal wavelets adapted to a class of constant coefficient differential equations, and a multigrid solution strategy, again subject to periodic boundary conditions.

The prevalence of periodic problems in the literature discussing wavelet Galerkin methods is, of course, not coincidental. General, analytic estimates of convergence rates and error are greatly simplified for periodic, in comparison to more general Dirichlet and Neumann, boundary conditions. Moreover, it is difficult to derive wavelets that retain orthonormality, varying degrees of smoothness and satisfy

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[1] *Proceedings of the 35th Structures, Structural Dynamics and Materials Conference*, Hilton Head, South Carolina, May, 1994, pp. 665–675
[2] codes and listings of the generalized connection coefficients and tensor product wavelet elements employed in this paper are available via email request to kurdila @ discovery. tamu. edu

prescribed boundary conditions. For example, the constructions of orthonormal wavelets on a closed interval in Jawerth (1994), while mathematically elegant, requires some fifty pages to describe. To circumvent the difficulties associated with adapting wavelets to enforce specific boundary conditions, several authors have advocated the use of domain embedding techniques whereby the problem of interest, with irregular boundary, is embedded in a larger domain with simpler topology and boundary conditions. This approach has been employed in Wells and Zhou (1992a, 1992b), and Glowinski, Pan, Wells and Zhou (1992) for classes of elliptic boundary value problems, as well as in Ko, Kurdila, Park and Strganac (1993), Ko, Kim, Kurdila and Strganac (1993) for control of distributed parameter systems and aeroelasticity. Its accuracy and efficiency relative to wavelet-based finite elements is addressed in Ko, Kurdila, Wells and Zhou (1994).

The purpose of this paper is to extend the class of problems that can be treated effectively using wavelet Galerkin methods by deriving tensor product wavelet-based finite elements. This task is considerably more difficult than usually encountered in conventional, polynomial based finite element methods in two respects: (i) many wavelet bases cannot be expressed in closed form, and consequently their corresponding Galerkin formulations cannot be expressed in closed form and (ii) many wavelet bases exhibit unusual smoothness properties, sometimes with a fractal character as shown in Daubechies (1992). Numerical integration via conventional quadratures would not be accurate in these cases, see Alpert (1992). Despite these difficulties, wavelet Galerkin finite elements can be calculated to any prescribed accuracy using specialized numerical techniques. Essentially, one can draw an analogy between the role of Gauss quadratures in conventional finite element formulations to *connection coefficients* defined in Latto, Resnikoff and Tenenbaum (1991) associated with derivatives of wavelets. Gauss quadratures provide numerically exact representations of integrals of polynomials, while the connection coefficients provide numerically exact representations of integrals of wavelets. The approach taken in this paper generalizes the methodology of Latto, Resnikoff and Tenenbaum (1991) and provides an algorithm for computing the wavelet elements. It should be noted that the uniqueness and existence of this class of elements is proven in the context of stationary subdivision refinement schemes in Dahmen and Micchelli (1993). The primary contributions of this paper are

(i) the derivation and presentation of a simplified algorithm for the calculation of the generalized connection coefficients. This work generalizes the approach taken by Latto, Resnikoff and Tenenbaum (1991), while treating an important case within the general theory derived in Dahmen and Micchelli (1993).
(ii) The derivation of tensor-product finite elements that are amenable to element processing and assembly techniques from the generalized connection coefficients.
(iii) The determination of quasi-optimal convergence rates in these elements.
(iv) The derivation of associated quadratures to insure quasi-optimal convergence rates.

The recent work of Dahlke and Kunoth (1993) has provided important background for tackling steps (i) and (ii) above, while Sweldens and Piessens (1994) study wavelet quadratures in general.

2 Wavelet and multiresolution analysis

Because of the early stage of development of multiresolution analysis, definitions of wavelets can vary considerably depending upon the generality of formulation desired. For the purposes of this paper the following definition will be sufficient:

Definition: A multiresolution analysis is a sequence of closed subspaces V_j

$$\cdots V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \cdots$$

that satisfy

(i) $\bigcup_j V_j = L^2(R), \bigcap_j V_j = \{0\}$

(ii) $f(x) \in V_j \Leftrightarrow f(2^j x) \in V_0$

(iii) $f(x) \in V_0 \rightarrow f(x-k) \in V_0, \quad \forall k$

(iv) *there exists a $\phi \in V_0$ such that its translates form a Riesz Basis of V_0*

It should be noted that function ϕ is referred to as the scaling function, or generator, of the multiresolution analysis. The first three properties of the multiresolution analysis are actually quite intuitive, and are consistent with many formulations and procedures employed in the finite element formulation as in Strang (1989). Property (i) simply requires that the subspaces can approximate any function that is square-integrable. Properties (ii) and (iii) embody the notion of "multiresolution": Property (ii) requires that the larger space V_{j+1} contains precisely scaled, finer versions of coarse functions found in V_j, while (iii) maintains that any one V_j space is shift-invariant. Property (iv) is more difficult to characterize simply. It requires that $\{\phi(x-k)\}_{k\in Z}$ form a basis (in the usual sense) for V_0, and in addition there must exist two constants C_1 and C_2 such that for any sequence $q = \{q_k\} \in l_2$

$$c_1 \|q\|_{l_2}^2 \leqq \left\| \sum_k q_k \phi(\bullet - k) \right\|_{L^2}^2 \leqq c_2 \|q\|_{l_2}^2 \tag{2.1}$$

In particular, if $\{\phi(x-k)\}_{k\in Z}$ is an orthonormal basis of V_0, it is clearly a Riesz basis with $C_1 = C_2 = 1$. For this paper, it is assumed that $\{\phi(x-k)\}_{k\in Z}$ is in fact an orthonormal basis. It should be noted, however, that the methodology derived in Section 3 is appropriate for much more general classes of wavelets than those whose scaling function generates an orthonormal basis of V_0.

