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A general method for multiple crack problems in a finite plate

Y. K. Cheung[1], C. W. Woo[2] and Y. H. Wang[3]

[1] Department of Civil and Structural Engineering, University of Hong Kong, Hong Kong
[2] Department of Mechanical Engineering, University of Hong Kong, Hong Kong
[3] Department of Mechanics, Huazhong University of Science and Technology, Wuhan, China

Abstract. A novel method for the multiple crack problems in a finite plate is proposed in this paper. The basic stress functions of the solution consist of two parts. One is the Fredholm integral equation solution for the crack problem in an infinite plate, and the other is that of the weighted residual method for general plane problems. The combined stress functions are used in the analysis and the boundary conditions on the crack surfaces and the boundary are considered. After the coefficients of the functions have been determined, the stress intensity factors (SIF) at the crack tips can be calculated. Some numerical examples are given and it was observed that when the cracks are very short, the results compare very favorably with the existing results for an infinite plate. Furthermore, the influence of the boundary can be considered. This method can be used for arbitrary multiple crack problems in a finite plate.

1 Introduction

Because of inherent defects that occur in the material or damage incurred during the service life of the components, cracks may appear in a plate. The number of cracks, sizes, locations, crack number and the boundary conditions may vary in practice. Some multiple crack problems in an infinite plate have been solved analytically, for example by complex variable method (Benthem et al. 1973) and integral equation method (Sneddon 1973). The solutions obtained are usually valid for some special cases, such as collinear, parallel or star cracks where the boundary conditions are usually rather simple. However, in practice, the fracture problems may have finite dimensions and complex boundary conditions, and it is difficult to use these analytical methods. Although the finite element method may also be used for multiple crack problems, it requires elaborate data preparation and long computing time. The accuracy of the results is usually not very satisfactory unless a large number of elements are used.

Meanwhile, the boundary collocation method has a series of advantages. It is especially suitable for the finite dimension cases. Because of the difficulty of choosing the stress functions, it is usually used for some simple geometric crack problems, such as an edge crack (Gross et al. 1964), central crack (Kobayashi et al. 1964) and double edge crack (Cheung et al. 1988) etc.

For the general multiple crack problems an efficient numerical method is therefore needed to deal with the complex geometry. In an infinite plate, Fredholm integral equation method (Chen 1984) may be used for some simple loading cases. Although it may also be applicable to some special finite plates, it is not easy to choose the additional stress functions. On the other hand, although the weighted residual method has been used for a general plane case (Xu et al. 1982), it is not suitable for analyzing crack problems. However, these two sets of stress functions are linearly independent, and they can be combined for the analysis of general multiple crack problems in a finite plate.

2 Elementary solutions

In this section the elementary solutions of the problem are discussed at first. They will then be combined to construct the stress functions of the general crack problems.

2.1 Solution for crack surface forces

According to the complex variable method of plane elasticity (Muskhelishvili 1975), the stresses are related to the two analytical functions, $\Phi(z)$ and $\Omega(z)$, as indicated by

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathrm{Re}\,\Phi(z) \tag{1}$$

$$\sigma_{yy} - \mathrm{i}\tau_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}. \tag{2}$$

As shown in Fig. 1, two pairs of forces are applied at a point of the crack, $(s, 0)$. The stress functions $\Phi(z)$ and $\Omega(z)$ can be defined as (Chen 1984).

$$\Phi(z) = \Omega(z) = -\frac{P - \mathrm{i}Q}{2\pi i}\frac{X(s)}{X(z)(z-s)}, \tag{3}$$

where

$$X(z) = \sqrt{z^2 - a^2}.$$

Using the elasticity theory, it can be shown that the equilibrium equations and the single-displacement condition (without rigid body motion) have been satisfied. To obtain a solution, only the boundary conditions needed to be considered. The normal and tangential stresses of a point, z, located on the $o'x'$ axis in Fig. 1 can be obtained as

$$\sigma_{y'y'} - \mathrm{i}\tau_{x'y'} = -\frac{P - \mathrm{i}Q}{2\pi i}X(s)[G(z) + e^{-2\mathrm{i}\alpha}\overline{G(z)}] - \frac{P + \mathrm{i}Q}{2\pi \mathrm{i}}X(s)[\overline{G(z)}(1 - e^{-2\mathrm{i}\alpha}) + e^{-2\mathrm{i}\alpha}(z - \bar{z})\overline{G'(z)}], \tag{4}$$

where

$$G(z) = \frac{1}{X(z)(z-s)}, \quad G'(z) = \frac{a^2 + sz - 2z^2}{(z-s)^2[X(z)]^3}. \tag{5}$$

