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Husserl's epistemology of mathematics and the foundation of platonism in mathematics

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Introduction[1]

In recent years the so-called platonistic conception of the nature of mathematical entities, according to which mathematical statements are about entities and relations in a similar way as statements that are concerned with real physical objects – with the only difference that the entities with which mathematical statements are concerned are neither physical, nor possess spatio-temporal properties in an essential way and, thus, are not sensuously perceived – has been the object of severe criticisms, both by propounders of some sort of constructivism[2] and by defenders of the old Anglo-American empiricist tradition recently rebaptized as 'causal'.[3] Certainly, the doctrines of the best known defenders of platonism, like Cantor, Frege and Gödel, have the common defect of not having developed an epistemology of mathematics that could explain in a satisfactory way how is it that we have access to the so-called mathematical entities. It is true that in the later writings of Frege (beginning with 'Der Gedanke', but especially in those written after 1920)[4] and in some of Gödel's writings[5] there are sketches of an explanation of our knowledge of mathematical truths, but its insufficient elaboration does not allow its use as an answer to the antiplatonistic criticisms.

Although not so well-known as a philosopher of mathematics, Edmund Husserl defended an essentially platonistic conception of mathematics; however, contrary to the authors just mentioned, Husserl was especially interested in epistemological problems, and his philosophy of mathematics, although it can be considered separately[6], has its epistemological foundation in what Husserl called categorial intuition.

Now, the term 'categorial intuition' is in Husserl – especially in the *Logische Untersuchungen* – a sort of generic term that requires some clarification if we want to penetrate to the depths of Husserl's epistemology of mathematics. Moreover, the contrast between categorial and sensible intuition in Husserl will not only serve the purpose of somehow demarcating the notion of categorial intuition, but will also serve to show that categorial intuition is nothing mysterious or metaphysical (in the negative sense of this last term), but, on the contrary, has its foundation in sensible intuition. Therefore, although mathematical entities are for Husserl deprived of any 'sensuousness', they are not completely disconnected from sensible intuition.

In this paper we will make a reconstruction and a systematisation of Husserl's epistemology of mathematics as based on the notion of categorial intuition. We will follow Husserl's discussion in the Sixth Logical Investigation and in *Erfahrung und Urteil.*

§ 1 *The problem of fulfillment of formal constituents of statements*

In the First Logical Investigation Husserl distinguished between acts in which meanings are constituted and acts in which those meanings are fulfilled or realized, and, correspondingly, between the meanings of expressions and the objectualities referred to by those expressions by means of their meanings. Although in the First Investigation Husserl was not completely explicit in this respect, it is clear, e.g., from § 11 of the Fourth Investigation and from § 60 of *Erfahrung und Urteil,* that statements whose meaning is a proposition (or thought) have as reference a state of affairs.

Now, as Husserl's interest in the First and Fourth Investigations is not epistemological, he postpones the discussion of the fulfillment (or realization) of the meanings of statements, together with other related problems, until the Sixth Investigation, which is a sort of culmination of his efforts in the rest of that philosophical masterpiece. In §§40, 42, 43 and 51 of the Sixth Investigation Husserl tells us that the formal constituents of statements, namely, particles like 'is', 'and', 'or', 'not', 'if', 'then', 'all', 'some', 'none', 'many', 'few', etc. and also numerical determinations and many other expressions (e.g., the relational expressions 'greater than' and 'at the side of') do not have any direct counterpart in sensible perception – although we speak of the fulfillment of the meanings of statements in which those constituents occur. No matter how closely a statement is linked to perception, only the material elements that are found in its terms can have their fulfillment in sensible perception or imagination, i.e. in sensible intuition.[7] On the other hand, formal constituents of statements do not obtain their fulfillment from sensible intuition. To the 'is', to the 'and' and to the 'or', to the 'some', to the 'not' and to the 'is greater than' there does not correspond anything in sensible perception, nor can we represent sensibly in the imagination the intuitive counterparts of those particles. We cannot apprehend with any of our senses, nor paint nor photograph objectual counterparts of such expressions (Hua XIX/2, 688).

However, that does not mean that the meanings of such formal constituents of statements cannot be fulfilled by a corresponding objectuality. Actually, if there were no possible fulfillment of the meanings of such formal constituents of statements, we could not clearly differentiate between the fulfillment of the meanings of the statements 'John and Peter are in the park' and 'John or Peter is in the park', although those two statements have different truth conditions.[8]

§2 *Sensible and categorial intuition*

Hence, although we do not sensibly intuit anything that could correspond to the formal constituents of statements, there must be some act of intuition, similar to but different from that of sensible intuition, in which such 'formal' expressions are fulfilled. Husserl calls this sort of intuition, in which the meanings of the formal constituents of statements are fulfilled and in which new categorially formed objectualities are constituted, categorial intuition. In this sort of intuition the categorially formed objectuality is not merely symbolically mentioned – as in the corresponding act in which the meaning of the expression used to refer to it is constituted – but actually intuited. It is in this sort of act that collections, indeterminate pluralities, totalities, numbers, states of affairs, etc. become objects (Hua XIX/2, 672). Such objectualities are based on the sensibly given, but are not to be reduced to the sensibly given. Similarly, the categorial act in which such objectualities of the understanding are constituted is based on acts of sensible intuition, but are not to be reduced to it. In an act of sensible intuition the constituted object is apprehended in a simple way, whereas in an act of categorial intuition the constituted object is constituted in founded acts that connect what is given in the founding (sensible) acts (Hua XIX/2, 674). Moreover, in an act of sensible intuition the object is receptively apprehended, whereas in an act of categorial intuition the object is never apprehended in a purely receptive way, but requires the intervention of the spontaneity of the understanding. Hence, Husserl calls the objects given in categorial intuition not only categorial or syntactical objectualities, but also objectualities of the understanding (EU, §§58f).

