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Stability and Attractivity in Associative Memory Networks

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Abstract. We focus on stable and attractive states in a network having two-state neuron-like elements. We calculate the connection matrix which guarantees the stability and the strongest attractivity of p memorized patterns. We present an analytical evaluation of the patterns' attractivity. These results are illustrated by some computer simulations.

1 Introduction

The ability to recall memorized patterns is a very important human feature. Many models of neural networks include it, and the capacity of memory is usually spatially distributed throughout the network. It is exactly contained in the "efficiencies" of synaptic junctions.

The study of Distributed Associative Memory Networks was initiated in the fifties. We refer e.g. to Rosenblatt (1958); Caianiello (1961); Kohonen (1970, 1972, 1976); Nakano (1972); Kohonen et al. (1974, 1977) etc. ....

In these models, the state of each neuron is represented by output spike frequencies. The memorization of patterns relies upon changes in the synaptic efficiencies according to the presentation of patterns. The further presentation of a perturbed pattern (or a part of it) leads to its recollection. This is the property of selective recall.

This theory is largely developed in Kohonen (1984).

This kind of models can also be used for constructing some associative memory which need not be a realistic model of neural network. They can be conceived for memorization and retrieval of patterns in another context, e.g. error corrections in transmissions. In this case, the like-neuron automata may have continuous – or discrete-valued states. They must just work as required. The simplest networks with two-state threshold "neurons" were studied by many authors: Little (1974), Hopfield (1982), Peretto (1984), Amit et al. (1985a, b), Weichbuch and Fogelman-Soulié (1985), Personnaz et al. (1985), etc. ....

The use of conceptual tools of statistical mechanics, especially spin glasses models, has allowed a good advance in understanding their behavior, and led to asymptotic results, when the number of units grows to infinity (Amit et al. 1985a, b).

In all these papers, the networks are used for recognizing, i.e. retrieving a given set of configurations referred to as patterns. However *these patterns*, whether deterministically or stochastically chosen, *are not always attractors, and not even stable states*, in the network defined by the classical connections suggested by Hebb (1949) and advocated by Cooper et al. (1978).

On the other hand, these connections are symmetric, and this is an unpleasant restriction, even if it allows to define one Hamiltonian whose local minima contain the patterns (Hopfield 1982; Peretto and Niez 1985).

All contributions have studied either a deterministic algorithm, with temperature $T=0$ (Hopfield 1982; Personnaz et al. 1985; Weichbuch and Fogelman-Soulié 1985) or a stochastic one with temperature $T>0$ (Amit et al. 1985a, b; Peretto 1984).

In this paper we shall study the deterministic algorithm. We consider a network consisting of N two-state units. We define a *configuration* or *network state* as an element $S=(S_1, S_2, ..., S_N)$ of the hypercube $\{-1, +1\}^N$, with $S_i=+1$ (or -1) if the i-th unit is *active* (or *inactive*). The configuration at time t is denoted by S^t.

The collective behavior of such a network is entirely specified by the strengths of the connections C_{ij}, between the source j and the receiver i, and by the threshold values. The $N \times N$ matrix $C=(C_{ij})$ acts as a decoding machine and will be called *connection matrix*.

Each unit receives inputs from all the others weighted by the strengths of connections. Like in neural networks, the sum of the weighted inputs represents the "membrane potential of the neuron": it becomes (or remains) active if this potential is higher than the threshold, and it becomes (or remains) inactive if this potential is smaller than the threshold.

There are p learned, previously known configurations referred to as *the patterns*, denoted by $S^1, \ldots, S^p$. We wish, starting from a configuration S, which contains some errors, to retrieve one of the S^m, $m=1, \ldots, p$, which is to be the nearest one.

Thus it is necessary to determine how to choose the (C_{ij}) matrix in order to get the patterns as attractive as possible. Moreover we have to use an iteration mode, either the serial one, or the parallel one.

Dynamic Description

At time 0, the initial configuration in S^0, issued from one of the patterns, but containing some errors (transmission errors, miscellaneous disturbances).

We call $S^t=(S_i^t)$ the configuration at time t.

At time t, the unit i receives the signal $\sum_j C_{ij}\left(\frac{S_j^t+1}{2}\right)$, the variable $\frac{S_j+1}{2}$ is equal to $+1$ if the unit j is active, 0 if not. If unit i receives a signal greater (resp. smaller) than the threshold, then it becomes active (resp. inactive). We shall choose, as usual, the threshold $\theta_i=\frac{1}{2}\sum_j C_{ij}$, for this choice implies that: $\sum_j C_{ij}\left(\frac{S_j^t+1}{2}\right)>\theta_i \Leftrightarrow \sum_j C_{ij}S_j^t>0$.

Then the two types of algorithm are:

a) Sequential Iteration Algorithm. At time t, pick at random a unit $i \in \{1, \ldots, N\}$, with uniform distribution, or in any case, in such a way that all the units can be selected. Then

$$\text{if } \sum_j C_{ij}S_j^t>0, \quad S_i^{t+1}=+1$$

$$\text{if } \sum_j C_{ij}S_j^t<0, \quad S_i^{t+1}=-1$$

$$\text{if } \sum_j C_{ij}S_j^t=0, \quad S_i^{t+1}=S_i^t.$$

At each step, only one unit is checked.

b) Parallel Iteration Algorithm. At time t, calculate all the sums $\sum_j C_{ij}S_j^t$ and set $S_i^{t+1}=+1, -1$, or S_i^t as $\sum_j C_{ij}S_j^t>0, <0$, or $=0$.

The system evolves by lining up the state S_i with the local field defined as $\frac{1}{2}\sum_j C_{ij}$.

For the two kinds of dynamics, two notions are interesting:

The stability. A configuration S^0 is stable iff $S^t=S^0$ for every t.

The attractivity. A configuration S^0 is a k-attractor (for $1 \leqq k \leqq N$) iff starting from a configuration S which presents k errors with respect to S^0, the dynamic leads to S^0.

Obviously the stability and the attractivity of a configuration S are defined for each kind of iteration mode and for each given matrix.

In this paper we attempt to solve the following problem: *Given patterns $S^1, \ldots, S^p$, build a matrix C (i.e. an algorithm) in such a way that the $S^1, \ldots, S^p$ are stable and k-attractors with k as great as possible.*

The paper is organized as follows:

In Sect. 2, we give exact definitions of stability, attractivity, domain of attraction of a configuration.

Section 3 is devoted to the study of orthogonal patterns.

In Sect. 4, following Personnaz et al. (1985), we give the general formulation of the connection matrix C which provides stability to patterns $S^1, \ldots, S^p$. Then in Sect. 5, we get the mathematical expression of patterns attractivity for a given matrix C.

Section 6 describes a construction of the matrix C, that maximizes all the patterns attractivity.

In Sect. 7, we come back to the case where all patterns are pairwise orthogonal, and stress the interest of that situation.