Now let's consider two basic cases. At first, let $P = 1$ and $Q = 0$, then the corresponding normal and tangential tractions at point z are (Chen 1984):

$$\sigma_{y'y'} - \mathrm{i}\tau_{x'y'} = f_{nn} - \mathrm{i}f_{nt} = -\frac{\sqrt{a^2 - s^2}}{2\pi}[G(z) + \overline{G(z)} + e^{-2\mathrm{i}\alpha}(z - \bar{z})\overline{G'(z)}]. \tag{6}$$

In f_{nn} and f_{nt} of the above equation, the first subscript n indicates that the tractions are due to a pair of normal forces acting at the point $(s, 0)$, the second subscript denotes the traction direction according to line $o'x'$, in the normal or tangential direction.

Secondly, let $P = 0$ and $Q = 1$, the corresponding normal and tangential tractions at point z are (Chen 1984):

$$\sigma_{y'y'} - \mathrm{i}\tau_{x'y'} = f_{tn} - \mathrm{i}f_{tt} = \frac{\sqrt{a^2 - s^2}}{2\pi \mathrm{i}}[\overline{G(z)}(1 - 2e^{-2\mathrm{i}\alpha}) - G(z) + e^{-2\mathrm{i}\alpha}(z - \bar{z})\overline{G'(z)}]. \tag{7}$$

Similarly, in f_{tn} and f_{tt} of the above equation, the first subscript t indicates that the tractions are due to a pair of tangential forces acting at the point $(s, 0)$, the second subscript denotes the traction direction according to line $o'x'$.

The above stress functions can be used for an infinite cracked plate with the vanishing remote stress condition. For a finite cracked plate, they must be superposed with other stress functions.

2.2 Solution by weighted residuals method

For the plane elastic problem, the stress components are related to the Airy stress function ϕ as

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y}. \tag{8}$$

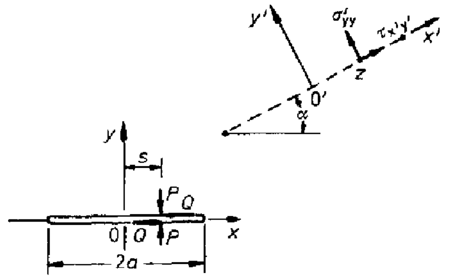


Fig. 1. Stresses at Point z according to the loading on the crack surface

A set of trial functions for plane problem (Xu et al. 1982) is taken as

$$\begin{aligned}\phi = \sum_{i=1}^{I} [&\sin \alpha x(C_{i,1} e^{\alpha y} + C_{i,2} e^{-\alpha y} + C_{i,3} y e^{\alpha y} + C_{i,4} y e^{-\alpha y}) \\ &+ \cos \alpha x(C_{i,5} e^{\alpha y} + C_{i,6} e^{-\alpha y} + C_{i,7} y e^{\alpha y} + C_{i,8} y e^{-\alpha y}) \\ &+ \sin \alpha y(C_{i,9} e^{\alpha x} + C_{i,10} e^{-\alpha x} + C_{i,11} x e^{\alpha x} + C_{i,12} x e^{-\alpha x}) \\ &+ \cos \alpha y(C_{i,13} e^{\alpha x} + C_{i,14} e^{-\alpha x} + C_{i,15} x e^{\alpha x} + C_{i,16} x e^{-\alpha x})],\end{aligned} \tag{9}$$

where

$$\alpha = \frac{i\pi}{d}$$

in which d is the maximum dimension in the x or y direction of the plate, C_{in} $(n = 1, 2, \ldots, 16)$ are the coefficients to be determined.

3 Calculation formulae

It can be verified that the two sets of stress functions given above are linearly independent. They can be combined as a general function for a multiple crack problem in a finite plate. It can be shown that the equilibrium equations and the compatibility equations have been satisfied for the assumed stress functions. Therefore, only the boundary conditions are to be considered.

