On the Representation of Curves in Descartes' Géométrie

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1. Introduction. Curves and their representation in 17th century mathematics

1.1 From antiquity until the beginning of the seventeenth century the collection of plane curves known to mathematicians did not change. It consisted in the conic sections, some higher algebraic curves such as the conchoid of Nicomedes and the cissoid of Diocles, and a few transcendental curves the most important of which were the Archimedean spiral and the quadratrix of Dinostratus. There was a marked change in this situation in the seventeenth century. Over a short period mathematicians vastly expanded the realm of curves open to mathematical treatment. Through the new analytic geometry of Fermat and Descartes the collection of mathematical curves came to include all algebraic curves, that is, all curves whose equation in rectilinear coordinates involves only the algebraic operations $+$, $-$, $\times$, $\div$ and $\sqrt[k]{\ }$ ($k > 1$, integer). The collection of transcendental curves, that is curves which do not admit an equation as above, was enlarged as well. The cycloid appeared on the mathematical scene around 1630, and the logarithmic curve in the 1660's. After that mathematicians encountered many more curves that depended algebraically on these two fundamental transcendentals. Most of these curves occurred as solutions of inverse tangent problems.

The new curves, like the earlier ones, could play three different roles. They could be an object of study, they could be a means to solve a problem and they could themselves be the solution of a problem. Thus Pascal, in his famous challenge to mathematicians of 1658, proposed the cycloid as an object of study and suggested that its area, the areas and centres of gravity of its segments and the contents and centres of gravity of solids arising by rotation of its segments be determined.

The "Cartesian parabola", introduced by Descartes in his *Géométrie*, was an example of a new curve that served as a means to solve a problem; the curve was used in the geometrical construction of the roots of equations of 5th and 6th degrees (*cf.* 5.2).

Finally, the curve with modern equation

$$ae^{-\frac{y}{a}} = a - y + x \qquad (1; 1)$$

is an example of a (transcendental) curve which originated as the solution of a problem, namely the famous problem of DEBEAUNE (1638) (*cf.* note 31).

1.2 The enormous increase in the number of curves forced 17^{th}-century mathematicians to think about how new curves could be introduced, described or defined. If the curves were to serve as an object of study, a means of solving a problem or as a solution, they had to be or to become *known*. In the previous period this was no problem since all curves were known and mathematicians could refer to any of them by name (ellipse, conchoid, spiral *etc.*), adding, if necessary, the values of their parameters.

But when is a new curve sufficiently known? Seventeenth-century mathematicians did not have a uniform definition of the concept of curve (nor apparently did they feel the need for such a definition) and therefore they had no standard form for specifying the curves they had in mind. In fact, there were many ways of specifying curves. One could, for instance, indicate how points on the curve could be constructed, one could describe a machine by which the curve could be traced, and (after analytic geometry had been introduced) one could give the equation of the curve. Some of these ways of describing curves were considered satisfactory, others less so, some not at all.

I shall use the term "representations of curves" to mean ways of specifying curves which were thought to make the curves sufficiently known. This term was not used in the 17^{th} century with that meaning; there was no term with that meaning then. Nevertheless, mathematicians did use the term "construction of curves" which has almost the same meaning but is more restricted.

The different ways in which curves were specified in 17^{th} century mathematics, the preferences that mathematicians expressed for certain of them and the reason given for these preferences form an important and interesting theme for historical study. These ways and preferences influenced the direction in which mathematics developed. Until now historians of mathematics have been hardly aware of this theme, mainly because a too rapid translation of 17^{th}-century mathematical arguments into modern analytical symbolism has obscured these aspects of the treatment of curves. The subject is also of more general interest because it touches on a wider mathematical, or perhaps metamathematical, question, namely, when is a mathematical entity "known" or when is a problem "solved"?

1.3 In this study I shall deal in particular with the representation of curves in DESCARTES' *Géométrie*[1]. At one stroke the *Géométrie* brought all the algebraic

[1] R. DESCARTES, *La Géométrie*, one of the essays in his *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences, plus la Dioptrique, les Meteores et la Geometrie qui sont des essais de cete methode*, Leiden, 1637. In references to the *Géométrie* I use the abbreviation *G* and I shall use the page numbers of the original edition (pp. 297–413). The original text is easily accessible in *The geometry of René Descartes with a facsimile of the first edition* (tr. and ed. by D. E. SMITH & M. L. LA-

curves into focus. But, as historians of mathematics have remarked[2] (with some surprise), DESCARTES did not consider the equation to be a sufficient representation of the curve; he used other kinds of representation instead.

Moreover, DESCARTES introduced a sharp distinction between admissible and inadmissible curves. The first he called "geometrical" the other "mechanical". The "geometrical" curves are what we now call algebraic curves (although DESCARTES did not explicitly say as much in the *Géométrie*, this can be inferred from what he did state); the "mechanical" curves are those which are now termed transcendental curves. But because DESCARTES did not consider the equation a sufficient representation of the curve, he could not establish any distinction between geometrical and non-geometrical curves on the basis of their equations; he had to reason about it on the basis of representations of curves which he did find acceptable. The acceptability of representations of curves is therefore a crucial concept in the *Géométrie*.

DESCARTES' distinction between "geometrical" and "mechanical" curves provided a serious issue in seventeenth-century mathematics. The increasing interest in transcendental curves (curves therefore that to DESCARTES were not admissible in geometry) forced mathematicians to use methods other than those expounded in the *Géométrie* and to take up a position with respect to the question of how far such curves could be considered "geometrical" or somehow admissible. Again, this question could only be dealt with in terms of the representation of these curves, and several of the representations used in these debates had been used in DESCARTES' *Géométrie*.

1.4 The complicated structure of the *Géométrie*, the different roles of curves in it, and DESCARTES' different criteria for geometrical acceptability of curves have not yet been satisfactorily unravelled[3]. A detailed analysis of the roles of

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THAM), New York (Dover) 1954. In my translations of texts from the *Géométrie* I have taken the English of SMITH & LATHAM as my starting point. However, their translation is very free and often unreliable, so in many cases I have had to modify it. The edition of the *Géométrie* in the *Œuvres de Descartes* (eds. C. ADAM & P. TANNERY, Paris 1897–1913) vol. **6**, pp. 367–485 also indicates the page numbers of the original. I shall use the abbreviation *A.T.* to refer to the *Oeuvres*.

[2] See, for instance, C. B. BOYER, *History of analytic geometry* (New York 1956) p. 88 and p. 102; see also M. S. MAHONEY, "Descartes: mathematics and physics", *Dictionary of scientific biography* (ed. C. C. GILLISPIE, New York 1970ff) vol. **4** (1971), pp. 55–61, footnote 7.

[3] There have been many studies on DESCARTES' *Géométrie*. Most of these, however, are unsatisfactory as far as the questions I discuss are concerned because they seek to answer the unfruitful question as to whether DESCARTES did or did not invent analytic geometry. The best source for the actual contents of the *Géométrie* is the *Géométrie* itself. The best summary of its intention, its development and its place within DESCARTES' mathematics is still G. MILHAUD, *Descartes savant*, Paris 1921. J. ITARD's *La géométrie de Descartes* (Conférences du Palais de la Découverte, série D, nr 39) Paris 1956 is very penetrating but also very dense. I have found A. G. MOLLAND's "Shifting the foundations, Descartes' transformation of ancient geometry", *Hist. Math.* **3** (1976), pp. 21–49 very helpful since it discusses the history of the concepts of construction and classification of curves in antiquity and DESCARTES' opinions on these classical ideas.

curves and their representation in the *Géométrie* sheds new light on the structure of DESCARTES' book, on its underlying programme and on the earlier development of DESCARTES' ideas about geometry. There is a conflict in the *Géométrie* between geometrical and algebraic methods of definition and criteria of acceptability. This conflict reflects a break in the development of DESCARTES' thought about geometry. In an early phase DESCARTES considered that the aim of geometry was to construct solutions of geometrical problems by means of curves traced by certain instruments; the instruments served as acceptable generalizations of ruler and compass. He tried to find new constructions in this way and to classify them. About 1630 that plan seemed to stagnate and DESCARTES also became fully aware of the power of algebraic methods. He then changed his programme. Algebra became the dominant tool, both for the solution of problems and for the classification of curves. But DESCARTES continued to believe in the principle of geometrical construction by means of curves traceable by instruments. As a result, there are conflicting elements in the *Géométrie.*

I shall show that it was impossible for DESCARTES to keep strictly to his earlier programme which was based on the use of instruments. But it was also impossible for him to work out a fully algebraic programme. If he had kept to his earlier plans, he would have lost the advantages of algebra; if he had adopted a fully algebraic approach, he could no longer have claimed that he was doing geometry. The contradictions in the *Géométrie* were indeed unavoidable. In order to understand DESCARTES' great contribution to geometry and algebra it is necessary to make these contradictions explicit and to explain how they influenced the structure of the *Géométrie.*

The later synthesis of algebraic and geometrical methods into what is now called analytic geometry was possible only because later mathematicians were not aware of (or forgot) the programmatic problems with which DESCARTES had struggled.

2. The problem of Pappus

2.1 In his *Géométrie* DESCARTES expounded a new programme for dealing with geometrical problems, and he used one problem as the key example: the problem of PAPPUS. I shall explain DESCARTES' programme in Section 3, but first I shall discuss the problem of PAPPUS and DESCARTES' solution of it. That discussion may illustrate the sort of geometrical problems which DESCARTES had in mind and explain the roles of geometrical constructions, curves and algebraic calculations in the *Géométrie.*

As far as the concepts of curves and constructions in DESCARTES' *Géométrie* are concerned one can compare J. VUILLEMIN, *Mathématiques et métaphysique chez Descartes,* Paris 1960 (in particular Ch. III "De la classification cartésienne des courbes", pp. 77ff.), G.-G. GRANGER, *Essai d'une philosophie du style,* Paris 1968 (in particular Ch. III "Style Cartésien, style Arguésien" pp. 43–70), and J. DHOMBRES, *Nombre, mesure et continu, épistémologie et histoire,* Paris 1978 (in particular the section pp. 134–143). My interpretation of the *Géométrie* differs in several points from the interpretations given in these articles.

In explaining the problem of PAPPUS[4] I shall use symbols for the lines, distances and numbers. DESCARTES presented the problem in prose accompanied by drawings. Let (see Figure 1) a number of lines L_i be given in the plane, and let φ_i be fixed angles.

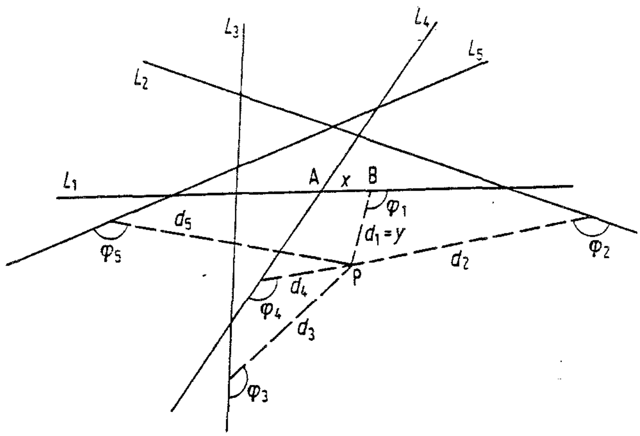

Fig. 1

Let d_i denote the length of the line segment from point P to L_i which makes an angle of φ_i with L_i. (If φ_i is 90°, d_i is the distance to L_i.) Let $\alpha : \beta$ be a given ratio and a a given line segment. It is required to find points with the following properties:

for three lines:

$$(d_1 \cdot d_2) : (d_3^2) = \alpha : \beta, \qquad (2;1)$$

for four lines:

$$(d_1 \cdot d_2) : (d_3 \cdot d_4) = \alpha : \beta, \qquad (2;2)$$

for an uneven number $(2n - 1)$ of lines, $n > 2$:

$$(d_1 \ldots d_n) : (d_{n+1} \ldots d_{2n-1} \cdot a) = \alpha : \beta, \qquad (2;3)$$

and for an even number $(2n)$ of lines, $n > 2$:

$$(d_1 \ldots d_n) : (d_{n+1} \ldots d_{2n}) = \alpha : \beta. \qquad (2;4)$$

PAPPUS gives the problem for three and four lines as well as its generalization to more lines[5]. What we have here is a locus problem; in each case there are infinitely many points which satisfy the condition; these points form a locus in the plane;

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[4] DESCARTES quoted the problem (*G*, pp. 304–306) in Latin from the edition by COMMANDINO, *Pappi Alexandrini Mathematicae Collectiones*, Bologna 1588, pp. 164^v–165^v. For the Greek text see F. HULTSCH (ed.) *Pappi Alexandrini Collectiones quae supersunt* Berlin 1876–1878, vol. **2**, pp. 677.