Thus, Husserl clearly distinguishes in a general way between sensible and categorial intuition, and calls the objects given in sensible intuition objects of lower level and the objects given in categorial intuition objects of higher level (Hua XIX/2, 674). Moreover, every act of sensible intuition (i.e., perception or imagination) can appear, either alone or together with other acts, as founding new acts based on it, in which new objectualities are constituted that could not have been given in any founding act. But it should be emphasized that the objectuality constituted in the categorial founded acts is 'built' on the objectualities given in the founding acts, and can only be given as such a founded objectuality. 'It is in such founded acts that the categorial in intuition and knowledge lies, and predicative thought finds its fulfillment' (Hua XIX/2, 675).

Now, it is important to underscore – as Husserl does in §61 of the Sixth Investigation – that the objectualities of the understanding are not objects in the primary sense of 'being', since they are not possible objects of sensible intuition. The categorial act in which they are constituted, neither modifies, nor affects, nor transforms the sensibly given, since that would be a sort of distortion; rather, it 'builds' on the sensibly given. Categorial intuition neither glues together nor links sensible objects to produce a new sensible whole. If this were the case, the originally given in sensible intuition would be modified, and categorial intuition would be a falsifying reorganizing of the sensibly given. In such a case, the result would be a new sensible object, although different from those of the founding acts. But what is constituted in a categorial act, although founded in the sensibly given, is not only an objectuality of a higher level than the sensible objects of its founding acts, but an objectuality of a different sort, a non-sensible objectuality.

§3 *Examples of categorial objectualities*

The two typical examples of categorial objectualities (or objectualities of the understanding) considered by Husserl both in the Sixth Investigation and in the second part of *Erfahrung und Urteil* (§§59ff.) are states of affairs and collections (or sets). With respect to the first of these, one can distinguish at least two cases, although the analysis is very similar in each case. Thus, a state of affairs can be constituted when we (mentally) detach a part or moment[9] from the whole to which it belongs, e.g., when we predicate of a book its being red or blue. In sensible intuition the whole is passively constituted with its parts and moments. But the relation that is constituted when we (mentally) detach a part or moment to bring out its connection to the whole, as, e.g., in the statements 'The book is blue' or 'Being blue is a property of the book' is of categorial nature. The sensibly given, namely, the book with its parts and moments, includes the passive proto-relation (in *Erfahrung und Urteil* Husserl prefers the expression 'situation of affairs' [Sachlage]) of the book with its blue moment, and on this proto-relation two states of affairs are founded, which are the different objectualities that are referred to by the statements 'The book is blue' and 'Being blue is a property of the book'.

Something similar occurs when we connect in a relation the objects of two (or more) sensible intuitions to form new states of affairs. Thus, e.g., we can link two objects A and B to form the states of affairs that A is bigger than B and B is smaller than A. In sensible intuition we have two objects A and B and the situation of affairs [Sachlage] that A has a greater size than B, and on the basis of the sensibly given two different states of affairs are constituted, namely, those which are referred to by the equivalent statements 'A is bigger than B' and 'B is smaller than A'. Something similar occurs in the case of two or more objects that lie side by side or one above the other. The sensibly given are the objects and the existing situation of affairs, whereas the corresponding states of affairs that are referred to by the statements 'A lies above B' and 'B lies below A' are objectualities of the understanding constituted in a categorial intuition founded on sensible intuitions.

It is important to underscore here that in some sense situations of affairs are also founded objectualities, since they are complexes of simple sensible objects. But they are not the object of any sensible intuition. Although they are not properly objectualities constituted by the spontaneity of the understanding, i.e., they are not categorial objectualities, in receptive sensible intuition we do not have them thematically as objectualities. Situations of affairs appear, rather, as mere passively constituted foundations of different states of affairs. Now, once the states of affairs are constituted and objectified in predications, they can be objectively apprehended as the situation of affairs underlying two or more states of affairs. For example, once a relation R and its inverse relation R^{-1} are constituted, one can apprehend as object the corresponding situation of affairs as the 'abstract' invariant proto-relation that underlies both of them.

Another typical example of an objectuality of the understanding considered by Husserl – especially in §61 of *Erfahrung und Urteil* – is that of a (finite) set or collection. In sensible intuition not one but many objects can be given at the same time and even as belonging together – although, as Husserl remarks (EU §62, p. 297), the objects brought together in a collection do not need to have any sensible link between them, e. g. that of similarity or spatio-temporal contiguity. In such a case, the plurality of objects is given, like the trees in a park, or the seats in a classroom, or the books on a shelf. But the collection (or set) of trees, or seats, or books is not sensibly given. The collection is not a sensibly perceived objectuality nor can it be represented sensibly in the imagination. It is an objectuality of the understanding constituted in a categorial intuition, built on sensible intuitions, in which the objects that belong to the collection are constituted. Sets – and similarly states of affairs – do not 'dissolve' into the sensible objects on which they are founded, but are objectualities of a different sort, namely, of a categorial sort, and precisely as such can only be given as founded objectualities, as founded on the objects that are their members. Certainly, as in the case of states of affairs and situations of affairs, there is in some sense a plurality given in receptivity, but it is only in a categorial intuition that the set as such is constituted (EU, §61, p. 292).