In any event, it is proven in Chui (1992) that condition (iv) implies that the scaling functions satisfy the two-scale, or refinement equations

$$\phi(x) = \sum_k a_k \phi(2x-k) \tag{2.2}$$

for some specific sequence $\{a_k\}$, or mask, associated with the scaling function ϕ. Now if $\{\phi(x-k)\}_{k\in Z}$ is an orthonormal basis for V_0, it is natural to ask how one would characterize its complement in the next larger space V_1. In other words, how does one represent $f_1 \in V_1$ as

$$f_1 = f_0 + g_0$$

where $f_0 \in V_0$ and

$$g_0 \in W_0 \equiv V_1 / V_0 ?$$

One of the most significant contributions within the field of wavelet analysis is that it is indeed possible to construct an orthonormal basis for W_0. In fact, under the conditions stated above, this basis is generated from a single function $\psi(x)$. The function $\psi(x)$ is referred to as the wavelet of the multiresolution analysis and defines the space

$$W_0 = \text{closure}\{\psi(x-k), k \in Z\}$$

By defining

$$W_j \equiv \{f \in L^2(R) : f(2^j x) \in W_0\}$$

one can write

$$V_{j+1} = V_j \oplus W_j$$

$$L^2(R) = \bigoplus_j W_j$$

By convention, the dilates and translates of the scaling function $\phi(x)$ and wavelet $\psi(x)$ are written as

$$\phi_k^j(x) \equiv 2^{(j/2)} \phi(2^j x - k) \tag{2.3}$$

$$\psi_k^j(x) \equiv 2^{(j/2)} \psi(2^j x - k) \tag{2.4}$$

while the r^{th} derivatives are written as

$$\frac{d^r \phi}{dx^r}(x) \equiv \phi^{(r)}$$

$$\frac{d^r \psi}{dx^r}(x) \equiv \psi^{(r)}$$

In the presentation in Section 3 it is important to distinguish between the r^{th} derivative of the k^{th} translate

$$\frac{d^r}{dx^r}\{\psi(x-k)\} \equiv \psi_k^{(r)}$$

and the k^{th} translate of the 2^r scaled function

$$2^{r/2} \psi(2^r x - k) \equiv \psi_k^r$$

3 Wavelet element construction

For purposes of constructing wavelet-based finite elements, this paper considers a typical variational boundary value problem, symbolically characterized by

$$a(u, v) = f(v), \quad \forall v \in H \tag{3.1}$$

where $a(\cdot,\cdot)$ is a coercive, symmetric bilinear form over the Hilbert space H. As is well-known, the governing equations for any of a number of common problems in computational mechanics can be expressed in this form. In the derivations that follow, it will be sufficient to consider the potential equation

$$\int_\Omega a_{11}(x,y) \nabla u \cdot \nabla v \, dxdy + \int_\Omega a_{00}(x,y)\, uv \, dxdy$$

$$= \int_\Omega f(x,y)\, v \, dxdy + \int_\Gamma gv \, ds \tag{3.2}$$

In fact, because the derivation of the one-dimensional wavelet element facilitates a straightforward discussion of multidimensional, tensor product elements, the solution is assumed for the present to consist of a superposition of scaling functions at resolution J and wavelets at the same and higher levels

$$u(x,y) = U(x)\, V(y) \tag{3.3}$$

where

$$U(x) = \sum_k \alpha_k^J \phi_k^J(x) + \sum_{i \geqq J, k \in Z} \beta_k^i \psi_k^i(x)$$

$$V(y) = \sum_k \xi_k^J \phi_k^J(y) + \sum_{i \geqq J, k \in Z} \eta_k^i \psi_k^i(y) \tag{3.4}$$

The coefficient data is assumed to be expressed as

$$a_{rr}(x) = \sum_{p \in Z} a_{rr,p} N_p(x) \tag{3.5}$$

in terms of functions $N(x)$ satisfying a two-scale refinement equation

$$N(x) = \sum_{p \in Z} c_p N(2x - p) \tag{3.6}$$

Upon substituting Eqs. (3.4), (3.5) and (3.6) into the governing Eq. (3.2), it is clear that the wavelet Galerkin approximation can be constructed from the following integrals:

$$\int_\Omega N_p \nabla \phi_q^J \cdot \nabla \psi_r^l \, dx$$

$$\int_\Omega N_p \nabla \phi_q^J \cdot \nabla \phi_r^J \, dx$$

$$\int_\Omega N_p \nabla \psi_q^l \cdot \nabla \psi_r^m \, dx$$

$$\int_\Omega N_p \phi_q^J \psi_r^l \, dx$$

$$\int_\Omega N_p \phi_q^J \phi_r^J \, dx$$

$$\int_\Omega N_p \psi_q^l \psi_r^m \, dx \tag{3.7}$$

In conventional finite element methods, these integrals would be calculated in closed form or by standard (Gauss) quadrature formulae. This is not feasible, however, for many wavelet bases. In many cases, there is no closed form expression for the bases, and they often cannot be integrated numerically due to their unusual smoothness characteristics. However, because the wavelet $\psi(x)$ is defined in terms of the scaling function ϕ, these integrals can be re-written in terms of the scaling function alone. To this end, define the *generalized connection coefficients*