As shown in Fig. 2, a finite plate contains L cracks. The loading may be applied on the crack surfaces and the boundaries. For a point on the boundaries, the total forces acting include: the forces acting on the point itself, the forces caused by action on the other cracks calculated by Eqs. (6) and (7), the forces calculated by Eq. (9). Using the principle of superposition, the force components at this point in the x and y directions (assume that the point is on the kth crack) can be obtained as:

$$\begin{aligned}&P_k(s_k) + {\sum_{l=1}^{L}}' \int_{-a_l}^{a_l} P_l(s_l) f_{nn,lk}(s_l, s_k)\, ds_l + {\sum_{l=1}^{L}}' \int_{-a_l}^{a_l} Q_l(s_l) f_{tn,lk}(s_l, s_k)\, ds_l \\ &\quad - \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)_{s=s_k} \cos(n, x) + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{s=s_k} \cos(n, y) = p_k(s_k), \quad (-a_k < s_k < a_k), \quad k = 1, 2, \ldots, L \\ &Q_k(s_k) + {\sum_{l=1}^{L}}' \int_{-a_l}^{a_l} P_l(s_l) f_{nt,lk}(s_l, s_k)\, ds_l + {\sum_{l=1}^{L}}' \int_{-a_l}^{a_l} Q_l(s_l) f_{tt,lk}(s_l, s_k)\, ds_l \\ &\quad - \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)_{s=s_k} \cos(n, y) + \left(\frac{\partial^2 \phi}{\partial y^2}\right)_{s=s_k} \cos(n, x) = q_k(s_k), \quad (-a_k < s_k < a_k), \quad k = 1, 2, \ldots, L,\end{aligned} \tag{10}$$

where the symbol Σ' means that $l = k$ has been excluded in the summation. The functions f can be found in Eqs. (6) and (7). $p_k(s_k)$ and $q_k(s_k)$ are the applied normal and tangential forces on the kth

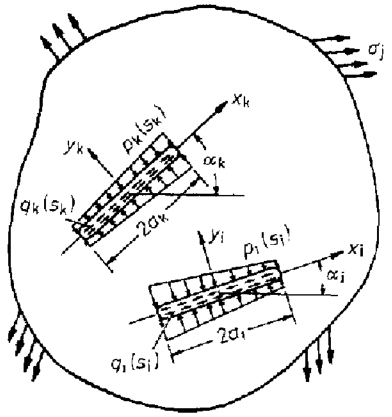


Fig. 2. Loading conditions of a multiple crack plate

crack. $P_k(s_k)$, $Q_k(s_k)$ and $P_l(s_l)$, $Q_l(s_l)$ are the forces which are to be determined on the crack surfaces. The $\cos(n,x)$ and $\cos(n,y)$ are the direction cosines of the point on the crack.

If a point on the boundary, a formula similar to Eq. (10) can be written, except that $P_k(s_k)$ and $Q_k(s_k)$ are taken to be zero, the summation terms should be from 1 to L and that there should be no exclusion.

The integrals in Eq. (10) can be written in Chebyshev integral form. For example,

$$\begin{aligned}\int_{-a_l}^{a_l} P_l(s_l) f_{nn,lk}(s_l, s_k)\, ds_l &= \sum_{m=1}^{M} P_l(s_{lm}) f_{nn,lk}(s_{lm}, s_{ki})\omega \\ \int_{-a_l}^{a_l} Q_l(s_l) f_{tn,lk}(s_l, s_k)\, ds_l &= \sum_{m=1}^{M} Q_l(s_{lm}) f_{tn,lk}(s_{lm}, s_{ki})\omega, \quad i = 1, 2, \ldots, M,\end{aligned} \tag{11}$$

where

$$s_{li} = a_l \cos\frac{(2i-1)\pi}{2M}, \quad s_{lm} = a_1 \cos\frac{(2m-1)\pi}{2M}, \quad \omega = \frac{\pi a_1}{M}\sin\frac{(2m-1)\pi}{2M}. \tag{12}$$

Therefore, the unknowns are $P_l(s_{lm})$, $Q_l(s_{lm})$ and C_{in}. If the crack number is L, the summation number of the function ϕ is I, then $2 \times L + 16 \times I$ unknowns are to be determined. For the crack surfaces, the number of points taken is $2 \times M$, the positions are determined by Eq. (12). On the boundary, the number of points can be chosen arbitrarily but the number must be larger than $8 \times I$. The points are arranged with equal intervals on every boundary for simplicity, and finer meshes are used for higher accuracy. The least square method is then used in the calculation.