§4 *Sorts of categorial intuition*

In all the examples of categorial intuitions and the corresponding categorial objectualities considered above, the objects of the founding sensible acts are in some sense incorporated as constituents in the objectuality that is constituted in categorial intuition. But in the Sixth Investigation (§§41, 47 and 52) Husserl also considers categorial objectualities which are founded on sensible objectualities but do not incorporate these as constituents. This is the case of what Husserl in the Sixth Investigation calls generalization, in which general objects or 'species' are constituted. In this case, the object given in the founding sensible intuition is only an instantiation or example of the species, but not one of its constituents. A similar situation occurs in the case of what Husserl calls (Hua XIX/2, 676) the singular indeterminate conception, for which the object given in sensible intuition, e.g., the concrete triangle sketched on the blackboard, is only a mere illustrative aid, not a constituent of the constituted objectuality. In both cases objectualities of higher level, founded on the sensibly given, are constituted, but this foundation is of a different sort than the foundation of the examples considered before.

In the act of generalization or generalizing abstraction a general object, the species, is constituted, and this species is *one*, in contrast to the unbounded plurality of singular objects of the 'same species' that can serve as basis of the categorial act and can be constituted in the founding sensible intuitions. By means of the comparative variation of the founding sensible acts and its corresponding referents, we become conscious of the identity of the species as the general object of which the objects of the plurality of possible founding acts are mere instantiations. On the other hand, in the act of singular indeterminate conception a singular but arbitrary object of a determinate species is given and, thus, not – as in the previous case – the species or idea of a triangle, nor any determinate singular triangle belonging to that species, but any triangle whatsoever.

It must be said here, however, that in *Erfahrung und Urteil* Husserl seems to use the expression 'objectuality of the understanding' (and thus 'categorial objectuality' and 'syntactical objectuality') in a somewhat more restricted sense that excludes the so-called general objects or species. In §64 of that work Husserl underscores the idea that the irreality of the objectualities of the understanding has to be distinguished from the generality of the species. Although – as we have seen – in the Sixth Investigation Husserl stresses some differences between the constitution of species and that of other non-sensible objectualities, in *Erfahrung und Urteil* the contrast between these sorts of objectualities is radicalized. A species is such that it can be instantiated in different objects – e.g., a color in different colored objects – and each of these objects has its individual moment of the species. The species – e.g., a color – can be apprehended only because a variety of different individual moments of the species is given; we compare them, and then we abstract the generic-universal by varying the examples perceived or imagined. But to apprehend a number (or state of affairs or collection) we do not need any comparison of supposed individual moments, nor any generalizing abstraction. The number 5 is identically the same object referred to in an unlimited plurality of acts, and not something obtained by comparing the objects referred to in those acts and then applying a so-called generic abstraction.

This restriction of the expression 'objectuality of the understanding' in *Erfahrung und Urteil*, although a clear modification of Husserl's views in *Logische Untersuchungen* – see, e.g., the Second Investigation – does not affect our discussion, since we are interested here almost exclusively in mathematical objectualities (see §§6 and 7 below) or, as we will also call them, pure categorial objectualities, and these are clearly different from species.

§5 *Levels of categoriality*

Now, the sorts of categorial acts can diverge in many directions. First of all, we have not so far emphasized the difference between the two basic sorts of sensible intuition, namely sensible perception and sensible imagination, since such a distinction is irrelevant for many purposes. It should be noticed, however, that an act founded on two or more simple acts can be founded either only on sensible perceptions, or only on sensible imaginations, or, in a mixed way, partly on sensible perceptions and partly on sensible imaginations (Hua XIX/2, 675). If the number of sensible acts is greater than two, this offers a diversity of combinations. Although Husserl is not here explicit in this regard, it is clear that for a categorial intuition to be a categorial perception, all its founding acts should be perceptions.

More interesting, however, is the complication that arises when categorial acts serve as founding acts of new categorial acts of higher level (Hua XIX/2, §§46, 59 and 60), as, e.g., when we establish a relation between two states of affairs or two sets. Since these new categorial acts of second level can also serve as founding acts of new categorial acts of third level, and so on indefinitely, we obtain a sequence of levels of foundation of categorial acts (Hua XIX/2, 675), that can be still more complicated if not all the inmediately founding acts of a categorial act belong to the same level. In this manner we obtain a whole hierarchy of types of categorial acts, in whose zero level lie the sensible acts and in whose first level lie those categorial acts that we have been considering, all of whose founding acts are sensible. This hierarchy of types of categorial acts is ruled by *a priori* laws that are 'the intuitive counterpart' of the logico-linguistic laws studied by Husserl in the Fourth Investigation. Actually, in either case the problem of truth does not play any role. The laws that govern the hierarchy of types of categorial intuitions 'do not directly say anything about the ideal conditions of possibility of an adequate fulfillment of meanings' (Hua XIX/2, 711). They are only laws of the pure doctrine of forms of intuitions, that regulate its primitive types, its forms of complication, and the sequence of ever more complicated forms of intuitions obtained by iterating the forms of complication (Hua XIX/2, 711). Restrictions enter the scene, however, as soon as one considers not only the mere syntactic possibility of forms of intuitions, but also the logical possibility of the objects of such intuitions.[10]

