M. A. (KEN) CLEMENTS

ANALYZING CHILDREN'S ERRORS ON WRITTEN MATHEMATICAL TASKS

Abstract. Methods developed by Newman and Casey for analyzing errors made by children attempting verbal arithmetic problems are described, with particular emphasis being given to Newman's hierarchy of error causes. Data obtained by Newman, Casey, and Clements are presented. These show that a large proportion of errors made by children in grades 5–7 in Victoria on verbal arithmetic problems are in the Newman categories 'Comprehension', 'Transformation', 'Process Skills', and 'Carelessness'.

1. THE SOURCES OF ERROR[1]

The sixth-grade class at the local parochial Catholic school had been asked to do a maths test which contained forty questions, some of which were multiple-choice. After ten minutes Jane raised her hand. When Sister Anastasi approached her and asked what she wanted, Jane pointed to the question shown in Figure 1 and asked: "Sister, when it says here 'Which angel is a right angel, does it mean that the wings should go this way, or that way?'"

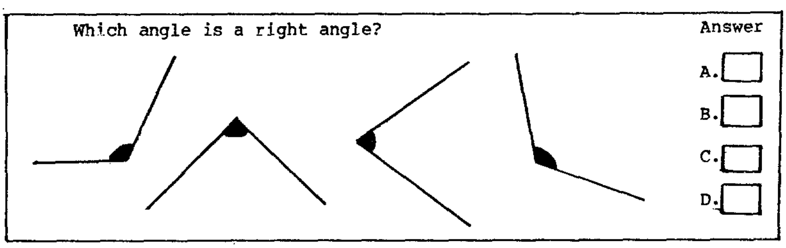

Fig. 1

Later, when correcting the tests, Sister Anastasi noticed that Billy had given the answer '96 hours' for the question "What does fifty-six minus forty equal?". Puzzled, Sister asked Billy how he had obtained his answer, and Billy replied: "Well, Sister, it says 'What does fifty-six minutes forty equal?' It didn't tell me what I had to do, so I added and got ninety-six. Now ninety-six is more than sixty, so the answer must be in hours".

Both Jane and Billy had made *reading* errors. They had misread important words in questions, and this had prevented them from proceeding further.

Charles, on the other hand, read the question in Figure 2 perfectly, but

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Sam goes to bed at 10 minutes to 9.
John goes to bed 15 minutes later than Sam.
What time does John go to bed?

Fig. 2.

wrote '15' for his answer. On being asked why he gave this answer he explained that "it says John goes to bed fifteen minutes later, so the answer must be '15'". Charles could read the words but had not grasped the meaning of all the information given in the question. He had not been able to proceed towards the solution of the problem because of a *reading comprehension* difficulty.

Here are some children.

I have 24 lollies and I want each child to have the same number of lollies. How many lollies will I give each child?

Fig. 3.

For the question shown in Figure 3, John wrote '144' as his answer. He explained that "there are twelve children, and twenty-four lollies; 12 into 24 goes 2, so we have two twelves; you multiply these two twelves; 12 times 12 is 144". When asked what the answer to the question was, John said: "Each child gets 144 lollies". John could read the question well, and knew that he had to find out how many lollies each child should get. He failed to solve the problem correctly because he did not formulate a correct sequence of mathematical steps: he did not *transform* from the written problem to an acceptable ordered set of mathematical procedures.

Percy obtained an answer of 'one' for the question shown in Figure 3. He reasoned: "I would give each child one lolly and keep twelve for myself". A careful analysis of the wording of the question suggests that Percy's solution should be regarded as correct, yet he was 'marked wrong'. Percy's 'error' arose because of the *form of the question*: in fact, the question is poorly worded because it allows two possible correct answers.

Elaine wrote down '$1.93' as her answer for the problem shown in Figure 4. She employed a faulty algorithm, and her error was due to a weakness in *process skills.* Kelvin, on the other hand, was 'marked wrong' because he simply wrote '93' for his answer. He made an *encoding* error, because he failed

Jim buys the box of flour.
He pays the shopkeeper \$2.00.
How much change should he get?

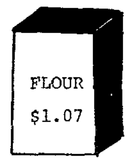

Fig. 4

to present his answer in an acceptable written form. Mary wrote down '83 cents' for her answer, but when she was asked how she got it she immediately said "Oh, I made a mistake. It should be 93 cents". Mary can be said to have made a *careless* error (which is just another way of saying we don't know why she made the initial error).

Five minutes before she had handed the mathematics test paper out to her class Sister Anastasi had reproached Jim, a bright pupil, for talking. Jim was sulking when he got the test paper and refused to attempt any questions seriously even though he could have done all of them if he had tried. Jim simply wrote down 'random' answers, and got no correct answers. His errors were caused by a lack of *motivation*.

2. THE NEWMAN AND CASEY HIERARCHIES

The description of errors in the previous section was based on the error categories defined by Newman (1977). Newman assumed that associated with any given word problem are a number of hurdles which have to be overcome if a correct solution is to be obtained, and that failure on any particular hurdle prevents a person from progressing to the next hurdle and from obtaining the correct solution. In this sense Newman defined a hierarchy of error causes which, she claimed, applies to one-step written mathematical problems. The hierarchy has five levels, as shown in Figure 5. Note that Newman is using the word 'hierarchy' in a slightly different sense from the way it is used in literature on 'learning hierarchies'.