[5] Note however that (2; 3) is not a generalization of (2; 1). In fact (2; 1), the problem in three lines, is an exceptional case which arises when two lines coincide in the problem for four lines.

this locus is generally a curve. PAPPUS states that for three and four lines the locus is a conic section and that for more than four lines nothing is known about the form of the locus.

2.2 DESCARTES sketches the general solution of the problem at the end of the first book of the *Géométrie* (pp. 309–314). His method is as follows. He sets

$$d_1 = y \tag{2; 5}$$

and he takes x to be the distance along L_1 from a fixed point A to the intersection of d_1 with L_1. He then shows by simple geometrical arguments that all d_i can be expressed linearly in x and y:

$$d_i = a_i x + b_i y + c_i . \tag{2; 6}$$

He notes that in the exceptional case when all lines are parallel, x does not occur in the expressions for the d_i.

He then remarks that the products $d_1 \dots d_n$, $d_{n+1} \dots d_{2n}$ and $d_{n+1} \dots d_{2n-1} \cdot a$ become expressions in x and y of degree at most n. The conditions (2; 1)–(2; 4) can therefore be rewritten as equations. Equation (2; 4) for instance becomes[6]

$$y(a_2 x + b_2 y + c_2) \dots (a_n x + b_n y + c_n) = \frac{\alpha}{\beta}(a_{n+1} x + b_{n+1} y + c_{n+1}) \dots (a_{2n} x + b_{2n} y + c_{2n}). \tag{2; 7}$$

For n lines the equation will be of degree at most n. For $n - 1$ lines the choice of $d_1 = y$ and the occurrence of a in the second product of lines implies that the highest power of x is at most $n - 1$. Thus the equation is of degree at most n, but the highest power of x is at most $n - 1$. This statement does not apply to the problem with only three lines, which makes that one exceptional. Finally for $2n$ and $2n - 1$ parallel lines the result is an equation in one unknown, namely y, of degree at most n; the locus then consists in a number of lines parallel to the given lines.

DESCARTES goes on to consider how the points satisfying the requirements of the problem (the points on the locus) can be constructed. He chooses arbitrary

[6] DESCARTES did not discuss the fact that as a result of this rewriting the d_i may have negative values, whereas the obvious interpretation of the original problem would require the d_i to remain positive. The effect of this is that DESCARTES found only one curve as locus, while the original interpretation would lead to a locus consisting in two curves. For instance in the four line locus (taking $\frac{\alpha}{\beta} = 1$) DESCARTES worked out

$$y(a_2 x + b_2 y + c_2) = (a_3 x + b_3 y + c_3)(a_4 x + b_4 y + c_4),$$

and found one conic section as the locus. But if the d_i were taken to be positive, the equation would become

$$|y|\,|a_2 x + b_2 y + c_2| = |a_3 x + b_3 y + c_3|\,|a_4 x + b_4 y + c_4|$$

or

$$y(a_2 x + b_2 y + c_2) = \pm (a_3 x + b_3 y + c_3)(a_4 x + b_4 y + c_4),$$

that is, two conics.

values for y and then constructs geometrically the corresponding values for x. In this way one can construct as many points as one wishes on the locus. In Section 6 I shall discuss this type of pointwise construction in more detail. DESCARTES remarks that for any chosen value of y, the corresponding x's are the roots of an equation the degree of which is, for $2n$ lines, at most n, and for $2n - 1$ lines at most $n - 1$. For three lines the result in general is an equation of degree 2. The exceptional case of $2n - 1$ parallel lines leads directly to an equation in y of degree n.

Thus the problem is reduced to the geometrical construction of roots of equations. DESCARTES then anticipates results which he will explain in the third book of the *Géométrie*. These results are the following: The roots of an equation of second degree can be constructed by ruler and compass. The roots of equations of third and fourth degrees can be constructed by the intersection of conics, in particular the intersection of a parabola and a circle. The roots of equations of fifth and sixth degrees generally cannot be constructed by the intersection of conics; more complex curves have to be used for them. It is possible to construct these roots by the intersection of a circle with a certain curve of third degree namely the "Cartesian parabola". On the basis of these results DESCARTES gives at the end of the first book the following classification of the cases of the problems of PAPPUS (*G* pp. 313–14):

a) 3, 4 or 5 lines, but not 5 parallel lines:
the equation in x is of degree $\leqq 2$ and therefore points on the locus can always be constructed with ruler and compass.

b) 5 parallel lines, 6, 7, 8 or 9 lines, but not 9 parallel lines:
the equation in x (or for 5 parallel lines, in y) is of degree $\leqq 4$ and therefore points on the locus can always be constructed by means of intersections of conics; in some cases, construction by ruler and compass only may be possible (namely if the equations happen to be of degree $\leqq 2$ or if they are reducible to such equations).

c) 9 parallel lines, 10, 11, 12, 13 lines but not 13 parallel lines:
the equation in x (or in y in the case of 9 parallel lines) is of degree $\leqq 6$; the construction by means of intersection of conic sections will in general not be possible and a more complicated curve will have to be used.

d) *etc.*

2.3 This classification refers to the construction of the locus. DESCARTES returns to the problem of PAPPUS in the second book. There he gives another classification, now according to what he calls the "genre" of the locus; I shall translate "genre" by "class". This relates to a classification of curves according to the degree of their equations, a classification that DESCARTES explains in the second book (*G* pp. 319–323). He argues there that all geometrical curves have algebraic equations (*cf.* Section 9.1). The curves of the first class are those with equations of the second degree: the circle, the parabola, the hyperbola and the ellipse. The second class contains the curves with equations of the 3^{rd} and 4^{th} degree; the third class those with equations of the 5^{th} and 6^{th} degree, and so forth.

I shall return to this classification in Section 3. It leads to the following classification of the cases of the problem of Pappus (*G* pp. 323–324):

a′) 3 or 4 lines:

The equation of the locus is of degree at most 2; the locus is of the first class.

b′) 5, 6, 7 or 8 lines:
The equation is of degree at most 4; the locus is of the second class or, in exceptional cases, of the first.

c′) 9, 10, 11 or 12 lines:
The equation is of degree at most 6; the locus is of the third class or of a lower class in exceptional cases.

d′) *etc.*

In this connection Descartes states that all (algebraic) equations can occur as equations of the locus of some problem of Pappus:

> And because the position of the given lines can vary in all sorts of ways and thereby change the given quantities and the signs + and − of the equations in all imaginable ways, it is evident that there is no curved line of the first class which would not be of use in this problem if it is proposed in four straight lines, nor one of the second which would not be of use if it is proposed in eight, nor of the third when it is proposed in twelve, and likewise with the others. Thus there is no curved line which is subject to calculation and which can be accepted in geometry, which is not of use for some number of lines. (*G* p. 324).

The statement is incorrect[7]. But it is important in Descartes' further classification of curves; I shall return to it in Section 9.

Descartes then gives a complete solution (*G* pp. 324–334) of the problem of Pappus in three and four lines, calculating the equations explicitly and discussing the positions of the resulting conics in the plane. This section is well known[8]; since it is not important for my present subject, I will not discuss it here.

Finally Descartes treats two special cases of the locus of the problem of five lines which will be discussed in Sections 5.2 and 8.1.

2.4 Descartes' solution of the problem of Pappus supplies a good illustration of the two different roles that curves can play in the solution of locus problems: a curve can occur as a locus; it can occur also as the means to construct points on the locus. Descartes treats the curves in totally different ways according to their roles. Consider for instance the case where conic sections occur as loci. This happens in problems with 3-lines and 4-lines and Descartes states that these problems are "plane". That implies that he considers the locus in this case to be constructible by ruler and compass. But with ruler and compass one can

[7] See the Appendix.

[8] For an extensive discussion of the section see D. T. Whiteside, "Patterns of mathematical thought in the later seventeenth century", *Arch. Hist. Ex. Sci.* **1** (1960–1962), pp. 179–388, especially pp. 290–295.

only construct points on a conic; one cannot construct or trace the conic as a whole. Nevertheless DESCARTES considers such a pointwise construction of a conic sufficient when it occurs as a locus. But when a conic is used as means construction DESCARTES sets stronger requirements. This occurs in the problem for 6, 7, 8 or 9 lines (Case b in Section 2.2), where points on the locus are constructed by means of intersections of conics with circles and straight lines. Apparently DESCARTES does not consider the conics involved here constructible by ruler and compass, because if that were so the whole construction could be performed by ruler and compass, and that is what DESCARTES denies. Hence pointwise constructions of conics are not acceptable if these conics themselves serve as means of construction; in that case their construction must satisfy stronger criteria. For higher-order curves too, the construction has to satisfy different criteria according to whether the curve occurs as a locus or serves as a means of construction of points on a locus.

The reason for a stronger requirement when the curve is used as a means of construction is that it is then supposed that the intersections of the curve with circles, straight lines or other conics can be found. But if a curve is given only through a pointwise construction as in the locus case, its intersections with other lines cannot be determined. To illustrate this let C_1 and C_2 be two conics (see Figure 2) whose equations are given (or equivalently whose vertices and diameters

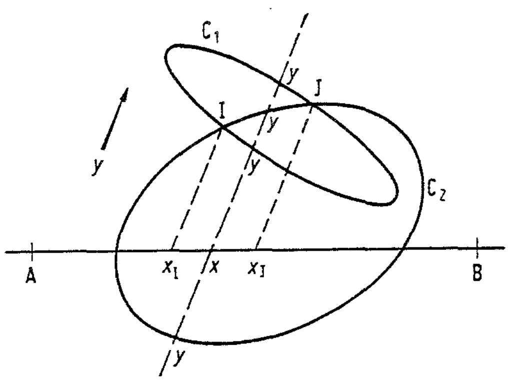

Fig. 2

are given) and whose intersections I and J we want to construct. Let AB be the axis of the x's and Y the direction of the y's. Points on C_1 and C_2 can be constructed by ruler and compass by taking arbitrary values for x and constructing the corresponding y's. But I and J cannot be accurately constructed in this way; we can approximate them but their exact position would only be found if accidentally we started our construction with x_I or x_J.

If a curve is used as means of construction, it must be possible to find its intersection with other curves. A pointwise construction is not sufficient for that purpose. Instead one obviously needs a method of tracing the curve by a continuous motion, so that the intersections with other lines are actually marked. We will see (Sections 4 and 5) that the requirement that curves be traceable by continuous motion is crucial to DESCARTES' *Géométrie*.

3. Descartes' Programme for Geometry

3.1 In his *Géométrie* DESCARTES presented a new approach to the solution of geometrical problems; it contrasted so strongly with earlier approaches that one can speak of a new paradigm. Before giving his own opinion on the programme of geometry DESCARTES explained what mathematicians before him, especially those from antiquity, had thought about this matter. He said (*G* pp. 315–317) that, traditionally, geometrical problems had to be solved by ruler and compass. However, classical mathematicians had already encountered problems which could not be solved in this way. They had solved these using intersections of conics or even more complicated curves such as the conchoid. But they called these curves mechanical, thereby implying that they did not consider them to be genuinely geometrical. DESCARTES went on to speculate about the reasons the ancients might have had for this, and he rejected these. His rendering of the classical arguments was oversimplified, if not inaccurate[9], but it served very well as an introduction and contrast to DESCARTES' own view.

DESCARTES' view can be summarized as follows: Construction of problems by ruler and compass is certainly simpler than, and therefore preferable to, construction by means of the intersection of conics or more complex curves. In the construction of problems one should always use the simplest possible curves. But this does not imply that more complex curves are necessarily less geometrical than the straight line and the circle, or that constructions by means of these curves are less geometrical than constructions by ruler and compass. There is a collection of curves of ever increasing complexity (circles, conics, conchoids, *etc.*) which are in principle acceptable in geometrical constructions. If a problem can be constructed by the intersection of two such curves and it cannot be constructed by simpler curves, then that construction is the right one to choose and it is no less geometrical a construction than one by ruler and compass.