In Sect. 8, we solve the settled problem, by considering the situation where all the patterns have the same degree of attractivity.

In Sect. 9, we introduce the usual notion of configuration energy, which is only defined when the matrix C is symmetric, and we give the relations between energy minima and attractors.

Section 10 is devoted to a discussion.

In Appendix 1, some properties of the Hamming distance are recalled, and in Appendices 2 and 3, numerical examples illustrate our results and show that it is impossible to get a better estimate of the attractivity than the given one.

2 Stability-Attractivity

Let C be a connection matrix, and $S=(S_i)$ a configuration. We consider each of the two kinds of algorithms (sequential or parallel ones).

We denote by $\delta(S, S')$ the Hamming distance of two configurations S and S', i.e. the number of components where S and S' differ, and by $d(S, S')$ the Euclidian distance of S and S', viewed as elements of $\mathbb{R}^N$.

One has $d(S, S')=2\sqrt{\delta(S, S')}$ (see Appendix 1 for properties of the Hamming distance).

Definition 2.1. A configuration S is *stable* (with respect to C) iff starting from S, the network state remains S, i.e. iff

$$\forall i \in \{1, \ldots, N\}, \quad S_i\left(\sum_j C_{ij} S_j\right) \geqq 0. \tag{2.1}$$

The set of stable configurations is denoted by E_0.

Definition 2.2. Let k be an integer. A configuration S is a *k-attractor* (with respect to C) iff starting from S' with $\delta(S, S') = k$, the network state evolves in one step towards S'' with $\delta(S, S'') \leqq k-1$ or leaves S' invariant (only in the sequential algorithm). This is equivalent to the condition:

$\forall i$, $\forall j_1, \ldots, j_k$ (with $j_1, \ldots, j_k$ all different)

$$S_i\left(\sum_j C_{ij} S_j - 2 \sum_{l=1}^{k} C_{ij_l} S_{j_l}\right) > 0. \tag{2.2}$$

The set of k-attractors is denoted by E_k.

Proof of the Above Equivalence. Let $S = (S_i)$ be a configuration which is a k-attractor and S' such that $\delta(S, S') = k$. Thus $S' = (S'_i)$ with

$$S'_i = -S_i \text{ for } i \in \{j_1, \ldots, j_k\}$$
$$S'_i = S_i \quad \text{otherwise}.$$

We start from S'. For i integer picked at random (sequential algorithm) or for every i (parallel algorithm) we compute

$$\alpha_i = \sum_j C_{ij} S'_j = \sum_j C_{ij} S_j - 2 \sum_{l=1}^{k} C_{ij_l} S_{j_l}.$$

If $\alpha_i > 0$, (resp. $\alpha_i < 0$) we set the component i to be $+1$ (resp. -1).

The condition $\alpha_i S_i > 0$ means that we line up the spin i with the corresponding value of S, and therefore at the next time for the sequential algorithm, the new configuration will be S'' with $\delta(S, S'') = k-1$ if $i \in \{j_1, \ldots, j_k\}$ or S' if not. For the parallel algorithm the new configuration will be S. □

Remarks 2.3. 1) S and $-S$ satisfy the same inequalities.

2) S is stable (resp. k-attractor) for the sequential algorithm iff it is for the parallel algorithm. Indeed, for both algorithms, the inequalities to check for every integer i picked at random are the same. This will appear as a consequence of the following proposition which ensures that the Definitions 2.1 and 2.2 are coherent.

Proposition 2.4. *For* $k < \frac{N}{2}$, *one has*

$$E_k \subset E_{k-1} \subset \ldots \subset E_1 \subset E_0.$$

Proof. Adding up the inequalities defining E_k, over all integers $j_1, \ldots, j_k$, we obtain that $E_k \subset E_0$. Then adding up over any subset of $(k-1)$ integers taken among $\{j_1, \ldots, j_k\}$, we get $E_k \subset E_{k-1}$, under the condition that $N > 2k$. This last condition is obvious if we note that $\delta(S, -S) = N$, and that $S \in E_k$ iff $-S \in E_k$: indeed if $2k \geqq N$, there would exist a state S' with $\delta(S', S) = k = \delta(S', -S)$, which is impossible since the spheres $B_\delta(S, k)$, $B_\delta(-S, k)$ are disjoined when $S \in E_k$ [sphere $B_\delta(S, k)$ is the set of configurations whose distance of S is less than k].

Moreover, note that the inequalities defining E_k are incompatible for $k \geqq \frac{N}{2}$. □

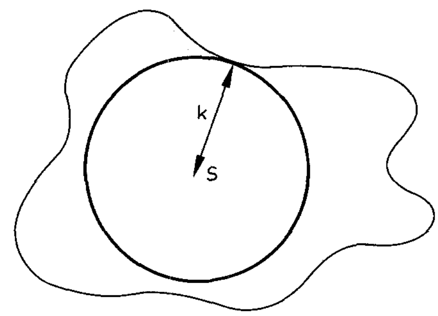

Fig. 1. Hamming radius

Let us now to define the domain of attraction of a configuration.

Definition 2.5. The *domain of attraction (DA)* of a configuration S is the (maybe empty) set of configurations S' such that, starting from S', the algorithm leads to S.

However, it is convenient to consider only circular domains, i.e. spheres for the Hamming distance.

Definition 2.6. The *Hamming radius of the DA* of a configuration S is the Hamming radius of the greatest sphere included in it.

Consequently, if $S \in E_k$, the radius of its domain of attraction is $\geqq k$.

Note that a Hamming sphere with center S and radius k, $B_\delta(S, k)$, is the intersection of the Euclidian sphere with center S and radius $2\sqrt{k}$, $B_d(S, 2\sqrt{k})$, and the hypercube $\{-1, +1\}^N$.

Remark 2.7. The maximal size of the DA of the patterns is necessarily bounded by the mutual Hamming distances of the patterns (and their opposite), since $S \in E_k$ and $S' \in E_{k'}$ require $k + k' < \delta(S, S')$ and $k + k' < \delta(S, -S') = N - \delta(S, S')$.

3 Case of Orthogonal Patterns

Two configurations S and S' are orthogonal iff their Euclidian product $\langle S, S' \rangle = \sum_i S_i S'_i = 0$. (Thus orthogonality is defined as usual in the Euclidian space $\mathbb{R}^N$.)

The case of pairwise orthogonal patterns is well known, and easy to study (see Hopfield 1982; Peretto 1984; Personnaz et al. 1985) especially because of the following properties:

First, all the mutual Hamming distances between the patterns (and their opposite) are equal to $\frac{N}{2}$, so the patterns are well separated.