According to Newman a person confronted with a one-step written problem has to read the problem, then comprehend what he has read, then carry out the transformation from the words to the selection of an appropriate mathematical 'model', then apply the necessary process skills, then encode the answer. Failure at any level of the hierarchy prevents the person from obtaining the correct answer, unless, of course, he happens, by chance, to arrive at the 'correct' answer by faulty reasoning.

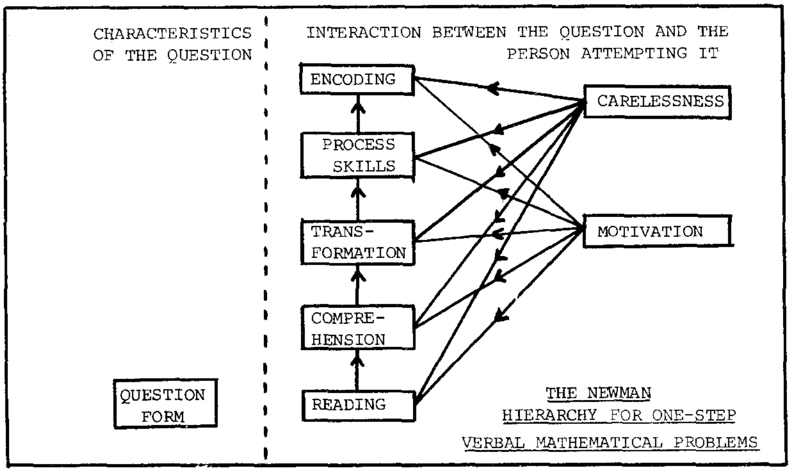

Fig. 5.[2]

Errors due to the form of the question would appear to be essentially different from errors in the other categories shown in Figure 5 because the 'fault' lies with the person who constructed the question rather than with the person attempting it. This difference is indicated in Figure 5 by the category labelled 'Question Form' being placed beside the five-stage hierarchy. Two other categories, 'Carelessness' and 'Motivation' have also been shown as separate from the hierarchy although, as indicated, these kinds of errors can be associated with any level of it. A careless error, for example, could be a reading error, or a comprehension error, and so on; similarly, someone might decide not to try to get the correct answer for a problem even after he has read and comprehended it satisfactorily, and carried out an appropriate 'transformation', while another person might refuse even to read the question properly.

Casey (1978), by modifying and extending Newman's hierarchy of error causes, has produced a more general 'hierarchy' which can be applied to the analysis of errors made on many-step verbal problems in mathematics. Casey's hierarchy is illustrated in Figure 6, in which the flow chart idea has been used rather loosely. Casey's approach emphasizes that anyone who attempts to solve a many-step problem has to identify, sequence, and solve an appropriate set of subproblems. In moving towards the overall solution the person often returns to lower stages of the hierarchy, not only after each subproblem has been solved but also while attempting to solve any particular subproblem. For example, in the middle of a complicated calculation someone might decide to re-read the question in order to check that all relevant information has been taken into account.

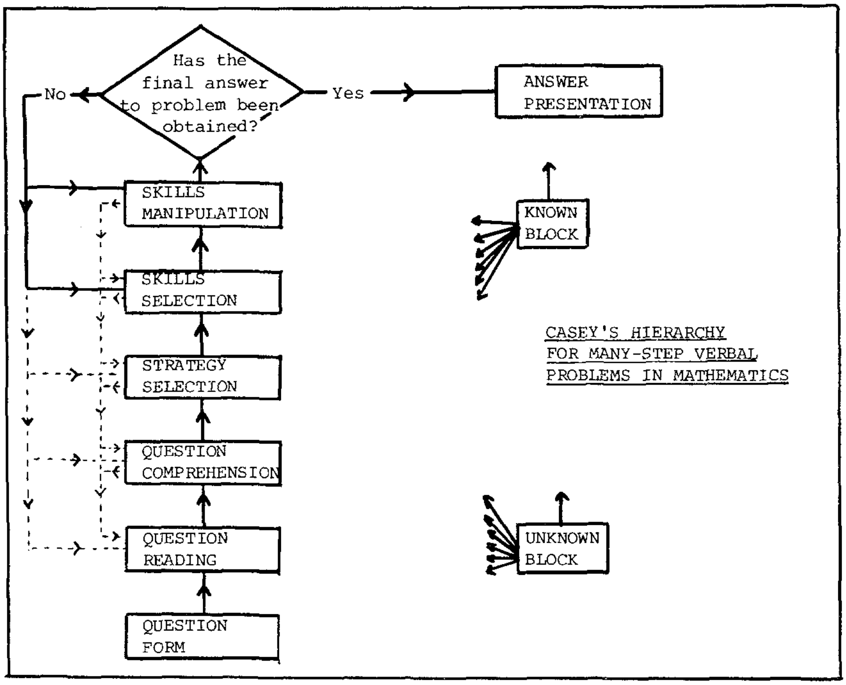

Fig. 6[3]

A comparison of Figures 5 and 6 will reveal that, unlike Newman, Casey prefers to include the category 'Question Form' in the hierarchy because this is the first point of interaction between the written task and the person attempting it. Casey also redefined Newman's 'Transformation' category in terms of 'Strategy Selection' and 'Skills Selection'; this was necessitated by the fact that he was concerned with many-step problems but Newman was concerned with one-step problems only. Casey also called the error categories outside his hierarchy by the names 'Known Block' and 'Unknown Block', whereas Newman used 'Motivation' and 'Carelessness' categories for those outside her hierarchy. Casey's 'Known Block' category could include Newman's 'Motivation', and his 'Unknown Block' would include 'Carelessness'.

