Probab. Th. Rel. Fields 71, 85-116 (1986)

Probability Theory and Related Fields

A Weighted Occupation Time for a Class of Measured-Valued Branching Processes*

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Summary. A weighted occupation time is defined for measure-valued processes and a representation for it is obtained for a class of measure-valued branching random motions on R^d. Considered as a process in its own right, the first and second order asymptotics are found as time $t \to \infty$. Specifically the finiteness of the total weighted occupation time is determined as a function of the dimension d, and when infinite, a central limit type renormalization is considered, yielding Gaussian or asymmetric stable generalized random fields in the limit. In one Gaussian case the results are contrasted in high versus low dimensions.

Introduction

A useful tool with which to study measure-valued stochastic processes is what we shall call the weighted occupation time. It is again a measure-valued process derived from the original one as follows. It X_t denotes some measure-valued process on a σ-algebra, $\mathscr{B}$, then the weighted occupation time Y_t is defined for $B \in \mathscr{B}$ by $Y_t(B) = \int_0^t X_s(B)\,ds$. Of course in order that Y_t be well defined, the sample paths of $X_{.}(B)$ should be sufficiently regular that the integral makes sense.

The significance of Y_t is that it is monotone in the amount of time that the set $B \in \mathscr{B}$ is charged by $X_{.}$ and also in the numerical value of the charge. In particular, if $Y_{t_2}(B) - Y_{t_1}(B) = \int_{t_1}^{t_2} X_s(B)\,ds = 0$ then for a.e. $s \in [t_1, t_2]$, $X_s(B) = 0$; and with a little regularity one can remove the a.e. qualification. The values $t_1 = 0$ or $t_2 = \infty$ are of special interest as they give information on the nature of the supports of the measures X_t, not at each instant but globally in time.

* Research supported in part by Natural Sciences and Engineering Research Council of Canada

The utility of the weighted occupation time process depends on the possibility of representing it in a manner which is accessible to various calculations. Such a representation can and will be obtained in this paper for a class of measure-valued critical branching random motions in R^d, d-dimensional Euclidean space. These are versions of the C.B. processes introduced by Jirina [15] and considered in Watanabe [22], but with possibly infinite total mass. In these examples, which can be obtained as weak limits of discrete branching random motions in R^d, it is natural to think of the weighted occupation time in units of "man-hours". That is, for $B \in \mathscr{B}(R^d)$, the Borel σ-algebra of R^d $Y_t(B)$ is a measure of the amount of individuals in the set B, weighted by the amount of time they each spend in B, during $[0, t]$.

Infinite particle systems have received much attention in recent years. An early survey for those involving interactions is Liggett [18], and one for interaction-free branching diffusions is Dawson and Ivanoff [6]. The article [11] of Holley and Stroock is related to some of the material of this present paper.

Initial studies on one of the examples considered here (the "$\beta=1$" case) were carried out in Dawson [4], and Dawson and Hochberg [5]. In [4], the limiting behaviour as $t \to +\infty$ was studied (local extinction versus existence of invariant distributions), and in [5] the local structure (Hausdorff dimension of the supports) of the states X_t of the process for fixed t was studied.

In this paper we shall study the weighted occupation time as a process in its own right since, as indicated above, it is physically meaningful. Applications to the nature of the supports of the original random measures will be given in a forthcoming article [13]. The organization of this article is as follows. In §0 we review briefly the relevant concepts and notation pertinent to our investigations. In §1 a construction is given for the class of processes considered; the proofs of the required estimates being delayed until §2. In §3 a representation is obtained for the weighted occupation time, and a time-dependent generalization thereof, in terms of its Laplace functional. In §4 we study the question of finiteness for the total weighted occupation time in bounded sets. The answer is found to be in the affirmative in "low" dimensions. Next, a theorem of central limit type is proved in "high" dimensions in §5. There the renormalized oscillations about the ever-growing mean of the weighted occupation time process are shown to converge to generalized random fields, with long range correlations in the finite variance case. In the final section, §6, the "intermediate" range of dimensions is studied in one case to illustrate the complete spatial correlation which arises in the limiting generalized random field. An appendix outlining the basic theory of the evolution equations occurring in this article, will be found at the end.

The material in this article is based in part on the author's doctoral thesis [12] which was written under the supervision of Professor D. Dawson, to whom gratitude is expressed. The definition and suggestion to use the spaces $M_p(\dot{R}^d)$ in §1 are due to him. Gratitude is also extended to M. Crandall for valuable conversations on evolution equations.

§ 0. Notation and Other Preliminaries

Firstly, we define the following (real-valued) function spaces. If M is a topological space then $C(M)$ will denote the Banach space of continuous bounded functions on M equipped with the usual sup norm $\|\cdot\|_\infty$. $C_c(M)$ will denote the subset of $C(M)$ whose members have compact support, and, in case M is locally compact, $C_0(M)$ will denote the subspace of $C(M)$ whose members vanish at infinity. When $M=R^d$, d-dimensional Euclidean space with the usual norm $|\cdot|$, we also define:

$$\bar{S}(R^d)=\{f\in C(R^d): f \text{ is rapidly decreasing}\}$$

$$S(R^d)=\{f\in\bar{S}(R^d): f \text{ is infinitely differentiable and } f^{(n)}\in\bar{S}(R^d),\ \forall n\in\mathbb{N}^d\}$$

$$C_p(R^d)=\{f\in C(R^d): \|f(x)\cdot|x|^p\|_\infty<\infty\},\ p\in R^1_+.$$

The non-negative members of each of these subsets will be denoted by the subscript "+"; e.g. $\bar{S}(R^d)_+$.

Dually, we have the following spaces. $M(R^d)$ will denote the space of positive Radon measures on $\mathscr{B}(R^d)$, the Borel σ-algebra of R^d. $M(R^d)$ carries the vague topology and is a Polish space (see [14]). $S'(R^d)$ is the space of tempered distributions. The non-negative elements of $S'(R^d)$, the tempered measures, can be given the following filtration. $S'(R^d)_+=\bigcup_{p\geq 0} M_p(R^d)$ where

$$M_p(R^d)=\{\mu\in M(R^d): (1+|x|^p)^{-1}\,d\mu(x) \text{ is a finite measure}\}.$$

For example, Lebesgue measure, which will always be denoted by λ, belongs to $M_p(R^d)$ for $p>d$. Also we will renotate $M_0(R^d)$, the finite measures, by $M_F(R^d)$.

Integration will be denoted by the pairing $\langle\cdot,\cdot\rangle$; for example, if $f\in C_p(R^d)$ and $\mu\in M_p(R^d)$ then $\langle f,\mu\rangle=\int_{R^d} f(x)\,d\mu(x)$.

A random measure is simply an $M(R^d)$-valued random variable. We refer the reader to Jagers [14] for a discussion of random measures. In particular, we shall have need for the Laplace and characteristic functionals of a random measure X defined by $E[\exp(-\langle\psi,X\rangle)]$ for $\psi\in C_c(R^d)_+$ (at least), and $E[\exp(i\langle\psi,X\rangle)]$ for $\psi\in C_c(R^d)$ (at least) respectively; either of them uniquely determine the law of X. Here E denotes expectation with respect to the law of X. In case we are considering an $M(R^d)$-valued stochastic process, X_t, expectations conditional on an initial value $X_0=\mu$ will be denoted by E_μ.

Two infinitessimal generators will be underlying the processes considered in this article. The first is the Laplacian, Δ, which generates the semigroup associated with a Brownian motion. The second is a fractional power of the Laplacian, $\Delta_\alpha=-(-\Delta)^{\alpha/2}(0<\alpha<2)$, (c.f. Yosida [23]) which generates the contraction semigroup S_t^α associated with a symmetric stable motion. The dependence on α will usually be suppressed and we write simply S_t. It is a convolution operator:

$$S_t\psi(x)=\int_{R^d} p_t^\alpha(x-y)\psi(y)\,dy, \qquad \psi\in C(R^d).$$

Various properties of the density p_t^α which we use can be found in, or inferred from their description in [8], for example. In general the domain of an operator A will denoted by $D(A)$.

Finally, if M is a topological space, we denote by $D(R_+^1, M)$ the set of functions from R_+^1 into M which are "cadlag" = right continuous, possessing limits from the left. $C(R_+^1, M)$ will denote the subset consisting of continuous functions.

§ 1. A Construction of the Processes

It was shown in [22] that there exists a multiplicative finite measure-valued (time-homogeneous) Markov process with sample paths in $D(R_+, M_F(R^d))$ whose transition functions have the following Laplace functionals:

$$E_\mu[\exp(-\langle\psi, X_t\rangle)] = \exp[-\langle U_t\psi, \mu\rangle], \qquad \mu\in M_F(R^d),\ \psi\in C_0(R^d)_+. \tag{1.1}$$

Here U_t is the strongly continuous semigroup with infinitesimal generator $A - g(\cdot)$ where A is the infinitesimal generator of a strongly continuous non-negative semigroup of bounded linear maps, $\{S_t\}_{t\geqq 0}$, on $C_0(R^d)$, and $g\colon R_+\to R_+$ is a cumulant generating function of an infinitely divisible positive real valued random variable. Thus for $\psi\in D(A)_+$, $U_t\psi$ is the solution of the evolution equation:

$$\begin{aligned} \dot{u}(t) &= Au(t) - g(u(t)) \\ u(0) &= \psi \end{aligned} \tag{1.2}$$

and for general $\psi\in C_0(R^d)_+$, $U_t\psi$ is the (so-called mild) solution of the associated integral equation:

$$u(t) = S_t\psi - \int_0^t S_{t-s}\, g(u(s))\, ds. \tag{1.3}$$

For an outline of the basic theory of these equations in the case $g'(0) = g(0) = 0$ see the appendix. For the sake of concreteness we shall assume from hereonin that the evolution equation (1.2) is of the following form:

$$\begin{aligned} \dot{u}(t) &= \Delta_\alpha u(t) - u^{1+\beta}(t), \qquad 0<\alpha\leqq 2,\ 0<\beta\leqq 1 \\ u(0) &= \psi, \qquad \psi\in D(\Delta)_+. \end{aligned} \tag{1.6}$$

The semigroup associated with (1.6) will continue to be denoted by U_t; $u(t)\equiv U_t\psi$, for $\psi\in C_0(R^d)_+$ satisfies:

$$u(t) = S_t^\alpha\psi - \int_0^t S_s^\alpha u^{1+\beta}(s)\, ds. \tag{1.7}$$

We would like to be able to start these processes at time $t=0$ with initial measures μ which are Radon but not necessarily finite; for example $X_0=\lambda$, Lebesgue measure, or X_0 a Poisson random field. Since we are dealing with branching processes, the most obvious strategy to obtain this extension is to decompose the initial measure into a series of finite measures with disjoint supports; the processes evolving from each of these finite measures could then be summed. However, in doing this summation there is no guarantee that regularity properties of the sample paths (e.g. continuity) would be preserved. We can overcome this problem by appealing to the following theorem from the general theory of Markov process (see 3.7 and Lemma 2.11 of [7]).

Theorem. *Let M be a locally compact space. If T_t: $C(M) \to C(M)$ is a Feller semigroup such that $T_t[C_0(M)] \subset C_0(M)$ and is strongly continuous (in t) on $C_0(M)$, then there exists an M-valued Markov process X with sample paths in $D(R^1_+, M)$ such that $E_m[F(X_t)] = T_t F(m)$ for all $F \in C(M)$, $m \in M$.*

In order to apply this theorem we will have to alter our setting since $M_p(R^d)$ endowed with the vague topology is not locally compact. We now give an outline of this setting.

The Spaces $M_p(\dot{R}^d)$. We define $\dot{R}^d = R^d \cup \{\tau\}$, τ being an isolated adjoined point. The role of τ will be to absorb the loss of temperance of tempered measures on $\mathscr{B}(R^d)$, in a sense to be made more precise; $\dot{R}^d$ plays the role here for tempered measures that the one point compactification of R^d does for finite measures. Also for $p \geqq 0$ we denote by $C_p(\dot{R}^d)$ (resp. $C_p(\dot{R}^d)_+$, $C_p(\dot{R}^d)_{++}$) the set of continuous (resp. non-negative, strictly positive) functions ψ on $\dot{R}^d$ such that $\lim_{|x| \to +\infty} |x|^p |\psi(x)| = c \in R^1$ and $\psi(\tau) = c$; $C_c(R^d) \subset C_p(\dot{R}^d)$ via extension by zero. The particular element $\phi_p(x) = (1 + |x|^p)^{-1}$ for $x \in R^d$, $\phi_p(\tau) = 1$ belonging to $C_p(\dot{R}^d)_{++}$ will be in constant use.

We denote by $M(\dot{R}^d)$ the set of non-negative Radon measures on $\mathscr{B}(\dot{R}^d)$; by extending $\mu \in M(R^d)$ through $\mu(\tau) \equiv \mu(\{\tau\}) = 0$ we have the inclusion $M(R^d) \subset M(\dot{R}^d)$. For $p \geqq 0$ we endow

$$M_p(\dot{R}^d) = \{\mu \in M_p(\dot{R}^d) \colon \langle \phi_p, \mu \rangle \equiv \int_{R^d} \phi_p \, d\mu + \mu(\tau) < +\infty\}$$

with the smallest topology rendering the maps $\{\mu \mapsto \langle \psi, \mu \rangle \colon \psi \in C_c(R^d) \cup \{\phi_p\}\}$ continuous. Specifically a basic open neighbourhood of $\mu \in M_p(\dot{R}^d)$ is a set of the form

$$N_\mu(\psi_1, \ldots, \psi_n, \phi_p; \varepsilon) = \{\nu \in M_p(\dot{R}^d) \colon |\langle \psi_i, \nu - \mu \rangle| < \varepsilon, 1 \leqq i \leqq n, |\langle \phi_p, \nu - \mu \rangle| < \varepsilon\}$$

where $(\psi_i)_{1 \leqq i \leqq n} \subset C_c(R^d)$ and $\varepsilon > 0$. As such $M_p(\dot{R}^d)$ is a locally compact Polish space. Indeed, if we denote a metric for the value topology on $M(R^d)$ (under which it is complete) by d_0 then $d(\mu, \nu) = d_0(\mu|_{R^d}, \nu|_{R^d}) + |\langle \phi_p, \mu - \nu \rangle|$ metrizes the topology on $M_p(\dot{R}^d)$. It is a tedious but routine matter to check that the collection of purely atomic measures, with rational masses at τ and points of R^d with rational coordinates, is a countable dense subset of $M_p(\dot{R}^d)$; we omit the details. The proof of completeness is similar to, but simpler than that of the local compactness of $M_p(\dot{R}^d)$, so we only indicate the latter.