This vision of the geometrical procedure of constructing problems determined a programme in three parts. First DESCARTES had to determine which curves were acceptable as genuinely geometrical means for the construction of problems. Secondly, he had to make it clear on which criteria some curves would be considered simpler than others; this would lead to a classification in order of simplicity within the collection of geometrically acceptable curves. Finally, a method had to be devised for finding the simplest possible curves by which each problem could be constructed. This is essentially the programme which DESCARTES worked out in his *Géométrie*.

3.2 The first point of the programme—differentiating between the curves which are acceptable in geometry and those which are not—caused DESCARTES (and his successors) the greatest number of conceptual problems. Basically DESCARTES took as geometrical curves those "which can be described by some regular motion" (*G* p. 369). But this is not a very clear criterion. Also DESCARTES wished to include in the collection of geometrically acceptable curves all curves that may occur as locus solutions of problems such as the problem of PAPPUS. This meant

[9] See the article by MOLLAND cited in Note 3.

that in fact—although DESCARTES never explicitly said so—he wanted to regard all algebraic curves as geometrical. But to do so he would have to prove that all algebraic curves could be traced by continuous and geometrically acceptable motions, or that they could be traced by other means which were just as geometrical as the tracing by continuous motion. In Sections 4–9 I shall discuss how DESCARTES dealt with this very complex part of his programme.

Algebra, in the sense of the existence of an algebraic equation of the curve, was the essential criterion in the first part of the programme. But the algebra had to remain implicit. DESCARTES could not simply take as "geometrical" all curves that admit an algebraic equation, because obviously that is not a geometrical criterion; if he were to adopt this criterion, DESCARTES could no longer claim that he was doing geometry.

3.3 In the second and the third part of the programme algebra could be used quite openly, and it formed the crucial tool. DESCARTES used the degree of the curve equations to classify the curves according to their simplicity. He divided them into classes ("genres"); the first class consisted of the curves with equations of 2nd degree; these are the conic sections. DESCARTES did not incorporate straight lines in his classification. Curves with equations of the 3rd and 4th degree were of the second class; those of the 5th and 6th degree were of the third class, *etc.* (*G* p. 319).

DESCARTES stressed elsewhere that in constructions we should always use curves of the lowest possible class (*G* p. 371). He noted that within one class some curves may be simpler than others, in the sense that one cannot construct such complicated problems with them as one can with the others. For instance, the circle is of the first class, but there are constructions that can be performed with the other curves of that class (the conic sections) but not with the circle. DESCARTES also mentioned the conchoid as an exceptional curve within the second class (*G* p. 323), but he did not discuss how to distinguish the exceptional curves from the other curves within one class.

It is not quite clear why DESCARTES, after taking the conic sections as the first class of curves, lumped all curves of the third and the fourth degrees together in the second class, those of the fifth and sixth degrees in the third class, and so forth. DESCARTES explained that he did so because there is a general rule whereby fourth-degree problems can be reduced to third-degree ones, and sixth-degree problems to fifth-degree ones (*G* p. 323). It seems likely that for problems of fourth degree and third degree he had in mind FERRARI's rule for reducing equations of fourth degree (in one unknown) to ones of third degree. But there is no such rule for equations of sixth and fifth degrees, so in this case DESCARTES was making rather a rash extrapolation.[10]

The classification may also have been connected with the methods used for constructing roots of equations by the intersection of curves. The roots of equations

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[10] DESCARTES tended to underestimate the dangers of extrapolating mathematical results, as is evident, for instance, from the penultimate sentence of the *Géométrie*: "For in the matter of mathematical progressions, once one has the first two or three terms, it is not difficult to find the others" (*G* p. 413).

of the third and fourth degrees can be constructed by circle and parabola, those of fifth and sixth degrees by circle and Cartesian parabola. In other words, by introducing one higher curve as a new means of construction one can construct roots of equations of two successive higher degrees. But there is a puzzling aspect here. DESCARTES' classification was for curves serving as means of construction, whereas this argument would classify problems rather than devices for construction. In fact, the degree of the constructing curves (parabola, degree 2, Cartesian parabola, degree 3, *etc.*) rises by steps of one.

In DESCARTES' classification, and especially in his arguments about the subdivision within one class, there is a contradiction between algebraic criteria of simplicity (the form of the equation, in particular its degree) and geometrical criteria of simplicity (the use of the curve as a device for construction). The special role that the circle plays in the first class shows that the classification is not really adequate for dinstinguishing the means of construction. I shall return to this contradiction in connection with pointwise constructions in Section 10.

3.4 For the third part of the programme—to find the simplest geometrical construction of the solution of a given geometrical problem – the crucial tool again was algebra. A problem should be reduced to an equation in one unknown (*G* pp. 300–302). Then the roots of this equation should be constructed geometrically by the intersection of certain curves, which should be as "simple" as possible, that is, of the lowest possible class. The simplicity of the curves by which a problem could be solved determined the class to which the problem belonged. Here DESCARTES followed classical usage and called problems *plane* if they could be solved by circles and straight lines, and *solid* if they also required a conic section. DESCARTES devoted most of the third book of the *Géométrie* to this point of the programme. He proved there that every equation of third or fourth degree could be constructed by the intersection of a circle and a parabola, and every equation of fifth or sixth degree equation by the intersection of a circle and a Cartesian parabola.[11]

3.5 At the beginning of the third book of the *Géométrie* DESCARTES gave a succinct formulation of the programme which I have been describing:

Although all curved lines which can be described by some regular movement must be admitted in geometry, this is not to say that for the construction of

[11] For details of this construction see *G* pp. 402–411 and WHITESIDE's note in *The mathematical papers of Isaac Newton* (ed. D. T. WHITESIDE, Cambridge, 1967–) vol. 1 (1967), p. 495, note 15. DESCARTES wrote the equation as $y^6 - py^5 + qy^4 - ry^3 + sy^2 - ty + v = 0$, where he intended p, q, r, s, t, and v to be positive. He mentioned as a condition for his construction that p^2 be smaller than $4q$. He stated that if this condition is not satisfied, a substitution $y \to y + c$ with c large enough will yield an equation which satisfies the condition. The substitution and its inverse correspond to straightforward geometrical constructions, so the construction of the roots of the resulting equation yields also the roots of the original equation. In fact DESCARTES' construction tacitly assumes some further conditions for the coefficients (such as for instance the arrangement with positive factors and alternating signs), but all these conditions can be satisfied by performing the substitution $y \to y + c$ with c large enough. The construction is therefore general and can be used to find the roots of any equation of sixth degree.

> any problem we may use indifferently the first one that occurs. We must always take care to choose the simplest through which the solution is possible. And it should be noted that by simplest curves one should understand not only those which can most easily be described, nor those which make the construction or the proof of the proposed problem easier, but primarily those of the simplest class which can be used to determine the required quantity. (*G* p. 369′–370)

Thereupon came an example of the construction of two mean proportionals between two given line segments by means of curves traced by a certain machine (which I shall discuss in Section 5.1). DESCARTES remarked that this construction might well be the easiest possible construction and might provide the clearest proof, but it used curves of a higher class than necessary and therefore

> ... it would be a mistake in Geometry not to use them [namely curves of a simpler genre]. On the other hand it is also a mistake to try in vain to construct a problem by a simpler class of lines than the nature of the problem allows. (*G* p. 371)

4. The representation of curves in Descartes' *Géométrie*

4.1 DESCARTES dealt with the fundamental question of his programme at the beginning of the second book of the *Géométrie*. He formulated that question in the marginal title as: "which are the curved lines that can be accepted in geometry" (*G* p. 315). He criticised the classical mathematicians for having called certain curves used in geometrical constructions "mechanical" rather than "geometrical". DESCARTES said that the fact that "mechanical" curves are described by certain machines does not make them less geometrical than the straight line and the circle, which, after all, are also traced by machines, namely by ruler and compass. DESCARTES did not wish to impose such strict requirements for geometrical curves; he accepted many more curves as geometrical:

> To trace all the curved lines which I wish to introduce here, nothing else need be supposed than that two or several lines can be moved one by the other, and that their intersections mark other lines ... . (*G* p. 316)

Such curves may be very complicated, but that need not make them less geometrical:

> It seems very clear to me that if we consider (as is customary) as geometrical that which is precise and exact, and as mechanical that which is not, and if we consider geometry as the science which furnishes a general knowledge of the measures of all bodies, we have no more right to exclude the more composite lines than the simpler ones, provided that we can imagine them as described by a continuous motion, or by several motions following each other, the last of which are completely regulated by those which precede. For in this way one can always have an exact knowledge of their measure. (*G* p. 316)

DESCARTES' criterion, then, to accept curves as geometrical was that they could be traced by continuous motion. The tracing of the curve is basic for an understanding of its nature; significantly, DESCARTES combined the word "tracing" with understanding and conceiving; he used expressions such as: "ways to trace and conceive curved lines" (*G* p. 319) and "to know and trace the line" (*G* p. 307).

4.2 Since DESCARTES considered curves primarily as traced by continuous motions generated by certain machines, he faced a number of difficult conceptual problems which are summarised below:

a) There are certain curves, such as the spiral and the quadratrix, which DESCARTES did not accept as geometrical but considered to be mechanical, in the sense that they were imprecise and inexact. These curves, however, can be traced by continuous motion (see Section 7.2). DESCARTES had therefore to specify which types of motion he accepted and which he rejected.

b) In the course of his studies DESCARTES came across several curves which he could not, or would not, present as traced by some continuous motion. Instead he presented them as constructed pointwise or as traced by machinery involving string. He had therefore to argue that such constructions or methods of tracing are just as acceptable in geometry as tracing by continuous motion.

c) Pointwise constructions and tracing machinery involving string can also be devised for curves which DESCARTES did not accept in geometry. Therefore he had to specify which pointwise constructions and which methods of tracing with string were acceptable.

d) Algebra was the crucial tool in DESCARTES' new programme for geometry, and the new curves he wished to introduce had to be amenable to algebraic treatment, that is, they had to have an algebraic equation. Thus DESCARTES had to consider whether his new curves had such equations, and conversely, whether equations resulting from the use of the algebraical methods would always correspond to geometrically acceptable curves.

The representation of curves is fundamental to these problems. The arguments about the acceptability of curves can only be formulated in terms of the representations of the curves, and the discussion is complex because three different methods of representation are involved: representation by specifying the continuous motion which traces the curve, representation by the method of constructing points on the curve, and representation by specifying the tracing machinery involving string. To these three one may add the fourth method—the representation of a curve by its equation. However, all these problems in fact arise because DESCARTES did not consider it acceptable in Geometry to represent a curve by its equation.

In the following sections I shall summarise DESCARTES' arguments with regard to the four problems mentioned above.

5. Curves described by continuous motion

5.1 DESCARTES said that to trace the curves that are acceptable in geometry, "nothing else need be supposed than that two or several lines can be moved one by the other and that their intersections mark other lines" (*G* p. 316). In the second book of the *Géométrie* he gave two examples to illustrate the kinds of motion he had in mind.

The first example concerned the famous instrument of Figure 3 (*G* p. 318 and p. 370).