3. THE ERROR ANALYSIS LITERATURE

In an important recent review Sheila K. Hollander (1978) has summarized the methods and results of a number of American researchers who have investigated the thought processes employed by children who are attempting verbal arithmetic problems. In most of the research described in Hollander's review, error categories were defined and pupils' errors analyzed in terms of these definitions. Typically, about 100 children in elementary or junior secondary grades were asked to try some one-step or two-step verbal arithmetic problems, and their responses to these were then examined. Often, structured interviews were conducted in order that pupils' thought processes could be clarified. While it would not be appropriate here to repeat Hollander's summaries, it will be in order to identify some of the researchers' conclusions which are most pertinent to the present paper.

In a study conducted in the 1920's C. S. Rice (see Hollander, p. 328) concluded that the main difficulty which children in grades 3 to 8 experienced when attempting verbal arithmetic problems was the choice of the appropriate mathematical operation. In 1930 L. John, who defined four error categories (Reasoning, Fundamentals, Reading, and 'Miscellaneous'), reported that 'Reasoning' errors were most common with pupils in grades 4 to 6, followed by errors due to an absence of 'Fundamentals' (see Hollander, p. 328). R. A. Doty, in 1940, claimed that the four main factors which affected the success of pupils in grades 4 to 6 on verbal arithmetic problems were (i) ability in computation, (ii) ability to gain mathematical implications from language forms, (iii) understanding of the various mathematical processes, and (iv) effective procedures with problems (see Hollander, p. 329). In the early 1950's C. G. Corle observed sixth-grade pupils as they worked one and two-step problems, and, after they had provided a written solution to the problem, noted their responses to the following questions:

1. What is your answer?
2. How did you work the problem?
3. Why did you work it that way?
4. Do you think your answer is reasonable?
5. Why do you think so?
6. Do you think your answer is right?
7. Why do you think so?

Corle classified data which were gathered into three major areas: insight, thought processes and number relationships, and computational skills. He concluded that good problem solvers tended to have superior insights, although some pupils who consistently got wrong answers were also capable of great insight; also, good problem solvers and poor problem solvers were similar in their employment of desirable computational procedures. Therefore, according to Corle, with the exception of the ability to solve more problems accurately, individual characteristics of good and poor problem solvers bear close resemblance (see Hollander, pp. 330–1). For reasons to be given later the present writer would disagree strongly with this claim.

Hollander praised Corle for avoiding limitations found in earlier studies in which researchers concentrated more on categorizing errors than on analyzing the thought processes which led to the errors. She criticized Corle, on the other hand, for the vagueness of her three major categories. In her own doctoral study, presented in 1973, Hollander investigated the strategies of sixth-graders reading and working verbal arithmetic problems, and concluded that successful problem-solving strategies could be attributed to five main factors:

1. A pupil's comprehension of mathematical relationships as expressed through the words and symbols within a problem;
2. the strength of a pupil's ability to employ abstract analytical reasoning;
3. the strength of a pupil's ability to reason insightfully;
4. the number of times a pupil refers to the text of the question, relative to his peers;
5. a pupil's ability to identify the minimum number of computational steps necessary for the solution of a problem.

Hollander maintains that a pupil's ability to note the information given or the information required in problems is *not* an important predictor of his problem-solving performance. Also, according to Hollander, the ability to read questions with a high degree of accuracy does not appear to be important (pp. 332–3).

There have been a number of American error analysis studies published in which researchers have relied completely on data obtained from pupils' written scripts. Papers by Arthur (1950), Roberts (1968), Cox (1975), and Knifong and Holtan (1976), are of this variety, but it is likely that such studies will never advance our understanding of why children make mistakes on written mathematical tasks. Consider, for example, the child who writes '11.55' as an answer to a question which required him to state the time 115 minutes ago, if it is now 1.00 p.m. Did this child convert 115 minutes to 1 hour and 55 minutes, take the 1 hour from 1.00 p.m. to get 12.00, and then, realizing that there was another 55 minutes to be accounted for, say, it must be before 12.00, so it's 11.55? This is a possible explanation, but there are many others, and it is obvious that any inferences about a child's thinking drawn from his written responses alone represent little more than guesswork on the part of the researcher. Written responses can suggest to a researcher, or teacher, reasons why a child is making errors, but structured interviews must be conducted with the child before consistent patterns of errors can be determined with any degree of certainty.

F. G. Lankford (1974) has reported a study in Virginia in which 176 seventh-grade pupils were interviewed for 40 to 50 minutes each for the purpose of determining each individual's pattern of thinking, or 'computational strategy', as Lankford called it (p. 26). Lankford concluded, among other things, that poor computers "have difficulty remembering the conventional operational algorithms, especially in fractions", and that "they have difficulty in matching those they do remember with the right exercise" (p. 29). Lankford also comments:

> Through individual oral interviews with several pupils in a class a teacher will soon become aware, maybe to his surprise, of wide variations in computational strategies employed by his pupils. He can then plan his teaching either to encourage and reinforce those variations that are acceptable or to try to achieve more uniformity in computational practice for his pupils by getting them to adopt and use a single strategy approved by him (p. 32).