$$\Gamma^{0,0}_{l;p,r} = \int_{\Omega} N(x-l)\,\phi(x-p)\,\phi(x-q)\,dx \tag{3.8}$$

and

$$\Gamma^{1,1}_{l;p,r} = \int_{\Omega} N(x-l)\frac{d\phi}{dx}(x-p)\frac{d\phi}{dx}(x-q)\,dx$$

$$= \int_{\Omega} N(x-1)\,\phi^{(1)}(x-p)\,\phi^{(1)}(x-q)\,dx \tag{3.9}$$

Once these integrals have been calculated, they can be combined to form all the integrals in Eq. (3.7), and they will eventually form the entries of the wavelet-based elements. It should be noted that these terms are derived via a generalization of the strategy employed in Latto, Resnikoff and Tenenbaum (1991) for the calculation of the usual two term *connection coefficients*

$$\int_{R} \phi'(x-p)\,\phi'(x-q)\,dx \tag{3.10}$$

or three term connection coefficients

$$\int_{R} \phi(x-n)\,\phi'(x-p)\,\phi'(x-q)\,dx \tag{3.11}$$

One should note that the definition of the generalized connection coefficients defined in Eq. (3.11) differs from that considered in Latto, Resnikoff and Tenenbaum (1991) in that the constituents of the integrand do not necessarily satisfy the same two-scale refinement equation. In addition, integrals of the form calculated in this paper are studied in Dahmen and Micchelli (1993) in the context of stationary subdivision schemes. The presentation that follows extends the approach taken in Latto, Resnikoff and Tenenbaum (1991) to derive a simple algorithm for computing the integrals studied in Dahmen and Micchelli (1993).

As noted in Dahmen and Micchelli (1993) and Latto, Resnikoff and Tenenbaum (1991), the calculation of connection coefficients consists of two phases. First, the connection coefficients are shown to satisfy a set of homogeneous equations that are derived from repeated application of the refinement equation. This procedure leads to an algebraic eigenvalue problem. Generally, the connection coefficients are not uniquely determined by this eigenvalue problem. Additional inhomogeneous normalization conditions are consequently derived from moment conditions in Latto, Resnikoff and Tenenbaum (1991), or from more general polynomial expressions that are "dual" to differentiation in a sense made precise in Dahmen and Micchelli (1993).

For the remainder of this paper, it is assumed that

$$N(x) = \chi_{[0,1]}(x), \tag{3.12}$$

In other words, the coefficient data $a_{1,1}(x)$ and $a_{0,0}(x)$ are represented in terms of characteristic functions $\chi_{[0,1]}(x)$. As will be shown, this choice leads to tensor product elements based on wavelet functions. To simplify notation, the generalized connection coefficients

$$\Gamma^{0,0}_{j;k,s} = \int_{-\infty}^{\infty} \chi_{[0,1]}(x-j)\,\phi(x-k)\,\phi(x-s)\,dx \tag{3.13}$$

are expressed more concisely as

$$\Gamma^{0,0}_{k,s} = \int_{-\infty}^{\infty} \chi_{[0,1]}(x)\,\phi(x-k)\,\phi(x-s)\,dx \tag{3.14}$$

$$\Gamma^{1,1}_{k,s} = \int_{-\infty}^{\infty} \chi_{[0,1]}(x)\,\phi'(x-k)\,\phi'(x-s)\,dx \tag{3.15}$$

3.1 Homogeneous equations

By substituting the two-scale relationship satisfied by the scaling functions into Eq. (3.14), $\Gamma^{0,0}_{k,s}$ becomes

$$\Gamma^{0,0}_{k,s} = \sum_{l,m} a_l a_m \int_{-\infty}^{\infty} \chi_{[0,1]}(x)\,\phi(2x-2k-l)\,\phi(2x-2s-m)\,dx \tag{3.16}$$

which can be written as

$$\Gamma^{0,0}_{k,s} = \sum_{l,m} a_l a_m \int_{-\infty}^{\infty} \chi_{[0,1]}(\tfrac{1}{2}\xi)\,\phi(\xi-2k-l)\,\phi(\xi-2s-m)\,\tfrac{1}{2}\,d\xi \tag{3.17}$$

under a change of variable in the integration. The characteristic function $\chi_{[0,1]}(x)$ satisfies a trivial two-scale equation. This can be seen by writing the definition

$$\chi_{[0,1]}(\tfrac{1}{2}\xi) = \begin{cases} 1, & 0 \leqq \xi \leqq 2 \\ 0, & \text{otherwise} \end{cases} \tag{3.18}$$

as

$$\chi_{[0,1]}(\tfrac{1}{2}\xi) = \chi_{[0,1]}(\xi) + \chi_{[1,2]}(\xi) \tag{3.19}$$

or

$$\chi_{[0,1]}(\tfrac{1}{2}\xi) = \chi_{[0,1]}(\xi) + \chi_{[0,1]}(\xi-1) \tag{3.20}$$