The possibility of taking no matter which categorial acts as founding acts of categorial acts of higher level and the corresponding possibility of 'expressing' these acts in corresponding meanings, lead to a relative distinction, both on the side of intuitions and on the side of meanings, between form and matter (Hua XIX/2, 711). In this relative sense, the objectuality constituted in a categorial act, on the basis of other objectualities that served as the material for its constitution, can serve as the material for the constitution of other objectualities of still higher level in new categorial acts. On the side of statements, the terms that are the matter of statements, correspond to the objects of founding acts, and it is to those terms that one has to look for any contribution made by sensibility. But since the objects of the founding acts can themselves be categorial objectualities, we have to inquire about their constitution. Actually, if we are interested in the fulfillment in intuition of the meaning of a statement, we have to inquire about the meanings of its terms, and if these are fulfilled by categorial objectualities, we have to inquire further about the objectualities that correspond to its founding acts. In this manner we have to continue descending the hierarchy of founded acts that serves as basis of the categorial act in which the state of affairs referred to by the statement under discussion is constituted, until we arrive at simple objects, which are the objects constituted in the simple acts of intuition that serve as ultimate foundations of such a categorial act (Hua XIX/2, 712).

§6 *Pure categoriality and mathematical intuition*

According to Husserl (Hua XIX/2, 712), everything categorial is based on sensible intuition, although the link with sensible intuition can vary in many ways. Now, Husserl calls 'sensible' only the simple acts, whereas he calls every founded act 'categorial', no matter if it is immediately or mediately based on sensibility. In the universe of categorial acts, however, Husserl distinguishes (Hua XIX/2, 713) between pure categorial acts, which are acts of the pure understanding, and mixed categorial acts, which are acts of the understanding mixed with sensibility.

Just as the generalizing abstraction, whose object is the species or idea, although necessarily based on individual intuition, does not refer to something individual, so there exists the possibility of general intuitions that refer neither to something individual nor to something sensible. Thus, Husserl distinguishes between sensible (generalizing) abstraction and pure categorial (generalizing) abstraction. The first gives us sensible concepts and sensible concepts mixed with categorial forms (or, briefly, mixed concepts). To the first group belong, e.g., the concepts of house, color and wish, whereas to the second group belong, e.g., the concepts of coloring, virtue and parallel axiom. Categorial abstraction, on the other hand, gives us purely categorial concepts like, e.g., the concepts of relation, set, number, and generally, all such concepts called by Husserl (see, e.g., Hua XVIII, 245 and Hua III/I, 27) formal-ontological categories and the derived concepts obtained from them. Sensible concepts – whether purely sensible (like *house*) or mixed (like *parallel axiom*) – have their ultimate foundation in sensible intuition. On the other hand, categorial concepts have their foundation in categorial intuitions, and with exclusive reference to the categorial form of the whole object categorially formed.

But this takes us to the origin of formal-ontological categories and to the problem of the intuition of mathematical entities. Given a categorial intuition of a relation, pure categorial abstraction directs itself to the form of the relation, leaving aside everything material in the related objects, considering them as mere indeterminate points of the relation. Thus, given a categorial intuition of the relation of 'being bigger than' between the sensible objects A and B, pure categorial abstraction directs itself to the relation, leaving the objects related as mere indeterminate points of the relation completely void of any individualizing traits. Similarly, given a categorial intuition of a set, pure categorial abstraction directs itself to the form of the collection, leaving the members of the set completely indeterminate. In this manner the formal-ontological categories of relation and set, respectively, are constituted. Thus, we can say that the intuition of mathematical entities, or, briefly, mathematical intuition, is categorial intuition purified by pure categorial abstraction. In this sense both pure logic and pure mathematics are purely categorial, since they do not contain any sensible concept in their whole theoretical foundation. In both of them the 'terms' remain purely indeterminate, and are usually represented by mere indicators (i.e., variables).

Once the formal-ontological categories, i.e., the primitive mathematical concepts, are constituted, the other mathematical entities are constituted in new pure categorial acts of higher level, in which the objectualities that serve as foundations are left completely indeterminate. In this manner the whole spectrum of mathematical entities is constituted, built on the basis of the formal-ontological categories which were constituted in a pure categorial abstraction that left indeterminate the sensible material on which a categorial intuition of first level was based. Thus, without contradiction, we can say that everything categorial is based on sensible intuition, and, on the other hand, that the concepts of pure mathematics are purely categorial, i.e., that they do not have any trace of something sensible in their constitution.

§7 *The Laws of categorial intuitions*

In theoretical thought categorial intuitions act as real or possible fulfillments or frustrations of meanings, and confer on the statements, depending on whether the first or second is the case, the truth value 'true' or 'false' (Hua XIX/2, 720). Thus, the pure laws that rule over categorial intuitions are, for Husserl, the pure laws of thought in the strict sense.