But, as Lankford himself recognizes, many busy teachers possess neither the time nor the knowledge of interview techniques to conduct effective interviews with their pupils.

The German mathematics educator, Hendrik Radatz (1979), in drawing attention to many European error analysis studies which have been rarely discussed in the American literature, commented that error analysis research in different countries has been characterized by very different starting points and interests. Radatz claims that American research on errors has long been oriented towards behaviourism whereas in Europe aspects of Gestalt theory and ideas of pedagogical reformers have been influential. Radatz himself proposed an information-processing classification of errors, and delineated five main categories, consisting of errors due to pupils'

1. language difficulties;
2. difficulties in obtaining spatial information;
3. deficient mastery of prerequisite facts and concepts;
4. incorrect associations or rigidity of thinking;
5. application of irrelevant rules or strategies.

According to Radatz it is often difficult to make a sharp separation among the possible causes of a given error because there is such a close interaction among causes. This view led him to conclude that since "the same error can arise from different problem-solving processes" a definite classification and hierarchy of error causes seems impossible to achieve (p. 171).

Despite Radatz's pessimism, this writer believes that Newman's error hierarchy and Casey's extension and refinement of it have provided data of a kind not to be found in earlier error analysis research. Some of this data will now be given.

4. DATA ARISING FROM ANALYSES BASED ON THE NEWMAN AND CASEY HIERARCHIES

4.1. *The Newman Data*

In 1976 Newman gave a 40-item mathematics test containing numerical, spatial, and logical questions, to 917 grade 6 pupils in 31 classes in 19 schools in Melbourne. The pupils' scripts were quickly marked, and within a fortnight of the tests being given four of the five lowest performing children in each of the 31 classes were interviewed by Newman or a Diploma in Education mathematics method students at Monash University (each of the latter was a mathematics graduate, and had been given four hours' training in interview technique by Newman). Altogether, then, 124 pupils were interviewed, the interviews being structured according to an 'error analysis guideline' drawn up by Newman. According to this guideline the interviewer would ask a pupil who had originally given an incorrect answer to a question to attempt the question once again; the interviewer would wait until the pupil, unassisted, had attempted the question once again, and would then ask the pupil to respond to the following questions or requests:

1. Please read the question to me. If you don't know a word leave it out.
2. Tell me what the question is asking you to do.
3. Tell me how you are going to find the answer.
4. Show me what to do to get the answer. Tell me what you are doing as you work.
5. Now write down the answer to the question.

Each of these requests corresponds to a level of the Newman hierarchy (see Figure 5). If a pupil who originally got a question wrong got it right when the interviewer asked him to do it once again the interviewer still made the five requests in order that information might be obtained on whether the original error had been due to carelessness or motivational factors.

The 124 low achievers interviewed by Newman and her assistants had originally made a total of 3002 errors on the 40-item test. The low achievers' teachers had agreed not to answer any questions from pupils in relation to the test until the diagnostic interviews had been conducted, and over seventy per cent of the errors originally made by pupils were repeated during the interview sessions. On average, each interview session lasted two hours, so almost 250 hours of interviewing took place.

For each error the interviewer's task was to determine where, it seemed,

TABLE I
Newman's classification of 3002 errors made by 124 sixth grade, low achievers (Melbourne, 1976)

Error Category	Number of errors in this category	Percentage of errors in this category
Reading	390	13
Comprehension	665	22
Transformation	361	12
Process skills	779	26
Encoding	72	2
Carelessness or motivation	735	25
Total	3002	100

the pupil first broke down in his progress towards the solution of the question. Thus, if a pupil demonstrated he could not read an essential word in a question and, in the opinion of the interviewer, this prevented the pupil from comprehending the question, then the error was classified as a 'Reading' error; if, on the other hand, the pupil read and comprehended a question, but could not identify an appropriate method for solving it the error was regarded as a 'Transformation' error. Notice, for example, that if a person's error was classified as a 'Reading' or 'Transformation' error then it is possible that the person could have correctly applied the process skills demanded by the question if he had got to the stage where he needed to apply them. In the Newman study the interviewers did not attempt to discover whether the pupils could have obtained the correct solutions to questions if they had not failed at particular hurdles on the way to the solutions. Newman classified the 3002 errors as shown in the second and third columns of Table I. She did not attribute any error to 'Question Form', and, for the purposes of her analysis decided not to discriminate between errors due to 'Carelessness' or 'Motivation'.

From Table I it can be seen that 47 per cent of the errors in the Newman study occurred before the pupils got to the point of using the process skills necessary to solve the problems. It should also be noted that Newman considered that only 16 of the 40 items on her test involved the ability to transform,[4] and that 25 per cent of the errors made on these 16 items first occurred at the transformation stage. Also, Newman, who is a lecturer in reading education at a tertiary college, did not include any questions on her test which, she believed, contained words or ideas which were obviously difficult for primary school children.