When the two-scale relationship for the characteristic function is replaced in Eq. (3.17), one obtains

$$\Gamma^{0,0}_{k,s} = \frac{1}{2}\sum_{l,m} a_l a_m \int_{-\infty}^{\infty} \{\chi_{[0,1]}(\xi) + \chi_{[0,1]}(\xi-1)\} \cdot \phi(\xi-2k-l)\,\phi(\xi-2s-m)\,d\xi \tag{3.21}$$

which can be expressed in terms of the original generalized connection coefficients as

$$\Gamma^{0,0}_{k,s} = \frac{1}{2}\sum_{l,m} a_l a_m \{\Gamma^{0,0}_{2k+l,2s+m} + \Gamma^{0,0}_{2k+l-1,2s+m-1}\} \tag{3.22}$$

By changing the summation limits, Eq. (3.22) becomes

$$\Gamma_{k,s}^{0,0} = \frac{1}{2}\left\{\sum_{p,r} a_{p-2k} a_{r-2s} \Gamma_{p,r}^{0,0} + \sum_{\hat{p},\hat{r}} a_{\hat{p}-2k} a_{\hat{r}-2s} \Gamma_{\hat{p},\hat{r}}^{0,0}\right\}$$

$$= \frac{1}{2}\sum_{p,r}\{a_{p-2k} a_{r-2s} + a_{p-2k+1} a_{r-2s+1}\} \Gamma_{p,r}^{0,0} \tag{3.23}$$

But Eq. (3.23) is just an eigenvalue problem having the form

$$\Gamma^{0,0} = [A^{0,0}]\,\Gamma^{0,0} \tag{3.24}$$

The same essential steps can be followed to generate the generalized connection coefficients $\Gamma_{k,s}^{1,1}$

$$\Gamma_{k,s}^{1,1} = \int_{-\infty}^{\infty} \chi_{[0,1]}(x)\,\phi'(x-n)\,\phi'(x-m)\,dx \tag{3.25}$$

By differentiating the two-scale relationship

$$\phi'(x) = 2\sum_{k=0}^{2M-1} a_k \frac{d\phi}{dx}(2x-k)$$

where M is the order of wavelet system, and substituting the result into Eq. (3.25), one can write

$$\Gamma_{k,s}^{1,1} = 2^2 \sum_{r,p} a_r a_p \int_{-\infty}^{\infty} \chi_{[0,1]}(x)\,\phi'(2x-2k-r)\,\phi'(2x-2s-p)\,dx \tag{3.26}$$

Now, by employing the two-scale relationship satisfied by the characteristic function $\chi_{[0,1]}(x)$ in Eq. (3.18) through (3.20), the expression above reduces to

$$\Gamma_{k,s}^{1,1} = 2^2 \sum_{r,p} a_r a_p \int_{-\infty}^{\infty} \{\chi_{[0,1]}(2x) + \chi_{[0,1]}(2x-1)\}$$

$$\cdot\,\phi'(2x-2k-r)\,\phi'(2x-2s-p)\,dx \tag{3.27}$$

by changing variable

$$\Gamma_{k,s}^{1,1} = 2^{2-1} \sum_{r,p} a_r a_p \cdot \left\{\int_{-\infty}^{\infty} \chi_{[0,1]}(\xi)\,\phi'(\xi-2k-r)\,\phi'(\xi-2s-p)\,d\xi\right.$$

$$\left. + \int_{-\infty}^{\infty} \chi_{[0,1]}(\xi-1)\,\phi'(\xi-2k-r)\,\phi'(\xi-2s-p)\,d\xi\right\} \tag{3.28}$$

With a change of indices, this equation takes the same general form as Eq. (3.23) above

$$\Gamma_{k,s}^{1,1} = 2^{2-1} \sum_{p,r} a_p a_r \{\Gamma_{2k+r,2s+p}^{1,1} + \Gamma_{2k+r-1,2s+p-1}^{1,1}\}$$

$$= 2^{2-1} \sum_{p,r} \{a_{p-2k} a_{r-2s} + a_{p-2k+1} a_{r-2s+1}\} \Gamma_{p,r}^{1,1} \tag{3.29}$$

In fact, the coefficients $\Gamma_{k,s}^{1,1}$ also satisfy an eigenvalue problem

$$\Gamma^{1,1} = [A^{1,1}]\,\Gamma^{1,1} \tag{3.30}$$

Considering the derivation of $\Gamma^{1,1}$ and $\Gamma^{0,0}$, it is not difficult to show that the elliptic partial differential equation of order $2d$ will require the solution of an eigenvalue problem having the form

$$\Gamma^{d,d} = [A^{d,d}]\,\Gamma^{d,d} \tag{3.31}$$

The entries of the matrix $[A^{d,d}]$ can be derived in an analogous manner

$$\phi^{(d)}(x) = 2^d \sum_{k=0}^{2M-1} a_k \phi(2x-k) \tag{3.32}$$

$$\Gamma_{k,s}^{d,d} = 2^{2d} \sum_{r,p} a_r a_p \int_{-\infty}^{\infty} \chi_{[0,1]}(x)\,\phi^{(d)}(2x-2k-r)$$

$$\cdot\,\phi^{(d)}(2x-2s-p)\,dx \tag{3.33}$$

$$\Gamma_{k,s}^{d,d} = 2^{2d-1} \sum_{r,p} a_r a_p \{\Gamma_{2k+r,2s+p}^{d,d} + \Gamma_{2k+r-1,2s+p-1}^{d,d}\} \tag{3.34}$$

$$\Gamma_{k,s}^{d,d} = 2^{2d-1} \sum_{r,p} \{a_{p-2k} a_{r-2s} + a_{p-2k+1} a_{r-2s+1}\} \Gamma_{p,r}^{d,d} \tag{3.35}$$