Furthermore, this case corresponds to the spin glasses one, in which case values S_i^m are picked at random, independently, with $\frac{1}{2}=\mathbb{P}(S_i^m=+1)=\mathbb{P}(S_i^m=-1)$ for all $i=1,\ldots,N$ and $m=1,\ldots,p$. The mean value of the Euclidian product $=\sum_i S_i^m, S_i^{m'}$ is then 0, (for $m\neq m'$) and the patterns are on average orthogonal, because the mean values of the number of components equal to $+1$ or to -1 are equal.

The usual connections (Hebbian connections) are then

$$C_{ij}=\frac{1}{N}\sum_{m=1}^{p} S_i^m S_j^m. \tag{3.1}$$

They are suggested by the principle of Hebb. In neural networks, following Hebb (1949), the synaptic efficacies are modified according to the neural activity, the strength of a synapse being proportional to the correlated activities of the neurons it connects. Here each coincidence of values for the patterns S^m in units i and j increases C_{ij}, and conversely.

With the above definition of matrix (C_{ij}), in the orthogonal case, numerical simulations show strong attractivity of patterns, thus good efficiency of information retrieval. In fact, we prove that:

Proposition 3.2. *If the patterns $S^1,\ldots,S^p$ are pairwise orthogonal, these configurations are k-attractors at least up to* $k=\left[\frac{N}{2p}\right]$ $\left(\textit{integer part of } \frac{N}{2p}\right)$, *for the Hebbian connection matrix.*

Proof. According to (2.2) and (3.1), for $m=1,\ldots,p$, S^m is a k-attractor for C iff

$$\frac{1}{N}S_i^m\left(\sum_{j=1}^{N}\left(\sum_{l=1}^{p}S_i^lS_j^l\right)S_j^m-2\sum_{\theta=1}^{k}\left(\sum_{l=1}^{p}S_i^lS_{j_\theta}^l\right)S_{j_\theta}^m\right)>0$$

for every $i=1,\ldots,N$, and distinct $j_1,\ldots,j_k$ in $\{1,\ldots,N\}$. Or iff

$$\frac{1}{N}\left[\sum_{l=1}^{p}S_i^lS_i^m\left(\sum_{j=1}^{N}S_j^lS_j^m\right)-2\sum_{\theta=1}^{k}\sum_{l=1}^{p}S_i^lS_{j_\theta}^lS_{j_\theta}^mS_i^m\right]>0.$$

The patterns $S^1,\ldots,S^p$ being pairwise orthogonal, $\sum_{j=1}^{N}S_j^lS_j^m=0$ if $l\neq m$, and $=N$ if $l=m$. So the condition becomes

$$\frac{1}{N}\left[N-2\sum_{\theta=1}^{k}\sum_{l=1}^{p}S_i^lS_{j_\theta}^lS_{j_\theta}^mS_i^m\right]>0.$$

Now

$$\left|2\sum_{\theta=1}^{k}\sum_{l=1}^{p}S_i^lS_{j_\theta}^lS_{j_\theta}^mS_j^m\right|$$

$$\leqq 2\sum_{\theta=1}^{k}\sum_{l=1}^{p}|S_i^l|\,|S_{j_\theta}^l|\,|S_{j_\theta}^m|\,|S_i^m|\leqq 2\mathrm{kp},$$

since $|S_i^l|=|S_{j_\theta}^l|=|S_{j_\theta}^m|=|S_i^m|=1$. The condition is satisfied if $N-2\mathrm{kp}>0$.

Since $2\left[\frac{N}{2p}\right]<\frac{N}{2}$ – Hamming distance of two orthogonal patterns – as soon as $p>2$, we see that each pattern is k-attractor at least for $k=\left[\frac{N}{2p}\right]$. □

4 Searching the Matrix C

We denote by A' the transposed of the matrix A, and we identify a vector of $\mathbb{R}^N$ with the column matrix of its components in the canonical basis. We denote by $\langle\ ,\ \rangle$ the usual Euclidian product, and $\|\cdot\|$ the associated norm. For a linear subspace E, we denote by $E^\perp$ the subspace of all vectors which are orthogonal to every vector in E.

We may express the stability of the patterns $S^1,\ldots,S^p$ (see Definition 2.1) by

$$\sum_j C_{ij}S_i^mS_j^m=\langle C_i, D_i^m\rangle>0 \tag{4.1}$$

for every $i=1,\ldots,N$ and every $m=1,\ldots,p$, where C_i is the i-th row of C, viewed as a vector of $\mathbb{R}^N$, and $D_i^m=S_i^mS^m$. The vector D_i^m is $\pm S^m$, so that its i-th component is equal to $+1$.

Thus, we want to determine N vectors $C_1,\ldots,C_N$ of $\mathbb{R}^N$, such that, for all i, the p vectors D_i^m are on the same side of hyperplane H_i, defined as the orthogonal space of C_i.

Such an hyperplane is not unique: more important is the volume left around each D_i^m, more performing will be the choice of C_i (and of H_i).

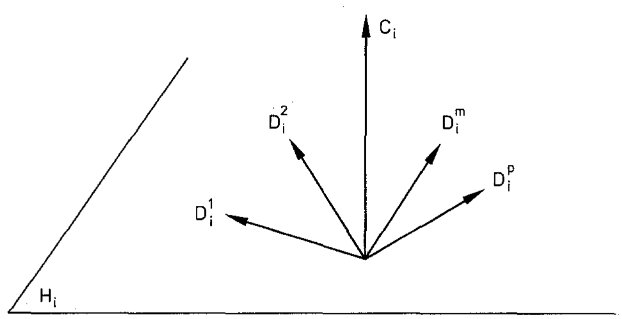

Fig. 2. Choice of hyperplane H_i

We see immediately that for $C_i=\frac{1}{p}\sum_{m=1}^{p} D_i^m$ (natural choice which corresponds to $C_{ij}=\frac{1}{p}\sum_{m=1}^{p} S_i^m S_j^m$, i.e. to Hebbian connections (3.1), apart from constant $\frac{N}{p}$), the S^m may not be stable, as can be observed by numerical simulations. (See also Personnaz et al. 1985.)

However this choice is often convenient, especially when the S^m are chosen at random, for then they are almost orthogonal (see Sect. 3).

In Personnaz et al. (1985), one can find the general expression for the matrix C ensuring stability of patterns $S^1,\dots,S^p$, under the condition that they are linearly independent.

Their formula is

$$C=(A_1S^1,\dots,A_pS^p)(\Sigma'\Sigma)^{-1}\Sigma'+\tilde{C}\,, \tag{4.2}$$

where Σ is the $(N\times p)$ matrix with columns $S^1,\dots,S^p$, $A_1,\dots,A_p$ are arbitrary positive diagonal N-matrices and $\tilde{C}$ is a $(N\times N)$-matrix such that $\tilde{C}\Sigma=0$. Indeed, the system to be solved is $\langle C_i, S_i^m S^m\rangle>0$ $(i=1,\dots,N;$ $m=1,\dots,p)$ or equivalently:

$$\langle C_i, S^m\rangle=\alpha_i^m S_i^m \quad \text{for arbitraries} \quad \alpha_i^m>0\,. \tag{4.3}$$

A more condensed form is: $CS^m=A_mS^m$, for $m=1,\dots,p$, where A_m is an arbitrary positive diagonal N-matrix.