To begin with the discussion of these laws, we should first of all notice with Husserl (Hua XIX/2, 716) that there is a great freedom in the application of categorial forms in the constitution of new objectualities of the understanding, since that which acts as the material to which a determinate categorial form is applied does not determine the categorial form.

But this does not mean that the formation of new categorial objectualities is not ruled by laws that in some sense restrict this freedom. First of all, one should notice that categorial objectualities are constituted only in founded acts and never in acts of the lowest level. Moreover, categorial objectualities cannot be constituted on the basis of just any foundation, even though 'we can think – understood as merely signifying – any relation between any points of reference, and, more generally, any form on the basis of any material' (Hua XIX/2, 717). As we shall see below, there is no complete parallelism between the laws of formation of categorial intuitions and its objects, and the laws of formation of meanings, since the former have restrictions that are totally foreign to the latter.

Now, the ideal laws that rule over the possibilities and impossibilities of categorial objectualities belong to them *in specie*, and, thus they belong to the formal-ontological categories and their derived concepts. Such laws regulate the possible variations to which the categorial objectualities can be submitted, while the material that corresponds to their foundations is left fixed. They restrict the variety of reorderings and transformations of categorial objectualities, and since the peculiarity of the pertinent materials is totally irrelevant, such laws have the character of purely analytic laws. Owing to this, on the side of expressions, algebraic symbols are used as bearers of totally indeterminate and arbitrary representations to express such materials. And for gaining insight into such laws, 'any categorial intuition would suffice that puts before our eyes the possibility of the categorial objectuality concerned' (Hua XIX/2, 718).

These purely analytic laws are precisely those that rule over the possibility of mathematical entities, since, as Husserl observes (Hua XIX/2, 718-9), 'the ideal conditions of the possibility of categorial intuitions are correlatively the conditions of the possibility of its objects, i.e., of categorial objectualities generally'. It is essentially the same thing to say that a determinate categorial objectuality is formed in such and such a manner, and to say that a categorial intuition is performable in which such an objectuality is constituted on the basis of such and such corresponding founding intuitions.

Now, such analytic laws do not say anything about the categorial acts performable on the basis of sensible intuitions that serve as ultimate foundations. All sorts of contents can serve as material for any categorial intuition. But such analytic laws can teach us that, when a given material assumes some given form, then there is a fixed set of other forms that such a material may assume. In other words, 'there is an ideally closed set of possible transformations of a given form into other forms' (Hua XIX/2, 720).

§8 *The parallelism with pure grammar and the possibility of paradoxes*

As we have said more than once (see §§5 and 7 above), the laws of formation of categorial intuitions, and correlatively the laws of categorial objectualities, run parallel to the laws of formation of meanings. To all categorial acts with their categorially formed objectualities there can correspond mere meaning acts. But, as we know from the First Investigation, a meaning can be constituted in a meaning act without there being an intuition that fulfills it. Moreover, the region of meanings is more inclusive than the region of intuitions in which their possible fulfillments can be constituted, since on the side of meanings there exists an unlimited plurality of complex meanings which are 'impossible', i.e., they are complexes of meanings linked in such a way as to be a complex unitary meaning, but such that the possibility of a corresponding objectuality in (categorial) intuition is excluded. Therefore, there does not exist a complete parallelism between the hierarchy of sorts of meanings and the hierarchy of sorts of objectualities of the understanding. Such a parallelism exists at the lowest level, i.e., that of sensible intuition and their corresponding meanings, but since there is a greater freedom for linking meanings to form complex meanings than for linking objectualities to form complex (categorial) objectualities, it is not the case that to every sort of meaning there corresponds a categorial objectuality. In other words, since one speaks of compatibility or incompatibility only in the region of the complex, to each simple meaning there corresponds an objectuality, but not to each complex meaning (Hua XIX/2, 729).

Thus, here originates the possibility of (analytic) contradictions and, especially, the possibility of the so-called paradoxes. From the point of view of the formation of meanings it is legitimate to form meanings such as 'the greatest ordinal' or 'the greatest cardinal number', or 'the set of all sets that do not contain themselves as elements'. But the objectualities that would correspond to such meanings are impossible, i.e., they cannot be constituted in any categorial intuition (see Appendix II below).

The divergence between the hierarchies of meanings and of categorial intuitions with their corresponding categorial objectualities leads Husserl (Hua XIX/2, §§63f.) to a distinction between what he calls proper and improper laws of thought. The laws of thought proper rule over the possibility of forming new categorial acts and their corresponding categorial objectualities. The laws of improper thought rule over the possibility of forming complex unitary meanings. Correspondingly, Husserl calls improper acts of thought all those acts in which meanings are constituted, and proper acts of thought all the corresponding (possible) intuitions. Now, since in the region of formation of meanings we are free to form any complex meaning, even when no object could correspond to it, if we want to avoid not only nonsense, but also formal (and material) countersense, we have to restrict the more extensive region of improper thought to conform to the objective possibility of fulfillment by some categorial intuition. If we restrict our consideration to formal countersense, the laws so obtained, namely, the pure laws of the validity of meanings, are precisely the pure logical laws in the strict sense, which, according to Husserl (Hua XIX/2, 723), have to run parallel to the pure analytic laws that rule over the formation of categorial objectualities, and are no less analytic than these. Such pure logical laws in the strict sense rule over the possibilities, determined in a purely categorial manner, of connection and transformation of meanings, without affecting the possibility of fulfillment of meanings, or, as we could also say, *salva veritate*. As in the case of the other two sorts of laws considered above, namely, those of proper thought and those of improper thought in its full, i.e., non-restricted sense, in such laws the material does not play any role, and 'material meanings' are substituted by algebraic signs that mean in an indirect and completely indeterminate way (Hua XIX/2, 724). It is in this precise Husserlian (neither Kantian, nor Fregean, nor Carnapian) sense that all such laws are analytic.[11] Here we are concerned with the pure conditions of the objective possibility of meanings, i.e., the possibility of meanings having a referent, and this leads us to the conditions of possibility of categorial intuitions. Thus, although these logical laws of the validity of meanings are not identically the same as the laws of categorial intuitions (the laws of the possibility of constitution of categorial objectualities), they follow closely in their steps.[12] Actually, the laws of the possibility of constitution of categorial objectualities are precisely the laws of mathematical existence.