3.2 Normalization equations

Unfortunately, the eigenvalue problems do not uniquely define the generalized connection coefficients required to form the constituent finite elements. As suggested in Dahmen and Micchelli (1993) and Latto, Resnikoff and Tenenbaum (1991), the polynomial reproducing property of wavelets is employed to generate a sufficient number of inhomogeneous equations to uniquely define the generalized connection coefficients. For example, it is well known that the wavelets derived by Daubechies (1992) satisfy

$$1 = \sum_{k\in Z} \phi(x-k) \tag{3.36}$$

By multiplying Eq. (3.36) by itself, and subsequently multiplying the product by the characteristic function of the interval [0, 1], one obtains

$$1 = \sum_{l,k\in Z} \phi(x-l)\,\phi(x-k) \tag{3.37}$$

$$\chi_{[0,1]}(x) = \sum_{l,k\in Z} \chi_{[0,1]}(x)\,\phi(x-l)\,\phi(x-k) \tag{3.38}$$

Now, a single integration yields a first normalization condition.

$$1 = \sum_{l,k} \Gamma_{l,k}^{0,0} \tag{3.39}$$

This procedure can be repeated inductively for many classes of wavelets. Specifically, Kurdila (1992) and Latto, Resnikoff and Tenenbaum (1991) note that the Daubechies wavelets of order $k+1$ can reproduce polynomials up to degree k

$$x^k = \sum_{r} M_r^k \phi(x-r) \tag{3.40}$$

$\Gamma_{k,l}^{1,1}$ for Daubechies wavelet system of order 3

k/l	−4	−3	−2	−1	0
−4	0.00019835924333573095	0.0015307947615466255	−0.004779464493845208	0.008407453346105678	−0.00535714285714285
−3	0.0015307947615466255	0.04602360988387823	−0.12829957129683345	0.20343833428322815	−0.12269316763181979
−2	−0.004779464493845208	−0.12829957129683345	0.4673646354266379	−1.0118172060370498	0.6775316064010917
−1	0.008407453346105678	0.20343833428322815	−1.0118172060370498	3.051861626311563	−2.2518902079038489
0	−0.00535714285714285	−0.12269316763181979	0.6775316064010917	−2.2518902079038489	1.7024089119917204

$\Gamma_{k,l}^{1,1}$ for Daubechies wavelet system of order 6

k/l	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
−10	4.7343460e-16	−1.5132171e-13	−2.1480720e-12	4.33702666e-11	−7.7183084e-11	−2.8848766e-10	1.264158329e-9	−2.39699804e-9	6.264319877e-9	−4.81952181e-9	1.26410529e-11
−9	−1.5132171e-13	1.00630726e-10	−5.9243478e-10	−3.49307204e-8	1.153015531e-7	5.070212545e-8	−7.44541830e-7	1.675602257e-6	−4.28732666e-6	3.247165523e-6	−2.14802891e-8
−8	−2.1480720e-12	−5.9243478e-10	9.018343374e-8	3.789094928e-7	−2.87438108e-6	4.844234533e-6	−1.18209388e-6	−5.56087201e-6	2.514474220e-5	−2.10527843e-5	2.126562011e-7
−7	4.33702666e-11	−3.49307204e-8	3.789094928e-7	2.313020814e-5	−4.41021153e-5	−2.28764308e-4	9.863110702e-4	−1.80316849e-3	3.465698116e-3	−2.47915438e-3	7.970588704e-5
−6	−7.7183084e-11	1.153015531e-7	−2.87438108e-6	−4.41021153e-5	2.624578973e-4	−4.88083308e-4	1.524550400e-4	1.099486573e-3	−4.50474045e-3	1.007256465e-3	2.518029062e-3
−5	−2.8848766e-10	5.070212545e-8	4.844234533e-6	−2.28764308e-4	−4.88083308e-4	7.646145971e-3	−2.40153852e-2	4.153782819e-2	−7.07569817e-2	5.724505575e-2	−1.09447099e-2
−4	1.264158329e-9	−7.44541830e-7	−1.18209388e-6	9.863110702e-4	1.524550400e-4	−2.40153852e-2	9.033227446e-2	−1.85328192e-1	3.557865848e-1	−2.36337717e-1	−1.57440468e-3
−3	−2.39699804e-9	1.675602257e-6	−5.56087201e-6	−1.80316849e-3	1.099486573e-3	4.153782819e-2	−1.85328192e-1	4.454822848e-1	−9.23238547e-1	5.221557233e-1	1.000984730e-1
−2	6.264319877e-9	−4.28732666e-6	2.514474220e-5	3.465698116e-3	−4.50474045e-3	−7.07569817e-2	3.557865848e-1	−9.23238547e-1	1.992128731948	−1.06423393399	−2.88667674e-1
−1	−4.81952181e-9	3.247165523e-6	−2.10527843e-5	−2.47915438e-3	1.007256465e-3	5.724505575e-2	−2.36337717e-1	5.221557233e-1	−1.06423393399	8.371792787e-1	−1.14518697e-1
0	1.26410529e-11	−2.14802891e-8	2.126562011e-7	7.970588704e-5	2.518029062e-3	−1.09447099e-2	−1.57440468e-3	1.000984730e-1	−2.88667674e-1	−1.14518697e-1	3.130090878e-1