After solving for $P_l(s_{lm})$, $Q_l(s_{lm})$($l = 1, 2, \ldots, L$) and C_{in} ($i = 1, 2, \ldots, I$, $n = 1, 2, \ldots, 16$), the stress functions $\Phi(z)$, $\Omega(z)$ and $\phi(x, y)$ can be determined. Note that only the functions $P_l(s_{lm})$, $Q_l(s_{lm})$ represent the stress singularity at the crack tips, and the SIF can be calculated by the formula:

$$K_I - \mathrm{i}K_{II} = \lim_{z \to \pm a} 2\sqrt{2\pi(z \mp a)}\,\Phi(z).$$

The SIF at the crack tips of the lth crack are

$$K_{I,l}^{\pm} = -\frac{1}{\sqrt{\pi a_l}} \int_{-a_l}^{a_l} P_l(s_l) \sqrt{\frac{a_l \pm s_l}{a_l \mp s_l}}\, ds_l, \quad K_{II,l}^{\pm} = -\frac{1}{\sqrt{\pi a_l}} \int_{-a_l}^{a_l} Q_l(s_l) \sqrt{\frac{a_l \pm s_l}{a_l \mp s_l}}\, ds_l, \tag{13}$$

where the upper and the lower signs refer to the right and left crack tips respectively.

In the calculation of Eq. (13), the following numerical integral formulae is used (Erdogan 1978):

$$\int_{-a_l}^{a_l} \frac{f(x)}{\sqrt{a_l^2 - x^2}}\, dx \cong \frac{\pi}{M} \sum_{m=1}^{M} f\left(a_l \cos\frac{(2m-1)\pi}{2M}\right). \tag{14}$$

4 Numerical examples

Three examples are presented to show the validity of the present method. The first one is for the crucifix crack in a plate with bi-axial loading. The second one is for two perpendicular cracks which do not cross. The third one is for the star-shaped crack in a finite plate.

4.1 Crucifix crack

As shown in Fig. 3, the problem of a crucifix crack in the form of a cross in a finite plate with bi-axial loading is analyzed. The half length and width of the plate are: $h = w = 0.5$, the half crack length a is varied from 0.05 to 0.40. The number of summation terms I for function ϕ is taken as 4, the number of the discrete points of the integral equation M is 16, the number of collocation points on the entire boundary N is 60. The computed results are shown in Table 1. Because of symmetry, the SIF values for the four crack tips are the same, and all the K_{II} are equal to zero. Therefore, only the results of K_I for one tip are presented.

For the crucifix crack in an infinite plate, the result obtained by an analytical method (Sneddon 1973) is given as:

$$K_I/K_0 = 0.8636, \quad K_{II}/K_0 = 0,$$

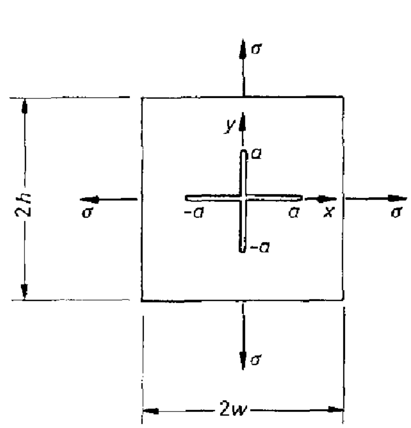
where $K_0 = \sigma\sqrt{\pi a}$.

When a/w is rather small, the computed results of a finite plate should approach those of the infinite case. This has been verified in Table 1 and shown graphically in Fig. 4. For the case $a/w = 0.1$, the computed values by the present method are:

$$K_I/K_0 = 0.8641, \quad K_{II}/K_0 = 0.$$

From the table and figure it can also be seen that the SIF values increase with increasing values of a/w. Therefore, if the crack length becomes large comparing with the plate dimension, it would be unsafe to use the SIF values from infinite plate.

In order to analyze the convergence of the present method, analyses have been carried out for various parameters involving the number of crack integral points, the number of summation terms of the function ϕ in Eq. (9), and the number of collocation points on the boundary. For convenience, only one of the parameters is examined at a time. The related dimensions are kept the same in the calculation: $w = h = 0.5$, $a/w = 0.1$. The computed results are presented graphically in Figs. 5 to 7. It can be seen that the SIF values converge satisfactorily although a small number of terms and collocation points are used.