TABLE II

Classification of 5158 errors made by 542 children (Grades 5–7), Melbourne, 1977–1979

	Grade 5 ($n = 55$)		Grade 6 ($n = 207$)		Grade 7 ($n = 280$)	
Error category	No. of errors	Percentage of errors	No. of errors	Percentage of errors	No. of errors	Percentage of errors
Reading	53	8	121	5	50	2
Comprehension	86	14	201	8	187	9
Transformation	170	27	636	25	538	27
Process skills	173	27	794	32	523	26
Encoding	12	2	40	2	19	1
Carelessness or motivation	143	22	707	28	705	35
Total	637	100	2499	100	2022	100

4.2. *The Clements Data*

Since 1977 the present writer has trained teachers attending many in-service education courses to use Newman's 'error analysis guideline'. Eighteen of these courses have been conducted in schools during school hours, and teachers attending the courses have been required to interview pupils in the schools with a view to classifying, according to the Newman hierarchy, the errors they had made on a 36-item test known as the *Monash Assessment of Mathematical Performance* (MAMP) test. This test, which was constructed by the present writer, contains 20 numeration items, 8 spatial items, and 8 logical items, and has been used in large research projects involving children aged between 9 and 14 years (the questions are mostly of the one-step variety, and the difficulty of the test is such that fifth-grade children in Victoria have averaged 23 questions correct out of 36, and eighth-graders have averaged 28 correct). The teachers attending the in-service courses interviewed 542 pupils altogether, 55 of whom were in grade 5, 207 in grade 6, and 280 in grade 7. The 542 pupils were from 21 classes, and in most cases every pupil in a class was interviewed. Thus, in contrast to the Newman data, which was based on interviews of low achievers, a normal range of children (so far as mathematical ability is concerned) provided the data shown in Table II.

Table II suggests that reading and reading comprehension difficulties cause fewer errors in higher grades, but errors occurring at the 'Transformation' and 'Process Skills' stages, and errors due to carelessness or motivation, are still common in grade 7.

The present writer has obtained another set of data over the period 1977–1979 from diploma, bachelor, and masters students taking a course work unit

TABLE III

Classifications of 1981 errors made by 92 low achievers and 92 average achievers (Grade 7), 1977–1979

	Low achievers (n = 92)		Average achievers (n = 92)	
Error category	No. of errors	Percentage of errors	No. of errors	Percentage of errors
Reading	117	8	18	3
Comprehension	225	16	32	6
Transformation	401	28	150	28
Process skills	351	24	126	23
Encoding	37	3	12	2
Carelessness or motivation	306	21	206	38
Total	1437	100	544	100

called 'Diagnostic and Remedial Procedures in Mathematics' in the Faculty of Education, Monash University. As part of this course students are trained to use the Newman 'error analysis guideline', and are then required to interview four grade 7 pupils, two low achievers and two average achievers, who have made errors on the MAMP test. During the interviews the pupils' errors were classified according to the Newman hierarchy. Altogether 184 pupils, consisting of 92 low achievers and 92 average achievers from 36 schools, were interviewed, and 1981 errors were classified. The classifications are summarized in Table III.

Entries in Table III show that reading and comprehension difficulties caused a smaller proportion of the errors made by average achievers, and that over a third of the errors made by average achievers were due to carelessness or motivation. If Tables II and III are compared it can be seen that the summary data for the grade 7 low achievers (Table III) and the grade 5 children (Table II) are similar.

4.3. *The Casey Data*

Casey's data set was different, in several important ways, from the data sets of Newman and Clements. Casey developed his own instrument for classifying errors; he also used a test which contained many-step problems only, and he analyzed the errors made by 120 grade 7 pupils at one school. In 1977 he trained thirty Diploma in Education mathematics method students at Monash University to interview children and classify errors according to an elaborate error classification instrument, involving detailed flow-charting procedures, which he constructed. He then arranged for each Diploma in Education student to interview four pupils attending an outer-suburban Catholic school in Melbourne, and to classify the errors made by the children on a fifteen-item test containing many-step problems of about average difficulty for grade 7 pupils. Casey himself was the normal mathematics teacher of all of the pupils, but he had not specifically taught them the topics covered by the questions on the test. A typical question was: 'If three tyres cost \$100.02, find the cost of five such tyres'. Casey's flow-chart for analyzing errors on this question occupied six pages, the first page of which is shown as Figure 7 (Casey, 1978, p. 301).

An important difference between the interview procedures used by Newman and Casey is that in Casey's study the interviewers were required to help pupils over errors. Thus, for example, if a child made a 'Question Reading' error the interviewer would record this and then inform the child that he had made a mistake when reading the question. After assisting him to read the question correctly the interviewer would then ask the pupil to continue efforts to solve the problem. If the pupil then made a 'Question Comprehension' error the interviewer would note this and explain the meaning of the question to him, and so on. Thus, in Casey's study, a pupil could make a number of errors on the one question. Casey has not yet published all details of his work, but in an article published in 1978 he reported that on 38 per cent of the occasions when a pupil gave an incorrect answer to a question the pupil had made three or more errors in attempting to find the solution (Casey, 1978, p. 299). He also reported that only 3.8 per cent of errors in his study were classified as 'Question Reading' or 'Question Comprehension' errors, but 41.3 per cent were 'Strategy Selection' or 'Skills Selection' errors. Interestingly enough, this means that 45 per cent of the errors were made at or below what Newman would call the 'Transformation' level. Casey also reported that 26.8 and 21.0 per cent of the errors in his study were classified as 'Skills Manipulation' and 'Unknown Block' errors, respectively (p. 300).