Quadratures for Various Order Daubechies wavelet system

Order	Quadratures					
	0	1	2	3	4	5
3	0.107970582852975	0.96665766648317	−0.07462824933614471			
4	0.01636789445764397	0.942821039835326	0.05986102351055656	−0.01904995780352725		
5	−0.00654676934866219	0.7273335649472294	0.39131003592382	−0.1323417073140668	0.02024487579167997	
6	−0.007211626366079303	0.4783825328750095	0.7349307603183212	−0.2675090764248883	0.07059355319669399	−0.00918614359905598

Explicit formula for calculating the coefficients M_r^k recursively can be found in Latto, Resnikoff and Tenenbaum (1991). By differentiating this expression once,

$$kx^{k-1} = \sum_r M_r^k \phi'(x-r) \tag{3.41}$$

squaring it,

$$klx^{k-1}x^{l-1}\chi_{[0,1]}(x) = \sum_r M_r^k M_s^l \chi_{[0,1]}(x)\phi'(x-r)\phi'(x-s) \tag{3.42}$$

and integrating

$$kl\int_0^1 x^{k+l-2}\,dx = \sum_{r,s} M_r^k M_s^l \Gamma_{r,s}^{1,1} \tag{3.43}$$

a second set of normalization conditions are derived.

$$\frac{kl}{k+l-1} = \sum_{r,s} M_r^k M_s^l \Gamma_{r,s}^{1,1} \tag{3.44}$$

This procedure can be extended to normalize the generalized connection coefficients of order d.

$$\Gamma_{l,k}^{d,d} = \int_{\Re^n} \chi_{[0,1]}(x)\phi^{(d)}(x-l)\phi^{(d)}(x-k)\,dx \tag{3.45}$$

By differentiating the polynomial expansion

$$x^k = \sum_s M_s^k \phi(x-s) \tag{3.46}$$

d times, one obtains

$$k\cdot(k-1)\cdots(k-d)\,x^{k-d} = \sum_s M_s^k \phi^{(d)}(x-s) \tag{3.47}$$

$$\frac{k!}{(k-d-1)!}x^{k-d} = \sum_s M_s^k \phi^{(d)}(x-s) \tag{3.48}$$

This expression can be multiplied by itself and integrated

$$\sum_{r,s} M_r^k M_s^l \Gamma_{r,s}^{d,d} = \frac{k!}{(k-d-1)!}\frac{l!}{(l-d-1)!}\int \chi_{[0,1]}(x)\,x^{k-d}x^{l-d}\,dx \tag{3.49}$$

to derive sufficient normalization conditions for the coefficients $\Gamma_{r,s}^{d,d}$

$$\sum_{r,s} M_r^k M_s^l \Gamma_{r,s}^{d,d} = \frac{k!}{(k-d-1)!}\frac{l!}{(l-d-1)!}\frac{1}{(k+l-2d+1)} \tag{3.50}$$

3.3
Wavelet element approximation properties

To employ the wavelet-based finite elements derived in the last section in practical computational mechanics problems, it is necessary to discuss their approximation properties. However, because of the simple geometry of the class of derived elements, a succinct characterization of their approximation power is possible using the Strang-Fix condition in Strang and Fix (1971).

Theorem 1. *Let $\phi(x)$ generate a multiresolution analysis and suppose its translates span polynomials of degree $\leqq M-1$. Suppose that Ω is the union of elemental domains*

$$\Omega = \bigcup_{r,s\in\Lambda^j} \Omega_{r,s}^j$$

where

$$\Omega_{r,s}^j = \{(x,y)\,|\,2^{-j}r \leqq x \leqq 2^{-j}(r+1)\ and\ 2^{-j}s \leqq y \leqq 2^{-j}(s+1)\}$$

Let $\mathscr{V}_j = span\{\phi_k^j(x)\}_{k\in Z} \cap \Omega$. Smooth functions can be approximated from $\mathscr{V}_j$ with error $O(2^{-j(M-s)})$ in $H^s(\Omega)$ using the wavelet-based elements derived in this section, where M is the order of the underlying Daubechies wavelet.

Proof. This theorem is a trivial application of the Jackson estimate for shift invariant spaces on R^2 and the fact that Ω is $C^{0,1}$. □

These error estimates will not be realized in practice, however, unless all right hand side terms are calculated with commensurate accuracy. This provides another slight difficulty for wavelet-based elements in that usual quadratures will not suffice. Beylkin, Coifman and Rohklin (1991) introduced a means of calculating these terms for the Coifman wavelets, and outline an extension that can be used for the wavelet-elements in this paper that are based on Daubechies wavelets. For a given function $f(x)$, the right hand terms of the wavelet-galerkin formulation that must be calculated typically have the form

$$\begin{aligned} L_k^j(f) &= 2^{j/2}\int f(x)\phi(2^jx-k+1)\,dx \\ &= 2^{j/2}\int f(\xi+2^{-j}(k-1))\phi(2^n\xi)\,d\xi \end{aligned} \tag{3.51}$$

where $L_k^j(f)$ denotes a linear functional on f defining a mapping from

$$L_k^j: C[a,b] \to R$$

The approximation of this family of linear functionals $\{L_k^j(f)\}_{j,k\in Z}$ can be achieved using the following