The general expression of C_i is

$$C_i=((\alpha_i^1S_i^1,\dots,\alpha_i^pS_i^p)(\Sigma'\Sigma)^{-1}\Sigma')'+\tilde{C}_i\,, \tag{4.4}$$

where $\alpha_i^1,\dots,\alpha_i^p$ are arbitrary positive scalars and $\tilde{C}_i$ is orthogonal to $S^1,\dots,S^p$. We notice that C_i is the sum of a linear combination of $S^1,\dots,S^p$, and of an orthogonal vector, and that it is defined up to a positive multiplicative constant.

We must determine how to choose the $N\times p$ constants α_i^m, and the vectors $\tilde{C}_i$ to optimize the attractivity of $S^1,\dots,S^p$. So, we shall study the size of the domain of attraction of the S^m, as a function of arbitrary coefficients of matrix C.

5 Size of the Domains of Attractivity (DA) of $S^1,\dots,S^p$

If we denote by k_m the radius of the DA of S^m (Definition 2.6), we have $k_m+k_{m'}<\inf(\delta(S^m,S^{m'}),$ $N-\delta(S^m,S^{m'}))$ for any pair m, m', with $m\neq m'$ (see Remark 2.7).

Furthermore k_m is limited by the "free room" around S^m, i.e. the subset of $\mathbb{R}^N$ which contains S^m and none of the other patterns.

Let us denote by $T_{j_1,\dots,j_k}$ the sign modifier in position $j_1,\dots,j_k$ of a configuration, i.e. the mapping defined by

$$T_{j_1,\dots,j_k}(S)=S^*$$

with

$$S_i^*=S_i \quad \text{for} \quad i\notin\{j_1,\dots,j_k\}$$
$$S_i^*=-S_i \quad \text{for} \quad i\in\{j_1,\dots,j_k\}\,.$$

Thus $S^m\in E_k$ (S^m is a k-attractor) iff

$$\langle C_i, T_{j_1,\dots,j_k}(D_i^m)\rangle>0 \tag{5.1}$$

for every $i\in\{1,\dots,N\}$ and for every subset $\{j_1,\dots,j_k\}$ of distinct integers of $\{1\dots N\}$. [Notations of (4.1) and Definition 2.2.]

We may interpret these inequalities in a geometric way: we denote by a_i^m, (resp. b) the endpoints of the vectors D_i^m [resp. $T_{j_i,\dots,j_k}(D_i^m)$].

The condition (5.1) means that S^m is a k-attractor iff for all i, the Hamming sphere with center a_i^m and radius k is entirely on the same side of $H_i=C_i^\perp$ (orthogonal space of vector C_i) see Fig. 3. It means that

$$\begin{aligned} d(a_i^m,b)&=2\sqrt{\delta(D_i^m,T_{j_1,\dots,j_k}(D_i^m))} \quad \text{(by Appendix 1)}\\ &=2\sqrt{k}\\ &<d(a_i^m,H_i)=\frac{\langle C_i,D_i^m\rangle}{\|C_i\|}\,. \end{aligned}$$

More precisely,

$$\begin{aligned} &|\langle C_i, T_{j_1,\dots,j_k}(D_i^m)\rangle-\langle C_i,D_i^m\rangle|\\ &\quad=|\langle C_i,O\mathbf{b}\rangle-\langle C_i,O\mathbf{a}_i^m\rangle|\\ &\quad=|\langle C_i,O\mathbf{b}-O\mathbf{a}_i^m\rangle|\\ &\quad=2\sum_{l=1}^{k}|C_{ij_l}|\leqq 2k\max_j|C_{ij}|\,, \end{aligned}$$

(because only k components differ). So (5.1) holds true as soon as

$$k<\frac{\langle C_i,D_i^m\rangle}{2\max_j|C_{ij}|} \quad \text{for every } i\,.$$

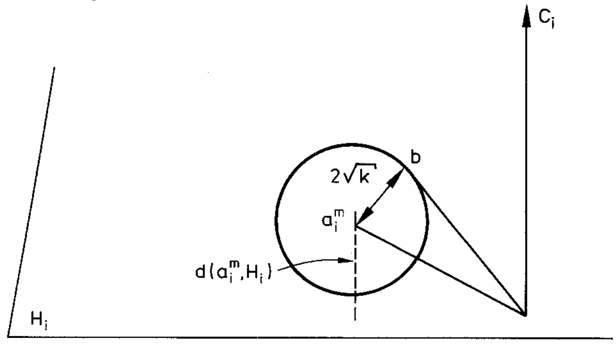

Fig. 3. Hamming sphere $B_\delta(a_i^m,k)$

Hence

Proposition 5.2. *For a connection matrix $C=(C_{ij})$, whose rows are $C_1, \dots, C_N$, each pattern S^m is k-attractor at least up to any k such that*

$$k<\tfrac{1}{2}\inf_i \frac{\langle C_i, D_i^m\rangle}{\max_j |C_{ij}|}=\gamma_m, \tag{5.2}$$

where $D_i^m = S_i^m S^m$.

So if k and k' are two integers satisfying (5.2), w.r.t. S^m (for k) and $S^{m'}$ (for k'), $S^m \in E_k$, $S^{m'} \in E_{k'}$, and $k+k' \leqq \inf(\delta(S^m, S^{m'}),\ N-\delta(S^m, S^{m'}))$.

Now we proceed to simplify the expression of γ_m [in (5.2)] when C is given by (4.2), (ensuring patterns stability).

The choice of the C_i's, in (4.4) has to ensure that $d(a_i^m, H_i)$ or γ_m are as big as possible.

First, we see that in (4.4) $\tilde{C}_i=0$ is the best choice. Indeed if $C_i=\hat{C}_i+\tilde{C}_i$ with $\hat{C}_i$ in the subspace $\mathscr{V}$ spanned by $\{S^1, \dots, S^p\}$ and $\tilde{C}_i$ in $\mathscr{V}^\perp$, we have

$$\langle C_i, D_i^m\rangle = \langle \hat{C}_i, D_i^m\rangle$$

and

$$\|C_i\|^2=\|\hat{C}_i\|^2+\|\tilde{C}_i\|^2, \quad \text{so} \quad d(a_i^m, H_i)=\frac{\langle C_i, D_i^m\rangle}{\|C_i\|}$$

will be greater for $\tilde{C}_i=0$.