Local compactness of $M_p(\dot{R}^d)$ follows easily from the characterization: $K \subset M_p(\dot{R}^d)$ is compact iff K is closed, and there is a $k > 0$ such that $K \subset \{\mu \in M_p(\dot{R}^d) \colon \langle \phi_p, \mu \rangle \leqq k\}$; the latter being itself compact. The necessity is easy, and the sufficiency will follow from the compactness of $K \equiv \{\mu \in M_p(\dot{R}^d): \langle \phi_p, \mu \rangle \leqq k\}$. To see this, let $(\mu_n)_{n\in\mathbb{N}} \subset K$. Then their restrictions to R^d form a vaguely precompact sequence since for each $\psi \in C_c(R^d)_+$ there exists a $c>0$ with $\psi \leqq c\phi_p$; so $\langle \psi, \mu_n \rangle \leqq c \langle \phi_p, \mu_n \rangle \leqq c \cdot k$. Therefore we can extract a vaguely convergent subsequence $(\mu_{n_j})_{j\in\mathbb{N}}$, $\lim_{j\to+\infty} \mu_{n_j} = \mu$, and without loss of generality we can assume that $\ell_1 = \lim_{j\to+\infty} \mu_{n_j}(\tau)$ and $\ell_2 = \lim_{j\to+\infty} \int_{R^d} \phi_p d\mu_{n_j}$ exist. Choose a sequence $(\rho_m)_{m\in\mathbb{N}} \subset C_c(R^d)$ such that $0 \leqq \rho_m \leqq 1$ and $\rho_m \uparrow 1$ as $m \to +\infty$. Then

$$\int_{R^d} \phi_p d\mu = \lim_m \int_{R^d} \rho_m \phi_p d\mu = \lim_m \lim_j \int_{R^d} \rho_m \phi_p d\mu_{n_j} \leqq \lim_m \lim_j \int_{R^d} \phi_p d\mu_{n_j} = \ell_2.$$

Therefore we can extend μ to $\{\tau\}$ by $\mu(\tau) = \ell_1 + \ell_2 - \int_{R^d} \phi_p d\mu \geqq 0$, whereupon $k \geqq \lim_{j\to+\infty} \langle \phi_p, \mu_{n_j} \rangle = \ell_1 + \ell_2 = \langle \phi_p, \mu \rangle$, which together with the vague convergence yields the convergence of $(\mu_{n_j})_{j\in\mathbb{N}}$ to $\mu \in K$, in $M_p(\dot{R}^d)$.

The Borel σ-algebra of $M_p(\dot{R}^d)$. Since we will be extending certain Markov transition kernels to $M_p(\dot{R}^d)$ it is appropriate to discuss $\mathbf{B_p}$, its Borel σ-algebra, here. Denote the (vaguely) Borel σ-algebra of $M(R^d)$ by $\mathbf{B_v}$. Throughout this subsection, $(\rho_n)_{n\in\mathbb{N}}$ will stand for a sequence in $C_c(R^d)_+$ such that $0 \leqq \rho_n \leqq 1$ and $\rho_n \uparrow 1$ as $n \to +\infty$. We note firstly that $M_F(R^d) = \bigcup_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \{\mu \in M(R^d): \langle \rho_k, \mu \rangle < n\} \in \mathbf{B_v}$. Also it is clear that $M_F(R^d) \subset M_p(\dot{R}^d)$. We will be showing that $M_F(R^d) \in \mathbf{B_p}$ and that the traces of the σ-algebras $\mathbf{B_v}$ and $\mathbf{B_p}$ on M_F coincide; but first we must examine the behaviour at τ.

It follows from the calculation done in the proof of the local compactness of $M_p(\dot{R}^d)$ that if $\mu_n \to \mu$ there then $\mu(\tau) \geqq \limsup_n \mu_n(\tau)$. Therefore for $c \geqq 0$, $\{\mu \in M_p(\dot{R}^d): \mu(\tau) \geqq c\}$ is closed, and so $\{\mu \in M_p(\dot{R}^d): \mu(\tau) \geqq c\}$ and $\{\mu \in M_p(\dot{R}^d): \mu(\tau) = c\}$ are Borel (i.e. belong to $\mathbf{B_p}$); the value $c=0$ is of particular importance.

Expressing

$$M_F(R^d) = [\bigcup_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \{\mu \in M_p(\dot{R}^d): \langle \rho_k, \mu \rangle < n\}] \cap \{\mu \in M_p(\dot{R}^d): \mu(\tau) = 0\}$$

yields that $M_F(R^d) \in \mathbf{B_p}$; replacing ρ_k with $\rho_k \phi_p$ yields $M_p(R^d) \in \mathbf{B_p}$. To see that the trace of $B_\mathbf{p}$ on $M_F(R^d)$ coincides with that of $\mathbf{B_v}$, we define $\mathscr{A} = \{B \in \mathbf{B_p}: B \cap M_F(\dot{R}^d) \in \mathbf{B_v}\}$. Evidently $\mathscr{A}$ is a σ-algebra, so to show that $\mathscr{A} = \mathbf{B_p}$ it suffices to show that $\mathscr{A}$ contains a neighbourhood subbasis of each point μ of $M_p(\dot{R}^d)$ (recall that an open subset of a separable metric space is Lindelöf: every open covering contains a countable subcovering). To this end let $\psi \in C_p(\dot{R}^d)$ and $\varepsilon > 0$, and denote

$$\begin{aligned} N &= \{\nu \in M_p(\dot{R}^d): |\langle \psi, \nu - \mu \rangle| < \varepsilon\} \\ &= \{\nu \in M_p(\dot{R}^d): -\varepsilon + \langle \psi, \mu \rangle < \langle \psi, \nu \rangle < \varepsilon + \langle \psi, \nu \rangle\}. \end{aligned}$$

If $\psi \in C_c(R^d)$ then clearly $N \cap M_F(R^d) \in \mathbf{B_v}$. If $\psi = \phi_p$ then N is of the form $\{\nu \in M_p(\dot{R}^d): c_1 < \langle \phi_p, \nu \rangle < c_2\}$ and from the representation:

$$\{\nu \in M_F(R^d): \langle \phi_p, \nu\rangle \gtrless c\} = \bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m>n} \{\nu\in M_F(R^d): \langle \rho_m \phi_p, \nu\rangle \gtrless c \pm 1/k\} \in \mathbf{B}_v,$$

we conclude that $N \cap M_F(R^d) \in \mathbf{B}_v$ in this case as well. Thus $\mathscr{A} = \mathbf{B}_p$ is established.

The Algebra $\mathscr{D} \subset C_0(M_p(\dot{R}^d))$. We denote by $\mathscr{D}$ the linear span of the functions $\{\exp(-\langle\psi, \cdot\rangle): \psi \in C_p(\dot{R}^d)_{++}\}$ on $M_p(\dot{R}^d)$. The purpose of this subsection is to show that $\mathscr{D}$ is dense in $C_0(M_p(\dot{R}^d))$.

We first note that if $\psi \in C_p(\dot{R}^d)$ then $\langle\psi, \cdot\rangle$ is continuous; for given $\mu \in M_p(\dot{R}^d)$ and $\varepsilon > 0$ we can find $R > 0$ such that $|x| > R$ implies $|\psi(x) - \psi(\tau)\phi_p(x)| < \varepsilon\phi_p(x)$; and thus we can find a $\rho \in C_c(R^d)$ with $0 \leqq \rho \leqq 1$, $\rho(x) = 1$ for $|x| \leqq R$, and $\rho(x) = 0$ for $|x| \geqq R+1$, so that $|(1-\rho)[\psi - \psi(\tau)\phi_p]| \leqq \varepsilon(1-\rho)\phi_p$. Setting $\psi_0 = \rho\phi_p$, $\psi_1 = \rho\psi$, $\delta = \min\left(1, \left(\frac{\varepsilon}{4}\right)[1+\psi(\tau)]^{-1}[1+2\langle\phi_p, \mu\rangle]^{-1}\right)$, and $N = N_\mu(\psi_0, \psi_1, \phi_p; \delta)$ we can calculate, after a few triangle inequalities, that for $\nu \in N$, $|\langle\psi, \mu\rangle - \langle\psi, \nu\rangle| < \varepsilon$.

If further $\psi \in (\dot{R}^d)_{++}$ then there is some $c > 0$ such that $\psi \geqq c\phi_p$. For $\mu \notin \{\nu \in M_p(\dot{R}^d): \langle\phi_p, \nu\rangle \leqq k\}$ (which is compact) $\exp(-\langle\psi, \nu\rangle) \leqq e^{-ck} \to 0$ as $k \to +\infty$. Therefore $\mathscr{D} \subset C_0(M_p(\dot{R}^d))$. It is easy to see that $\mathscr{D}$ separates points of $M_p(\dot{R}^d)$ and vanishes at no point in $M_p(\dot{R}^d)$, so by the Stone-Weierstrass theorem ([2], p. 28) $\mathscr{D}$ is dense in $C_0(M_p(\dot{R}^d))$ (with the supnorm topology).

Construction of the Feller Semigroup. Our starting point will be the Markov transition kernel $P_t(\mu, d\nu)$ on $(M_F(R^d), \mathbf{B}_v)$ such that

$$\int_{M_F} \exp(-\langle\psi, \nu\rangle) P_t(\mu, d\nu) = \exp(-\langle U_t\psi, \mu\rangle) \quad \text{for } \mu \in M_F(R^d) \text{ and } \psi \in C_p(R^d)_+;$$

the existence of P_t follows from the work of Watanabe [22], and others in special cases. Since $M_F(R^d) \cap \mathbf{B}_p \subset \mathbf{B}_v$ we can extend $P_t(\mu, \cdot)$ to a probability measure on $\mathbf{B}_p$ for each $\mu \in M_F(R^d)$ by $P_t(\mu, B) = P_t(\mu, B \cap M_F(R^d))$ for $B \in \mathbf{B}_p$. We now extend $P_t(\mu, d\nu)$ to a Markov transition kernel on $(M_p(\dot{R}^d), \mathbf{B}_p)$ as follows. Throughout this subsection $p > d$; and $p < d + \alpha$ if $\alpha < 2$. The semigroups S_t^α, U_t are extended to $C_p(\dot{R}^d)_+$ by $S_t^\alpha\psi(\tau) = U_t\psi(\tau) = \psi(\tau)$[1].

If $\mu = \mu(\tau)\delta_\tau$ then we set $P_t(\mu, \cdot) = \delta_\mu$. If $\mu(\tau) = 0$, then we decompose R^d into a disjoint union of bounded Borel subsets, $R^d = \bigcup_{n\in\mathbb{N}} B_n$, and define the finite measures $\mu_n(B) = \mu(B \cap B_n)$, for $B \in \mathscr{B}(R^d)$, and $\mu^n = \sum_{i=1}^n \mu_i$. Clearly $\mu = \lim_{n\to+\infty} \mu^n$ setwise; moreover $\mu_n \to \mu$ in $M_p(\dot{R}^d)$. Indeed if $\psi \in C_c(R^d)$ then $\langle\psi, \mu^n\rangle = \langle\psi, \mu\rangle$ for sufficiently large n; also

$$|\langle\phi_p, \mu^n\rangle - \langle\phi_p, \mu\rangle| = \int_{R^d \setminus \bigcup_{i\leqq n} B_i} \phi_p \, d\mu \to 0 \quad \text{as} \quad n \to +\infty.$$

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[1] That S_t^α, $U_t: C_p(\dot{R}^d) \to C_p(\dot{R}^d)$ follows from Proposition 2.3

We claim that $P_t(\mu^n, \cdot)$ converges weakly, and proceed in two steps:

(i) tightness: Given $\varepsilon > 0$ choose $k > \langle S_t \phi_p, \mu\rangle/\varepsilon$ and let K be the compact set $\{v \in M_p(\dot{R}^d): \langle \phi_p, v\rangle \leqq k\}$. Then

$$P_t(\mu^n, K^c) \leqq (1/k) \cdot \int_{M_F} \langle \phi_p, v\rangle P_t(\mu^n, dv) = (1/k) \cdot \langle S_t \phi_p, \mu^n\rangle < \varepsilon.$$

The evaluation of the integral over $M_F(R^d)$ is effected as in Proposition 5.1.

(ii) uniqueness: Let P be any subsequential weak limit of $(P_t(\mu^n, \cdot))_{n\in\mathbb{N}}$. Then we calculate its Laplace functional for $\psi \in C_p(\dot{R}^d)_+$ as:

$$\begin{aligned}\int_{M_p} \exp(-\langle \psi, v\rangle) P(dv) &= \lim_{k\to+\infty} \int_{M_p} \exp(-\langle \psi, v\rangle) P_t(\mu^{n_k}, dv)\\ &= \lim_{k\to+\infty} \exp(-\langle U_t \psi, \mu^{n_k}\rangle) = \exp(-\langle U_t \psi, \mu\rangle)\end{aligned}$$

since $\mu^{n_k} \to \mu$ and by Prop. 2.3 of the next section $U_t \psi \in C_p(R^d)$. Since $\mathscr{D}$ is dense in $C_0(M_p(\dot{R}^d))$, P is uniquely determined; we denote it by $P_t(\mu, dv)$.

At this point we make the important observation that $P_t(\mu, \{v \in M_p(\dot{R}^d): v(\tau) > 0\}) = 0$. Indeed if $(\rho_n)_{n\in\mathbb{N}} \subset C_c(R^d)$ with $0 \leqq \rho_n \leqq 1$ and $\rho_n \uparrow 1$ as $n \to +\infty$ then setting $\sigma_n = 1 - \rho_n$:

$$\begin{aligned}P_t(\mu, \{v(\tau) > 0\}) &= \int_{M_p} \langle 1_\tau, v\rangle P_t(\mu, dv)\\ &\leqq \liminf_{n\to+\infty} \int_{M_p} \langle \sigma_n \phi_p, v\rangle P_t(\mu, dv), \text{ by Fatou's lemma}\\ &= \liminf_{n\to+\infty} \int_{R^d} S_t^\alpha(\sigma_n \phi_p) d\mu\\ &= 0, \quad \text{by dominated convergence.}\end{aligned}$$

For the general case $\mu \in M_p(R^d)$ we decompose $\mu = \mu_0 + \mu_1$ with $\mu_0(\tau) = 0$, $\mu_1 = \mu(\tau)\delta_\tau$ and set $P_t(\mu, \cdot) = P_t(\mu_0, \cdot) * P_t(\mu_1, \cdot)$. Then for $\psi \in C_p(\dot{R}^d)_+$:

$$\int_{M_p} \exp(-\langle \psi, v\rangle) P_t(\mu, dv) = \exp(-\langle U_t \psi, \mu_0\rangle) \cdot \exp(-\langle \psi, \mu_1\rangle) = \exp(-\langle U_t \psi, \mu\rangle)$$

since $U_t \psi(\tau) = \psi(\tau)$ (see Prop. 2.3).