Theorem 2. *Let the linear functional $L: C[a,b] \to R$, and the quadrature points and weights*

$$a = x_0 < x_1 \cdots x_k < x_{k+1} = b$$

$$a_0, a_1 \cdots a_k, a_{k+1}$$

be given. Suppose that the linear functional is polynomial reproducing in that

$$L(p) - \sum_{i=0}^{k+1} a_i p(x_i) = 0$$

for all $p \in \mathscr{P}_{M-1}$. *Then for all* $f \in C^M[a,b]$, *remainder form*

$$L(f) - \sum_{i=0}^{k+1} a_i f(x_i) = \int_a^b K(u) f^{(M)}(u)\, du$$

where the Peano kernel $K: [a,b] \to R$ *is defined by*

$$K(u) = \frac{1}{r!} L(\cdot - u)_+^{M-1} - \sum_{i=0}^{k+1} a_i (x_i - u)_+^{M-1}$$

for all $u \in [a,b]$.

When applied to calculating the right hand terms in Eq. (3.2), this theorem yields a family of functionals characterized by

$$L_k^n(f) = 2^{-n/2} \sum_{l=0}^{M-1} c_l (2^{-n} l + 2^{-n}(k-1))$$

$$x_{l;n,k} = 2^{-n} l + 2^{-n}(k-1) \tag{3.52}$$

$$a_{l;n} = 2^{-n/2} c_l$$

The equations that must be satisfied to define the quadrature weights reduce to

$$\sum_{l=0}^{M-1} c_l = \int_R \phi(x)\, dx = M_0^0$$

$$\sum_{l=0}^{M-1} l c_l = \int_R x \phi(x)\, dx = M_0^1 \tag{3.53}$$

$$\sum_{l=0}^{M-1} l^2 c_l = \int_R x^2 \phi(x)\, dx = M_0^2$$

$$\vdots$$

To demonstrate the utility of these quadrature formulae, the quasi-interpolation projection in de Boor (1993) is

$$Q^j f = \sum_k L_k^j(f) \phi_k^j(x)$$

Because the family of functionals defines a "good quasi-interpolant sequence of order M" in the sense of de Boor (1993), it is possible to obtain an explicit bound for the Peano kernel $K(x)$. One can write

$$\| f(x) - Q^j f(x) \|_\infty \leqq C 2^{-jM} \| f^{(M)} \|_\infty$$

It is important to note that this convergence rate is consistent with that guaranteed by the Strang-Fix condition for the Daubechies wavelet element of order M.

4
Numerical examples

The first example has been selected solely to verify the approximation properties of the Daubechies wavelet-based elements and quasi-interpolation formula derived in the last section. For this study, a one dimensional, second order Neumann problem has been considered. The problem is to find $u \in H^1(0,1)$ such that

$$-\frac{d^2 u}{dx^2} + u = f, \quad \forall x \in [0,1]$$

where

$$f(x) = (1 + 4\Pi^2) \cos(2\Pi x)$$

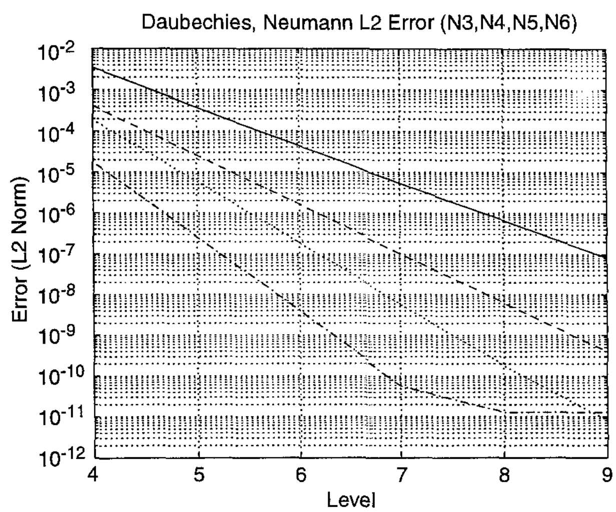
and

$$u'(0) = 0$$

$$u'(1) = 0$$

As shown in Figs. 2 and 3, the rate of convergence matches that predicted by the Strang-Fix condition and the quasi-interpolation order. One point of interest in these results is the sub-optimal rate of convergence for the Daubechies wavelet-based element corresponding to $M = 6$. This is expected, however, due to the numerical error induced by truncating the generalized connection coefficients to a fixed number of accurate digits. This phenomenon is studied in more detail in Ko and Kurdila (1993).

The second example considers a two dimensional Neumann problem on a multiply connected, irregular domain. Again, order $M = 3$ Daubechies wavelet-based elements are utilized, although in this case tensor product elements are employed. We consider a potential flow over two closely placed Joukowski



;. 1. L_2 Error versus multiresolution error, Daubechies order 3,4,5,6 ments

Rate of decrease of error in L_2 norm(2^{xj})

Wavelet Order	Expected	Computed
3	−3	−3.0655
4	−4	−3.9959
5	−5	−4.9241
6	−6	−6.0805

Fig. 2. Daubechies wavelet elements, order 3,4,5,6 L_2 Error approximation rate, theoretical versus calculated