Using (4.3),

$$\langle C_i, D_i^m\rangle = S_i^m \langle C_i, S^m\rangle = S_i^m(\alpha_i^m S_i^m)=\alpha_i^m. \tag{5.3}$$

As to the vector C_i, we write, from (4.4),

$$C_i=\Sigma(\Sigma'\Sigma)^{-1}\begin{pmatrix}\alpha_i^1 & S_i^1\\ & \vdots & \\ \alpha_i^p & S_i^p\end{pmatrix} \tag{5.4}$$

and

$$\|C_i\|^2 = C_i' C_i = (\alpha_i^1 S_i^1, \dots, \alpha_i^p S_i^p)(\Sigma'\Sigma)^{-1}\begin{pmatrix}\alpha_i^1 & S_i^1\\ & \vdots & \\ \alpha_i^p & S_i^p\end{pmatrix}$$

$$=(\alpha_i^1, \dots, \alpha_i^p)\operatorname{diag}(S_i^1 \dots S_i^p)(\Sigma'\Sigma)^{-1} \times \operatorname{diag}(S_i^1 \dots S_i^p)\begin{pmatrix}\alpha_i^1\\ \vdots\\ \alpha_i^p\end{pmatrix}$$

[where $\operatorname{diag}(S_i^1 \dots S_i^p)$ is the matrix whose diagonal elements are $S_i^1, \dots, S_i^p$ and others are 0]

$$=\alpha_i' W_i \alpha_i \text{ (obvious notations)}.$$

We remark that if we denote by $D_i=(D_i^1, \dots, D_i^p) = (S_i^1 S^1, \dots, S_i^p S^p)$, i.e. the matrix of the patterns normalized to $+1$ in the i-th component, we have

$$W_i=(D_i' D_i)^{-1}.$$

So

$$d^2(a_i^m, H_i)=\frac{(\alpha_i^m)^2}{\alpha_i' W_i \alpha_i} \tag{5.5}$$

and

$$\gamma_m=\tfrac{1}{2}\inf_i \frac{\alpha_i^m}{\max_j |C_{ij}|}. \tag{5.6}$$

The inequality (5.2) does not give exact values of the sizes $k_1, \dots, k_p$ of the domains of attraction (DA) of $S^1, \dots, S^p$, because it is only a sufficient condition.

However, the pattern S^m is attractive at least up to γ_m, for $m=1, \dots, p$.

Using geometric notations let us consider patterns S^1, S^2. Let us assume that $\delta(S^1, S^2)$ is small with respect to N. Since for every i such that $S_i^1=S_i^2$, the distance $d(a_i^m, H_i)$ is "great", whereas the sizes of the DA of S^1 and S^2 are small, [since less than $\delta(S^1, S^2)$] this seems contradiction. But we must notice that on the contrary, for i such that $S_i^1=-S_i^2$, we have $D_i^1=-S^1$ or $D_i^2=-S^2$ and $d(a_i^m, H_i)$ small, which leads to a small value of γ_m for $m=1, 2$.

Now, Proposition 5.2 can be completed by:

Proposition 5.7. *For a connection matrix $C=(C_{ij})$, whose rows $C_1, \dots, C_N$ are given by (4.4) with $\tilde{C}_i=0$, each pattern S^m is attractor at least up to Hamming distance*

$$\gamma_m=\tfrac{1}{2}\inf_i \frac{\alpha_i^m}{\max_j |C_{ij}|}. \tag{5.7}$$

6 Optimal Matrix

Let us start from an initial configuration S^0, obtained by distorting one of the patterns, e.g. S^{m_0}. There will not be any identification error if S^0 belongs to the domain of attraction (DA) of S^{m_0}.

Hence the next definition:

Definition 6.1. A matrix C for which the patterns $S^1, \dots, S^p$ are attractors, is *optimal* if it maximizes the minimum radius (Definition 2.6) of the DAs of the patterns.

Since we have not the exact value of these radii, we try to determine a matrix C, called semi-optimal, and which maximizes the minimum distance $d(a_i^m, H_i)$ (Fig. 3).

Thus we look for positive constants α_i^m, $m=1, \dots, p$, $i=1, \dots, N$, which maximize for each i, $\inf_m d^2(a_i^m, H_i)$

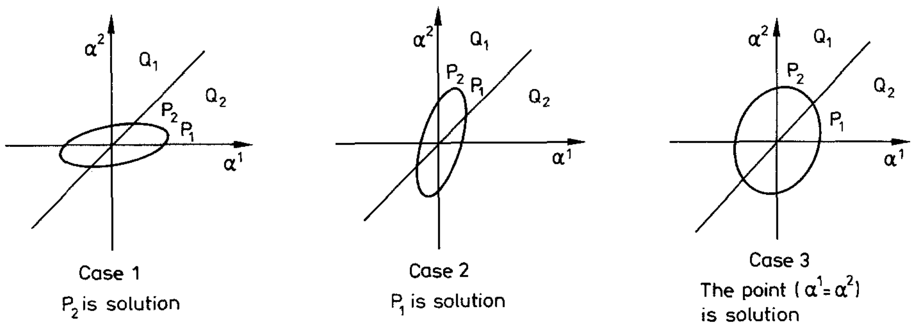

Fig. 4. Case $p=2$

$=\inf\limits_m \frac{(\alpha_i^m)^2}{\alpha_i' W_i \alpha_i}$ [by (5.5)]. Let us sketch a construction of the α_i^m for fixed i.

a) We look for a vector $\alpha_i=(\alpha_i^1,\ldots,\alpha_i^p)$ in $(\mathbb{R}^+)^p$, with $\alpha_i' W_i \alpha_i=1$.

b) We cut out $(\mathbb{R}^+)^p$ into quadrants Q_j defined by $\alpha_i^j=\min\limits_n(\alpha_i^m)$, for $j=1,\ldots,p$.

In each quadrant Q_j, we want to maximize $(\alpha_i^j)^2$, i.e. α_i^j. For instance, if $p=2$, we consider the ellipse whose equation is $\alpha_i' W_i \alpha_i=1$,

c) We compute the p points $P_j=(\alpha_i^1,\ldots,\alpha_i^p)$, $j=1,\ldots,p$, solutions of $\alpha_i' W_i \alpha_i=0$ and $\frac{\partial \alpha_i^j}{\partial \alpha_i^m}=-\frac{(W_i\alpha_i)_m}{(W_i\alpha_i)_j}=0$ for $m \neq j$, and keep the points P_j belonging to the corresponding quadrant Q_j. We have $\max\limits_{\alpha_i}\left(\min\limits_m \frac{(\alpha_i^m)^2}{\alpha_i' W_i \alpha_i}\right)=\max\limits_j\{(\alpha_i^j)^2/P_j=(\alpha_i^1,\ldots,\alpha_i^p)\in Q_j\}$ under the condition that the above set is not empty.

d) If for all j, $P_j \notin Q_j$ (case 3), we restrict ourselves in the quadrants of $(\mathbb{R}^+)^{p-1}$, $(\mathbb{R}^+)^{p-2}$, etc. ... defined by inequalities such as:

$$\alpha_i^j=\alpha_i^{j'}=\min(\alpha_i^m)$$

$$\alpha_i^j=\alpha_i^{j'}=\alpha_i^{j''}=\min(\alpha_i^m)\ldots \text{and so on}.$$

If none of the P_js successively found belongs to the convenient domain, we get the solution $\alpha_i^1=\alpha_i^2=\ldots=\alpha_i^p$ (see Sect. 8).