$P_t(\mu, \cdot)$ is weakly continuous in μ, for if $\mu_n \to \mu$ in $M_p(\dot{R}^d)$ then as before $(P_t(\mu_n, \cdot))_{n\in\mathbb{N}}$ is tight and any weak subsequential limit can be identified as $P_t(\mu, \cdot)$ through its Laplace functional. Also the Chapman-Kolmogorov property: $P_{t+s}(\mu, B) = \int_{M_p} P_t(\mu, dv) P_s(v, B)$ for $B \in \mathbf{B_p}$, can be checked using Laplace functionals.

Thus we have, in defining T_t: $C(M_p(\dot{R}^d)) \to C(M_p(\dot{R}^d))$ by $T_t F(\mu) = \int_{M_p} F(v) P_t(\mu, dv)$, a Feller semigroup. Modulo some estimates, which will be given in the next section, we can now establish the existence of the branching processes.

Theorem 1.1. *Let $0 < \alpha \leqq 2$, $0 < \beta \leqq 1$, $p > d$; and $p < d + \alpha$ in case $\alpha < 2$. Then there exists an $M_p(R^d)$-valued multiplicative Markov process X_t, with sample paths in*

$D(R^1_+, M_p(R^d))$ a.s., *such that*

$$E_\mu[\exp(-\langle\psi, X_t\rangle)] = \exp[-\langle u(t), \mu\rangle], \qquad \mu\in M_p(R^d),\ \psi\in C_p(R^d)_+$$

where $u(t)$ *is the mild solution of the evolution equation:*

$$\dot{u}(t) = \Delta_\alpha u(t) - u^{1+\beta}(t)$$
$$u(0) = \psi$$

($u(t)$ *being a strong solution if* $\psi\in D(\Delta)$ *as well).*

Proof. By the theorem quoted earlier in this section (Theorem 3.7 and Lemma 2.11 of [7]) it suffices to show that T_t: $\mathscr{D}\to\mathscr{D}$ and is strongly continuous (in t) on $\mathscr{D}$, since $\mathscr{D}$ is dense in $C_0(M_p(\dot{R}^d))$; that X_t is necessarily $M_p(R^d)$-valued if $X_0\in M_p(R^d)$, has already been observed. Proposition 2.3 implies that $T_t\mathscr{D}\subset\mathscr{D}$. We now verify the strong continuity of T_t on $\mathscr{D}$; that is for $\psi\in C_p(R^d)_{++}$

$$\lim_{t\downarrow 0}\sup_{\mu\in M_p}|T_t[\exp(-\langle\psi,\cdot\rangle)](\mu) - \exp(-\langle\psi,\mu\rangle)| = 0.$$

Fix $\psi\in C_p(\dot{R}^d)_{++}$ and choose some constant $c_1>0$ such that $\psi\geqq c_1\phi_p$ and $S_t^\alpha\psi\geqq c_1\phi_p$; the latter being possible for sufficiently small $t<\eta_1<1$ by Lemma 2.4. Let $\varepsilon>0$ be given and restrict $t\in[0,\eta_1]$. Choose $k>0$ such that $\exp[-e^{-c}\cdot c_1\cdot k]<\varepsilon/2$, where c is the constant in Lemma 2.1 (ii), and define $K=\{\mu\in M_p(\dot{R}^d)\colon \langle\phi_p,\mu\rangle\leqq k\}$. For $\mu\notin K$ both $\exp(-\langle\psi,\mu\rangle)$ and $T_t[\exp(-\langle\psi,\cdot\rangle)](\mu) = \exp(-\langle U_t\psi,\mu\rangle)$ are less than $\varepsilon/2$.

Now by Lemma 2.4 we can find an η, $0<\eta<\eta_1$, such that for $t\in[0,\eta]$, $|U_t\psi-\psi|<\delta\phi_p$, where δ chosen so that $e^{\delta k}-1<\varepsilon$. Then for $\mu\in K$:

$$|\exp(-\langle U_t\psi,\mu\rangle) - \exp(-\langle\psi,\mu\rangle)| = |\exp(-\langle U_t\psi-\psi,\mu\rangle)-1|<\varepsilon$$

(note: if $a,b\in R^1$ with $|a|\leqq b$ then $|e^{-a}-1|\leqq e^b-1$).

Thus the strong continuity, and hence the theorem, is proved. Q.E.D.

§ 2. Estimates and Asymptotics for the Semigroups S_t, U_t

Lemma 2.1. *Let* $\psi\in C_0(R^d)_+$ *and* $c\equiv\|\psi\|_\infty^\beta$. *The following inequalities hold for all* $t\geqq 0$.

(i) $0\leqq S_t\psi - U_t\psi\leqq tS_t\psi^{1+\beta}$

(ii) $U_t\psi\geqq e^{-ct}S_t\psi$.

Proof. (i) $u(t)\equiv U_t\psi = S_t\psi - \int_0^t S_{t-s}u^{1+\beta}(s)\,ds$. Since $u(t)\geqq 0$, (Theorem **A** of the Appendix) $U_t\psi\leqq S_t\psi$. Therefore

$$\begin{aligned} 0\leqq S_t\psi - U_t\psi &\leqq \int_0^t S_{t-s}[S_s\psi]^{1+\beta}\,ds \\ &\leqq \int_0^t S_{t-s}S_s\psi^{1+\beta}\,ds, \text{ by Jensen's inequality} \\ &= tS_t\psi^{1+\beta}. \end{aligned}$$

(ii) First note that $0 \leqq u(t) = U_t\psi \leqq \|U_t\psi\|_\infty \leqq \|S_t\psi\|_\infty \leqq \|\psi\|_\infty = c^{1/\beta}$. Letting $W_t = e^{-ct}S_t$ and $w(t) = W_t\psi$, we wish to show that $0 \leqq v(t) \equiv u(t) - w(t)$. By continuity it suffices to show this for $\psi \in D(\Delta)_+$.

Then w satisfies: $\dot{w}(t) = \Delta_\alpha w(t) - c w(t)$

$w(0) = \psi$

and v satisfies: $\dot{v}(t) = \Delta_\alpha v(t) - u^{1+\beta}(t) + c w(t)$

$= \Delta_\alpha v(t) - c\, v(t) + [c - u^\beta(t)]\, u(t)$

$v(0) = 0.$

Casting the last equation into integral (mild) form:

$$v(t) = \int_0^t W_{t-s}([c - u^\beta(s)]\, u(s))\, ds,$$

which is non-negative since $0 \leqq u(s) \leqq c^{1/\beta}$ for all s. Q.E.D.

In the next lemma we summarize some standard facts concerning the symmetric stable densities p_t^α. We omit the proofs here; the proof of (i) is given in [8], and details for (ii) and (iii) were given in [12].

Lemma 2.2. (i) *For* $0 < \alpha \leqq 2$, $t > 0$, p_t^α is smooth, symmetric and unimodal.

(ii) *For* $0 < \alpha \leqq 2$, $t > 0$, $x \in R^d$: $p_t(t^{1/\alpha}x) = t^{-d/\alpha} p_1^\alpha(x)$.

(iii) *For* $0 < \alpha < 2$, $x \in R^d$ *with* $|x| \geqq 1$:

$$p_1^\alpha(x) \leqq \frac{c}{|x|^{d+\alpha}},\ c \geqq 0 \text{ a constant, depending on } \alpha.$$

For $\alpha = 2$, $x \in R^d$: $p_1^2(x) = (4\pi)^{-d/2} \exp(-x^2/4)$.

Proposition 2.3. *Let* $p > d$; *and* $p < d + \alpha$ *in case* $\alpha < 2$. *If* $\psi \in C_p(R^d)_+$ *with* $\lim_{|x| \to +\infty} |x|^p \psi(x) = \ell \in R^1$ *then also* $\lim_{|x| \to +\infty} |x|^p v(t,x) = \lim_{|x| \to +\infty} |x|^p u(t,x) = \ell$ *where* $v(t,x) = S_t\psi(x)$ *and* $u(t,x) = U_t\psi(x)$.

Proof.

$$v(t,x) = \int_{R^d} p_t^\alpha(x-y)\psi(y)\,dy = \int_{R^d} p_t^\alpha(y)\psi(x+y)\,dy,$$

so Fatou's lemma implies that $\liminf_{|x| \to +\infty} |x|^p v(t,x) \geqq \ell$. To obtain an upper bound, fix $0 < k < 1/2$ and decompose $R^d = B_1 \cup B_2$ where $B_1 = \{y \in R^d : |x-y| \leqq k|x|\}$ and $B_2 = B_1^c$. Note that for $y \in B_1$: $|y| \geqq (1-k)|x|$. Given $\varepsilon > 0$ choose R such that $|y|^p \psi(y) \leqq \ell + \varepsilon$ for $|y| \geqq R/2$. Then for $|x| > R$, $|x|^p v(t,x) = I_1 + I_2$ where

$$I_1 = \int_{B_1} p_t^\alpha(x-y)(|x|/|y|)^p(|y|^p\psi(y))\,dy \leqq (1-k)^{-p}(\ell + \varepsilon)$$

and

$$I_2 = |x|^p \int_{B_2} p_t^\alpha(x-y)\psi(y)\,dy \leqq |x|^p \cdot \begin{cases} c\,|kxt^{-1/\alpha}|^{-d-\alpha} \cdot \|\psi\|_1, & 0 < \alpha < 2 \\ (4\pi t)^{-d/2} \exp(-kx^2/4t) \cdot \|\psi\|_1, & \alpha = 2 \end{cases}$$

by Lemma 2.2, where $\|\psi\|_1=\int_{R^d}\psi(y)\,dy$. Therefore $\limsup_{|x|\to+\infty} I_1\leqq(1-k)^{-p}(\ell+\varepsilon)$ and $\limsup_{|x|\to+\infty} I_2=0$. As k,ε are arbitrary $\limsup_{|x|\to+\infty}|x|^p v(t,x)\leqq\ell$, and together with the lower bound we conclude that $\lim_{|x|\to+\infty}|x|^p v(t,x)=\ell$.

By Lemma 2.1 (i), $u(t,x)\leqq v(t,x)$ so $\limsup_{|x|\to+\infty}|x|^p u(t,x)\leqq\ell$. From the same lemma $u(t,x)\geqq v(t,x)-tS_t\psi^{1+\beta}(x)$. From the first part of the proof $\lim_{|x|\to+\infty}|x|^{p(1+\beta)}S_t\psi^{1+\beta}(x)=\ell^{1+\beta}$; therefore $\liminf_{|x|\to+\infty}|x|^p u(t,x)\geqq\ell$. Thus

$$\lim_{|x|\to+\infty}|x|^p u(t,x)=\ell. \qquad \text{Q.E.D.}$$

Lemma 2.4. *Let $p>d$; and $p<d+\alpha$ in case $0<\alpha<2$. If $\psi\in C_p(R^d)_+$ is such that $\lim_{|x|\to+\infty}|x|^p\psi(x)$ exists, then for all $\delta>0$ there exists an $\eta>0$ such that for $0\leqq t\leqq\eta$: $|U_t\psi-\psi|\leqq\delta\cdot\phi_p$ where $\phi_p(x)=(1+|x|^p)^{-1}$. This estimate is also valid for S_t, in place of U_t.*

Proof. We give an outline of the proof; the details were given in [12]. First we make some reductions. From Lemma 2.1 (i):

$$|U_t\psi-\psi|\leqq|U_t\psi-S_t\psi|+|S_t\psi-\psi|\leqq ctS_t\psi+|S_t\psi-\psi|$$

so it suffices to prove the lemma with S_t in place of U_t. Next, ψ/ϕ_p is continuous with a limit at ∞, so for any $\delta>0$ we can find a smooth δ-uniform approximation ψ_1 of it having bounded first derivatives; set $\tilde\psi=\psi_1\phi_p$. Then $|\tilde\psi-\psi|=\phi_p|\psi_1-\psi/\phi_p|<\delta\phi_p$ and

$$\begin{aligned}|S_t\psi-\psi|&\leqq|S_t\tilde\psi-\tilde\psi|+S_t|\tilde\psi-\psi|+|\tilde\psi-\psi|\\&\leqq|S_t\tilde\psi-\tilde\psi|+\delta S_t\phi_p+\delta\phi_p.\end{aligned}$$

Therefore it suffices to prove the lemma with S_t in place of U_t and ψ smooth with $|\nabla\psi|\leqq c\phi_p$ for some constant c (∇ denotes the gradient), as was the case for $\tilde\psi$.

In this case we write:

$$|(S_t\psi(x)-\psi(x)|\leqq\int_{B_1}+\int_{B_2}+\int_{B_3}p_t^\alpha(x-y)|\psi(y)-\psi(x)|\,dy$$

where

$$B_1=\{y\in R^d\colon |x-y|<\ell t^{1/\alpha}\}$$
$$B_2=\{y\in R^d\colon \ell t^{1/\alpha}<|x-y|<|x|/2\}$$
$$B_3=\{y\in R^d\colon |x|/2<|x-y|\}$$

and where ℓ is chosen so that $\int_{\{|y|>\ell\}}p_1(y)\,dy$ is sufficiently small.

Then given $\delta>0$, the integrals over B_1 (using the mean value theorem on ψ) and B_2 can be majorised by $\delta\phi_p$ for t sufficiently small; as well as that over B_3 for $|x|\geqq 2$ using Lemma 2.1, and the stochastic continuity of p_t^α then extends the estimate to all $x\in R^d$. Q.E.D.