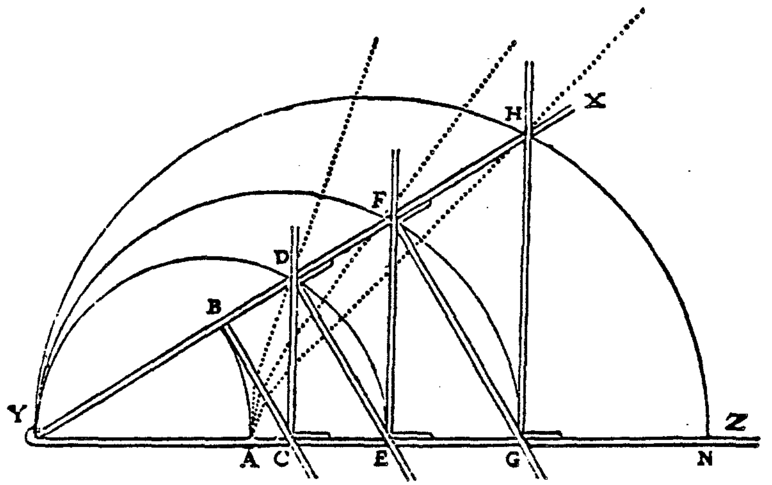

Fig. 3

It is a system of linked rulers. The rulers *YX* and *YZ* are connected in *Y* by a pivot. Ruler *BC* is fixed perpendicular to *YX* in *B*. The rulers *CD*, *EF* and *GH* are made in such a way that they can slide along *YZ* and still remain perpendicular to it. Similarly the rulers *DE* and *FG* slide along *YX* while remaining perpendicular to it. At the beginning of the motion of the instrument angle *XYZ* is assumed to be zero and all the rulers coincide in point *A*. Now the angle *XYZ* is opened by keeping *YZ* fixed and rotating *YX*. Ruler *BC* pushes *CD* outwards, *CD* pushes *DE*, *DE* pushes *EF*, *etc.* Point *B* (fixed on *XY*) describes a circle; points *D*, *F* and *H*, sliding along *YX*, describe other curves (dotted in the figure). DESCARTES argued that these curves, although described by more and more complicated combinations of motions, should all be accepted in geometry:

> But I do not see what could prevent us from conceiving the description of the first [*i.e.* the curve described by *D*] as clearly and distinctly as that of the circle, or at least as that of the conic sections, nor what could prevent us from conceiving the second one and the third one and all the others, which one can describe equally well as the first one; nor therefore what could prevent us from accepting all these curves in the same manner, to serve the speculations of geometry. (*G* pp. 318–19)

The instrument of Figure 3 is found even in DESCARTES' very early studies; I shall discuss its use and origin in Section 10. It is interesting to note that DESCARTES' discussion of the instrument in this passage did not serve primarily to explain which motions he had in mind for the tracing of acceptable curves; he added another example which explained this better. Rather, the instrument served to show that, however composite a motion is, the resulting curve can be conceived in a clear and distinct way, and is therefore acceptable in geometry. The instrument was a precise illustration of the description, given above, of acceptable tracing motions:

> a continuous motion, or(-) several motions following each other the last of which are completely regulated by those which precede. (*G* p. 316)

Here the first motion is the rotating motion of the rulers *YX* and *BC*, the subsequent motions are those of the rulers *CD, DE, EF, etc.*; *BC* regulates the motion of *CD*, *CD* that of *DE* and so forth.

The text, and especially the use of the key words, clear and distinct ("nettement", "distinctement", *G* p. 318), show that Descartes saw a parallel between the series of interdependent motions in the machine, all regulated by the first motion, and the "long chains of reasoning" in mathematics, discussed in the *Discours de la Methode*, which, provided each step in the argument is clear, yield results as clear and certain as their starting point.[12]

5.2 But the example of the instrument of Figure 3 did not cover all the combinations of motions which Descartes had in mind, because it involved only straight lines as moving parts. When Descartes wrote "nothing else need be supposed than that two or several lines can be moved the one by the other, and that their intersections mark other lines", he also had moving curved lines in mind. This becomes clear in a series of further examples. These concerned a tracing process whereby new, more complicated curves are generated from the motion of simpler curves and straight lines. Descartes proceeded as follows (*G* pp. 319ff; see Figure 4):

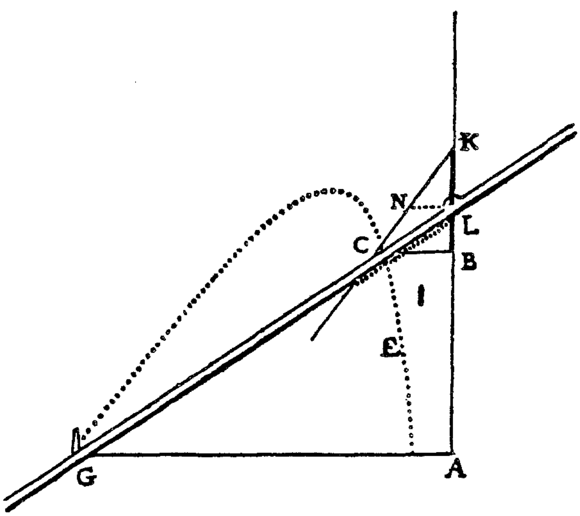

Fig. 4

[12] Descartes *Œuvres* (see Note 1) vol. 6, p. 19.

A ruler GL pivots at G. It is linked at L with a device NKL which can be moved along the vertical axis while the direction of the line KN is kept constant. When L is moved along the vertical axis, the ruler turns around G and the line KN is moved downwards remaining parallel to itself. The intersection C of these two moving straight lines describes the curve GCE. DESCARTES derived the equation of this curve

$$y^2 = cy - \frac{c}{b}xy + ay - ac \qquad (5;\ 1)$$

(where $GA = a$, $KL = b$, $NL = c$, $CB = y$ and $AB = x$) and concluded that it was a curve of the first class; he added that it was in fact a hyperbola. Thus the straight line KN in the machine produced a curve of the first class.

Next DESCARTES asserted (G p. 322) that if the straight line in the machine were replaced by a curve of the first class, the resulting curve would be of the second class. He mentioned the case where KN is a circle with centre L; the resulting curve will then be the conchoid of NICOMEDES. (See Figure 5; on all the lines through a fixed point (G), the intercepts between the conchoid and an axis (KA) are equal.) The conchoid is of a higher class than the conics.

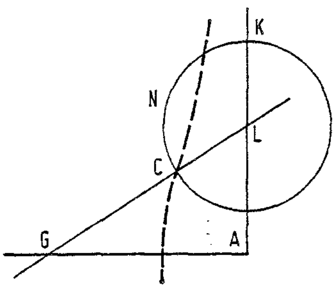

Fig. 5

Then DESCARTES replaced the circle by a parabola (see Figure 6), and stated that the resulting curve would be the "first and simplest curve for the problem of PAPPUS if there are only five lines given in position" (G p. 322). The curve played a central role in DESCARTES' *Géométrie*; it became later known as the

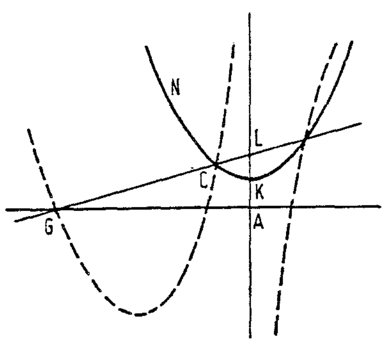

Fig. 6

"Cartesian parabola".[13] Further on in the second book (*G* pp. 335–337) he showed that this curve was indeed the solution of a special simple case of the five line locus problem[14], and he gave the equation of the curve:

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy. \tag{5; 2}$$

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[13] See G. Loria, *Spezielle algebraische und transzendente ebene Kurven*, Leipzig (2d ed.) 1910–1911, vol. 1, pp. 51–52. Other names for the curve are "trident" (Newton) and "parabolic conchoid".

[14] Namely the problem (*cf.* Section 2.1)

$$d_1 \cdot d_2 \cdot d_3 = d_4 \cdot d_5 \cdot a$$

in the case where L_1, L_2, L_4 and L_3 (in that order) are equidistant and parallel, and L_5 is perpendicular to the other lines; a is the distance between L_1 and L_2, and all the distances d_i are taken perpendicular to L_i. The equation of the curve is:

$$y^3 - 2ay^2 - a^2y + 2a^3 = axy.$$

Compare the figure (from *G*, p. 336). The lines L_i are *GF*, *ED*, *IH*, *AB* and *GA* respectively. The distances are taken perpendicular to the lines, $d_1 = CF$, $d_2 = CD$, $d_3 = CH$, $d_4 = CB = y$, $d_5 = CM = x$. *GL* is the ruler moving around *G*. *CKN* is the parabola moving vertically along its axis *AB*. *GEC* is the branch of the "Cartesian parabola" described by *C*, *NIO* is the branch described by the other intersection *N*, *cGc* and *oIn* are the branches of the other "Cartesian parabola" which occurs if the distance to L_5 is taken to be positive in the other direction.

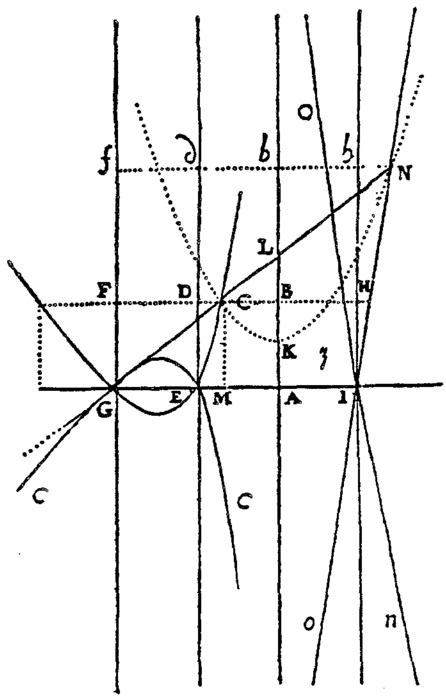

Figure for Footnote 14

In the third book DESCARTES explained how this curve can be used for finding the roots of a sixth degree equation (*G* p. 403); he discussed there in more detail how the curve was traced by the combined motions of a ruler and a parabola. In the passage from the second book where he introduced this curve DESCARTES said that it was of the second class. Furthermore he claimed that if a curve of the second class was used in the tracing, the resulting curve would be of the third class *etc.* (*G* p. 322), but he did not prove this.[15]

5.3 The linkage machines and the device of a moving curve whose intersection with a ruler traces new curves are the examples which DESCARTES gave to illustrate his concept of tracing curves by combination of motions. It is a fundamental concept because DESCARTES stated that he would introduce new curves only if they were traceable in this way. This means that there were other curves which he would not accept as geometrical because they could not be traced in this way. As examples DESCARTES mentioned the spiral and the quadratrix. However, both the spiral and the quadratrix can be traced by a combination of continuous motions; they were in fact defined in such a way. DESCARTES had therefore to specify a further requirement for the motions in order to rule out these curves. Before discussing this requirement I shall indicate the ways in which the two curves mentioned can be traced by continuous motions.

The Archimedean spiral[16] (see Figure 7) is described by two motions, one rotatory motion of a ruler *OR* which turns uniformly around *O*, and one rectilinear motion of a point *P* which moves uniformly along the ruler *OR*. The point *P* traces the spiral.

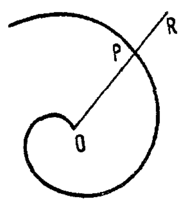

Fig. 7

[15] DESCARTES claimed that it is easy to prove this by actual calculation. But this is not true. A counter example arises when one takes $y^3 = c^2z$ for the moving curve (taking z as vertical coordinate measured from K, and y as horizontal coordinate). This curve has degree 3 and therefore belongs to the second class. The new curve described by the intersection of this curve and the ruler GK has equation $(a - y)\, y^3 = c^2(xy - by + ab)$ (where x is the vertical coordinate taken from A, $a = GA$, $b = KL$). This curve has degree 4 and therefore also belongs to the second class. In general, if the equation of the moving curve is linear in z, the degree of the resulting curve will be only one higher than the degree of the moving curve (comme il est fort aysé a connoistre par le calcul), and hence in that case the resulting curve may belong to the same class as the original curve.

[16] See ARCHIMEDES, *On Spirals*, definition 1, in *The works of Archimedes* (ed. T. HEATH, New York, Dover reprint), p. 165.

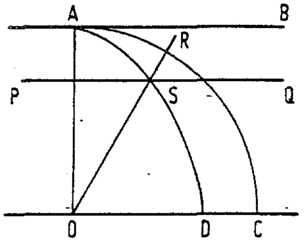

Fig. 8

The quadratrix[17] (see Figure 8) can be described by a combination of separate motions, namely by one rotatory motion of a ruler OA which turns uniformly around A, from position OA to position OC, and one rectilinear motion of a ruler PQ which moves uniformly downwards from position AB to position OC, during the same time that OR turns from OA to OC. The intersection S of both rulers traces the quadratrix ASD. The construction implies that during the motion always

$$AP : \widehat{AR} = AO : \widehat{AC}. \tag{5; 3}$$

DESCARTES said about the spiral, the quadratrix and similar curves, that "they are conceived as described by two separate movements, between which there is no relation ("raport") that can be measured exactly", and that for that reason they "really only belong to Mechanics". (*G* p. 317) The absence of a measurable "raport" is the essential point here, for in both cases the two movements could in principle be linked in such a way that the one determines the other, namely by a string mechanism which I shall discuss in Section 7. In view of the methods of tracing the quadratrix and the spiral we may conclude that when DESCARTES spoke about the measures of the motions, he meant their velocities. Indeed these measures have no exactly measurable "raport", as the comparison of the velocities involves the comparison of the lengths of straight and curved lines, in particular the ratio $AO : \widehat{AC}$ of the radius of a circle to the quarter arc of that circle. The argument returns some pages later in the *Géométrie* in connection with the tracing of curves by machines involving strings (see Section 7). There DESCARTES wrote:

> ... the proportion between straight lines and curves is not known and I even believe that it can never be known by man. (*G* p. 340).