We can sum up the results as follows:

Proposition 6.2. *By iterating this approach for every $i=1,\ldots,N$, we construct a matrix semi-optimal C, which leaves as much volume as possible, around each point a_i^m, for every i, with*

$$C_i=\Sigma(\Sigma'\Sigma)^{-1}(\alpha_i^1 S_i^1\ldots\alpha_i^p S_i^p)' \tag{6.2}$$

for $\Sigma=(S^1,\ldots,S^p)$ and $(\alpha_i^1,\ldots,\alpha_i^p)$ positive yielding the maximum of $\min\limits_m \frac{(\alpha_i^m)^2}{\alpha_i' W_i \alpha_i}$, with $W_i=(D_i' D_i)^{-1}$ and $D_i=(S_i^1 S^1,\ldots,S_i^p S^p)$.

7 Case of Orthogonal Patterns

In this section we assume that the patterns $S^1,\ldots,S^p$ are pairwise orthogonal.

In this case, $\Sigma'\Sigma=N\,\mathrm{Id}_p$, and for every i, $D_i' D_i=N\,\mathrm{Id}_p$, since the vectors D_i^m are also pairwise orthogonal.

So the ellipses of equations $\alpha_i' W_i \alpha_i=1$ are spheres and the research of the semi-optimal α_i^m, leads to the solution $\alpha_i^1=\ldots=\alpha_i^p$, (see the two-dimensioned example, case 3, in Sect. 6).

Since the α_i^m are defined (for each i) up to a positive multiplicative constant, we may choose $\alpha_i^m=1$ for every i and every m.

So from (5.4),

$$C_i=\Sigma\left(\frac{1}{N}\mathrm{Id}_p\right)(S_i^1\ldots S_i^p)' \quad\text{and}\quad C_{ij}=\frac{1}{N}\sum_{m=1}^{p} S_i^m S_j^m. \tag{7.1}$$

In other words, in the orthogonal case, the semi-optimal solution corresponds to the equality of the α_i^m, and to the classical Hebbian connections. [In that case C is the projection matrix introduced by Kohonen (1970).]

Thus, we may compute γ_m and $\inf\limits_i d^2(a_i^m, H_i)$, using (5.5) and (5.6)

$$d^2(a_i^m, H_i)=\frac{(\alpha_i^m)^2}{\alpha_i' W_i \alpha_i}=\frac{N}{p}, \tag{7.2}$$

and

$$\gamma_m=\tfrac{1}{2}\inf_i \frac{\alpha_i^m}{\max\limits_j |C_{ij}|}=\frac{N}{2}\inf_i \frac{1}{\max\limits_j\left|\sum\limits_{m=1}^{p} S_i^m S_j^m\right|}.$$

But $\max\limits_j\left|\sum\limits_m S_i^m S_j^m\right|=p$ is obtained for $j=i$, since all terms of the sum are then equal to $+1$.

Hence we get

$$\gamma_m=\frac{N}{2p}\,(\text{independent of } m). \tag{7.3}$$

So we find again the result of Sect. 3, which we now enounce as follows:

Proposition 7.4. *In the case of pairwise orthogonal patterns, the Hebbian connection matrix* (3.1) *is at least semi-optimal and ensures the attractivity of the patterns at least up to the Hamming distance* $\frac{N}{2p}$.

In the not orthogonal case, we remark that if $\delta(S^m, S^{m'}) > \frac{N}{2}$ for some (m, m'), then $\delta(S^m, -S^{m'}) < \frac{N}{2}$; therefore the volume "winned" by the component i will be "lost" by component i' such that $D_{i'}^m = D_i^m$ and $D_{i'}^{m'} = -D_i^{m'}$.

The sizes of the domains of attraction depend on the minima over i; we see that the favourable case is the orthogonal one.

Hence the practical interest of

– picking at random [with $\mathbb{P}(S_i^m = +1) = \mathbb{P}(S_i^m = -1) = \frac{1}{2}$], as in spin glasses, ensuring, on average, orthogonality.

– a deterministic encoding of the objects to be recognized, by means of pairwise orthogonal patterns.

8 Domains of Attraction with Equal Sizes

The research of the optimal matrix C sketched in Sect. 6 is tedious. We shall simplify it by taking all constants $\alpha_i^m = +1$ (Personnaz et al.'s method), which is equivalent to equal attractivity of each pattern.

We need now a geometric interpretation and a lower bound of the common size of the domains of attraction (DA).

Since $\alpha_i^m = \langle C_i, D_i^m \rangle$, by (5.3), in the case $\alpha_i^m = 1$ for every $m = 1, \ldots, p$, we have to determine a hyperplane H_i, orthogonal to C_i at equal distance of all the points a_i^m for $m = 1, \ldots, p$ (see Figs. 2 and 3). The vector C_i is orthogonal to the affine space containing all points $a_i^1, \ldots, a_i^p$.

We write, from (5.4),

$$C_i = \Sigma(\Sigma'\Sigma)^{-1} \begin{pmatrix} S_i^1 \\ \vdots \\ S_i^p \end{pmatrix} \quad \text{i.e.} \quad C = \Sigma(\Sigma'\Sigma)^{-1}\Sigma', \tag{8.1}$$

which is the orthogonal projection matrix of $\mathbb{R}^N$ on the vector space $\mathscr{V}$, spanned by $S^1, \ldots, S^p$ (introduced by Kohonen 1970). The matrix C is symmetric, and its columns (equal to its rows) are images of the vectors $e_1, \ldots, e_N$ (canonical basis of $\mathbb{R}^N$) by this projection.

So we have $C_i = \mathrm{proj}_{\mathscr{V}}(e_i)$, $D_i^m \in \mathscr{V}$, and $\langle C_i, D_i^m \rangle = \langle e_i, D_i^m \rangle = +1$ by the definition of the vectors D_i^m, whose i-th component is equal to $+1$.

Hence, noting that $\max_{i,j} |C_{ij}| = \max_i (C_{ii}) = \max_i \|C_i\|^2$ since C is a projection matrix ($C^2 = C$), we get

Proposition 8.2. *For the matrix* $C = \Sigma(\Sigma'\Sigma)^{-1}\Sigma'$ *with* $\Sigma = (S^1, \ldots, S^p)$, *each pattern* S^m *is attractor at least up to Hamming distance*

$$\gamma = \frac{1}{2} \frac{1}{\max_i \|C_i\|^2} = \frac{1}{2} \frac{1}{\max_i C_{ii}}. \tag{8.2}$$

See numerical examples of evaluation of γ in Appendix 2.