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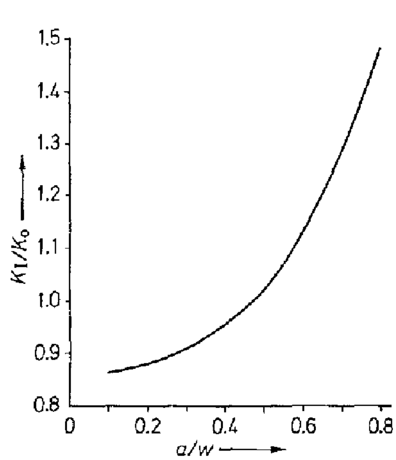

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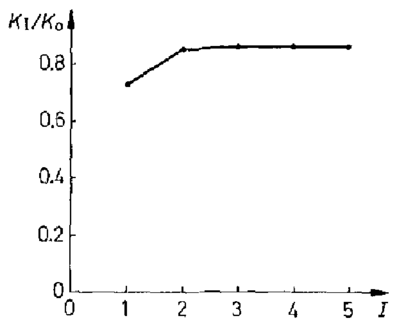
Figs. 3 and 4. 3 Crucifix crack in a finite plate. 4 SIF for the crucifix crack

Table 1. Calculated results for the crucifix crack

$\frac{a}{w}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$\frac{K_I}{K_0}$	0.8641	0.8800	0.9092	0.9537	1.0223	1.1300	1.2866	1.4857



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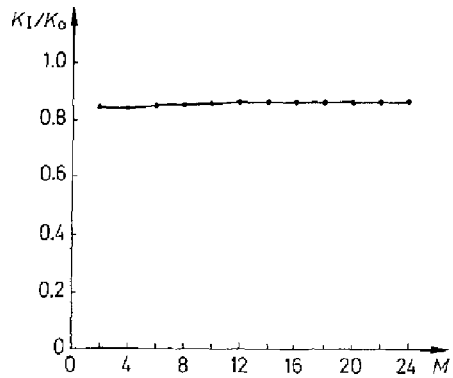

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Figs. 5 and 6. K_I/K_0 as a function **5** of summation terms of ϕ; **6** of crack integral points

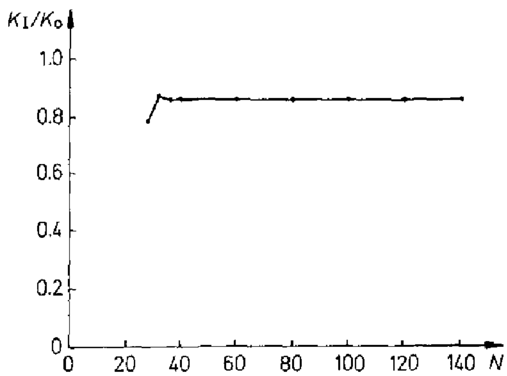
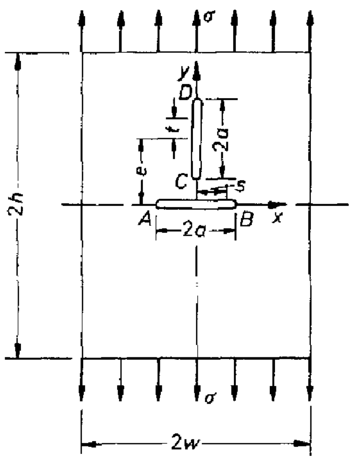


Fig. 7. K_I/K_0 as a function of boundary collocation points

4.2 Two perpendicular cracks

As shown in Fig. 8, two perpendicular cracks in a plate is analyzed. One of cracks is on the x-axis, with the crack center coincides with the origin, and the other is on the y axis with an eccentric distance to the origin. This example is chosen to represent an asymmetric case.