Thus the separation between geometrical and non-geometrical curves, which was fundamental in DESCARTES' vision of geometry, rested ultimately on his conviction that proportions between curved and straight lengths cannot be found exactly. This, in fact, was an old doctrine, going back to ARISTOTLE.[18] The central role of the incomparability of straight and curved in DESCARTES' geometry ex-

[17] Often called the quadratrix of DINOSTRATUS, although the names of HIPPIAS and NICOMEDES are also connected with the curve. PAPPUS discusses the curve in his *Mathematical collections*. See I. BULMER THOMAS, "Dinostratus", *Dictionary of scientific biography* (ed. C. C. GILLISPIE, New York 1970ff.) vol. 4, pp. 103–105.

[18] See T. L. HEATH, *Mathematics in Aristotle* (Oxford 1949), pp. 140–142.

plains why the first rectifications of algebraic (*i.e.* for DESCARTES geometrical) curves[19] in the late 1650's were so revolutionary: they undermined a cornerstone of the edifice of DESCARTES' geometry.

5.4 These, then, were DESCARTES' arguments about tracing curves by continuous motion. A curve could be accepted as geometrical if there was an acceptable way of tracing it. Obviously, this criterion was connected with the use of the curve as a means of construction (see Sections 2.4 and 10); the intersections of the curve with other lines could be considered constructable only if the curve was actually traced. But in the *Géométrie* DESCARTES also accepted other ways of representing curves. I shall discuss these, and their relation to tracing by continuous motion, in the following sections.

6. Pointwise construction of curves

6.1 As we have seen (Section 2.2), DESCARTES solved the problem of PAPPUS by constructing arbitrarily many points on the locus. The method was as follows: first derive the equation of the locus in indeterminates x and y; then choose an arbitrary value η for y and form the equation in one unknown for the corresponding value or values of x; then solve this equation geometrically, that is construct the root or roots ξ; and finally construct the point or points with coordinates ξ,η on the locus. By repeating this process, taking other values for y, one can find arbitrarily many points on the locus. However, it is not at all obvious that this construction can be regarded as a satisfactory construction for the whole curve which forms the locus. It is not a construction by continuous motion. The process yields only a finite number of points on the curve. And generally it is not possible to use this construction for determining the intersection of the locus with a given curve (*cf.* Section 2.4).

In his discussion of the problem of PAPPUS in the first book of the *Géométrie* DESCARTES did not say whether this pointwise construction could be considered as a construction of the locus as a curve. In the case of the three-line and four-line problems, where the locus is a conic, DESCARTES did not stop after giving the pointwise construction; he also indicated how in each case the position of the vertices, axes, *latus rectum* and *latus transversum*[20] could be found, thus giving a representation of the locus curve by naming it (ellipse, hyperbola *etc.*) and giving its basic parameters (*G* pp. 327–332). However, later on in the second book DESCARTES returned to pointwise constructions of curves and stated that, in certain cases, curves constructed pointwise should be accepted in geometry.

[19] Rectifications of algebraic curves were found around 1658, independently, by VAN HEURAET, NEILE and FERMAT. See for instance M. E. BARON, *The origins of the infinitesimal calculus* (Oxford 1969), pp. 223–228.

[20] *Latus rectum* and *latus transversum* are the classical terms for certain line segments occurring in the defining properties of conic sections. If the vertex of the conic section is taken as origin and the X-axis is along the diameter, then the *latus rectum* a and the *latus transversum* b occur in the analytical formulas for the conics in the following way:

$y^2 = ax$ (parabola); $y^2 = ax - \frac{a}{b}x^2$ (ellipse); $y^2 = ax + \frac{a}{b}x^2$ (hyperbola).

He made this stament when dealing with the five-line locus problem

$$d_1 \cdot d_2 \cdot d_3 = d_4 \cdot d_5 \cdot a. \qquad (6;\,1)$$

DESCARTES solved the problem for two special cases, both involving four equidistant parallel lines and the fifth line perpendicular to the others. He took the position of the lines as in Figure 9a and 9b, respectively. In the first case (*G* pp. 335–339; *cf.* section 5.2 and note 14) he found that the locus was the "Cartesian parabola" which he had introduced earlier as the curve described by a combined motion of a ruler and a parabola. For the second case DESCARTES gave only a property of the locus (formulated in a rather obscure way[21]), from which at best a pointwise construction could be derived (*G* p. 339); he did not explain how that locus could be traced by a continuous motion. He then decided not to give any more details because he had already indicated in the first book how points on the locus could generally be constructed:

> As to the lines serving in the other cases, I shall not bother to distinguish them into different kinds, for I have not undertaken to say everything. And now

[21] DESCARTES' cryptic description of the curve is as follows: The curve is such that if all the straight lines applied consecutively to its diameter [*i.e.* the ordinates] are taken equal to those of a conic section, then the segments of the diameter between the vertex and these lines [*i.e.* the abscissae] have the same ratio to a given line as that line has to the segments of the diameter of the conic section to which these lines are applied consecutively. (*G* p. 339)

Following C. RABUEL, *Commentaires sur la Géométrie de M. Descartes* (Lyon 1730) p. 271, we may interpret the passage as follows. If we take the origin in the centre of the figure (*cf.* Figure 9b),

$$d_1 \cdot d_2 \cdot d_3 = d_4 \cdot d_5 \cdot a$$

leads to

$$x\left(y + \frac{3}{2}a\right)\left(y - \frac{3}{2}a\right) = \left(y + \frac{a}{2}\right)\left(y - \frac{a}{2}\right)a$$

as the equation for the required curve. Taking

$$w = \frac{1}{2}a^{-1}\left(y^2 - \frac{9}{4}a^2\right),$$

we find

$$w : a = a : (x - a).$$

If we now take the "vertex" in DESCARTES' text to be the point $V(x = a, y = 0)$, and draw the parabola

$$2aw = y^2 - \frac{9}{4}a^2$$

with w taken along the X-axis from V, then the required curve and the parabola are related in such a way that for points (x, y) and (z, y) on either curve with equal ordinates y, the abscissae $x - a$ and w (taken from V) satisfy

$$w : a = a : (x - a).$$

This corresponds to what DESCARTES says, but he does not specify that in this case the conic section is a parabola. If the conic section and the position of the vertex are given, DESCARTES' description implies a pointwise construction of the curve.

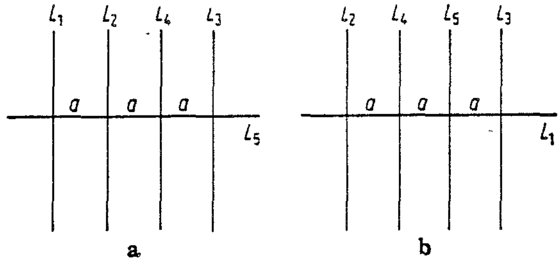

Fig. 9

that I have explained the way to find an infinity of points through which they pass, I think that I have sufficiently explained the way to describe them. (*G* p. 339)

Thus DESCARTES stated that pointwise constructions are sufficient to describe the curves.

6.2 DESCARTES said more about the acceptability of these constructions in the next section, the margin title of which is:

> Which are the curved lines that one describes by finding many of their points and that can be accepted in geometry. (*G* p. 340)

As the title indicates, DESCARTES accepted pointwise constructions under certain conditions as sufficient constructions for curves. As was the case in the tracing of curves by continuous motion, these conditions must exclude curves such as the spiral and the quadratrix. However, as DESCARTES said, there are pointwise constructions for these curves as well. DESCARTES probably had in mind here the following pointwise construction for the quadratrix[22] (see Figure 10). Divide arc *AC*

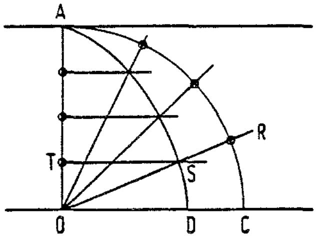

Fig. 10

in 2, 4, 8, 16 *etc.* parts (this can be done with ruler and compass) and do the same with the radius *OA*. Then draw radii such as *OR* to the points of division on *ARC* and draw horizontals such as *TS* through the points of division of *OA*. The intersections, such as *S*, of corresponding radii and horizontals are on the quadratrix. In this way arbitrarily many points on the quadratrix, lying arbitrarily close

[22] This pointwise construction follows immediately from the description of the quadratrix by continuous motion; see Note 17. T. L. HEATH mentions the construction in his *A history of Greek mathematics* (2 vols, Oxford 1921), vol. 1, p. 230, as a means to find points on the curve near *D* (Figure 9) and thus to approximate *D*. But he gives no reference to classical sources containing this construction.

to each other can be geometrically constructed. There is a similar construction for the spiral.

Since DESCARTES had to exclude these pointwise constructions, he had to explain the difference between these unacceptable constructions and the acceptable pointwise constructions of, for instance, the loci for the problem of PAPPUS. According to DESCARTES the difference lay in the fact that for curves such as the quadratrix the constructable points were special points. For the quadratrix as constructed above, they are the points with ordinates $\frac{k}{2^n} OA$; generally one finds only those points of the quadratrix which correspond to a division of the angle which is possible by Euclidean constructions. In the case of acceptable pointwise constructions every point is in principle constructable because the construction may start from any given value of one of the coordinates. DESCARTES explained this as follows:

> It is worthy of note that there is a great difference between this method of finding several points to trace a curved line, and that used for the spiral and similar curves. For with the latter one does not find indifferently all points of the required curve, but only those points which can be determined by a simpler measure than is required for the composition of the curve. Therefore, strictly speaking, one does not find any one of its points, that is, not one of those which are so properly points of the curve that they cannot be found except by means of it. On the other hand there is no point on the curves which are of use for the proposed problem [the PAPPUS problem] that could not occur among those which are determined by the method explained above. And because this method of tracing a curved line by finding a number of its points taken at random is only applicable to curves that can also be described by a regular and continuous motion, one may not exclude it entirely from geometry. (*G* pp. 339–340)

Thus DESCARTES stated firmly, but without any attempt at proof, that curves admitting a pointwise construction in which every point on them can, in principle, be constructed, can also be traced by continuous motion and are therefore geometrical. The passage suggests that DESCARTES saw a correspondence between the complete arbitrariness of the constructed points on the curve and the continuity of the motion.

6.3 After this passage DESCARTES repeatedly used pointwise construction to represent curves. For instance he introduced the famous *ovals*[23], which are curves with certain optical properties, by giving a pointwise construction. As an illustration of such a representation by pointwise constructions I summarize DESCARTES' introduction of the first oval (*G* p. 352, "this is how I describe them"):

[23] The ovals which DESCARTES discusses on pp. 352–368 of the *Géométrie* are curves whose surfaces of revolution provide shapes of lenses with the property that light rays coming from one point converge, after passing through the lens, to another point (and variants of this property). DESCARTES explains how these ovals can be constructed when the positions of the light source and the converging point, and the refractive index of the lens material, are given.

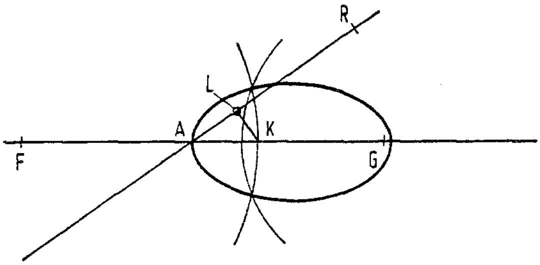

Fig. 11

Let two lines (see Figure 11) be given, intersecting in A at a given angle. A lies between the points F and G on the one line; the ratio of AF to AG is given. R lies on the other line, $AG = AR$. To construct points on the oval, take an arbitrary point K on AG. Draw a circle with centre F and radius FK. Draw KL perpendicular to AR. Draw a circle with centre G and radius RL. The two intersections of the two circles lie on the oval. By repeating this construction starting from other points K on AG, arbitrarily many points on the oval can be found. The construction yields a geometrical curve, because the choice of K is completely arbitrary.