5. DISCUSSION

Data reported in this paper confirm the view that many errors made by children on written mathematical tasks are due to reading, reading comprehension, and transformation difficulties, and that this often means a child uses inappropriate process skills in an attempt to find a solution. Obviously, the frequency and type of errors a child makes when attempting a verbal problem in mathematics depends on the interaction between 'question variables' (such as the vocabulary and syntax used in the question (see Linville, 1976), the

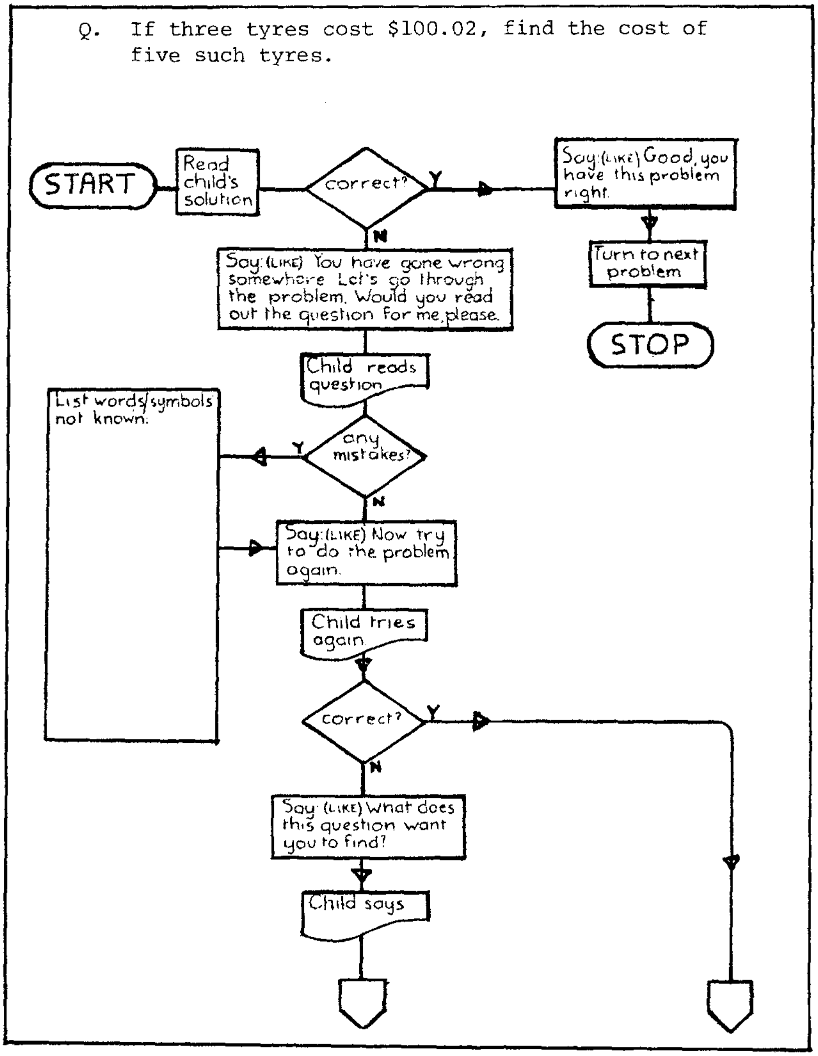

Fig. 7

complexity of the ideas in the question, and the level of mathematics needed to solve it), and 'person variables' (such as intelligence, reading ability, mathematical knowledge and ability, persistence). Because of this, it is inevitable that children will make errors on written mathematical tasks for a variety of reasons, and the error analysis procedures developed by Newman and Casey should enable an individual's pattern of errors to be determined.

The Clements data presented in this paper suggest that if representative groups of fifth and seventh grade pupils attempt the same verbal arithmetic problems then the following statements are likely to be true:

1. The seventh graders will make fewer errors than the fifth graders;
2. Compared with the fifth graders, a greater proportion of the errors made by seventh graders will be due to carelessness or motivation, and a smaller proportion due to faulty reading or reading comprehension;
3. At both grade levels most errors will occur at the 'Transformation' or 'Process Skills' stages, or will be due to 'Carelessness'.

The Clements data also suggest that if representative groups of seventh grade low achievers and average achievers (in mathematics) attempt the same verbal arithmetic problems three statements similar to those given above for fifth and seventh graders apply, with 'fifth graders' being replaced by 'low achievers' and 'seventh graders' by 'average achievers'.

The Casey data suggest that if seventh grade children attempt verbal arithmetic problems of a kind found in typical mathematics textbooks for seventh graders, then about 40 per cent of their errors will be due to difficulties arising from what Casey calls 'Strategy Selection' and 'Skills Selection' (and Newman calls 'Transformation').