The plate dimensions and the parameters are the same as in the above example. The crack length $2a$ is varied from 0.1 to 0.40, and the eccentric distance $e = 1.2a$. The computed results are summarized in Table 2 and Fig. 9. For the case of small crack length, the present results agree very well with those of the infinite plate (Chen 1984). Because of the symmetry of the plate about the y axis, the SIF for crack tips A and B have the same values, and K_{II} for the crack tips C and D are equal to zero. As long as the crack length increases, the K_I values at the crack tips A, B and the absolute values at C become larger. For the crack tip D, it decreases slightly at first and then



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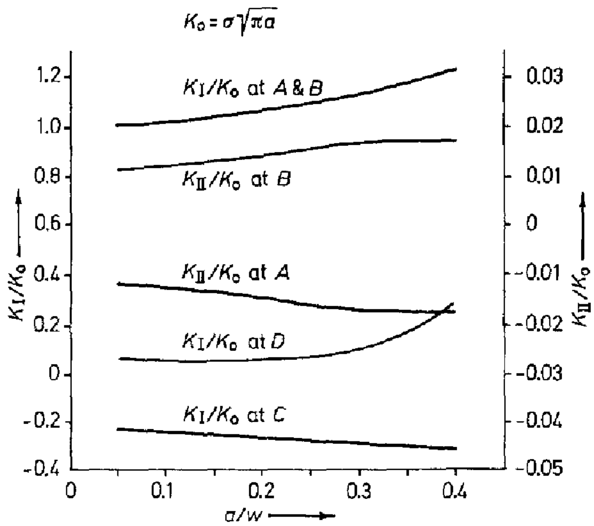

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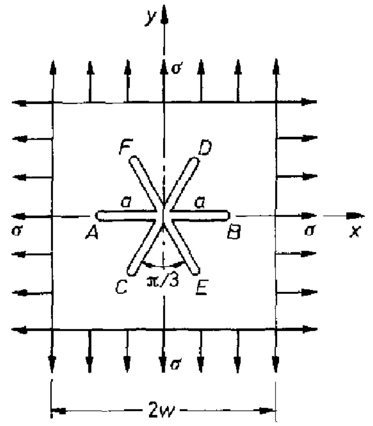
Figs. 8 and 9. 8 Two perpendicular cracks in a finite plate. 9 SIF for the two perpendicular cracks

Table 2. Calculated results for two perpendicular cracks

Crack tip	$\frac{a}{w}$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
A	$\frac{K_I}{K_0}$	1.0085	1.0193	1.0374	1.0626	1.0947	1.1337	1.1801	1.2338
	$\frac{K_{II}}{K_0}$	−0.0118	−0.0124	−0.0134	−0.0146	−0.0159	−0.0169	−0.0171	−0.0175
B	$\frac{K_I}{K_0}$	1.0085	1.0193	1.0374	1.0626	1.0947	1.1337	1.1801	1.2338
	$\frac{K_{II}}{K_0}$	0.0118	0.0124	0.0134	0.0146	0.0159	0.0169	0.0171	0.0175
C	$\frac{K_I}{K_0}$	−0.2305	−0.2369	−0.2472	−0.2608	−0.2766	−0.2920	−0.3030	−0.3091
	$\frac{K_{II}}{K_0}$				0.0				
D	$\frac{K_I}{K_0}$	0.0544	0.0520	0.0497	0.0507	0.0605	0.0886	0.1532	0.2773
	$\frac{K_{II}}{K_0}$				0.0				



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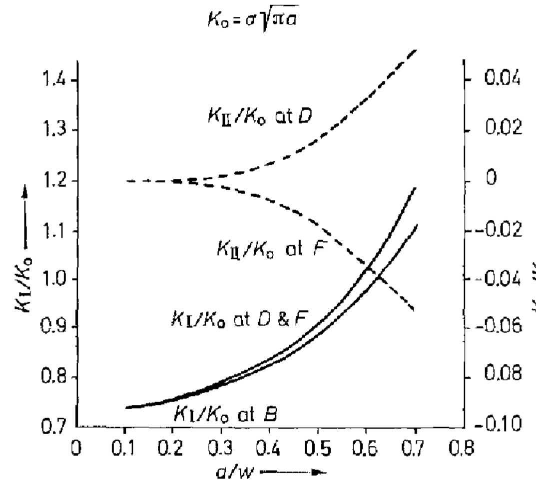

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Figs. 10 and 11. **10** Star-shaped cracks in a finite plate. **11** SIF for the star-shaped crack

increases. For K_{II} values at A and B, the absolute values increase as long as the crack length increases.