7. Curve construction using string

7.1 The passage in the second book on the geometrical acceptability of curves given by pointwise constructions was followed by a passage about a third way of representing curves, namely tracing them with machines involving strings. The title in the margin of that section is:

> And which curves that one describes by means of a string can be accepted. (*G* p. 340)

DESCARTES then referred to his *Dioptrique*[24], in which he had given constructions by strings for the ellipse and the hyperbola. The construction for the ellipse is the well known "gardener's construction" (see Figure 12): A string is fixed in the points A and B. It is stretched by a tracing pin T which is moved around A and B, the strings being kept straight. It then traces an ellipse with foci A and B (*A.T.* *6*, p. 166).

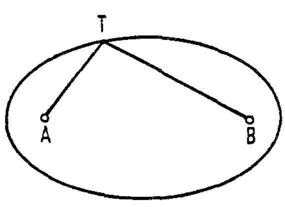

Fig. 12

[24] *La Dioptrique*, one of the three essays of the *Discours*; in DESCARTES' *Œuvres* (see Note 1), vol. 6, pp. 79–228.

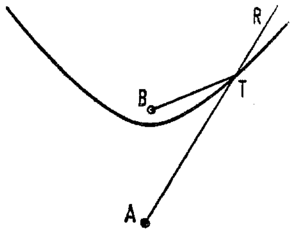

Fig. 13

For the hyperbola (see Figure 13) a ruler *AR* pivots at *A*; a string is fixed at *B* and at point *R* on the ruler. The string is stretched by a tracing pin *T* which is kept against the ruler. When the ruler is turned around *A* with *T* kept fixed to the ruler and *AT* stretched, *T* describes one arm of a hyperbola with foci *A* and *B* (*A.T.* 6, p. 176).

It should be noted that in the *Dioptrique* DESCARTES called this construction of the ellipse "rather rough and not very exact" (*A.T.*, 6, p. 166) but thought that it was a better means for understanding the nature of an ellipse than the section of a cone or a cylinder. In discussing the construction of the hyperbola DESCARTES pictures "a gardener who uses it to mark off the border of some flower bed" (*A.T.* 6, p. 176). Nevertheless, in the *Géométrie* DESCARTES accepted these constructions as genuinely geometrical representations of curves. This shows that he was more concerned that his constructions should be clear and comprehensible in principle than that they should be accurate in practice.

7.2 But constructions by strings could be used also to trace curves which DESCARTES did not accept as geometrical. DESCARTES mentioned this but did not give examples. He may have had in mind a method similar to the one which HUYGENS in 1650 suggested for tracing the spiral[25] (see Figure 14, which is HUYGENS' sketch): A ruler *AB* pivots in *B*. Around *B* there is a circular disk *EH*

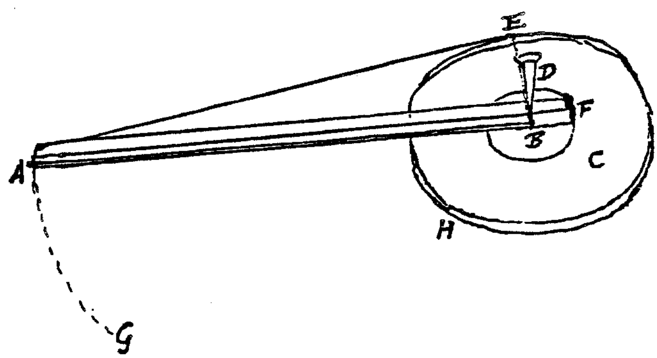

Fig. 14

[25] See C. HUYGENS, *Œuvres Complètes* (22 vols. The Hague 1888–1950) vol. **11**, p. 216; a note from 1650.

fixed to the plane. A string *EAD* is fixed to the rim of the disk in *E*, slung around a small pulley at *A* and led along the ruler to the centre. A drawing pin is fixed at the end *D* of the string. If the ruler is moved around counter-clockwise, the string winds up round the disk, the pin *D* is drawn along the ruler and describes an Archimedean spiral on the plane. DESCARTES was certainly able to devise machinery of this kind for tracing the quadratrix.

DESCARTES had to exclude this way of using string in tracing curves. He did so by excluding the cases where the string is partly curved and partly straight and where during the motion curved parts change into straight ones or vice versa. His reason for excluding these cases was, as I have discussed above (Section 5.3), his conviction that ratios between straight and curved lines cannot be given exactly. DESCARTES argued as follows:

> Nor should we reject the method in which a string or a loop of thread is used to determine the equality of or the difference between two or more straight lines which can be drawn from each point of the required curve to certain other points or towards certain other lines at certain angles. We have used this method in the Dioptrique to explain the ellipse and the hyperbola. It is true, though, that one cannot accept in geometry any lines which are like strings, that is, which are sometimes straight and sometimes curved, because the proportion between straight lines and curved lines is not known and I even believe that it can never be known by man, so one cannot conclude anything exact and certain from it. Nevertheless, because in these constructions one uses strings only to determine straight lines whose lengths are perfectly known, this should not be a reason for rejecting them. (*G* pp. 340–341)

7.3 Further on in the *Géométrie* DESCARTES used string constructions as an alternative to pointwise constructions to represent ovals. By way of illustration I summarise the string construction of the first oval (see Figure 15): *FE* is a ruler that pivots at *F*. A string is fixed at *E* on the ruler and at *G* on the axis *FAG*. It is slung around a pin *K* on the axis and it is kept straight by a tracing pin at *C* against the ruler. Thus the string is kept to *E–C–K–C–G*. Now the ruler is

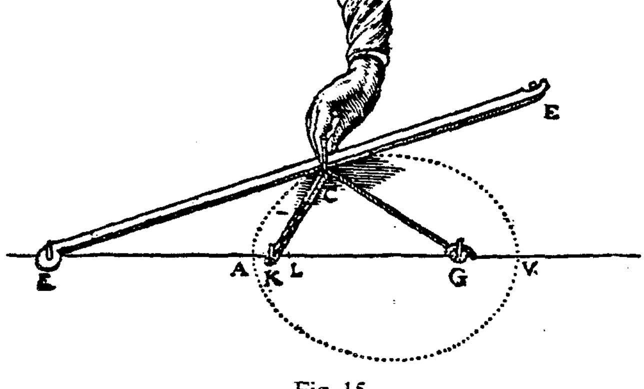

Fig. 15

turned around F and in that motion the tracing pin C traces the oval. The points F, A, K and G on the axis can be chosen such as to give the oval the required optical properties.

8. Equations of curves in the Géométrie

8.1 We have seen that DESCARTES used three different kinds of representation of curves: tracing machines, pointwise constructions and tracing machines involving strings. In each case further conditions (which exclude the transcendental curves) have to be satisfied if the resulting curves are to be acceptable in geometry. In the first and the third kind of representations these conditions have to do with the axiom of incommensurability of the straight and the curved, and in the second with the randomness of the constructible points on the curve. It is noteworthy that DESCARTES did not try to connect these two types of condition. In fact they relate to different aspects of curve tracing by continuous motion. Incommensurability of the straight and the curved relates to the condition that the combined motions which trace the curve regulate each other in a measurable way (*cf.* Section 4.1). The randomness of the constructable points relates to the continuity of the tracing motions (*cf.* Section 6.2).

We must now consider the role of equations as representations of curves: to what extent did DESCARTES consider the equation to be a sufficient representation of a curve? DESCARTES was convinced that the equation of a curve incorporates all information on its properties; He wrote:

> Now if one knows the relation that all points of a curved line bear to all points of a straight line in the way I have explained [*i.e.* as soon as the equation is known]. it is also easy to find their relation to all the other given points and lines; and subsequently to find the diameters, axes, centres and other lines or points to which each curve has some special relation, or a more simple relation than to others, and in that way to conceive various ways of describing the curves, and to choose the easiest. (*G* p. 341)

The passage suggests that finding the description of the curve from its equation still requires an effort, so the equation itself is not an appropriate representation of the curve.

This is in keeping with the fact that nowhere in the *Géométrie* did DESCARTES use an equation to introduce or represent a curve. In several cases he treated curves without giving their equations; in other cases he gave the equation almost casually in the course of his arguments. The solution of the problem of PAPPUS with five lines, four of which are parallel, equidistant and perpendicular to the fifth (*cf.* Section 6.1 and Note 21), was given in Book II by a prose description of a defining property of the locus. The description could have been translated into an equation, and would certainly have been more informative if it had been. Equations of the curves traced by the machine discussed in Section 5.1 were not given in the *Géométrie,* nor did DESCARTES present the equations for the ovals. The Cartesian parabola, so fundamental to the *Géométrie,* was introduced in Book II as the curve traced by the intersection of a parabola and a ruler. Its equation was given afterwards, and clearly not as a representation of the curve but as a means of proving that the curve solves the five-line locus problem (*G* p. 337), or as a means to determine its tangents (*G* p. 344). For readers to whom the description of the curve by ruler and parabola "seems difficult" DESCARTES added as an alternative representation a pointwise construction, but not the equation (*G* p. 407).

The conclusion from these facts must be that for DESCARTES the equation of a curve was primarily a tool and not a means of definition or representation. It was part of a whole collection of algebraic tools which in the *Géométrie* he showed to be useful for the study of geometrical problems. The most important use of the equation was in classifying curves into classes and in determining normals to curves. Here the equation must actually be written out. In many other cases DESCARTES could get through his calculations about problems without writing down the equation of the curve explicitly.

9. Geometrical curves

9.1 Within his programme for geometry, DESCARTES did not, and could not, simply state that geometrical curves are those which admit algebraic equations. But how did DESCARTES see the class of geometrical curves? Did he really consider this class to be the same as the class of curves admitting algebraic equations, and did he think that every such equation could occur as the equation for a geometrical curve? And was he aware of the extension of the class of curves which he decided to banish from geometry? I shall deal with these questions in this section.

DESCARTES stated firmly that all geometrical curves have equations. After explaining the curve-tracing machine discussed in Section 5.1 he wrote:

> I could give here several other ways of tracing and conceiving curved lines, which would be more and more complicated by degrees to infinity. But to understand the totality of all curves that are in nature and to distinguish and order them in certain classes, I do not know a better way than to say that all points of those that can be called geometrical, that is those which admit some precise and exact measure, necessarily have some relation to all points of a straight line, which can be expressed by some equation, the same equation for all points. (*G* p. 319)

He went on to explain how these equations can be found for the curves traced by the machines discussed in Section 5.2.

The converse question, namely whether all algebraic equations describe geometrical curves, is a much more difficult one and DESCARTES did not answer it explicitly. Taken in its strict sense the question is whether for every algebraic equation a tracing machine, or a combination of continuous motions in the sense explained in Section 5, can be found which describes the curve having that equation. DESCARTES did not deal explicitly with that question anywhere. However, it is such a fundamental question in the whole Cartesian programme of geometry that it seems very unlikely that DESCARTES was unaware of it. His silence on this question must be due to his inability to answer it. It is not surprising that DESCARTES could not answer it; a proof that the answer to the question is positive was found only in the 19th century.[26]

Implicitly, DESCARTES' answer to the question was positive. An equation of a curve implies a pointwise construction; one takes successive fixed values for one of the variables, say for y, and constructs geometrically the corresponding values for x as the roots of the resulting equation in x. DESCARTES was convinced that this could always be done. In the third book of the *Géométrie*, he showed that the roots of equations (in one unknown) up to the sixth degree can be found by the intersection of geometrically acceptable curves, and he claimed that the same can be done for equations of higher degree (*G* p. 413; *cf.* Section 3.4). This is how DESCARTES solved the problem of PAPPUS and he even claimed that every equation can arise as the equation for the locus in a problem of PAPPUS in some number of lines. Hence algebraic equations yield pointwise constructions for the curves they describe and these constructions are acceptable in geometry because one has a completely free choice of starting point for the construction of the points (namely, the choice of the y; see Section 6.2). Moreover, DESCARTES claimed that such pointwise constructions of curves are equivalent to tracing by continuous motion, and hence, implicitly, he claimed that all algebraic curves are geometrical in the sense of being traceable by continuous motion.