Corollary 2.4. *For each $T>0$ there is a constant $C(T)>0$ such that for all $t\in[0,T]$: $S_t\phi_p \leqq C(T)\cdot\phi_p$.*

Proof. Choose η as in Lemma 2.4 corresponding to $\delta=1$, say, and an integer $n>T/\eta$. Then for $t\in[0,T]$:

$$S_t\phi_p=(S_{\frac{t}{n}})^n\phi_p\leqq(1+\delta)^n\phi_p=2^n\cdot\phi_p;$$

take $C(T)=2^n$. Q.E.D.

§ 3. A representation for the Weighted Occupation Time

It was shown in § 1 that there exists for $p>d$ an $M_p(R^d)$-valued multiplicative Markov process X_t, with cadlag sample paths, whose Laplace functional is given by

$$E_\mu[\exp(-\langle\psi,X_t\rangle)]=\exp(-\langle U_t\psi,\mu\rangle);\qquad \psi\in C_p(R^d)_+,\ \mu\in M_p(R^d). \tag{3.1}$$

Here U_t is the semigroup on $C_p(R^d)_+$ associated with the evolution equation:

$$\begin{aligned}\dot{u}(t)&=\Delta_\alpha u(t)-u^{1+\beta}(t)\\ u(0)&=\psi\end{aligned} \tag{3.2}$$

where $0<\alpha\leqq 2$, $0<\beta\leqq 1$; and $p<d+\alpha$ in case $\alpha<2$. In other words, $U_t\psi$ satisfies the mild form of Eq. (3.2), namely $u(t)=S_t\psi-\int_0^t S_{t-s}u^{1+\beta}(s)\,ds$. The constraint $p>d$ will be tacitly assumed throughout the rest of the paper, and also $p<d+\alpha$ if $0<\alpha<2$, whenever the space $M_p(R^d)$ occurs.

Definition 3.1. We define the weighted occupation time process Y_t (permanent notation) by

$$\langle\psi,Y_t\rangle=\int_0^t\langle\psi,X_s\rangle\,ds,\qquad \text{for } \psi\in C_p(R^d).$$

Note that the temporal integral can be taken a.s. in the sense of Riemann since we have enough regularity on the sample paths of X_t.

With the qualifications on p as stated above, we have the following representation for the Laplace functional of (X_t, Y_t).

Theorem 3.1. *Let $\mu\in M_p(R^d)$ and $\phi,\psi\in C_p(R^d)_+$, then*

$$E_\mu[\exp(-\langle\psi,X_t\rangle-\langle\phi,Y_t\rangle)]=\exp[-\langle U_t^\phi\psi,\mu\rangle],\qquad t\geqq 0, \tag{3.3}$$

where $U_t^\phi(\psi)$ is the strongly continuous semigroup associated with the evolution equation

$$\begin{aligned}\dot{u}(t)&=\Delta_\alpha u(t)-u^{1+\beta}(t)+\phi\\ u(0)&=\psi\end{aligned} \tag{3.4}$$

($U_t^\phi(\psi)$ will satisfy (3.4) provided $\phi,\psi\in D(\Delta_\alpha)_+$ as well).

Proof. The existence of U_t^ϕ can be established as in the appendix. Since ϕ is fixed here, we shall suppress the dependence of U_t^ϕ on ϕ and write simply U_t. Taking a Riemann sum approximation:

$$E_\mu\left(\exp\left[-\langle\psi, X_t\rangle - \int_0^t \langle\phi, X_s\rangle\, ds\right]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-1}\langle\phi, X_{\frac{n}{N}t}\rangle \frac{t}{N} - \left\langle \frac{t}{N}\phi + \psi, X_t\right\rangle\right]\right).$$

At this point we introduce two more semigroups. The first, V_t, is the one associated with the homogeneous version of Eq. (3.4) i.e. $V_t\psi$ is the mild solution of the equation

$$\dot v(t) = \Delta_\alpha v(t) - v^{1+\beta}(t)$$
$$v(0) = \psi$$

(V_t also has another notation: U_t^0).

The second one, W_t is given simply by $W_t\psi = \psi + t\phi$. $W_t\psi$ is thus the solution of the ordinary differential equation:

$$\dot w(t) = \phi,$$
$$w(0) = \psi.$$

Using the Markov property and the relation (3.1) we can continue to calculate:

$$E_\mu\left(\exp\left[-\langle\psi, X_t\rangle - \int_0^t \langle\phi, X_s\rangle\, ds\right]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-1}\left\langle\frac{t}{N}\phi, X_{\frac{n}{N}t}\right\rangle\right]\right.$$
$$\left.\cdot E_\mu\left[\exp(-\langle W_{\frac{t}{N}}\psi, X_t\rangle) \mid X_s, 0 \leqq s \leqq \frac{N-1}{N}t\right]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-1}\left\langle\frac{t}{N}\phi, X_{\frac{n}{N}t}\right\rangle\right] \cdot E_{X_{\frac{N-1}{N}t}}[\exp(-\langle W_{\frac{t}{N}}\psi, X_{\frac{t}{N}}\rangle)]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-1}\left\langle\frac{t}{N}\phi, X_{\frac{n}{N}t}\right\rangle\right] \cdot \exp[-\langle V_{\frac{t}{N}} W_{\frac{t}{N}}\psi, X_{\frac{N-1}{N}t}\rangle]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-2}\left\langle\frac{t}{N}\phi, X_{\frac{n}{N}t}\right\rangle\right] \cdot \exp\left[-\left\langle\frac{t}{N}\phi + V_{\frac{t}{N}} W_{\frac{t}{N}}\psi, X_{\frac{N-1}{N}t}\right\rangle\right]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-2}\left\langle\frac{t}{N}\phi, X_{\frac{n}{N}t}\right\rangle\right]\right.$$
$$\left.\cdot E_\mu\left[\exp(-\langle W_{\frac{t}{N}} V_{\frac{t}{N}} W_{\frac{t}{N}}\psi, X_{\frac{N-1}{N}t}\rangle) \mid X_s, 0 \leqq s \leqq \frac{N-2}{N}t\right]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-2}\left\langle\frac{t}{N}\phi, X_{\frac{n}{N}t}\right\rangle\right] \cdot E_{X_{\frac{N-2}{N}t}}[\exp(-W_{\frac{t}{N}} V_{\frac{t}{N}} W_{\frac{t}{N}}\psi, X_{\frac{t}{N}}\rangle)]\right)$$
$$= \lim_{N\to\infty} E_\mu\left(\exp\left[-\sum_{n=1}^{N-2}\left\langle\frac{t}{N}\phi, X_{\frac{n}{N}t}\right\rangle\right] \cdot \exp[-\langle V_{\frac{t}{N}} W_{\frac{t}{N}} V_{\frac{t}{N}} W_{\frac{t}{N}}\psi, X_{\frac{N-2}{N}t}\rangle]\right).$$

Repeating this procedure consisting of: borrowing the last term of the first exponential, conditioning up until the previous time, using the Markov property and the relation (3.1), we finally arrive at the expression

$$E_\mu\left(\exp\left[-\langle\psi, X_t\rangle-\int_0^t\langle\phi, X_s\rangle\,ds\right]\right)=\lim_{N\to\infty}\exp[-\langle(V_{\frac{t}{N}}W_{\frac{t}{N}})^N\psi,\mu\rangle]$$
$$=\exp[-\langle U_t\psi,\mu\rangle].$$

The Trotter-Lie product formula $U_t=\lim_{N\to\infty}(V_{\frac{t}{N}}W_{\frac{t}{N}})^N$ can be proven as in Chorin et al. ([3], §2 or §5) on $C_0(R^d)_+$ (strong convergence). The interchange of limit and integral above is justified by dominated convergence as follows. With $\phi,\psi\leqq c\cdot\phi_p$:

$$(V_{\frac{t}{N}}W_{\frac{t}{N}})^N\psi\leqq(S_{\frac{t}{N}}W_{\frac{t}{N}})^N\psi=S_t\psi+\frac{t}{N}\sum_{m=1}^{N}S_{\frac{mt}{N}}\phi\leqq c\cdot C(t)\cdot[1+t]\cdot\phi_p,$$

by Corollary 2.4. Q.E.D.

A more useful tool is the following time dependent generalization of the weighted occupation time process Y_t: $\int_0^t\langle\phi(s), X_s\rangle\,ds$. We can also obtain a representation for it, using the previous theorem.

Theorem 3.2. *Let $\mu\in M_p(R^d)$, and $\phi: R^1_+\to C_p(R^d)_+$ be right continuous and piecewise continuous such that for each $T>0$ there is a $k>0$ such that $\phi(s)\leqq k\cdot(1+|x|^p)^{-1}$ for all $s\in[0,T]$. Then*

$$E_\mu\left[\exp\left(-\int_0^T\langle\phi(s), X_s\rangle\,ds\right)\right]=\exp[-\langle U(T,0)(0),\mu\rangle]\tag{3.5}$$

where $U(t,s)$ is the propagator associated with (i.e. $U(t,s)\psi$ satisfies in the mild sense) the evolution equation:

$$\begin{aligned}\dot u(t)&=\Delta_\alpha u(t)-u^{1+\beta}(t)+\phi(T-t),\qquad s\leqq t\leqq T\\ u(s)&=\psi.\end{aligned}\tag{3.6}$$

Proof. The proof is somewhat similar to the previous one so the details will be omitted. One first notes that $U(t,s)\psi$ depends continuously on ϕ so that it suffices to verify (3.5) in the case that ϕ is a right continuous step function. In that case one can condition backwards as before and apply Theorem 3.1 on each interval of constancy of ϕ. Conditioning backwards forces one to consider the time-reversed forms of (3.4) and (3.6). In effect, if $[0,T]$ is partitioned into $0=s_0<s_1<\ldots<s_N=T$ with $\phi=\phi_n$ on $[s_n,s_{n+1})$ then we obtain (3.5) with $U(T,0)(0)$ replaced by $U^{\phi_1}_{s_1-s_0}U^{\phi_2}_{s_2-s_1}\ldots U^{\phi_N}_{s_N-s_{N-1}}(0)$ which is, term by term, equal to $U(T-s_0,T-s_1)\,U(T-s_1,T-s_2)\ldots U(T-s_{N-1},T-s_N)(0)$ which in turn collapses to $U(T,0)(0)$ by the propagator property. Q.E.D.

In the next section we shall need a description of $u=\lim_{t\to\infty}u(t)$ where $u(t)$ satisfies Eq. (3.4) with $\psi=0$. The following theorem demonstrates the expectation that u satisfies the steady state equation corresponding to (3.4).

Theorem 3.3. *Let $\phi \in C_c(R^d)_+$. Then the solution $u(t)$ of Eq.* (3.4), *with $\psi=0$, increases with t to a function u. Moreover the convergence is uniform and u satisfies the equation*

$$\Delta_\alpha u - u^{1+\beta} + \phi = 0 \tag{3.7}$$

if $d>\alpha$ in the case $0<\alpha<2$ and without restriction if $\alpha=2$.

Proof. That $u(t)$ increases to a function u, as $t\to\infty$, can be seen from (3.3) with $\mu=\delta_x$, a point mass at $x\in R^d$. Also $0<u(t)\leqq(\|\phi\|_\infty)^{1/1+\beta}$, which results from a simple application of the maximum principle. Thus u is uniformly bounded and positive.

Now it follows from the Appendix that $\dot{u}(t)$ is a (non-negative) solution of the mild equation:

$$\dot{u}(t) = S_t\phi - \int_0^t S_{t-s}[(1+\beta)\,u^\beta(s)\cdot\dot{u}(s)]\,ds.$$

Since S_t is a non-negative semigroup, $0\leqq\dot{u}(t)\leqq S_t\phi\leqq K_{d,\alpha}t^{-d/\alpha}\cdot\|\phi\|_1$ where $\|\phi\|_1$ is the L^1-norm of ϕ and $K_{d,\alpha}$ is some constant. To obtain the last inequality observe that

$$\begin{aligned} t^{d/\alpha} S_t\phi(x) &\equiv t^{d/\alpha}\int_{R^d} p^\alpha(t,y)\,\phi(x-y)\,dy \\ &= t^{d/\alpha}\int_{R^d} t^{-d/\alpha}p^\alpha(1,t^{-1/\alpha}y)\,\phi(x-y)\,dy \qquad \text{(c.f. Lemma 2.2(ii))} \\ &\uparrow p^\alpha(1,0)\cdot\int_{R^d}\phi(x-y)\,dy, \quad \text{as } t\uparrow\infty \qquad (p^\alpha(1,\cdot)\text{ is unimodal}). \end{aligned}$$

Thus if $d>\alpha$ and $K=K_{d,\alpha}\cdot\|\phi\|_1$ then for $t>s$:

$$u(t)-u(s) = \int_s^t \dot{u}(r)\,dr \leqq K\int_s^t r^{-d/\alpha}\,dr \leqq [K/(d/\alpha-1)]\,s^{1-d/\alpha}.$$

This establishes the uniform convergence of $u(t)$ to u as $t\to\infty$. But from the equation $\Delta_\alpha u(t) = u^{1+\beta}(t)-\phi+\dot{u}(t)$ we then obtain the uniform convergence of $\Delta_\alpha u(t)$ to $u^{1+\beta}-\phi$ as $t\to\infty$. Since $u,\phi\in C_0(R^d)$, the space of continuous functions on R^d vanishing at infinity, and Δ_α is a closed operator there, we conclude that u satisfies Eq. (3.7).

In the case $\alpha=2$, $d=1$ or 2 we must argue indirectly. Note that by the argument of the previous paragraph, it suffices to show that $u(t)\to u$ uniformly as $t\to\infty$. In turn it suffices to show that the convergence is uniform on compact subsets of R^d, and that $u(x)\to 0$ as $|x|\to\infty$. The latter follows from the estimate $u(t,x)\leqq c|x|^{-2/\beta}$, uniform in $x\in R^d$ and $t\in R^1_+$, for sufficiently large c depending only on β,ϕ. This estimate is easily derived by a comparison argument: for $|x|\geqq R$ and c large, $\Delta(u(t)-v)>u^{1+\beta}(t)-v^{1+\beta}$, and $u(t)-v$ for $|x|\leqq R$; so the maximum of $u(t)-v$ on $|x|\geqq R$ cannot be positive.