4.3 Star-shaped crack in a finite plate

As shown in Fig. 10, the star-shaped crack crossing at a point consists of three equal length cracks. Assuming that a pressure p act on the crack surfaces, the SIF values for an infinite plate may be calculated using the formula (Finn Ouchterlong 1976):

$$K_I/K_0 = 0.7454, \quad K_{II}/K_0 = 0,$$

where $K_0 = p\sqrt{\pi a}$.

Table 3. Calculated results for star-shaped crack

Crack tip	$\frac{a}{w}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7
B	$\frac{K_I}{K_0}$	0.74078	0.75699	0.78459	0.82551	0.88515	0.97581	1.11419
	$\frac{K_{II}}{K_0}$				0.0			
D	$\frac{K_I}{K_0}$	0.74083	0.75779	0.78840	0.83649	1.90876	1.01817	1.19360
	$\frac{K_{II}}{K_0}$	0.00003	0.00045	0.00224	0.00700	0.01684	0.03381	0.05292
F	$\frac{K_I}{K_0}$	0.74083	0.75779	0.78840	0.83649	0.90876	1.01817	1.19360
	$\frac{K_{II}}{K_0}$	−0.00003	−0.00045	−0.00224	−0.00700	−0.01684	−0.03381	−0.05292

This problem is analyzed assuming $a/w = 0.1$, and the other calculation parameters are: $I = 4$, $M = 30$, $N = 80$. The computed SIF values are:

$$K_I/K_0 = 0.7408, \quad K_{II}/K_0 = 0,$$

It can be seen that the proposed method gives good accuracy.

For bi-axial loading as shown in Fig. 10, the problem is analyzed with the crack lengths a/w varying from 0.1 to 0.7. The results are summarized in Table 3 and plotted graphically in Fig. 11. Because of symmetry, the SIF values for the opposite sides of each crack are identical. It can be seen that K_I values are slightly higher for the two inclined cracks *CD* and *EF*. The K_I values increases significantly with increasing crack length. The K_{II} values for the horizontal crack *AB* are obviously equal to zero while the two inclined cracks *CD* and *EF* have opposite values.

5 Conclusions

In this paper two kinds of the stress functions are combined and used for the multiple crack problems in a finite plate. By using Fredholm integral equation method and boundary collocation method, the unknown coefficients of the functions can be determined. Then the SIF at every crack tip can be calculated by the formulae.

Some examples have been used to show the advantages of the method:

(1) The method provides a general treatment to different multiple crack problems, including crack number, crack position, boundary shape and loading conditions.

(2) The calculation procedure is relatively simple, and involves little data preparation to define the plane dimensions and calculation parameters. It can be programmed and executed on a PC computer very efficiently.

(3) Satisfactory accuracy and convergence are achieved with small number of collocation points and summation terms.

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References

Benthem, J. P.; Koiter, W. T. (1973): Asymptotic approximations to crack problems. In: Sih, G. C. (ed): Methods of Analysis and Solutions of Crack Problems. Noordhoff International, Leyden, 131–178

Chen, Y. Z. (1984): Multiple crack problems of antiplane elasticity in an infinite body. Eng. Fract. Mech. 20, 767–775

Cheung, Y. K.; Woo, C. W.; Wang, Y. H. (1988): The stress intensity factor for a double edge cracked plate by boundary collocation method. Int. J. Fract. 37, 217–231

Erdogan, F. (1978): Mixed boundary-value problems in mechanics. In: Mechanics Today. 4, 1–86

Finn Ouchterlong (1976): Stress intensity factors for the expansion loaded star crack. Eng. Fract. Mech. 8, 447–448

Gross, B.; Srawley, J. E.; Brown, W. F. (1964): Stress intensity factor for a single-edge-notch tension specimen by boundary collocation method. NASA TN D-2395

Kobayashi, A. S.; Cherepy, R. B.; Kinsel, W. C. (1964): A numerical procedure for estimating the stress intensity factor of a crack in a finite plate. J. Appl. Mech. 86, 681–684

Muskhelishvili, N. I. (1975): Some Basic Problems of Mathematical Theory of Elasticity. Second English ed., Noordhoff

Sneddon, I. N. (1973): Integral transform methods. In: Sih, G. C. (ed): Methods of Analysis and Solutions of Crack Problems, Noordhoff International, Leyden, 315–378

Xu Wenhuan; Chen Qin (1982): Analysis of two dimensional problems by the least square boundary collocation method. China Civil Engineering Journal. 13, 37–46

Communicated by S. N. Atluri, March 17, 1992