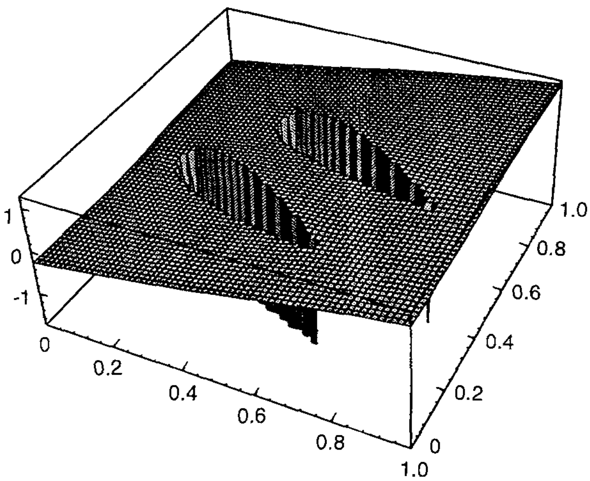


Fig. 3. Neumann problem, two dimensions, Daubechies order 3 wavelet elements, 64 × 64 DOF

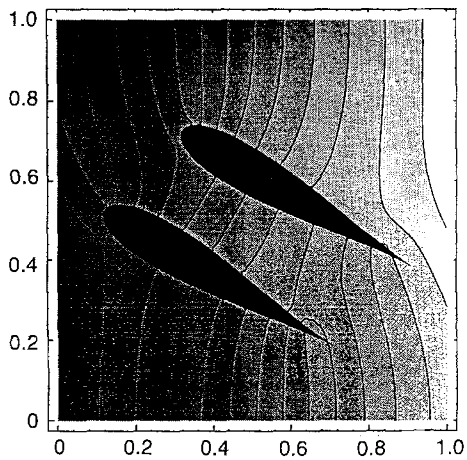


Fig. 5. Neumann problem, two dimensions, Daubechies order 3 wavelet elements, 64 × 64 DOF, potential contours

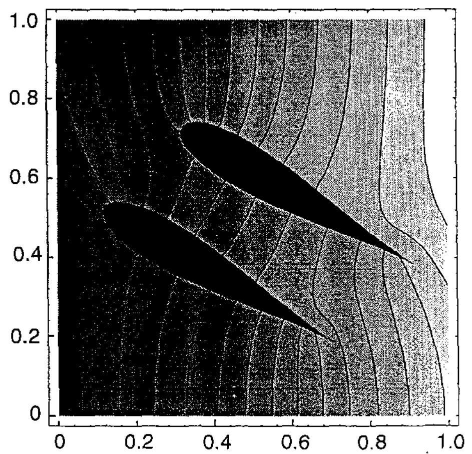


Fig. 6. Neumann problem, two dimensions, Daubechies order 3 wavelet elements, 128 × 128 DOF, potential contours

airfoils. The airfoils are constructed so that they are contained in $[0, 1] \times [0, 1]$ square. The governing equation is

$$\nabla^2 \Phi = 0$$

and the boundary conditions are

$$\frac{\partial \Phi}{\partial n} = \begin{cases} -1, & \text{on } x = 0 \\ 1, & \text{on } x = 1 \\ 0, & \text{on } y = 0, y = 1, \text{ and on airfoils} \end{cases}$$

where Φ is the velocity potential of the flow. The numerical results are shown in Figs. 3 through 6. Figures 3, and 5 are the results of 64 × 64 square elements, and Figs. 4, and 6 are of 128 × 128 square elements.

In as much as these examples have been selected solely to illustrate that wavelet bases can be implemented as conventional finite elements, the Kutta condition has not been enforced in these simulations. That is, the model problem has zero circulation as shown at the trailing edge in Figs. 5 and 6. Clearly this is a feature of the simplicity of the model problem, and not of the wavelet elements.

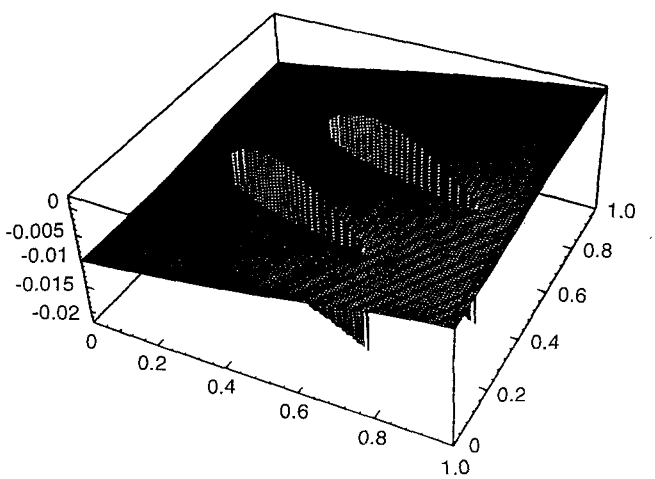


Fig. 4. Neumann problem, two dimensions, Daubechies order 3 wavelet elements, 128 × 128 DOF

5
Conclusions

This paper has presented the construction of a new class of finite element techniques that are formulated in terms of wavelet basis functions. These elements can be extended using tensor products to represent a class of irregular domains in higher dimensions. Because of the unusual smoothness properties of the Daubechies wavelets employed, specialized techniques for element evaluation and quadratures are derived. In addition to the derivation of a construction technique for wavelet elements, a characterization of the approximation properties of the elements is derived using the Strang-Fix condition. The accuracy of quadratures for the corresponding element calculations is established by constructing a quasi-interpolation scheme associated with the elements.

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