To establish the former objective we can appeal to some standard estimates in the theory of Sobolev spaces. For any bounded region $\Omega\subset R^d$ we have the compact embedding $W^{1,q}(\Omega)\hookrightarrow C^{0,\gamma}(\bar{\Omega})$, where $q>d$ and $0<\gamma<1-d/q$ (c.f. Ladyzhenskaya and Ural'tseva [17; Ch. 2, Theorem 2.1]. Since $u(t)$ and $\Delta u(t)$

are uniformly bounded we can conclude from the inequalities (2.22) and (8.6) of p. 48 and 171, respectively, in [17] that $\{u(n)\}_{n=1}^{\infty}$ is bounded in $W^{1,q}(\Omega)$ and hence precompact in $C^{0,\gamma}(\bar{\Omega})$. Therefore a subsequence $\{u(n_j)\}_{j=1}^{\infty}$, and hence $u(t)$, converges uniformly on $\bar{\Omega}$. Since Ω was arbitrary we are done. Q.E.D.

§ 4. Total Weighted Occupation Time

An obvious question one should ask is whether or not the total weighted occupation time, $\int_0^\infty X_t(B)\,dt$, for a bounded Borel set $B \subset R^d$, is finite. In this section we find that the answer depends only upon the values of α, β and d, in the case when the initial condition behaves like λ = Lebesgue measure, and where B has non-empty interior. In this case we can clearly reduce the consideration to the quantity $\int_0^\infty \langle \phi, X_t \rangle\,dt$ where $\phi \in C_c(R^d)_+$.

For the remainder of this section we restrict attention to the case $\alpha = 2$. Now, from Theorem 3.1 we know that

$$E_\lambda\left[\exp\left(-\int_0^t \langle \phi, X_s \rangle\,ds\right)\right] = \exp(-\langle u(t), \lambda \rangle), \qquad \phi \in C_c(R^d)_+ \cap D(\Delta)$$

where $u(t)$ is the solution of the evolution equation

$$\begin{aligned} \dot{u}(t) &= \Delta u(t) - u^{1+\beta}(t) + \phi \\ u(0) &= 0. \end{aligned} \tag{4.1}$$

Therefore

$$E_\lambda\left[\exp\left(-\int_0^\infty \langle \phi, X_s \rangle\,ds\right)\right] = \exp(-\langle u, \lambda \rangle) \tag{4.2}$$

where $u = \lim_{t \to \infty} u(t)$, which exists by Theorem 3.3. Thus $\int_0^\infty \langle \phi, X_s \rangle\,ds = \infty$ a.s. if $\langle u, \lambda \rangle = \infty$. In Theorem 4.4 we will see that the converse is also true.

Since Δ and λ are translation invariant we can assume that the origin of R^d belongs to B and also that ϕ is a radial function i.e., $x, y \in R^d$ with $|x| = |y|$ implies $\phi(x) = \phi(y)$. Thus we have reduced the question of almost sure integrability of $X_s(B)$ to that of $u(|x|) \equiv \lim_{t \to \infty} u(t, |x|)$, $x \in R^d$, where $u(t)$ satisfies the radially symmetric version of (4.1):

$$\begin{aligned} \dot{u}(t, r) &= u''(t, r) + \frac{(d-1)}{r} u'(t, r) - u^{1+\beta}(t, r) + \phi(r), \qquad r \in R_+^1,\ \phi \in C_c^2(R_+^1)_+ \\ u(0, r) &= 0 \end{aligned} \tag{4.3}$$

($C_c^2(R_+)_+$ is the class of non-negative, twice continuously differentiable functions on R_+^1, having compact support) so that by Theorem 3.3 $u(r)$ satisfies:

$$u''(r) + \frac{(d-1)}{r} u'(r) - u^{1+\beta}(r) + \phi(r) = 0.$$

Finally, since u is bounded, the integrability of u is equivalent to its integrability in the exterior of any bounded region on R^d. In particular we can test for integrability off the support of ϕ. There $u(r)$ is a strictly positive and bounded solution of the ordinary differential equation:

$$u''(r)+\frac{(d-1)}{r}\,u'(r)-u^{1+\beta}(r)=0, \qquad r\geqq r_0>0. \tag{4.4}$$

The integrability of $u(|\cdot|)$ can be decided from its asymptotic behaviour which will be described shortly. Thus the original problem is reduced to one in the theory of ordinary differential equations, into which we now make a detour. We first dispose of the one dimensional case which is exactly solvable. The derivation is easy and so will be omitted.

Proposition 4.1. *The positive bounded solutions of Eq.* (4.4), *in dimension* $d=1$, *form a one parameter family:*

$$\left[c+\frac{\beta}{2}\left(\frac{2}{2+\beta}\right)^{1/2}\cdot r\right]^{-2/\beta}$$

depending upon the constant c.

For $d>1$ (in fact for $d=1$ as well), the asymptotics of the solution of Eq. (4.3) can be handled as in Sawyer and Fleischman [20] where the results of the this section were obtained for the classical branching Brownian motion. There were many details omitted there. These were accounted for in [12] (see also the historical remark at the end of this section). We record the result in the form of a theorem.

Theorem 4.2. *The solution of Eq.* (4.3) *tends in ratio (denoted by* "$\sim$") *as* $r\to\infty$ *to the following asymptote:*

$$\begin{aligned} &d=1: && u(r)\sim[c+(\beta/2)[2/(2+\beta)]^{1/2}\cdot r]^{-2/\beta}\\ &d\geqq 2: (d-2)\,\beta<2: && u(r)\sim c_{\beta,d}\,r^{-2/\beta}\\ &\qquad (d-2)\,\beta=2: && u(r)\sim c_d[r^2\log r]^{-1/\beta}\\ &\qquad (d-2)\,\beta>2: && u(r)\sim c_{\beta,d}\,r^{2-d}. \end{aligned} \tag{4.5}$$

From the description in (4.5) we can answer the question posed in this section almost immediately.

Theorem 4.3. *The measure-valued branching random motion* X_t *with* $\alpha=2$ *and* $X_0=\lambda$ *spends an infinite total weighted occupation time in bounded Borel subsets of* R^d*, having non-empty interior, iff* $\beta d\geqq 2$. *(If* $\beta d<2$ *the condition on the interior is clearly superfluous.)*

Proof. We have seen from (4.2) that $\int_0^\infty \langle\phi, X_t\rangle\,dt=\infty$ a.s. if $\langle u,\lambda\rangle=\infty$ where u is the solution to Eq. (4.1). From (4.5), with r replaced by $|x|$, we see that this is only the case when $\beta d\geqq 2$.

To deal with the complimentary case $\beta d<2$, we return to (4.2) and observe that:

$$P_\lambda\left[\int_0^\infty \langle\phi, X_t\rangle\, dt<\infty\right]=\lim_{\theta\to 0^+} E_\lambda\left[\exp\left(-\theta\int_0^\infty \langle\phi, X_t\rangle dt\right)\right]=\lim_{\theta\to 0^+}\exp(-\langle u_\theta, \lambda\rangle).$$

Here u_θ satisfies Eq. (4.1) with ϕ replaced by $\theta\phi$. It is easy to see, via (4.2) with λ replaced by an arbitrary point mass δ_x, $x\in R^d$, that u_θ decreases with θ. Moreover it decreases to 0 since solutions of Eq. (4.1) are bounded from above by $\|\theta\phi\|_\infty^{1/1+\beta}$. Furthermore, if $\beta d<2$ then $\langle u, \lambda\rangle<\infty$. Thus by Lebesgue's Dominated Convergence theorem, $\lim_{\theta\to 0^+}\exp(-\langle u_\theta, \lambda\rangle)=1$. Q.E.D.

Remark 4.4. It is clear from the nature of the proof that the only information needed about X_0 is its "temperence" i.e. its asymptotic growth rate on expanding balls. If this is known then an analogous result can be obtained with an inequality involving β, d and the temperence. As such, Theorem 3.3 should be valid if $X_0=N$ is a Poisson random field with intensity λ, Lebesgue measure; and it is. Indeed, for $u\in C_0(R^d)_+$ $E[\exp(-\langle u, N\rangle)]=\exp(-\langle 1-e^{-u}, \lambda\rangle)$ and for $0<u<1$: $0<u-u^2/2<1-e^{-u}<u$. From (3.5) u^2 is always integrable, and so again $\int_0^\infty \langle\phi, X_t\rangle\, dt<+\infty$ iff $\langle u, \lambda\rangle<+\infty$.

Remark 4.5. It is not unreasonable to conjecture that when $\alpha<2$, Theorem 3.3 remains valid with the inequality $\beta d\geqq 2$ replaced by $\beta d\geqq\alpha$. This would follow if an analogue of (4.5) could be established with all 2's there replaced by α's.

Historical Remark 4.6. Equations of type (4.4) seem to have first arisen in the kinetic theory of gases, in a cosmological context, around the turn of the century. Investigations into the qualitative behaviour of the solutions began with the work of Emden and continued in a series of papers by Fowler (see Fowler [9] and the references therein). In [9], Eq. (4.4) is generalised and analyzed for what would correspond to rational values of β. However the 2-dimensional ($d=2$) case is not covered by the methods used there.

Also, the asymptotics of non-radial solutions to $\Delta u=u\cdot|u|^\beta$ in an exterior domain has been worked out with methods entirely different from those of Fowler [9], and Sawyer and Fleischman [20], by L. Veron (private communication).

§5. Second Order Asymptotics: High Dimensions

In the previous section we established that for $\alpha=2$ and $\phi\in C_c(R^d)_+$, $\int_0^\infty\langle\phi, X_t\rangle dt<\infty$ iff $\beta d<2$. In the present section we investigate the situation $(d-\alpha)\beta>\alpha$ more closely, leaving the final case consisting of $\beta d\geqq 2$, $(d-2)\beta\leqq 2$ for later. More specifically we shall prove a limit theorem for the random oscillations of $\int_0^t\langle\phi, X_s\rangle\, ds$ about its mean (for $\phi\in\bar{S}(R^d)_+$) which will reveal an interesting spatial structure, as concerns ϕ. Before proceeding, we first calculate the value of this mean.

Lemma 5.1. *λ being Lebesgue measure on R^d,*

$$E_\lambda\left[\int_0^t \langle \phi, X_s\rangle\, ds\right] = \left\langle \int_0^t S_{t-s}\phi\, ds, \lambda\right\rangle = t\langle \phi, \lambda\rangle, \ \phi \in C_p(R^d)_+.$$

Proof. As usual, if $\phi \in D(\Delta)$:

$$\begin{aligned} E_\lambda\left[\int_0^t \langle \phi, X_s\rangle\, ds\right] &= -\frac{d^+}{d\theta}\bigg|_{\theta=0} E_\lambda\left[\exp\left(-\theta \int_0^t \langle \phi, X_s\rangle\, ds\right)\right] \\ &= -\frac{d^+}{d\theta}\bigg|_{\theta=0} \exp(-\langle u_\theta(t), \lambda\rangle) \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \dot{u}_\theta(t) &= \Delta_\alpha u_\theta(t) - u_\theta^{1+\beta}(t) + \theta\phi \\ u_\theta(0) &= 0 \end{aligned}$$

by Theorem 3.1.

Now $u_0(t) \equiv 0$ and by Theorem B of the Appendix, $v(t) = \frac{d^+}{d\theta}\big|_{\theta=0} u_\theta(t)$ satisfies:

$$\begin{aligned} \dot{v}(t) &= \Delta_\alpha v(t) - (1+\beta)\, u_0^\beta(t) + \phi \\ v(0) &= 0. \end{aligned}$$

Therefore $v(t) = \int_0^t S_{t-s}\phi\, ds$ and from (5.1):

$$E_\lambda\left[\int_0^t \langle \phi, X_s\rangle\, ds\right] = \left\langle \frac{d^+}{d\theta}\bigg|_{\theta=0} u_\theta(t), \lambda\right\rangle = \left\langle \int_0^t S_{t-s}\phi\, ds, \lambda\right\rangle. \quad \text{Q.E.D.}$$

Before coming to the main theorems of this section we dispose of two more lemmas of which we shall have need in the proofs.

Lemma 5.2. *Let $x, y, y_1 \in R^1$ satisfy $-y_1 < x < 0$ and $x + y > 0$. Then with $0 < \beta \leqq 1$:*

$$0 < y^{1+\beta} - (x+y)^{1+\beta} < y y_1^\beta + y_1 y^\beta.$$

Proof. It is easily established by means of the calculus that for $z \geqq 1$, $0 < z^\beta - (z-1)^\beta \leqq 1$. Setting $z = y/(-x)$ we obtain the inequality: $0 < y^\beta - (x+y)^\beta \leqq (-x)^\beta$. Therefore,

$$\begin{aligned} &0 < y^{1+\beta} - (x+y)^{1+\beta} \\ &\quad = y[y^\beta - (x+y)^\beta] - x(x+y)^\beta < y(-x)^\beta + (-x)\, y^\beta < y y_1^\beta + y_1 y^\beta. \quad \text{Q.E.D.} \end{aligned}$$

Lemma 5.3. *If ψ is a bounded measurable function on R^d which is $O(|x|^{-p})$ as $|x| \to \infty$ for some $p > d$. Then*

$$\phi(x) = \int_{R^d} \frac{\psi(y)}{|x-y|^{d-\alpha}}\, dy \quad (d > \alpha)$$

is also bounded, and $O(|x|^{\alpha-d})$ *as* $|x|\to\infty$. *Moreover,*

$$\|(1+|x|^{d-\alpha})\,\phi(x)\|_\infty \leqq K\cdot\|(1+|x|^p)\,\psi(x)\|_\infty$$

for some $K>0$.

Proof. For the boundedness, one can split the integral over the unit ball in R^d and its complement, and then make the obvious estimates. For the asymptotic result, one can split the integral over $\left\{y\in R^d\colon |y-x|\leqq\frac{|x|}{2}\right\}$ and its complement, and make the obvious estimates. We omit the straightforward details. Q.E.D.