It is clear that the crucial step in this argument is the equivalence of pointwise constructions and constructions by continuous motion. Through this equivalence, curves described by equations acquire a status in geometry equal to that of curves traced by continuous motion. But we have seen that DESCARTES' arguments for the equivalence were weak (Section 6.2). He must therefore have had strong reasons for incorporating the equivalence in his geometry. In Section 10 I shall say something more about his reasons and about some conclusions which may be drawn from these concerning the formation of DESCARTES' ideas in geometry in the years before the publication of the *Géométrie*.

9.2 It is noteworthy that DESCARTES' basic argument in rejecting the transcendental curves was the incommensurability of straight and curved lines. This argument applied only to transcendental curves depending on the quadrature of the circle, such as the quadratrix and the spiral, which were the only ones DESCARTES mentioned explicitly. But how many transcendental curves did he know, and, what is more important, did he know curves depending on logarithmic relations, and which arguments did he use to exclude these from geometry?

The idea that curves generated by motions not mutually subordinate are to be rejected from geometry occurred earlier in DESCARTES' letter to BEECKMAN of 26 March 1619[27] (see Section 10.2); he mentioned the quadratrix as an example. By that time he had also hit upon the logarithmic relation in connection with the problem "de reditu redituum" (income on income, *i.e.* compound interest). He considered two axes, one divided in equal parts, the other in proportional

[26] A. B. KEMPE, "On a general method of describing plane curves of the n^{th} degree by linkwork", *Proc. London Math. Soc.* **7** (1876) pp. 213–216.

[27] *A.T.* **10**, pp. 154–158.

parts. In one study[28] the idea of a curve representing the relation between corresponding parts on each axis seemed to underlie his argument. In another study[29] he actually drew that curve, called it the *linea proportionum* and recognized it as belonging to the same class as the quadratrix:

> The line of proportions is to be put in the same class as the quadratrix for it is generated by two motions, one circular and one straight, which are not subject to each other. (*A.T. 10* pp. 222—223)

It is not clear how DESCARTES got the (wrong) idea that the line of proportions is generated by a combination of a straight motion and a circular motion; perhaps he was thinking only of the quadratrix when he mentioned these motions. The figure in the published text suggests that he had no clear idea about the form of the curve.

There is no evidence that DESCARTES before 1637 actively studied transcendental curves other than the quadratrix and the spiral. But shortly after the publication of the *Géométrie* we find DESCARTES discussing the logarithmic spiral in a letter to MERSENNE[30] and another logarithmic curve in connection with one of the problems set by DEBEAUNE[31]. Around this time he also studied the cycloid.[32] In the case of DEBEAUNE'S problem DESCARTES did not explicitly recognize that the curve was connected with logarithms, although he may well have seen the link. He worked out two motions which together describe that curve and he found that these two motions

> are so incommensurable that they cannot be regulated by each other in an exact way; and therefore that this line belongs to those which I have rejected from my Geometry as being only mechanical. (*A.T. 2*, p. 517)

From the little information we have, then, it seems that before the publication of the *Géométrie* DESCARTES may have had the idea that by rejecting the quadratrix, the spiral "and the like" (*G* p. 317) he was not really rejecting any interesting curves but only those originating from motions which involve the relation between curved and straight lines. Shortly after 1637 he came upon several other "non-geometrical" curves; some of them obviously did not depend on the relation between curved and straight, and some of them were indeed quite interesting.

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28 *A.T.* **10**, pp. 77–78; the study dates from before December 1618.

29 *A.T.* **10**, p. 222–223, from 1619–1621.

30 DESCARTES to MERSENNE 12-9-1638; *A.T.* **2**, pp. 352–362, in particular p. 360.

31 For DESCARTES' solution of DEBEAUNE'S problem see his letter to DEBEAUNE of 20-2-1639; *A.T.* **2**, pp. 510–519, and C. J. SCRIBA, "Zur Lösung des 2. Debeauneschen Problems durch Descartes", *Arch. Hist. Ex. Sci.* **1** (1960–1962), pp. 406–419.

32 MERSENNE mentioned the cycloid and ROBERVAL'S studies on the quadrature in his letter to DESCARTES of 28-4-1638 (*A.T.* **2** pp. 116–122). In his answer of 27-5-1638 (*A.T.* **2** pp. 134–153) DESCARTES said that he had never thought of the curve before (p. 135). He discussed the curve and its properties in several subsequent letters to MERSENNE.

10. Once more: Descartes' programme

10.1 We have seen that in DESCARTES' programme for geometry as expounded in the *Géométrie* there was a contradiction in the criteria for the geometrical acceptability of curves. On the one hand DESCARTES claimed that he accepted curves as geometrical only if they could be traced by certain continuous motions. This requirement was to ensure that intersections with other curves could be found, and it was induced by the use of the curve as means of construction in geometry. On the other hand DESCARTES stated that, under certain conditions, curves represented by pointwise constructions were truly geometrical. Pointwise constructions were related to curve equations in the sense that an equation for a curve directly implied its pointwise construction. Pointwise construction was used primarily for curves that occurred as solutions to locus problems.

The link between the two criteria is DESCARTES' argument that pointwise constructible curves can be traced by continuous motion. We have seen that that argument, and hence also the link, is very weak (*cf.* Section 9.1). This makes the criteria themselves all the more interesting. DESCARTES needed both criteria; he could not restrict his attention to one of them. On the one hand he could not keep strictly to the criterion of continuous motion because he wished to regard all loci for problems of PAPPUS as geometrical curves. In studying the five-line locus he came upon such a locus for which he could not, or would not, give a construction by continuous motion (*cf.* Section 6.1 and Note 21). Certainly he could not prove in general that all curves with algebraic equations can be traced by continuous motion. On the other hand, DESCARTES also could not simply state that a curve is geometrical when it can be constructed pointwise. Pointwise construction presupposes means of construction and in DESCARTES' system these means can only be curves traced by continuous motion. Hence if he would adopt pointwise constructibility as a criterion for a curve to be geometrical, DESCARTES would still have to show that the necessary curves for performing this pointwise construction can be traced by continuous motion. This he also could not do in general.

Why then did DESCARTES not cut this Gordian knot in the most obvious way, namely by defining geometrical curves as those which admit algebraic equations? Why did he not simply state that all such curves are acceptable means of construction and that the degrees of their equations determine their order of simplicity? That principle would have removed the contradictions mentioned above. But DESCARTES did not accept the principle. In order to understand why we have to look at the development of DESCARTES' ideas on geometry.

10.2 Even in 1619 DESCARTES had a programme for his geometrical research. We know this from what he wrote in his letter to BEECKMAN of 26 March 1619:

> I hope to prove (–) that certain problems can be solved with straight and circular lines only; that others can only be solved with other curved lines which originate in one single motion, and which therefore can be traced by the new compasses, which I do not think are less certain and geometrical than the ordinary ones with which circles are drawn; and that finally other

(problems) can be solved only by curved lines originating from different motions that are not subordinate to each other and that certainly are only imaginary (imaginariae); such a curve is the well known quadratrix. I think that one cannot imagine problems that cannot be solved by at least these lines; but I hope to be able to demonstrate which questions can be solved by the first or the second method but not by the third; so that in geometry nothing remains to be found. (*A.T. 10* p. 157)

The passage shows that by 1619 DESCARTES had formed the conception of geometry which he adhered to all his life. He did not consider geometry primarily as an axiomatic, deductively ordered corpus of knowledge about points, lines *etc.*, but as the science of solving geometrical problems. Once all such problems could be solved there was nothing further to do in geometry.

The compasses DESCARTES had in mind here were two linkage machines[33] for solving problems or tracing curves. One of these we have met in the *Géométrie*; it is the machine illustrated in Figure 3. It was designed to find mean proportionals between two given lines segments. The similarity of the triangles involved immediately yields

$$YB : YC = YC : YD = YD : YE = YE : YF = YF : YG = YG : YH. \quad (10; 1)$$

Hence to find, for instance, two mean proportionals between YA and some given line λ, one opens the compass until $YE = \lambda$; YC and YD are then the required proportionals. Alternatively, one can first trace the curve AD by continuously opening the compass. Then when λ is given, one intersects AD with the circle with diameter $YE = \lambda$. The point of intersection is D; YC (the abscissa of D) and YB are the required proportionals. In the same way the curve AF traced by F serves to determine four mean proportionals.

The other compass was not mentioned in the *Géométrie*. Just as the first compass was based on a very simple geometric device to find proportionals, this one employed a very simple device for dividing an angle into any given number of equal parts. For the trisection of the angle the machine is as in Figure 16. There are four rulers AB, AC, AD and AE, all pivoted at A. On each of them, at fixed

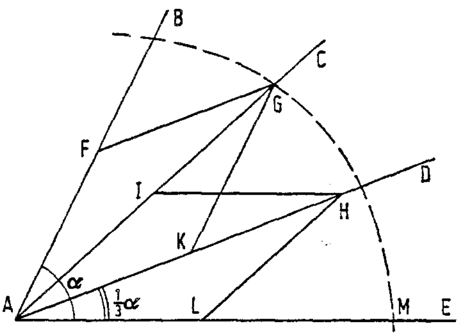

Fig. 16

[33] *A.T.* **10**, p. 234 and p. 240.

and equal distances from A, there are adjusted links FG, IH, KG, and LH, all equal in length. FG and KG are joined such that G can move along ruler AC; IH and LH are joined such that H can move along ruler AD. If a given angle α has to be trisected the compass is opened until $\sphericalangle BAE$ is equal to α; then $\sphericalangle BAC$ is equal to $\frac{1}{3}\alpha$. Alternatively (see Figure 17) let the point G trace a curve MG by opening the compass while leaving ruler AE fixed. If angle α has to be trisected one draws the angle at A with one arm along AE, chooses a point F' on the other arm such that $AF' = AF$ and draws a circle around F' with radius FG. Through the intersection G' of the circle and the curve one draws AG'. Then $\sphericalangle F'AG' = \frac{1}{3} \sphericalangle F'AE = \frac{1}{3}\alpha$.

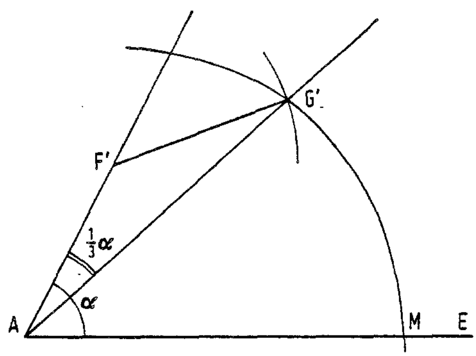

Fig. 17

Around this time DESCARTES had experimented with other compasses similar to these, and in his letter to BEECKMAN he mentioned that he had devised compasses for the construction of all types of cubic equations. It was nothing new to use instruments such as DESCARTES' compasses for various special constructions; they were used in classical mathematics, and PAPPUS' *Collections* mention several such machines. Also the idea of considering the curves traced by these machines was not new; the conchoid of NICOMEDES, for instance, is a curve traced by a special kind of instrument for certain constructions, the so-called neusis constructions[34]. It should be noted, however, that in formulating his programme DESCARTES considered the curves themselves rather than the compasses as the means of construction.

10.3 If we compare the programme which DESCARTES outlined in his letter of 1619 to BEECKMAN with the programme of the *Géométrie* we find significant differences. These differences concern the role of algebra and pointwise constructions. We see that by 1619 DESCARTES' programme contained the following ideas: Geometry is the science of solving or constructing geometrical problems. Construction by means more complex than the circle and the straight line (the compass

[34] See the chapter "On the problems known as neuseis" in *The works of Archimedes* (ed. T. L. HEATH, Dover edition), pp. c-cxxii, in particular p. cvii. For the conchoid see also Figure 5.

and the ruler) need not be less geometrical. Curves that can be traced by one single continuous motion such as that provided by the compasses are acceptable means for geometrical construction. There are also problems that can be constructed only by curves traced by a combination of motions that are not subordinate to each other. Such curves are "imaginary"; an example is the quadratrix. All problems can be solved with such curves, but DESCARTES wishes to classify the problems that can be solved by acceptable geometrical curves. These elements of the programme of 1619 were all to be found in the *Géométrie* as well. But several points of the programme of the *Géométrie* were still lacking in 1619. Most notable was the absence of algebra. It is true that DESCARTES envisaged geometrical solutions of algebraic equations by means of compasses, but algebra did not yet play a role in the classification of geometrical means of construction according to their simplicity, nor in a method for finding the simplest possible constructions. It seems likely that by 1619 DESCARTES envisaged classifying the constructing curves according to the simplicity of the compasses used to draw them. A trace of this is found in the *Géométrie* where DESCARTES, in discussing the compass for mean proportionals, said

> I do not believe that there could be an easier method to find as many mean proportionals as one wishes, nor one whose proof would be more evident, than to use the curved lines traced by the instrument *XYZ* ... . (*G* p. 370)

But DESCARTES went on to say that the curves traced by that instrument are of a higher class than necessary and that therefore they should not be used in a truly geometrical solution of the problem to find mean proportionals (*G* p. 371). We may conclude that by 1637 DESCARTES' algebraic criterion for the simplicity of curves, namely their class, defined via the degree of the equation, had replaced, and indeed was in conflict with an earlier criterion for simplicity, namely the simplicity of the compass and of the resulting proof of the construction.