Remark 8.3. The matrix C obtained when $\alpha_i^m = 1$, for every i, m, is symmetric. However for what concerns the algorithm, it is equivalent to the matrix obtained by multiplying each row by a positive arbitrary constant: this matrix is no more symmetric.

Neither it is in the optimal case of Sect. 6.

9 Energy

The patterns $S^1, \ldots, S^p$ span the subspace $\mathscr{V}$, and the matrix

$C = \Sigma(\Sigma'\Sigma)^{-1}\Sigma'$ [see (8.1)]

is the symmetric projection matrix on $\mathscr{V}$, whose elements are in interval $[-1, +1]$, with $C = C^2 = C'C$, hence

$C_{ij} = \langle C_i, C_j \rangle = \langle \mathrm{proj}_{\mathscr{V}}(e_i), \mathrm{proj}_{\mathscr{V}}(e_j) \rangle$.

In that case, we define an energy function

$$E(S) = -\tfrac{1}{2} H(S) \tag{9.1}$$

with

$$H(S) = \sum_{i,j} C_{ij} S_i S_j. \tag{9.2}$$

We have the following theorem (with C symmetric).

Theorem 9.3. i) *$H(S)$ increases when the system evolves (in sequential or parallel algorithm).*

ii) *The k-attractor states are local maxima of H. More precisely, if S is k-attractor, and $S^* = T_{j_1, \ldots, j_k}(S)$ (notations of Sect. 5), then*

$H(S) > H(S^*)$.

iii) *The learned patterns are absolute maxima of H and any absolute maximum is a stable state.*

Demonstration. *i)* is clear. We prove *ii)* noting that $H(S) - H(S^*) = 4 \sum_{i \in I} S_i \left(\sum_{j \notin I} C_{ij} S_j \right)$ for $I = \{j_1, \ldots, j_k\}$. Since S is k-attractor, it is stable and for all i,

$S_i\left(\sum_j C_{ij}S_j\right) > 0$ (2.1) and $S_i\left(\sum_{j\notin I} C_{ij}S_j - \sum_{j\in I} C_{ij}S_j\right) > 0$ (2.2), hence $H(S) > H(S^*)$. For *iii)*, we write $H(S) = S'CS = (CS)'CS = \|CS\|^2 = \|\mathrm{proj}_{\mathscr{V}}(S)\|^2$ for all configuration $S \in \{-1, +1\}^N$.

Since $\|S\|^2 = N$, $H(S) \leq N$ for all S with equality iff $S \in \mathscr{V} = \mathrm{Vect}(S^1, \ldots, S^p)$. So patterns $S^1, \ldots, S^p$ are absolute maxima, and also the opposite $-S^1, \ldots, -S^p$.

Any linear combination of the learned patterns, element of $\{-1, +1\}^N$, (if exists) will be absolute maxima and spurious stable state: $H(S) = N$ iff $S \in \mathscr{V}$, i.e. iff $CS = S$ which is the stability. □

Note that if the patterns $S^1, \ldots, S^p$ are orthogonal,

$$H(S) = \sum_{m=1}^{p} \|\mathrm{proj}_{S^m}(S)\|^2 = \frac{1}{N} \sum_{m=1}^{p} \langle S, S^m \rangle^2 .$$

See in Appendix 3 examples of spurious stable and attractor states, which are local maxima of H. At temperature $T = 0$ (studied here) the algorithm may reach some of these states, but when $T > 0$ using the annealing method, only the absolute maxima of H will be reached, i.e. the spurious stable states belonging to $\mathscr{V}$.

10 Provisional Conclusions

1) To choose the sizes of the domains of attraction (DA) in such a way they are equal for all patterns $S^1, \ldots, S^p$ enables us to give no preference to any pattern.

Assume that we start from an initial configuration S^0 distorted from S^{m_0}. We modelize this disturbance: an error occurs in each component $i = 1, \ldots, N$, independently with a small probability q. The number of errors $\delta(S^0, S^{m_0})$ is a Binomial distribution $\mathscr{B}(N, q)$.

Starting from S^0, the algorithm acts and the probability that it gives a good answer (S^{m_0}), is bounded from below by the probability that S^0 belongs to the DA of S^{m_0}, with radius k_{m_0}, i.e. by $\mathbb{P}(\delta(S^0, S^{m_0}) \leq k_{m_0})$.

Of course, we can approximate this probability, by substituting the Normal Distribution $\mathscr{N}(Nq, Nq)$ to the Binomial distribution. (N is great and q small.)

So selecting an algorithm, i.e. a matrix C such that all the k_m are equal, does yield the same lower bounds for all the probabilities of correct identification, whatever is the configuration S^m to identify.

2) Now let us assume that the choice of the configuration S^m to identify, is made with a probability $(p_m)\left(\sum_m p_m = 1\right)$. Then the probability of wrong identification is less than $\varepsilon = \sum_m p_m \mathbb{P}(\delta(S^0, S^m) > k_m / S^0$ arises from $S^m)$ and if an identification error for S^m costs g_m, the mean error cost is less than $G = \sum_m g_m p_m \mathbb{P}(\delta(S^0, S^m) > k_m / S^0$ arises from $S^m)$.

In that case, the choice of the semi-optimal matrix C, described in Sect. 6, yields a reasonable lower bound ε and G.

It remains that these calculations are approximations (apart from the fact we cannot calculate the exact size k_m of the DA of the patterns S^m, w.r.t. a given matrix C).

3) If $\delta(S^0, S^m) > k_m$, there is an error if S^0 falls in the DA of another configuration, but we do not know what happens if S^0 does not belong to a DA of the S^m. So S^0 may be attracted by one of the spurious configurations made attractive by the matrix C, for instance a configuration $-S^m$, or other linear combination of the S^m. (See Appendix for numerical examples.)

4) The complete study of the deterministic algorithm (Temperature $T = 0$) enables us to see that only the spurious stable state belonging to the subspace $\mathscr{V}$ spanned by the patterns, remain when the temperature $T > 0$ using the annealing method. This result confirms the results of Amit et al. (1985a, b), and shows it is true for all N, and not only when $N \to +\infty$.

5) The algorithm ensures a perfect retrieval of patterns if the initial state S^0 satisfies $\delta(S^0, S^m) < \gamma$ [defined in (8.2)] for some S^m. But, of course, the algorithm ensures a very good retrieval if S^0 is more distant from the patterns. Let be $\mathscr{B}$ the set of initial states which lead to some pattern. The simulations show that $\mathscr{B}$ occupies a great portion of the hypercube [see numerical evaluations in Peretto and Niez (1985), which agree with examples of Appendix 3]. Of course, the size of $\mathscr{B}$ decreases when p increases, for a fixed N, like the attractivity γ.