We now prove convergence of the finite dimensional distributions of $t^{-1/1+\beta}[Y_t-E_\lambda Y_t]$. To this end we define for fixed $\phi\in\bar{S}(R^d)_+$ the numerical process

$$z_T=T^{-1/1+\beta}\left(\int_0^T\langle\phi,X_t\rangle\,dt-E_\lambda\left[\int_0^T\langle\phi,X_t\rangle\,dt\right]\right).$$

Theorem 5.4. *If* $(d-\alpha)\beta>\alpha$ *then* z_T *converges weakly, as* $T\to\infty$, *to a stable law of index* $1+\beta$:

$$E_\lambda[e^{-\theta z_\infty}]=\exp\left[\theta^{1+\beta}\cdot\int_{R^d}\left[c_{d,\alpha}\int_{R^d}\frac{\psi(y)}{|x-y|^{d-\alpha}}\,dy\right]^{1+\beta}dx\right],$$

$$c_{d,\alpha}=\Gamma\left(\frac{d-\alpha}{2}\right)\cdot\left[2^\alpha\pi^{d/2}\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1},\qquad \theta\geqq 0.$$

(Γ *denotes the usual gamma function*).

Proof. By Theorem 3.1, the Laplace transform of $\int_0^T\langle\phi,X_t\rangle\,dt$ is given through

$$E_\lambda\left[\exp\left(-\int_0^T\langle\phi,X_t\rangle\,dt\right)\right]=\exp(-\langle u(T),\lambda\rangle)$$

where $u(T)$ is the solution of the evolution equation:

$$\begin{aligned}&\dot u(t)=\Delta_\alpha u(t)-u^{1+\beta}(t)+\phi,\qquad \phi\in D(\Delta)_+,\\ &u(0)=0.\end{aligned}$$

Therefore by Lemma 5.1, the bilateral Laplace transform of z_T is given by:

$$E_\lambda[e^{-z_T}]=\exp(-\langle v_T(T),\lambda\rangle),\qquad v_T(t)\equiv u_T(t)-w_T(t)$$

where $u_T(t)$ is the solution of the evolution equation

$$\begin{aligned}&\dot u_T(t)=\Delta_\alpha u_T(t)-u^{1+\beta}(t)+T^{-1/1+\beta}\cdot\phi\\ &u_T(0)=0\end{aligned}\tag{5.2}$$

and

$$w_T(t)\equiv T^{1/1+\beta}\int_0^t S_{t-s}\,\phi\,ds,$$

so that $w_T(t)$ is the solution of

$$\begin{aligned}\dot{w}_T(t) &= \Delta_\alpha w_T(t) + T^{-1/1+\beta}\phi \\ w_T(0) &= 0.\end{aligned} \tag{5.3}$$

With the aid of Lemmas 5.2 and 5.3 we shall calculate $\lim_{T\to\infty} \langle v_T(T), \lambda\rangle$. From (5.2) and (5.3) it follows that $v_T(t)$ satisfies the equation

$$\begin{aligned}\dot{v}_T(t) &= \Delta_\alpha v_T(t) - [v_T(t) + w_T(t)]^{1+\beta} \\ v_T(0) &= 0.\end{aligned}$$

Therefore

$$v_T(t) = -\int_0^t S_{t-s}[v_T(s) + w_T(s)]^{1+\beta}\, ds \tag{5.4}$$

and this holds, by continuity, even if $\phi \notin D(\Delta)$. Thus

$$\begin{aligned}\langle v_T(T), \lambda\rangle &= -\int_0^T \langle S_{T-t}[v_T(t) + w_T(t)]^{1+\beta}, \lambda\rangle\, dt \\ &= -\int_0^T \langle [v_T(t) + w_T(t)]^{1+\beta} - w_T^{1+\beta}(t), \lambda\rangle\, dt - \int_0^T \langle w_T^{1+\beta}(t), \lambda\rangle\, dt. \end{aligned} \tag{5.5}$$

The second integral in (5.5) converges, as $T \to \infty$, to the right quantity:

$$\begin{aligned}\lim_{T\to\infty} \int_0^T \langle w_T^{1+\beta}(t), \lambda\rangle\, dt &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left\langle \left[\int_0^t S_s \phi\, ds\right]^{1+\beta}, \lambda \right\rangle dt \\ &= \left\langle \left[\int_0^\infty S_s \phi\, ds\right]^{1+\beta}, \lambda \right\rangle\end{aligned}$$

by l'Hospital's rule and the Monotone Convergence Theorem,

$$= \int_{R^d} \left[c_{d,\alpha} \int_{R^d} \frac{\phi(y)}{|x-y|^{d-\alpha}}\, dy\right]^{1+\beta} dx.$$

The integral is finite by Lemma 5.3 provided $(d-\alpha)(1+\beta) > d$, i.e., $(d-\alpha)\beta > \alpha$.

Since we can always replace ϕ by $\theta\phi$ the main task is to show that the first integral in (5.5) tends to zero as $T \to \infty$.

Now, $u_T(t) \leqq w_T(t)$ follows from the integral form of (5.2). Therefore $v_T(t) \leqq 0$ and clearly $v_T(t) + w_T(t) = u_T(t) \geqq 0$. From (5.4) it follows that

$$v_T(t) \geqq -\int_0^t S_{t-s} w^{1+\beta}(s)\, ds.$$

Therefore we can apply Lemma 5.2 with $x = v_T(t)$, $y = w_T(t)$ and

$$y_1 = \int_0^t S_{t-s} w^{1+\beta}(s)\, ds,$$

yielding

$$0 \leqq \int_0^T \langle w_T^{1+\beta}(t) - [v_T(t) + w_T(t)]^{1+\beta}, \lambda \rangle \, dt$$
$$\leqq \int_0^T \left\langle w_T(t) \cdot \left[\int_0^t S_{t-s} w_T^{1+\beta}(s) \, ds \right]^\beta, \lambda \right\rangle dt$$
$$+ \int_0^T \left\langle w_T^\beta(t) \cdot \left[\int_0^t S_{t-s} w^{1+\beta}(s) \, ds \right], \lambda \right\rangle dt. \quad (5.6)$$

Each of the integrals in (5.6) can be shown to tend to zero as $T \to \infty$, using l'Hospital's rule and Lemma 5.3. We omit the details.

To conclude the proof a little care must be taken since we used a bilateral Laplace transform – although $\int_0^T \langle \phi, X_t \rangle \, dt$ is positive, in defining z_T we have subtracted a positive (non-random) term which increases to infinity as $T \to \infty$. Thus z_T takes on some negative values and z_∞, if it exists, possibly has values in all of R^1.

Let $\{t_n\}_{n=1}^\infty \subset R_+^1$ with $t_n \to \infty$ as $n \to \infty$. By a well-known selection principle [8] there exists a subsequence $\{t_{n_k}\}_{k=1}^\infty$ such that F_k converges vaguely to, say F, as $k \to \infty$, where F_k is the distribution of $z_{t_{n_k}}$ and F is some (possibly defective) distribution. Define, for $N \in \mathbb{N}_+$, the continuous function $c_N: R^1 \to [0, 1]$ which is 1 on $[-N, \infty)$, 0 on $(-\infty, -N-1]$ and affine on $[-N-1, -N]$. Then

$$\int_{R^1} e^{-\theta x} c_N(x) \, dF_k(x) \to \int_{R^1} e^{-\theta x} c_N(x) \, dF(x)$$

as $k \to \infty$. Therefore given $\varepsilon > 0$ there exists a $k(\varepsilon)$ such that if $k > k(\varepsilon)$ then

$$-\varepsilon + \int_{R^1} e^{-\theta x} c_N(x) \, dF_k(x) < \int_{R^1} e^{-\theta x} c_N(x) dF(x) < \int_{R^1} e^{-\theta x} c_N(x) \, dF_k(x) + \varepsilon. \quad (5.7)$$

Therefore

$$\int_{R^1} e^{-\theta x} c_N(x) \, dF(x) < \int_{R^1} e^{-\theta x} \, dF_k(x) + \varepsilon. \quad (5.8)$$

Now, we have already established that $\int_{R^1} e^{-\theta x} \, dF_k(x) \to e^{c\theta^{1+\beta}}$ for some $c \in R_+^1$ independent of $\{n_k\}_{k=1}^\infty$. Therefore letting $k \to \infty$ in (5.8) we obtain

$$\int_{R^1} e^{-\theta x} c_N(x) \, dF(x) \leqq e^{c\theta^{1+\beta}} + \varepsilon. \quad (5.9)$$

By the Monotone Convergence Theorem, we can let $N \to \infty$ in (5.9), yielding

$$\int_{R^1} e^{-\theta x} \, dF(x) \leqq e^{c\theta^{1+\beta}} + \varepsilon. \quad (5.10)$$

In particular $\int_{R^1} e^{-\theta x} \, dF(x)$ is finite.

Also, $0 < \theta < \theta_1$ and $x \leqq -N$ imply that $(\theta - \theta_1) x \geqq -(\theta - \theta_1) N$ and $e^{-(\theta - \theta_1) x} \leqq e^{(\theta - \theta_1) N}$. Therefore,

$$\int_{-\infty}^{-N} e^{-\theta x}\,dF_k(x)=\int_{-\infty}^{-N} e^{-(\theta-\theta_1)x}\cdot e^{-\theta_1 x}\,dF_k(x)\leqq e^{(\theta-\theta_1)N}\int_{R^1} e^{-\theta_1 x}\,dF_k(x). \quad (5.11)$$

The integral in (5.11) converges, as $k\to\infty$, to $e^{c\theta_1^{1+\beta}}$. In particular it is bounded by M, say, which is independent of N.

Therefore given $\varepsilon>0$ choose $N>0$ (and any $\theta_1>\theta$) such that $e^{(\theta-\theta_1)N}\cdot M<\varepsilon/3$ and $\int_{-\infty}^{-N} e^{-\theta x}\,dF(x)<\varepsilon/3$. Then choose k sufficiently large such that

$$\left|\int_{R^1} e^{-\theta x}c_N(x)\,dF_k(x)-\int_{R^1} e^{-\theta x}c_N(x)\,dF(x)\right|<\varepsilon/3.$$

With these values of k, N, θ_1 we have

$$\begin{aligned}&\left|\int_{R^1} e^{-\theta x}\,dF_k(x)-\int_{R^1} e^{-\theta x}\,dF(x)\right|\\ &\quad\leqq \int_{-\infty}^{-N} e^{-\theta x}\,d(F_k+F)(x)+\left|\int_{R^1} e^{-\theta x}c_N(x)\,dF_k(x)-\int_{R^1} e^{-\theta x}c_N(x)\,dF(x)\right|\\ &\quad\leqq 3\cdot\varepsilon/3=\varepsilon.\end{aligned}$$

Therefore $\int_{R^1} e^{-\theta x}\,dF(x)=e^{c\theta^{1+\beta}}$, which is independent of the subsequence $\{n_k\}_{k=1}^{\infty}$. It then follows that $\int_{R^1} e^{-\theta x}\,dF_t(x)\to e^{c\theta^{1+\beta}}$ as $t\to\infty$.

For $\beta<1$,

$$\begin{aligned}e^{c\theta^{1+\beta}}&=\exp\left[\int_0^{\infty}[e^{-\theta x}-1+\theta x]\,c_1\frac{dx}{x^{1+(1+\beta)}}\right]\\ &=\exp\left[\theta^{1+\beta}\cdot c_1\cdot\int_0^{\infty}[e^{-y}-1+y]\,\frac{dy}{y^{1+(1+\beta)}}\right]\end{aligned}$$

is the Lévy Representation of an asymmetric stable law of index $1+\beta$, where $c_1=c\left[\int_0^{\infty}(e^{-y}-1+y)\,\frac{dy}{y^{1+(1+\beta)}}\right]^{-1}$. For $\beta=1$, $e^{c\theta^2}$ is the Laplace transform of a normal law. In all cases, $e^{c\theta^{1+\beta}}$ is the transform of a unique distribution. This can be seen from an analytic continuation argument, yielding the characteristic function (as in the proof of Theorem 5.6).

Thus in both cases F_t, and hence z_t, converge weakly as stated in this theorem. Q.E.D.

In identifying the limiting generalized random fields, the Gaussian ($\beta=1$) and non-Gaussian ($\beta<1$) cases will be dealt with separately since in the first case the covariance structure is not readily available from Theorem 5.4.

Theorem 5.5. *In the case $\beta=1$, $d>2\alpha$, the signed-measure valued process Z_T:*

$$\langle\phi, Z_T\rangle=T^{-1/2}\left(\int_0^T\langle\phi, X_t\rangle\,dt-E_\lambda\left[\int_0^T\langle\phi, X_t\rangle\,dt\right]\right),\qquad \phi\in S(R^d)$$

converges in $S'(R^d)$, as $T\to\infty$, to a Gaussian random field, Z_∞, with covariance structure:

$$E[\langle\phi, Z_\infty\rangle\langle\psi, Z_\infty\rangle] = c_{d,\alpha} \iint_{(R^d)^2} \frac{\phi(x)\psi(y)}{|x-y|^{d-2\alpha}}\, dx\, dy,\ c_{d,\alpha} \text{ a constant; } \phi, \psi\in S(R^d). \quad (5.12)$$

Proof. In the previous theorem it was shown that for fixed $\phi\in\bar{S}(R^d)_+$ the numerical process $z_T=\langle\phi, Z_T\rangle$ converged, as $T\to\infty$, to a Gaussian random variable z_∞ with variance

$$\Gamma\left(\frac{d-\alpha}{2}\right)\cdot\left[2^\alpha\pi^{d/2}\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1}\int_{R^d}\left[\int_{R^d}\frac{\phi(y)}{|x-y|^{d-\alpha}}\,dy\right]^2 dx.$$

In order that this expression be recognized as that in (5.12) (up to a constant factor) we return to the calculation just following (5.5) of the previous proof, and proceed a little differently. Using the notation of that proof:

$$\begin{aligned}
\lim_{T\to\infty}\int_0^T\langle w_T^2(t),\lambda\rangle\,dt &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\left\langle\left[\int_0^t S_s\phi\,ds\right]^2,\lambda\right\rangle dt\\
&=\left\langle\int_0^\infty\int_0^\infty (S_s\phi)\cdot(S_r\phi)\,dr\,ds,\lambda\right\rangle\\
&=\int_0^\infty\int_0^\infty\int_{R^d}\int_{R^d} p_s^\alpha(x-z)\,\phi(x)\cdot\int_{R^d} p_r^\alpha(y-z)\,\phi(y)\,dy\,dr\,ds\,dz\\
&=\iint_{(R^d)^2}\left[\int_0^\infty\int_0^\infty p_{r+s}^\alpha(x-y)\,dr\,ds\right]\phi(x)\cdot\phi(y)\,dx\,dy.
\end{aligned}$$