Now, usually our thought operates partly intuitively and partly symbolically (i.e., without corresponding intuition). Husserl calls (Hua XIX/2, 725) complex acts that are partly intuitive and partly symbolical improper categorial intuitions. In such a case the objectual correlate of such an act is represented only improperly and its possibility is not secured. But this takes us once more to the problem of the possibility of contradictions and, especially, of paradoxes, since only when the *a priori* possibility of the constitution of each of the founding objectualities that serve as 'material' foundations of different levels of a complex categorial objectuality is established, is the possibility of the constitution of such an objectuality also established. Hence, although our thought operates partly symbolically, the fulfillment of every complex non-contradictory meaning must in principle be possible.

Appendix I

The assessment of categorial objectualities in Erfahrung und Urteil

As is well known, from 1905 onwards there is a reorientation of Husserl's philosophy in the direction of transcendental phenomenology, and it is pertinent to examine if this reorientation somehow affects our present discussion.[13] It is reasonable to think that even when in the *Logische Untersuchungen* – some would like to say in its first volume only – Husserl had sustained a platonist conception of mathematics, in his later philosophy he must have inclined himself to a sort of constructivism that had remained latent since his *Philosophie der Arithmetik.* It seems appropriate to try to illuminate this situation by considering Husserl's discussion of the objectualities of the understanding in §64 of *Erfahrung und Urteil,* a text on which Landgrebe was working under Husserl's supervision at the time of the latter's death and which would thus include any modification of Husserl's conception of mathematical entities produced by any of Husserl's reorientations of his philosophy after 1900.[14] We will see, however, that Husserl's assessment of mathematical (and other categorial) objectualities in *Erfahrung und Urteil* does not lead to any sort of constructivism, but at most to a refinement of his platonistic conception. Since what is true in general of the objectualities of the understanding – even in the restricted sense of this expression in *Erfahrung und Urteil*[15] – is also true in particular of mathematical entities, in what follows we will often refer only to mathematical objectualities, which are our main interest and for which Husserl's assessment seems easier to defend than for categorial objectualities in general.

For Husserl in *Erfahrung und Urteil* the principal difference between individual objects given in sensible intuition and categorial objectualities lies in their different relation to temporality. Certainly, for Husserl all objects have a determinate relation to the internal time of consciousness, in which all acts of consciousness are constituted. But, whereas individual objectualities are also linked to physical objective time and to its objective temporal points, the objectualities of the understanding are not so linked. They are unreal objectualities, if by 'real' we understand 'real physical'. Real individual objects of sensibility are individualized by their appearance in the objective point (or interval) of physical time, which is represented in internal time. But the time points that individualize objects of the lowest level do not play a similar role in the case of objects of higher level (immediately or mediately) founded on them, and *a fortiori* do not play any role in the case of mathematical objectualities, whose link with sensible individual objects – as explained in §6 above – is particularly thin. Mathematical objectualities are not bounded to any temporal point or interval of points. In contrast to real objects, they are irreal in the sense of being anywhere and nowhere. They can appear in any spatio-temporal coordinates – even in different spatial coordinates at the same time – and be the same objectuality. One can say that mathematical objectualities existed before they were discovered, and that the laws based on them were valid before being discovered, in the sense that both mathematical objectualities and the laws that rule over them can be generated at any time point by subjects with the capacity to constitute the first and to recognize the validity of the second. In this way the existence of mathematical objectualities is omnitemporal, and one can say generally that mathematical (and other unreal) objectualities existed even before someone had constituted them. Certainly, when they are actualized or constituted, they enter spatio-temporal facticity, i.e. they locate themselves spatio-temporally, but this insertion in spatio-temporal facticity does not individualize them.

The intemporality, the everywhere and nowhere of mathematical objectualities, is a special form of temporality, namely, omnitemporality, and this form of temporality is clearly different from that of real objects. Mathematical objectualities exist in any time, and do not have any link to objective spatio-temporal points or intervals of points that correspond to the internal time in which the categorial acts that constitute them are themselves constituted.

Appendix II

Some non-objectualities of the understanding

In §5 we have shown how categorial objectualities are constituted, and in §8 we have underscored the thesis that the parallelism between the formation of complex meanings and the formation of complex categorial objectualities breaks down, since some restrictions intervene in the latter case that are completely unknown in the first. In this Appendix we will make use of such restrictions to show that some supposed mathematical entities that have originated well-known paradoxes in set theory cannot be constituted in any categorial intuition and, thus, cannot have any mathematical existence on the basis of the epistemology of mathematics that we have extracted from Husserl's texts.