The analyses of errors by Newman, Casey, and the present writer do *not* confirm C. G. Corle's claim that 'with the exception of the ability to solve more problems accurately, individual characteristics of good and poor problem solvers bear close resemblance' (see Hollander, 1978, pp. 330–1). Indeed, the data obtained demonstrate the futility of attempts to make valid general statements about the characteristics of good and poor problem solvers. All that can be said is that an individual's error pattern can throw considerable light on why that individual makes mistakes on mathematical tasks.

6. STRENGTHS AND LIMITATIONS OF THE NEWMAN TECHNIQUE

In order to illustrate how the error analysis procedures described can provide useful information for teachers, a summary of an interview conducted by the writer is now given. The subject interviewed was John, a fourteen year-old boy in grade 8, who, prior to the interview, had gained a score of 22/36 on the MAMP test. The writer used the Newman Technique to try to establish John's 'pattern of errors'.

One of the fourteen questions which John answered incorrectly was:

> Here are three fractions: $\frac{1}{3}, \frac{1}{4}, \frac{2}{5}$.
> Write these fractions in order of size, from *smallest* to *largest*.

When doing the MAMP test John had given the answer as $\frac{2}{5}$, $\frac{1}{4}$, $\frac{1}{3}$. Before conducting the interview the writer hypothesized that John's error had been due to his holding a belief that 'the larger the denominator, the smaller the fraction'. Here is the transcript of the analysis of John's error as revealed by the interview.

Interviewer: John, will you do this one for me.

John: [Attempts question – mumbles to himself, and after about a minute writes down: $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{4}$]. I'm no good at fractions. Is that right?

Interviewer: Read the question to me.

John: Here are three fractions . . . Ah . . . one-third . . . Ah . . . one-fourth . . . Ah . . . two-five. Write down the fractions in order of size from smallest to largest.

Interviewer: Good. What is the question asking you to do?

John: [Pointing to the three fractions on the MAMP question sheet] To work out the smallest and the largest.

Interviewer: Good. How are you going to do that, John?

John: I haven't the foggiest. I'm no good on fractions.

Interviewer: [Pointing to the numeral for $\frac{2}{5}$ on the MAMP question sheet] What does that mean, John?

John: I haven't got a clue. Can't do fractions.

Interviewer: [Pointing to the answer which John had given earlier] How did you get that answer?

John: Dunno. Guessed. Was I right?

I told John that he had been wrong and that I would show him 'how to do fractions' later. I then proceeded with the analysis of his errors on other questions.

It was decided that John's error on the above question on fractions should be classed as a 'Comprehension' error. Certainly, he had not read the numeral $\frac{2}{5}$ accurately, but I did not think *this* had prevented him from understanding what the question was asking him to do. John's main problem was that he did not know the meaning of the symbol $\frac{2}{5}$.

When John's fourteen errors on the MAMP test had been analyzed it was found that two of them had first occurred at the 'Reading' stage, four at the 'Comprehension' stage, four at the 'Transformation' stage, three at the 'Process Skills' stage, and one had been due to 'Carelessness'. By contrast, Charles, another grade 8 boy who had also made fourteen errors on the MAMP test, was found to have made six at the 'Process Skills' stage and eight because of 'Carelessness'. Unlike John, Charles had little difficulty reading and comprehending questions, and carrying out necessary transformations. The usefulness of the Newman technique had been demonstrated, for although both had scored 22/36 on the MAMP test, they clearly needed different kinds of remedial assistance.

However, it should be clear from the transcript of the interview with John that the extent of John's knowledge in the area of fractions had not been revealed during the interview. The day after I interviewed John I spoke to him, once again on the subject of fractions, with the intention of probing further his difficulties in this topic. After preliminary discussion on the meaning of 'one-third', and 'one-fifth', I wrote the numeral $\frac{2}{7}$ and asked him to read it. He read: 'two-seven'. I told him that it should be read 'two-sevenths', and that $\frac{1}{4}$ was usually read 'a quarter'. After a while he was able to read numerals such as $\frac{2}{5}$, $\frac{4}{7}$, $\frac{3}{8}$ correctly. I then asked him 'to find the number in the box' if

$$\tfrac{3}{8} \text{ of } 16 = \square .$$

He said he could not do it because he did not know what $\frac{3}{8}$ meant. I drew a rectangle and asked him to shade one-eighth of it. He did so, correctly. I drew another rectangle, and asked him to shade three-eighths of it. He had no idea what to do. I explained that $\frac{3}{8}$ is 'three lots of $\frac{1}{8}$', and showed him how to shade three-eighths of the rectangle. He appeared to be happy with the explanation, but when asked to shade five-sixths of another rectangle he could not do so. He could not shade one-eighth of a circle. When showed sixteen marbles he could separate one-eighth of them from the others, but did not know how many marbles he would have if he had three-eighths of the sixteen marbles. My strong impression was that any numeral of the form m/n where $m \geqslant 2$ evoked no visual imagery in John's mind.

John's case, as described above, demonstrates that although the Newman technique is invaluable for gross diagnostic purposes, more detailed probing is essential if the aim is to discover how a child thinks about a given area of mathematics.