The last step involved the Chapman-Kolmogorov equation for transition functions. We consider the cases $\alpha<2$ and $\alpha=2$ separately. For $\alpha=2$

$$\int_0^\infty\int_0^\infty p_{r+s}^2(x-y)\,dr\,ds=\int_0^\infty\int_0^m p_m^2(x-y)\,dn\,dm,$$

under the transformation $(r,s)\mapsto(m,n)\equiv(r+s,s)$;

$$\begin{aligned}
&=\int_0^\infty m\cdot\frac{e^{-(x-y)^2/4m}}{(4\pi m)^{d/2}}\,dm\\
&=\frac{1}{4\pi}\int_0^\infty\frac{e^{-(x-y)^2/4m}}{(4\pi m)^{(d-2)/2}}\,dm\\
&=\frac{\Gamma\left(\frac{d-4}{2}\right)}{16\pi^{d/2}}\cdot\frac{1}{|x-y|^{d-4}}\equiv c_{d-2}\cdot\frac{1}{|x-y|^{d-4}}.
\end{aligned}$$

For $\alpha<2$ we use the subordination formula [1, p. 18, 19] and the fact [1, p. 264] that for the laws η_t^γ of a one-sided stable process of index $0<\gamma<1$:

$$\int_0^\infty \eta_t(b-a)\,dt=\frac{1}{\Gamma(\gamma)}(b-a)^{\gamma-1}\cdot 1_{[a,\infty)}(b), \quad (5.13)$$

(a one-dimensional potential kernel) Γ being the usual gamma function. Therefore,

$$\int_0^\infty \int_0^\infty p_{r+s}^\alpha(x-y)\,dr\,ds = \int_0^\infty \int_0^\infty \int_0^\infty p_t^2(x-y)\,\eta_{r+s}^{\alpha/2}(t)\,dt\,dr\,ds$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^t p_t^2(x-y)\,\eta_r^{\alpha/2}(t-u)\cdot\eta_s^{\alpha/2}(u)\,du\,dt\,dr\,ds,$$

since $\eta_{r+s}^{\alpha/2} = \eta_r^{\alpha/2} * \eta_s^{\alpha/2}$, (convolution)

$$= \frac{1}{[\Gamma(1-\alpha/2)]^2} \int_0^\infty \int_0^t p_t^2(x-y)\,\frac{1}{(t-u)^{1-\alpha/2}}\cdot\frac{1}{u^{1-\alpha/2}}\,du\,dt,$$

by (5.13)

$$= \frac{1}{[\Gamma(1-\alpha/2)]^2} \int_0^\infty \int_0^\infty p_t^2(x-y)\,\frac{1}{(t-u)^{1-\alpha/2}}\cdot\frac{1}{u^{1-\alpha/2}}\,dt\,du$$

$$= \frac{1}{[\Gamma(1-\alpha/2)]^2} \int_0^\infty \int_0^\infty p_{t+u}^2(x-y)\,\frac{1}{t^{1-\alpha/2}}\cdot\frac{1}{u^{1-\alpha/2}}\,dt\,du. \tag{5.14}$$

We now make the transformation $(t,u)\mapsto(m,n)\equiv(t+u,t\cdot u)$ which has Jacobian $|t-u| = [m^2-4n]^{1/2}$. Accordingly we continue:

$$= \frac{1}{[\Gamma(1-\alpha/2)]^2} \int_0^\infty \int_0^{m^2/4} p_m^2(x-y)\cdot\frac{1}{n^{1-\alpha/2}}\cdot\frac{dn\,dm}{[m^2-4n]^{1/2}}$$

$$= \frac{4^{1-\alpha/2}}{[\Gamma(1-\alpha/2)]^2}\cdot\int_0^1 \frac{dn'}{(n')^{1-\alpha/2}\cdot(1-n')^{1/2}}\cdot\int_0^\infty \frac{p_m^2(x-y)}{m^{(2-2\alpha)/2}}\,dm;$$

under the scaling $n = \frac{m^2}{4}\cdot n'$;

$$= c'_{d,\alpha}\int_0^\infty \frac{e^{-(x-y)^2/4m}}{(4\pi m)^{d/2}\cdot m^{(2-2\alpha)/2}}\,dm$$

$$= \frac{4^{1-\alpha}}{\pi^{d/2}}\cdot c'_{d,\alpha}\int_0^\infty \frac{e^{-(x-y)^2/4m}}{(4m)^{(d-2\alpha+2)/2}}\,dm$$

$$= c_{d,\alpha}\cdot\frac{1}{|x-y|^{d-2\alpha}}.$$

Here $c'_{d,\alpha} = B(\alpha/2, 1/2)$ (B, the Beta function) and

$$c_{d,\alpha} = 4^{-\alpha}\pi^{(2-d)/2}\cdot\Gamma(\alpha/2)\,\Gamma((d-2\alpha)/2)\cdot\Gamma^{-1}((\alpha+1)/2).$$

The upshot of these calculations is that for all $0<\alpha\leq 2$, and each $\phi\in\bar{S}(R^d)_+$, $z_\infty \equiv \lim_{T\to\infty}\langle\phi, Z_T\rangle$ has the Gaussian moment generating function $\exp[\theta^2/2\cdot K(\phi,\phi)]$ where the bilinear form K is defined on $\bar{S}(R^d)$ by

$$K(\phi,\psi) = 2c_{d,\alpha}\iint_{(R^d)^2} \frac{\phi(x)\,\psi(y)}{|x-y|^{d-2\alpha}}\,dx\,dy:\ \phi,\psi\in\bar{S}(R^d).$$

Denote the 2×2 matrix $\begin{bmatrix} K(\phi,\phi) & K(\phi,\psi) \\ K(\psi,\phi) & K(\psi,\psi) \end{bmatrix}$ by $\kappa(\phi,\psi)$ for $\phi,\psi\in\bar{S}(R^d)$.

For a general $\phi\in S(R^d)$, write $\phi=\phi_1-\phi_2$ where $\phi_1=\max\{\phi,0\}$ and $\phi_2=-\min\{\phi,0\}$; both $\phi_1,\phi_2\in\bar{S}_+(R^d)$. Then $(\langle\phi_1,Z_T\rangle,\langle\phi_2,Z_T\rangle)$ converges jointly, in the weak sense, as $T\to\infty$ to a bivariate Gaussian random variable with covariance matrix $K=\kappa(\phi_1,\phi_2)$. Indeed, let $\theta_1,\theta_2\geqq 0$; then

$$\begin{aligned} &E_\lambda[\exp(-\theta_1\langle\phi_1,Z_T\rangle-\theta_2\langle\phi_2,Z_T\rangle)] \\ &=E_\lambda[\exp(-\langle\theta_1\phi_1+\theta_2\phi_2,Z_T\rangle)] && (5.15)_a \\ &\to\exp[\tfrac{1}{2}\cdot K(\theta_1\phi_1+\theta_2\phi_2,\theta_1\phi_1+\theta_2\phi_2)] \quad \text{as} \quad T\to\infty && (5.15)_b \\ &=\exp[\tfrac{1}{2}(\theta_1,\theta_2)\cdot\kappa\cdot(\theta_1,\theta_2)^*],^1 && (5.15)_c \end{aligned}$$

using the bilinearity of K.

Therefore the characteristic function of $(\langle\phi_1,Z_T\rangle,\langle\phi_2,Z_T\rangle)$ converges to the corresponding characteristic function, namely $\exp[-\frac{1}{2}(t_1,t_2)\cdot\kappa\cdot(t_1,t_2)^*]$ where $t_1,t_2\in R^1$, and $\kappa=\kappa(\phi_1,\phi_2)$. In particular we can let $t_1=-t_2=t\in R^1$ to obtain

$$\begin{aligned} \lim_{T\to\infty} E_\lambda[\exp(it\langle\phi,Z_T\rangle)] &= \lim_{T\to\infty} E_\lambda[\exp[i(t,-t)\cdot(\langle\phi_1,Z_T\rangle,\langle\phi_2,Z_T\rangle)]] \\ &=\exp[-\tfrac{1}{2}\cdot(t,-t)\cdot\kappa\cdot(t,-t)^*] \\ &=\exp\left[-\frac{t^2}{2}K(\phi,\phi)\right], \quad \text{as in } (5.17)_{b,c}. \qquad (5.16) \end{aligned}$$

Setting, in (5.16), $t=1$ and replacing ϕ with $t_1\phi+p_2\psi$, where $\phi,\psi\in S(R^d)$ and $t_1,t_2\in R^1$, we obtain as in (5.16) that $(\langle\phi,Z_T\rangle,\langle\psi,Z_T\rangle)$ converges as $T\to\infty$ to a bivariate Gaussian random variable with covariance matrix $K(\phi,\psi)$.

Finally, we observe that the quadratic form $K(\phi,\phi)$, $\phi\in S(R^d)$, is positive definite since we have, from Theorem 5.4, the alternate expression (up to a constant factor) for it:

$$\int_{R^d}\left[\int_{R^d}\frac{\phi(y)}{|x-y|^{d-\alpha}}\,dy\right]^2 dx.$$

Also by Lebesgue's Dominated Convergence theorem, $K(\cdot,\cdot)$ is continuous on $S(R^d)$. The Bochner-Minlos theorem [10] allows us to infer the existence of a unique probability measure P on $S'(R^d)$ corresponding to a Gaussian random field Z_∞, whose characteristic functional is given by:

$$E[\exp(i\langle\phi,Z_\infty\rangle)]=\exp[-\tfrac{1}{2}K(\phi,\phi)], \quad \phi\in S(R^d). \quad \text{Q.E.D.} \qquad (5.17)$$

Remark 5.1. The covariance kernel in (5.17) reveals a high degree of long range correlations since the kernel is not globally integrable. In the next section we will see that in lower dimensions the spatial correlations are complete in the sense that the covariance kernel will be constant. Of course the weighted occupation time process will be renormalized differently in order that a non-trivial limiting distribution be captured.

---

1 "*" = transpose

We now turn to the case $0<\beta<1$ where we have the following limit theorem.

Theorem 5.6. *In the case $0<\beta<1$, $(d-\alpha)\beta>\alpha$, the signed measure-valued process Z_T:*

$$\langle\phi, Z_T\rangle = T^{-1/(1+\beta)}\left(\int_0^T \langle\phi, X_t\rangle\, dt - E_\lambda\left[\int_0^T \langle\phi, X_t\rangle\, dt\right]\right), \qquad \phi\in S(R^d)$$

converges in $S'(R^d)$, as $T\to\infty$, to a stable generalised random field Z_∞ with characteristic functional:

$$\begin{aligned}\exp\{&-c\int_{R^d}|\langle\phi, G(x)\rangle|^{1+\beta}\,dx + ic\tan[(1+\beta)(\pi/2)]\\ &\cdot\int_{R^d}\langle\phi, G(x)\rangle\cdot|\langle\phi, G(x)\rangle|^{\beta}\,dx\}\end{aligned} \tag{5.18}$$

where $\phi\in S(R^d)$, $G(x)=|x-\cdot|^{\alpha-d}$, and $c=-c_{\alpha,d}\cos[(1+\beta)(\pi/2)]$, with $c_{\alpha,d}$ as in Theorem 5.4. Moreover one can specify a Hilbert subspace H', with

$$S(R^d)\hookrightarrow H\hookrightarrow L^2(R^d)\hookrightarrow H'\hookrightarrow S'(R^d),$$

containing the support of the law of Z_∞ such that (5.18) can be rewritten in standard form (see Kuelbs [16]):

$$\begin{aligned}\exp\{&-\int_{S^{H'}}|\langle\phi, s\rangle|^{1+\beta}\,\gamma(ds) + i\tan[(1+\beta)(\pi/2)]\\ &\cdot\int_{S^{H'}}\langle\phi, s\rangle\cdot|\langle\phi, s\rangle|^{\beta}\,\gamma(ds)\}, \qquad \phi\in H,\end{aligned} \tag{5.19}$$

where $S^{H'}$ is the unit sphere in H', γ is a finite Borel measure on $S^{H'}$, and $\langle\cdot,\cdot\rangle$ is the natural pairing on $H\times H'$.

Proof. The expression in (5.18) is easily seen, as for any stable law, to satisfy the hypotheses of the Bochner-Minlos theorem [10]. We are thus assured the existence of Z_∞. To carve out of $S'(R^d)$ a Hilbert subspace H' as stated in the present theorem, we shall apply Theorem 3.1 of Hida [10].

To this end we note that $S(R^d)$ is a countably Hilbert nuclear space relative to the scale of Hilbert norms (see Theorem 7 of Ch. I, Sect. 3.6 of [24]):

$$\|\phi\|_n^2 = \sum_{|k|\leq n}\int_{R^d}(1+x^2)^{(q\cdot n)/2}\,|\phi^{(k)}(x)|^2\,dx$$

where k is a multi-index, $n\in\mathbb{N}$, $\phi^{(k)}$ is a k-th order non-mixed partial derivative and $q>0$. Then as in Lemma A.1 of Hida [10] (with misprints appropriately corrected) we have that $\|(1+x^2)^{p/2}\,\phi(x)\|_\infty \leq K_1\cdot\|\phi\|_d$, if q is larger than $\frac{2p}{d}+1$. Here K_1 is some constant depending on p.

Now for each $x\in R^d$ define an element $G(x)\in S'(R^d)$ through the locally integrable kernel $|x-y|^{\alpha-d}$ i.e.

$$\langle\phi, G(x)\rangle = \int_{R^d}\phi(y)\cdot|x-y|^{\alpha-d}\,dy, \qquad \text{for } \phi\in S(R^d).$$

From the inequality in the previous paragraph and Lemma 5.3 it follows that $G(x)$ extends to an element of H', the dual of H which is the completion of $S(R^d)$ in the Hilbertian norm $\|\cdot\|_d$. Indeed,

$$\sup_{\|\phi\|_d \leqq 1} |\langle\phi, G(x)\rangle| \leqq \sup_{\|\phi\|_d \leqq 1} K_2 \cdot \|(1+|y|^p)\cdot\phi(y)\|_\infty \cdot [1+|x|^{d-\alpha}]^{-1},$$

K_2 a constant;

$$\leqq K_1 \cdot K_2 \cdot \|\phi\|_d \cdot [1+|x|^{d-\alpha}]^{-1} \tag{5.20}$$

$$\leqq K_3 \cdot \|\phi\|_d, \quad K_3 = K_1 \cdot K_2. \tag{5.21}$$

Denoting the norm in H' by $\|\cdot\|_{-d}$ we thus have that $\|G(x)\|_{-d} \leqq K_3$, for all $x \in R^d$.