1. Russell's Paradox: If it were true – as was thought to be true in naive set theory – that every property determines a set, one could consider the set R of all sets x such that $x \notin x$. Thus, $R=\{x/x \notin x\}$. One could then ask if the set R is or is not an element of itself, i.e., one could ask if $R \in R$ or $R \notin R$. Now, if $R \in R$, i.e., if $R \in \{x/x \notin x\}$, then $R \notin R$. On the other hand, if $R \notin R$, then R satisfies the condition required for membership in R, since $R=\{x/x \notin x\}$. Hence, $R \in R$. Thus, if $R \in R$ then $R \notin R$, and if $R \notin R$, then $R \in R$.

Now, it is easy to observe that the Russell set R cannot be constituted at any of the levels in the hierarchy of categorial intuitions, since, although new sets can be constituted at every level of the hierarchy, they have as members only objectualities constituted at lower levels of the hierarchy. Hence, it can never be the case that a set is a member of itself. More generally, for any categorial (and, in particular, for any mathematical) objectuality, it is never the case that it applies to itself, since its 'arguments' have to be constituted at lower levels of the hierarchy.

2. Cantor's Paradox: As is well known, two sets have the same cardinality if there exists a bijection between them, and a set A has a greater cardinality than a set B if there exists a bijection between B and a (proper) subset of A but no bijection between A and B. Cantor's Theorem establishes that the power set of a set C (i.e. the set of all subsets of C) has a greater cardinality than C. Thus, beginning with the set of natural numbers, the power set construction allows the formation of sets of ever greater infinite cardinality. Now, in naive set theory one assumes that there exists a set V of all sets. Since V is the set of all sets, it must have cardinality equal to or greater than that of any set C, since any other set D is a subset of V. Now, by the power set construction, one can obtain the power set P(V) of V, and by Cantor's Theorem the cardinality of P(V) is greater than the cardinality of V. But since P(V) is a set, its cardinality must be equal to or less than the cardinality of V. Thus, we obtain a contradiction.

Now, a set like V, i.e., the set of all sets, cannot be constituted at any stage of the hierarchy of categorial objectualities, since new sets can be constituted at each stage of the hierarchy, no matter if the hierarchy continues in the transfinite. The process of constitution of categorial objectualities – and, particularly, that of sets – is iterable without limit.[16] Thus, since the set of all sets cannot be constituted in any categorial intuition, Cantor's Paradox is also blocked.

In the same vein one can show that the supposed entities that originate the rest of the paradoxes of naive set theory cannot be constituted in any categorial intuition.

Appendix III

Some objectualities of the understanding

According to Husserl's philosophy of mathematics, mathematics – geometry excluded – is formal ontology, i.e., an ontology based on the completely formal concept of 'something in general' [*Etwas überhaupt*]. The fundamental concepts of mathematics, the so-called formal-ontological categories, are formal variations of the concept of 'something in general' that give rise to the fundamental mathematical structures, each based on one of those categories. Among these categories Husserl includes the concepts of set and relation, together with, e.g., those of whole and part, and of cardinal and ordinal number.

As is well known, in current discussions of set-theoretical foundations of mathematics not all such notions are considered equally fundamental.[17] The notion of set is usually considered as the most fundamental notion of mathematics and all other basic notions are defined by means of it. Such fundamentality of the notion of set has been recently questioned by mathematicians working in category theory.[18] However, since our interest here is to show how it is that mathematical entities are constituted in mathematical intuition, we can avoid discussions about the foundations of mathematics, and will simply take as fundamental mathematical objectualities two of Husserl's favorite examples, namely, sets and relations. Thus, we will give some sketchy indications of how some other mathematical entities are constituted in mathematical intuition on the basis of those two fundamental ones.

As was shown in §3 above – see also §6 – both collections or sets and relations can be constituted in categorial intuitions of the first level, and since we can abstract from the peculiar nature of the members of the set and of the terms of the relation, the formal-ontological categories of set and relation are constituted in a mathematical intuition of the first level. Thus, given two or more sets, relations between them, e.g., bijective correspondences between sets or between sets and subsets of other sets can be constituted in mathematical intuitions of the second level – mathematical since the peculiarity of the members of the correspondence pairs is here abstracted from. Finite cardinal numbers can then be constituted in mathematical intuitions of the third level as equivalence classes of sets whose members are related by such bijective correspondences.

Once sets have been constituted in mathematical intuitions of the first level, non-empty intersections, unions, relative complements and symmetric differences of sets can be constituted in mathematical intuitions of the first level. On the basis of the relative complement, the empty set can be constituted as the relative complement of a set to itself in a mathematical intuition of the first level. But once the empty set is constituted, intersections in the more general sense – that does not exclude the possibility of being empty – can also be constituted in mathematical intuitions of the first level.[19] Moreover, families of sets or of subsets of a given set, and, in particular, the power set of a given set, can be constituted in mathematical intuitions of the level immediately higher than the level in which the given set (or sets) is (are) constituted, and such a process of formation of new sets in acts of mathematical intuition of ever higher level can be iterated indefinitely. But once a set, its power set, the empty set, and the intersection and union of sets have been constituted, filters and ultrafilters, and ideals and prime ideals can be constituted in new mathematical intuitions of the same level as the power set of which they are subfamilies.