Appendix 1: Hamming Distance

For $S, S' \in \{-1, +1\}^N$, we denote by $\delta(S, S')$ the number of distinct components of S and S': it is the Hamming distance of S and S'.

The following properties are easy to check

1) δ is a distance in $\{-1, +1\}^N$,

2) $\delta(S, -S) = N$; if $\sum_{i=1}^{N} S_i S'_i = 0$, i.e. if S and S' are orthogonal (only with N pair), $\delta(S, S') = \frac{N}{2}$.

3) $\delta(S, -S') = N - \delta(S, S')$.

If d is the Euclidian distance in $\mathbb{R}^N$, $\langle \cdot \rangle$ the Euclidian product, $\|\cdot\|$ the Euclidian norm, cos and sin the usual trigonometric functions, we have

4) $\|S\| = \sqrt{N}$, $\delta(S, S') = \frac{1}{2}(N - \langle S, S' \rangle) = N \sin^2 \frac{(S, S')}{2}$ and $d(S, S') = 2\sqrt{\delta(S, S')}$.

Appendix 2: Evaluation of Gamma

For different values of N and p, we compare γ in various cases

a) orthogonal case $\gamma = N/2p$

b) random choice of patterns

c) random choice of $(S^2, \ldots, S^p)$ and $S^1 = (1, \ldots, 1)$

d) a non orthogonal case.

$N=20$	a	b	c	d
$p=3$	3.3	1.9	2.4	1.5
$p=4$	2.5	1.4	1.9	1.1
$p=5$	2	1.2	1.5	0.7

$N=30$	a	b	c	d
$p=3$	5	3.7	3.8	2
$p=4$	3.7	2.6	2.7	1
$p=5$	3	1.8	1.9	0.7

$N=40$	a	b	c	d
$p=3$	6.7	5.3	4.9	3.1
$p=4$	5	4	3.9	1.2
$p=5$	4	3.1	3.1	0.8
$p=6$	3.3	2.3	2.4	0.7

$N=50$	a	b	c	d
$p=4$	6.3	5.8	5.1	1.4
$p=5$	5	3.8	3.9	0.9
$p=6$	4.2	3.0	3.1	0.8

Note that if we want to retrieve exactly a pattern transmitted with 10% errors at most, we must choose $p \ll N$, approximately $p \sim 0.10N$ (for random case). This agrees with numerical results of Hopfield (1982) or Amit et al. (1985a, b) for instance. (Take account they use a connection matrix which does not ensure the stability of all patterns.)

Appendix 3

We indicate numerical results of various simulations.

We determine all the stable states, their order of attractivity, their Hamming distance from patterns, and their energy.

In each case, 1000 trials are performed with initial state at random. The system evolves very quickly (at most 4 parallel iteration steps) to one of the stable states. We indicate the number of trials ending into each stable states. See that the value of γ above calculated is not subestimated, and gives a good estimation of the BA′ size.

Example 1

$N=20$, $p=3$, $\gamma=2.4$ (case c of Appendix 2)

$S^1 = \;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1$

$S^2 = \;1\;\;1\;\;1\;-1\;\;1\;-1\;-1\;-1\;-1\;\;1\;\;1\;-1\;\;1\;-1\;\;1\;-1\;-1\;-1\;\;1\;-1$

$S^3 = -1\;-1\;\;1\;\;1\;-1\;\;1\;-1\;\;1\;\;1\;\;1\;-1\;-1\;-1\;\;1\;-1\;-1\;-1\;-1\;\;1\;\;1$

Example 2

$N=20$, $p=3$, $\gamma=1.9$

S^1, S^2, S^3 are random (case b)

Example 3

$N=20$, $p=4$, $\gamma=1.4$

S^1, S^2, S^3, S^4 are random (case b)

Example 4

$N=20$, $p=4$, $\gamma=1.1$

S^1, S^2, S^3, S^4 are not orthogonal ($\delta(S^1, S^3) = \delta(S^1, S^4) = 3$)

Example 1	Nb of trials ending into one of the patterns (or opposite) = 67%													
State	S^1	$-S^1$	S^2	$-S^2$	S^3	$-S^3$	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}
Att. order	2	2	2	2	2	2	1	1	0	0	0	0	0	0
$\delta(S^m)$	0	20	11	9	11	9	15	5	17	3	14	6	6	14
	11	9	0	20	12	8	14	6	6	14	15	5	17	3
	11	9	12	8	0	20	14	6	6	14	3	17	5	15
$H(S)$	20	20	20	20	20	20	15.4	15.4	15.4	15.4	15.4	15.4	15.4	15.4
% n.b. trials ending	11.7	10.7	10.2	11.6	11.1	12.2	6.7	5.8	5.4	4.0	1.9	3.1	3.3	2.3

Example 2	Nb of trials ending into one of the patterns (or opposite) = 54%													
State	S^1	$-S^1$	S^2	$-S^2$	S^3	$-S^3$	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}
Att. order	1	1	1	1	1	1	2	2	0	0	0	0	0	0
$\delta(S^m)$	0	20	12	8	12	8	14	6	18	6	6	2	14	14
	12	8	0	20	12	8	14	6	6	18	6	14	2	14
	12	8	12	8	0	20	14	6	6	6	18	14	14	2
$H(S)$	20	20	20	20	20	20	16	16	16	16	16	16	16	16
% n.b. trials ending	9.6	8.8	9.9	8.1	8.3	9.6	9.9	8.2	4.7	6.0	4.3	5.0	2.8	4.8

Example 3. In that case S^1, S^2, S^3, S^4 and their opposite are 1-attractors (compare with $\gamma=1.4$). And there are 40 stable states whose energy are 14.1, 15.1, 15.2, 15.6 or 16. The Hamming distance between a pattern and a spurious stable state can be 2, and that shows that the evaluation of γ is exact.

Among 1000 trials, the system ends into one of the pattern in 52% of the cases

Example 4. There are 8 1-attractors, S^1, S^2, S^3, S^4, their opposites and 4 configurations more which have the same energy ($H(S)=20$). There are 4 stable states, and 4 1-attractors, with $H(S)=16$

State	S^1	$-S^1$	S^2	$-S^2$	S^3	$-S^3$	S^4	$-S^4$	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
Att. order	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
$\delta(S^m)$	0	20	11	9	3	17	3	17	6	14	6	14	12	9	8	11
	11	9	0	20	14	6	10	10	13	3	17	7	5	2	15	18
	3	17	14	6	0	20	6	14	3	11	9	17	9	12	11	8
	3	17	10	10	6	14	0	20	3	13	7	17	15	12	5	8
$H(S)$	20	20	20	20	20	20	20	20	16	16	16	16	16	16	16	16

The system ends into one of the pattern is 52% of the cases

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