To obtain the representation of (5.19) we simply set

$$s(x) = G(x)/\|G(x)\|_{-d} \quad \text{and} \quad \gamma(B) = \int_{T^{-1}(B)} c\cdot\|G(x)\|_{-d}^{1+\beta}\,dx,$$

for B a Borel subset of $S^{H'}$, where the transformation T: $R^d \to S^{H'}$ by $x \mapsto s(x)$; it is not hard to see that G, and hence T, is continuous. Finally from (5.20) we see that $\gamma(S^{H'}) \leqq \int_{R^d} c[1+|x|^{d-\alpha}]^{1+\beta}\,dx < \infty$ since $(d-\alpha)\beta > \alpha$ is equivalent to $(d-\alpha)(1+\beta) > d$.

The proof of the stated convergence of Z_T as $T \to \infty$ is similar to that of Theorem 5.5 so we shall omit some of the details. In Theorem 5.4 the convergence was established for $\phi \in \bar{S}(R^d)_+$. In that case, to go from the Laplace functional to the characteristic functional one should extend the domain of θ (c.f. Theorem 5.4) to the right half of the complex plane where $\theta^{1+\beta} = \exp\{(1+\beta)\log\theta\}$, $-\pi/2 < \arg\theta < \pi/2$, is the unique analytic continuation of $\theta^{1+\beta}$ for positive θ. The characteristic functional is obtained in the limit as $\theta \to -i$.

In order to extend the convergence of Z_T as $T \to \infty$ to $\phi \in S(R^d)$ it suffices, as in Theorem 5.5, to show that $(\langle\phi_1, Z_T\rangle, \langle\phi_2, Z_T\rangle)$ converges as $T \to \infty$ to the appropriate bivariate stable law, where $\phi_1, \phi_2 \in \bar{S}(R^d)_+$ play the roles of $\max\{\phi, 0\}$ and $-\min\{\phi, 0\}$ respectively, for $\phi \in S(R^d)$. To this end we define, in analogy to the transformations G and T, the continuous map t: $R^d \to S^{d-1}$, the unit sphere in R^d, through $t(x) = |g(x)|^{-1}(g_1(x), g_2(x))$ where $|\cdot|$ denotes the norm in R^2 and $g = (g_1, g_2)$ is defined by $g_j(x) = \langle\phi_j, G(x)\rangle$, $j = 1, 2$; $x \in R^d$.

For $\theta = (\theta_1, \theta_2) \in R_+^2$ we know from Theorem 5.4 that ("$\cdot$" denotes the dot product in R^2):

$$\begin{aligned} &E_\lambda[\exp\{-\theta\cdot(\langle\phi_1, Z_T\rangle, \langle\phi_2, Z_T\rangle)\}] \\ &= E_\lambda[\exp\{-\langle\theta_1\phi_1 + \theta_2\phi_2, Z_T\rangle\}] \\ &\to \exp\{\int_{R^d} c_{d,\alpha}[\theta\cdot g(x)]^{1+\beta}\,dx\}, \quad \text{as} \quad T\to\infty; \\ &= \exp\{\int_{R^d} c_{d,\alpha}[\theta\cdot t(x)]^{1+\beta}\cdot|g(x)|^{1+\beta}\,dx\}. \end{aligned} \tag{5.22}$$

By defining a Borel measure γ_2 on S^{d-1} through $\gamma_2(B) = \int_{t^{-1}(B)} c_{d,\alpha}|g(x)|^{1+\beta}\,dx$, for B a Borel subset of S^{d-1}, we can rewrite (5.22) as $\exp\{\int_{S^{d-1}} [\theta\cdot s]^{1-\beta}\gamma_2(ds)\}$. Note that the support of γ_2 lies in $S^{d-1} \cap R_+^2$. The finiteness of $\gamma_2(S^{d-1})$ follows from Lemma 5.3. Thus we have the required convergence to the appropriate bivariate stable law.

The switch to the characteristic functional description can be carried out as before, and the extension to $\phi \in S(R^d)$ has already been inicated. Q.E.D.

Remark 5.2. The choice of H' in Theorem 5.6 was certainly not optimal in the sense of minimality. There are many other possible choices. For instance we could also use the dual of the completion of $S(R^d)$ in the graph norm $\|(\Delta - x^2)^k \phi\|_{L^2}$ for suitable k. These spaces figure in the so-called N-representation of $S'(R^d)$ (c.f. Reed and Simon [19] or Simon [21]).

§ 6. Second Order Asymptotics: Intermediate Dimensions

In the case $\alpha=2$, $\beta=1$ it has been established that the total weighted occupation time in bounded regions is a.s. finite in one dimension but not in dimensions greater than one. Also the second order behaviour has been studied in dimensions greater than four. In the present section we concentrate on the intermediate range of dimensions $d=3, 4$, for which the second order asymptotics can be calculated and stand in striking contrast to those of higher dimensions. The proof of the following theorem is too similar to those of Theorems 5.4 and 5.5 to warrant its demonstration here. The pertinent calculations were carried out in [12]. Note that the borderline case $d=2$ is not convered here.

Theorem 6.1. *In the case $\alpha=2$, $\beta=1$, $d=3$ the signed-measure valued process Z_T:*

$$\langle \phi, Z_T \rangle = T^{-3/4} \left[\int_0^T \langle \phi, X_t \rangle \, dt - E_\lambda \left[\int_0^T \langle \phi, X_t \rangle \, dt \right] \right]; \qquad \phi \in S(R^d)$$

converges in $S'(R^d)$, as $T \to \infty$, to a Gaussian random field, Z_∞, with covariance structure:

$$\begin{aligned} &E[\langle \phi, Z_\infty \rangle \cdot \langle \psi, Z_\infty \rangle] \\ &\quad = \frac{(2-\sqrt{2})}{3\pi^{3/2}} \iint_{(R^3)^2} \phi(x) \psi(y) \, dx \, dy \left(= \frac{(2-\sqrt{2})}{3\pi^{3/2}} \langle \phi, \lambda \rangle \cdot \langle \psi, \lambda \rangle \right); \qquad \phi, \psi \in S(R^d). \end{aligned} \tag{6.1}$$

Remark 6.1. There is a similar result, with a different constant in (6.1), in dimension $d=4$ where the factor $T^{-3/4}$ is replaced by $[T \log T]^{-1/2}$.

It would thus seem that in three dimensions the temporal, rather than spatial, aspect of the weighted occupation time process is the more important of the two. In this direction the following theorem is valid. Its proof is almost identical to that of the previous theorem and will thus be omitted.

Theorem 6.2. *In the case $\alpha=2$, $\beta=1$, $d=3$ with $\phi \in S_+(R^d)$ and $t \in R_+^1$ fixed, the numerical process*

$$z_t^T = T^{-3/4}\left[\int_0^{T\cdot t} \langle \phi, X_s\rangle\, dt - E_\lambda\left[\int_0^{T\cdot t} \langle \phi, X_s\rangle\, ds\right]\right]$$

converges weakly as $T\to\infty$ to a normal law with variance

$$\left(\frac{2-\sqrt{2}}{3\pi^{3/2}}\right)\langle \phi, \lambda\rangle^2 \cdot t^{3/2}.$$

Moreover for $0 < t_1 \leqq t_2$

$$\lim_{T\to\infty} E_\lambda[z_{t_1}^T \cdot z_{t_2}^T] = \frac{\langle \phi, \lambda\rangle^2}{3\pi^{3/2}}\,(t_1^{3/2} + t_2^{3/2} - \tfrac{1}{2}[(t_1+t_2)^{3/2} - (t_2-t_1)^{3/2}]). \tag{6.2}$$

Remark 6.2. (i) In proving Theorem 6.2, appeal must be made to Theorem 3.2.

(ii) With a little more work, tightness can be established as well as the convergence of higher finite dimensional distributions, allowing one to speak of a limiting Gaussian process Z_t^∞ on $C([0,\infty), R)$.

Appendix

We collect here some facts concerning the basic properties of the evolution equation appearing in this paper. Proofs of the assertions made here will only be sketched since the techniques involved are standard ones from the theory of evolution equations but may not be well known to the intended audience. Much of the details were carried out in [12].

The object of interest is the eqution

$$\begin{aligned} \dot{u}(t) &= Au(t) - g(u(t)) \\ u(0) &= \psi \end{aligned} \tag{I}$$

where A is a linear operator from its domain $D(A) \subset C_0(R^d)$ into $C_0(R^d)$ which generates a strongly continuous semigroup $\{S_t\}_{t\geqq 0}$ of non-negative contractions on $C_0(R^d)$, and $g: R_+^1 \to R_+^1$ is continuously differentiable with $g'(0) = g(0) = 0$, $\psi \in C_0(R^d)_+$.

By a strong solution of (I) we understand a continuously differentiable curve $u: R_+^1 \to D(A)_+$ satisfying (I); apriori $\psi \in D(A)_+$. By a mild solution of (I) we understand a continuous curve $u: R_+^1 \to C_0(R^d)_+$ satisfying:

$$u(t) = S_t\psi - \int_0^t S_{t-s}[g(u(s))]\, ds. \tag{II}$$

Theorem A. *If $\psi \in C_0(R^d)_+$ then Eq.* (I) *possesses a unique mild solution. If, in addition, $\psi \in D(A)$ then this solution is also strong and conversely every solution of Eq.* (I) *satisfies* (II). *Thus Eq.* (I) *has a unique solution.*

Proof. Extend g to a continuously differentiable function $\tilde{g}: R \to R_+$ by setting $\tilde{g}(x) = g(\max[x, 0])$. As g is locally Lipschitz continuous and S_t is a contraction, Eq. ($\widetilde{\text{II}}$) ((II) with g replaced by $\tilde{g}$) can be solved locally (in t) by the usual Picard iteration scheme. This also yields continuous dependence of the solution on the initial datum ψ.

If in addition $\psi \in D(A)$ then $u: [0, t_0) \to C_0(R^d)$ is a strong (local) solution. To see this, differentiate (I) formally and cast into mild form; define $v: [0, t_0) \to C_0(R^d)$ ("$v(t) = \dot{u}(t)$") through:

$$v(t) = S_t[A\psi - \tilde{g}(\psi)] - \int_0^t S_{t-s}[\tilde{g}'(u(s)) \cdot v(s)]\, ds. \qquad \text{(III)}$$

Equation (III) can also be solved by Picard iteration.

Setting $v_h(t) = [u(t+h) - u(t)]/h$ for $h > 0$ and $t \in [0, t_0)$, we can derive from Eq. ($\tilde{\text{II}}$), by an application of Gronwall's inequality, that

$$\|v_h(t)\| \leqq \|v_h(0)\| \cdot c_1 \exp(c_2 t)$$

for some constants $c_1, c_2 > 0$ depending on ψ and t, and sufficiently small $h > 0$. Also $v_h(0) \to v(0)$ as $h \to 0^+$ so that $\|v_h(t)\|$ is bounded in h for each $t \in [0, t_0)$.

From Eqs. ($\tilde{\text{II}}$) and (III) we can derive, again by Gronwall's inequality, that $\|v_h(t) - v(t)\| \leqq \|v_h(0) - v(0)\| \cdot c_3 \exp(c_4 t)$ for sufficiently small $h > 0$; c_3, c_4 are constants depending on ψ and t. Thus $\frac{d^+}{dt} u(t) = v(t)$ for $t \in [0, t_0)$.

By a standard lemma (see [23, p. 239]), u is actually continuously differentiable on $[0, t_0)$ and it is straightforward to check that u satisfies ($\tilde{\text{I}}$) (($\tilde{\text{I}}$) is (I) with g repalced with $\tilde{g}$).

Conversely if u satisfies ($\tilde{\text{I}}$) then $w(t) \equiv S_t\psi - \int_0^t S_{t-s}[\tilde{g}(u(s))]\, ds$ satisfies $\dot{w}(t) = Aw(t) - g(u(t))$. Thus $[u(t) - w(t)]^{\cdot} = A[u(t) - w(t)]$ and $[u(0) - w(0)] = 0$, so that $u(t) - w(t) \equiv 0$ and u satisfies ($\tilde{\text{II}}$).

To see that the local solution $u: [0, t_0) \to C_0(R^d)$ is non-negative we can argue by contradiction as follows. Assuming $\psi \in D(A)_+$, fix $0 < t_2 < t_0$ and choose $(t_1, x_1) \in R^1_+ \times R^d$ such that $u(t_1, x_1) = \min\{u(t, x): (t, x) \in [0, t_2] \times R^d\}$. If $u(t_1, x_1) < 0$ then the non-negativity of S_t implies that $Au(t_1, x_1) \geqq 0$, and of course $\tilde{g}(u(t_1, x_1)) = 0$. Thus $\dot{u}(t_1, x_1) \geqq 0$ which implies that $u(s, x_1) = u(t_1, x_1)$ for s sufficiently close and less than t_1. By a connectedness argument we are led to the absurdity that $\psi(x_1) = u(0, x_1) < 0$. Using the continuous dependence of u on ψ we can lift the restriction that $\psi \in D(A)_+$ which is dense in $C_0(R^d)_+$. Since t_2 was arbitrary, $u(t) \geqq 0$ for $t \in [0, t_0)$.

The upshot of the non-negativity result is that Eq. (I) and (II) are equivalent to ($\tilde{\text{I}}$) and ($\tilde{\text{II}}$) respectively. In particular from (II) we see immediately that $0 \leqq u(t) \leqq S_t\psi$ so that we actually have global existence (in t) of u. Q.E.D.

Using similar techniques, the following theorem can be proven.

Theorem B. *If, in Eq.* (I), ψ *depends on a real parameter* θ *in a continuously differentiable manner, then the solution* u *does as well. Moreover,* $v(t, \theta) \equiv \frac{d}{d\theta} u(t, \theta)$ *is a mild solution of the equation:*

$$\dot{v}(t,\theta)=Av(t,\theta)-g'(u(t,\theta))\cdot v(t,\theta)$$
$$v(0,\theta)=\frac{d}{d\theta}\,\psi(\theta). \tag{IV}$$

Remark. It is clear that various parts of the previous two theorems can be proven in greater generality under less stringent hypotheses; we have taken a quick route here for the sake of brevity.

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Received August 15, 1981; in revised form March 19, 1984 and in final form May 6, 1985