In this manner all (legitimate) mathematical entities of classical mathematics should in principle be constituted in mathematical intuitions of ever higher level. The legitimacy of a mathematical entity would be established by tracing the genesis of its constitution in the hierarchy of types of mathematical intuitions, since, as we have seen above, paradoxical entities cannot be constituted in any mathematical intuition. This program, even if successful, would not, however, eradicate the possibility of contradictions in mathematics, since it is always possible for a mathematician to attribute contradictory properties to a legitimate mathematical entity, but the danger of contradictions would be considerably diminished.

Notes

1. References to Husserl's works published in the Husserliana edition are the standard ones. EU is used as an abbreviation of *Erfahrung und Urteil*, fifth revised edition (Hamburg: F. Meiner, 1976).
2. See, e.g., M. Dummett's 'The Philosophical Basis of *Intuitionistic Logic*', in P. Benacerraf and H. Putnam, eds., *Philosophy of Mathematics*, second revised edition (Cambridge et al.: Cambridge University Press 1983), pp. 97–129.
3. See, e.g., P. Benacerraf's 'Mathematical Truth', in P. Benacerraf and H. Putnam, op. cit., pp. 403–420.
4. See G. Frege, *Nachgelassene Schriften*, second edition (Hamburg: F. Meiner, 1983), pp. 282–302. For 'Der Gedanke' see G. Frege, *Kleine Schriften* (Darmstadt: Wissenschaftliche Buchgesellschaft, 1967), pp. 342–362.
5. See K. Gödel's 'Russell's Mathematical Logic' in P. Benacerraf and H. Putnam, op. cit., pp. 447–469.
6. For an exposition of Husserl's philosophy of mathematics, see the present author's doctoral dissertation *Edmund Husserls Philosophie der Logik und Mathematik im Lichte der gegenwärtigen Logik und Grundlagenforschung* (Bonn 1973). See also R. Schmit, *Husserls Philosophie der Mathematik* (Bonn: Bouvier, 1981).
7. Perception and imagination are, for Husserl, the two principal sorts of intuition, but their difference is unimportant for most of our discussions.
8. Concerning the issues discussed in this and the next two §§ the interested reader can compare our treatment with that of Adolf Reinach in 'Zur Theorie des negativen Urteils', (1911) translated in Barry Smith, ed., *Parts and Moments* (München and Wien: Philosophia Verlag, 1982), pp. 315–377. See especially Part II, pp. 332–354.
9. For Husserl's distinction between part and moment, see Hua XIX/1, U. III and Barry Smith, ed., op. cit., especially the first of the 'Three Essays in Formal Ontology' by P. M. Simons and the introductory essay 'Pieces of a Theory' by B. Smith and K. Mulligan.
10. Such restrictions will be discussed in §§7 and 8 above. In this context Husserl is considering the possibility of a morphology of intuitions in analogy to the morphology of meanings considered in pure logical grammar (see, e.g., Hua XIX/1, U. IV and Hua XVII, Kap. I, §13), and for which even the formal possibility of fulfillment (and, thus, the avoidance of countersense) is unimportant. However, although the distinction in the realm of meanings between the levels determined by the laws that avoid nonsense and those that avoid countersense (the logical laws) has been particularly fruitful and generally accepted, the analogous distinction in the realm of intuitions seems somewhat artificial.
11. For Husserl, a statement is analytically true if it is true and can be formalized *salva veritate*. See Hua XIX/1, U. III, §12.
12. For Husserl, there is (or should be) a sort of parallelism between logic and

mathematics. See Hua XVII, Kap. 2–3 and Hua XVIII, Kap. XI, and the present author's dissertation cited in note 6 above.

13. Supposedly in the 1930s there occurred another reorientation of Husserl's thought.
14. Although *Erfahrung und Urteil* was prepared for publication, possibly with stylistic alterations, by Ludwig Landgrebe on the basis of Husserlian manuscripts, the present writer does not have any misgivings concerning the authorship of the ideas expounded in that work. In particular, Husserl's conception of ideal entities in *Erfahrung und Urteil* is, in the present writer's opinion, a 'natural consequence' of a reexamination of his conception in *Logische Untersuchungen* in the context of his later philosophy.
15. See §4 above.
16. The first set of infinite level would be obtained by constituting the set of all categorial objectualities constituted at any finite level.
17. The concepts of part and whole are not even generally acknowledged as mathematical concepts. See, however, the book edited by Barry Smith cited in note 9 above.
18. Both for set-theoretical and for categorial foundations of mathematics, see W. S. Hatcher's excellent book *The Logical Foundations of Mathematics* (Oxford et al.: Pergamon Press, 1982). (This book is a revised version of Hatcher's former *Foundations of Mathematics* [London et al.: W. B. Saunders & Co., 1968].)
19. We do not intend to be completely rigorous here, especially when considering intuitions of the same level. Thus, we are not excluding any other more 'natural' ordering in the generic constitution of such objectualities. Alternative orderings – even those for which the notions of set and relation are not fundamental – are allowed, provided that type restrictions in the hierarchy of mathematical objectualities are not violated. Moreover, it should be underscored that in this (and the former) appendix we use freely the term 'set', although only the constitution of finite sets has been discussed in the main text.