7. SOME WARNINGS, AND SUGGESTIONS FOR FURTHER RESEARCH

The data presented by Newman, Casey, and the present writer show that many children cannot select and order the skills they will need to solve simple mathematical problems. It appears that as Rice (Hollander, 1978, p. 328), Doty (Hollander, 1978, p. 329), Talton (1973), Van Engen (1959), and others have previously suggested, many children fail to solve word problems because they cannot decide *how* they should be tackled. This raises the vexed question whether children can be taught to 'transform'. In a carefully designed, empirical study (as yet unpublished) recently conducted by E. Chamberlain and P. Munro[5] (1979), 52 children in grades 3 to 6 in a suburban primary school in Melbourne were identified as having difficulty 'transforming', and 26 matched pairs were then formed. A child in each pair was randomly allocated to an experimental group, which was given ten hours' training, spread over ten weeks, aimed specifically at developing the children's abilities to transform (in the Newman sense). After the training sessions were completed it was found that the experimental group obtained a significantly higher mean score ($p < 0.05$) on a test consisting of verbal arithmetic problems than the 'control' group. When (late in 1978) a parallel 'retention' test was given a year after the previous test, it was found that although differences were no longer statistically significant, the results tended to favour children who had been in the experimental group. Chamberlain and Munro concluded that the narrowing of the difference between the two groups emphasized the need for continual reinforcement of the problem-solving strategies they had been taught. They also recognized the possibility that although attempts to teach problem-solving methods are likely to result in immediate gains, these may not be permanent (Chamberlain and Munro, 1979, p. 27). Clearly, much more careful research is still needed on the matter.

It is, perhaps, wise to sound two notes of warning before concluding this paper. First, it should not be imagined that if two or more children have been identified as being especially prone to 'Transformation' errors (or 'Process Skills' errors, etc.) they need similar remedial treatments. This point is well illustrated in a recent unpublished paper by Marriott (1976), who analyzed the written responses made by 2826 pupils in grades 5 to 8 in Victorian schools to a question on the Monash MAMP test. The question was:

Carry out the following subtraction:

$$\begin{array}{r} 940 \\ -\underline{586} \\ \underline{} \end{array}$$

Marriott's paper, which is entitled 'Two Hundred Ways to Subtract, Most of Them Wrong', reveals that exactly two hundred answers were given, only one of which was correct! '446' was, as experienced teachers would predict, easily the most common error. Marriott's analysis demonstrates that even children with error profiles which seem to be similar are likely to make very different errors on the same problem.

The second warning is that the Newman hierarchy does not imply that a verbal arithmetic problem is necessarily more difficult than the corresponding arithmetic problem involving the direct application of the relevant process skills. This is illustrated by the following investigation carried out by the present writer. A twenty-item arithmetic test was constructed, and Questions 5 and 18 were:

Question 5: Write in the answer $1 - \frac{1}{4} =$ ______ (Answer).

Question 18: A cake is cut into four equal parts and Bill takes one of the parts. What *fraction* of the cake is *left*?

When the test was given to the 126 grade 6 pupils in a Melbourne school in 1978, 57 obtained the correct answer to Question 5 and 98 obtained the correct answer to Question 18. When another test, which was identical to the first test except that the positions of Questions 5 and 18 were interchanged, was given to the 105 grade 6 pupils in another Melbourne school in 1978, 96 obtained the correct answer to the 'cake' problem, and 55 for the '$1 - \frac{1}{4}$' problem. Thus, it appears that the verbal arithmetic problem is easier than the '$1 - \frac{1}{4}$' problem, despite the fact that the verbal problem involves more reading, comprehension, and transformation. The present writer interviewed six children (three from each school) who got the verbal problem correct but '$1 - \frac{1}{4}$' incorrect, and these interviews revealed:

1. The *imagery* evoked by the cake problem helped pupils;
2. Some pupils feel they cannot proceed with 'fraction sums' if they are presented in numeral form.

Research is needed to clarify those factors which make some real-world verbal arithmetic problems easier than the corresponding arithmetical 'sums'. Radatz's information-processing classification of errors could provide a useful basis for such research.

Finally, many teachers have informed the present writer that they have found that the Newman hierarchy provides an excellent basis for one-to-one teacher–pupil encounters in mathematics classrooms. Often when a teacher sits down to help a pupil with a mathematics problem he is tempted 'to show the pupil how to do it'. Instead of this, it is recommended, especially with junior pupils, the teacher begins by asking the pupil to read the question aloud. Then the teacher can ask: 'What is the question asking you to do?', and 'How are you going to do it?', and so on. It appears that Newman's creation of an error hierarchy has not only changed, for the better, the direction of error analysis research, but has also proved useful to classroom teachers who, when assisting individual pupils, follow a routine suggested by the 'error analysis guideline'.

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NOTES

[1] While the discussion in the text is not based on an actual classroom testing situation, the errors given are taken from case studies conducted by Newman and the present writer.
[2] Although Newman regards Figure 5 as an accurate representation of the ideas inherent in her hierarchy it was constructed by the present writer.
[3] Although Casey regards Figure 6 as an accurate representation of the ideas inherent in his 'hierarchy' it was constructed by the present writer.
[4] Many of Newman's questions involved the direct application of arithmetical algorithms.

For example: Write in the answer: $\begin{array}{r} 554 \\ -\ \underline{108} \\ \underline{} \end{array}$

[5] Chamberlain and Munro's study has been awarded the 1979 G. S. Browne prize for educational research in Victoria. (This prize is awarded by the Victorian Institute of Educational Research.) The Newman study won the G. S. Browne prize for 1977.

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