The other element not in the programme of 1619 concerned loci and pointwise constructions. In 1619 DESCARTES did not wish to introduce new curves in geometry for purposes other than constructions. The problems to be constructed had one solution or a finite number of solutions. In 1619 DESCARTES did not consider the case where the solutions are infinite in number forming a locus which, by the nature of the process of solving the problem, is constructed pointwise. Hence he was not faced with the problem of whether or not such curves should be accepted in geometry and according to which criteria.

10.4 Evidently, therefore, DESCARTES' programme of geometry changed between 1619 and 1637. The stages of this change are in fact fairly well known.[35]

[35] See G. MILHAUD, *Descartes savant* (Paris 1921), Chs. 1, 3 and 6. JOHN SCHUSTER has recently published a detailed study of the development of DESCARTES' ideas about universal mathematics in relation to his metaphysics and his programmes for philosophy; see J. A. SCHUSTER, "Descartes' mathesis unversalis; 1619–1628" in *Descartes; philosophy, mathematics and physics* (ed. S. W. GAUKROGER, Hassocks (Sussex) (Harvester Press), 1980), pp. 41–96. He argues that one of the reasons why DESCARTES after 1628 abandoned his programme for universal mathematics as formulated in the *Regulae ad directionem ingenii* was that he encountered difficulties in working out the geometrical

It was probably shortly after his letter to BEECKMAN of March 1619 and before November 1620 that DESCARTES studied the construction of problems through the intersection of conics and found the solution of all equations of third and fourth degree through the intersection of a parabola and a circle. This must have given him the idea that the conics are the class of constructing curves immediately following the circle and the straight line.

This idea may have led him to search for a construction of all equations of fifth and sixth degrees through the intersections of a circle with one special curve more complicated than the conics. He succeeded in finding this construction; the curve is the Cartesian parabola; the construction is explained at the end of Book 3 of the *Géométrie* (*G* pp. 402–411; *cf.* Section 3.4). But we do not know the date of this discovery[36]. These results must have induced DESCARTES to consider that the degree of the equation of the curve, rather than the simplicity of the tracing machine, was the criterion for the geometrical simplicity of curves used in constructions.

The other new aspect, loci and pointwise constructions, probably was incorporated into DESCARTES' programme in 1631 when GOLIUS suggested that he might try his hand at the problem of PAPPUS. We know that DESCARTES solved the problem in a number of weeks and that the solution appearing in the *Géométrie* is essentially the one he sent to GOLIUS in January 1632[37]. This study must have turned DESCARTES' attention more to algebra, to the equation as embodying all the information about the curve, to the need to incorporate all curves admitting algebraic equations in geometry and to the need to admit pointwise construction for curves.

10.5 However, more important than the chronology of the changes in DESCARTES' geometrical ideas is the fact that these changes explain the basic contradiction in DESCARTES' programme in the *Géométrie*. The programme of 1619

theory of equations. This fits in well with the chronology of DESCARTES' changing ideas about geometry. Also SCHUSTER's study provides an illustration of DESCARTES' attitude to contradiction and failure in his programmes (*i.e.* the programme of universal mathematics) which seems to correspond well with what I find about DESCARTES' attitude toward the failure of his earlier programme for geometry.

[36] It seems likely that this discovery was made later than 1628, for we have a note by BEECKMAN about his interview with DESCARTES in October 1628. DESCARTES had explained to BEECKMAN the construction of the roots of any equation of the fourth degree by the intersection of a circle and a parabola. BEECKMAN noted that "M. DESCARTES made so much of this invention that he confessed never to have found anything superior himself and even that nobody else had ever found anything better" (*A.T.* **10**, p. 346). It is not likely that DESCARTES would have made this kind of comment if by that time he had already known the general construction of the roots of equations of 5^{th} and 6^{th} degrees.

[37] See DESCARTES' letter to GOLIUS of January 1632, *A.T.* **1**, pp. 232–236. DESCARTES refers to an "écrit" sent earlier to GOLIUS; this "écrit" has been lost. In the letter DESCARTES adds a definition of classes ("genres") of curves. The definition is not clear and the terminology is quite different from that used in the *Géométrie*. Still, from the further indications in the letter, it seems likely that DESCARTES had by that time found the essential elements of the solution of the problem of PAPPUS as it appeared in the *Géométrie* and that the "écrit" sent to GOLIUS contained a condensed version of this solution.

may have been impracticable, but it was consistent. It provided a demarcation between geometrical and non-geometrical constructions, and the criterion used in that demarcation was a geometrical one: the constructions had to be performed with machines that were generalizations of the ruler and the compass.

In the programme of 1637 algebra had become dominant. DESCARTES now classified curves according to the degree of their equations and a large part of the *Géométrie* (especially the third book) is devoted to algebraical techniques relating to the roots and coefficients of equations (reduction of equations, sign rule, removal of terms from the equation, change of negative roots into positive ones, *etc.*). But despite all this algebra, what had remained was DESCARTES' conception that geometry was the science of solving geometrical problems by the construction of points through the intersection of curves. Therefore the main aim of the third book was the construction of roots of equations through the intersection of curves.

This aim determined the structure of the third book and the nature of the algebraic techniques presented in it. The reduction of equations to other equations of lower degree was necessary for finding the construction by the simplest possible constructing curves. The techniques relating to the roots and coefficients of the equation served to reduce the equations to standard forms, for which DESCARTES then gave standard constructions. For equations of third and fourth degrees this was the construction by the intersection of circle and parabola; for equations of fifth and sixth degrees the construction by the intersection of circle and Cartesian parabola.

Thus, although algebra occupied a dominant position in DESCARTES' programme of 1637, it was the geometrical aim of the work that determined its structure and provided the motivation.

10.6 We now have the answer to the question raised in Section 10.1, namely why DESCARTES kept the criterion of tracing by continuous motion for the geometrical curves, and why he did not simply define geometric curves as those which have algebraic equations. As we have seen, the whole structure of his *Géométrie* depended on the conception of construction by the intersection of geometrical curves. For DESCARTES, these intersections were actually found or constructed only if the curves could be traced by continuous motion. In that case one can conceive clearly and distinctly that the intersections are found. If he were to renounce his criterion of tracing by continuous motion and at the same time keep to his programme of construction by the intersection of curves, he would have to state as an axiom that for all curves having an algebraic equation the intersections are given or constructible.

It is evident that DESCARTES could not do this. An axiom which states that the intersections of curves are constructible is by no means clearly and distinctly evident, so it would not satisfy DESCARTES' criterion for accepting a statement as a basis for further argument.

Moreover, by adopting this approach DESCARTES could no longer claim that he was doing geometry; he would be doing some kind of algebra. But that would mean giving up the principal aim of his work; to bring order into the science of geometry.

Finally, the whole structure of the *Géométrie*, which was based on finding the simplest constructing curves for a given problem, would lose much of its meaning. If the intersections of all algebraic curves are by axiom constructible, there is no evident reason for finding the simplest curves for a given problem, and hence there is not much point in finding constructions for roots of equations. The roots of an equation $x^n + ax^{n-1} + \ldots = 0$ are the intersections of the curve $y = x^n + ax^{n-1} + \ldots$ with the straight line $y = 0$; thus they are already given as intersections of curves with algebraic equations.

We see that DESCARTES could not give up his definition of geometrical curves by continuous motion because then he would have lost the claim of doing geometry and hence the rationale of the whole structure of his work would have been destroyed.

11. Conclusion

11.1 As I hope I have shown, the representation of curves is the key to understanding the structure of DESCARTES' *Géométrie* and its underlying programme. Although there were contradictions in the structure and the programme, there was an underlying unity of vision. DESCARTES had this vision as early as 1619, but it did not find its clear expression until the *Géométrie* of 1637. According to this vision geometry can and should be structured, and the bewildering jumble of problems, methods and solutions, in which it is impossible to know where the problems end and the solutions begin, can and should be cleared up. DESCARTES' view, in short, was that geometry concerns a surveyable, "orderable" collection of well defined problems, well defined also in the sense that there are clear criteria of adequacy for their solutions. DESCARTES left it to his successors to work out the programme, to find its limitations and to come to terms with its contradictions.

Appendix

On the curves that occur as loci of problems of PAPPUS

In the passage quoted in Section 2.3 DESCARTES asserts that every curve can arise as the locus of some problem of PAPPUS. In this Appendix I shall discuss the possible interpretations of this assertion and show that in each case the assertion is wrong. First of all I shall derive some properties of polynomials occurring in the equations of loci of problems of PAPPUS.

Let V_n be the $(6n + 1)$-dimensional real vector space consisting of vectors

$$v = (a_1, b_1, c_1, a_2, b_2, c_2, \ldots, a_{2n}, b_{2n}, c_{2n}, \alpha).$$

Consider the mapping P_n defined by

$$P_n(v)(x, y) = \prod_{i=1}^{n} (a_i x + b_i y + c_i) - \alpha \prod_{i=n+1}^{2n} (a_i x + b_i y + c_i).$$

P_n maps V_n into the space of polynomials in two variables with real coefficients.

If none of the factors $a_i x + b_i y + c_i$ is constant (*i.e.* if for all i, $a_i \neq 0$ or $b_i \neq 0$), then

$$P_n(v)(x, y) = 0$$

is the equation of the locus of a problem of PAPPUS in $2n$ lines. Let these lines be called L_i; they have as equations

$$a_i x + b_i y + c_i = 0.$$

Taking into account the possibility that some of the factors may be constant, we see that

$$P_n(v)(x, y) = 0$$

is the equation of the locus of the following generalized problem of PAPPUS:

> Given $k + l$ lines, with $k \leqq n$ and $l \leqq n$, find the locus of points in the plane such that the ratio of the product of their (oriented) distances to the first k lines and the product of their (oriented) distances to the last l lines is constant.

This is a natural generalization of the original problem of PAPPUS. All $2m$-line and $(2m - 1)$-line problems with $m \leqq n$ occur among the $P_n(v)(x, y)$. I shall call a polynomial that can be written as $P_n(v)$ for some n and some $v \in V_n$, a *Pappus polynomial*. PAPPUS polynomials are those polynomials which can be written as the difference between two polynomials each of which can be decomposed into linear factors. DESCARTES' assertion therefore is about whether or not every polynomial can be written as such a difference.

I shall need a number of properties of PAPPUS polynomials, derived in the following Theorems 1–3.

Theorem 1. Let $v \in V_n$ and let the lines L_i corresponding to the nonconstant factors $a_i x + b_i y + c_i$ be not all parallel to each other. Then there are points (x, y) in the plane such that $P_n(v)(x, y) = 0$.

Proof. Because not all the lines are parallel, there is a pair of lines L_i, L_j with $i \leq n$ and $j > n$, which intersect each other. Let (x_0, y_0) be their point of intersection. Then

$$a_i x_0 + b_i y_0 + c_i = a_j x_0 + b_j y_0 + c_j = 0$$

and therefore

$$P_n(v)(x_0, y_0) = 0.$$

If we are dealing with a proper $2n$-line problem, *i.e.* if all the factors in $P_n(v)$ are non-constant, the degree of $P_n(v)$ will generally be equal to n. But because $P_n(v)$ is the difference of two polynomials it may happen that the terms of high degree cancel each other, so that a $2n$-line problem may lead to an equation of degree less than n. An example is the 8-line problem

$$(y + 1)(y - 1)(x + 1)(x - 1) - (y + 2)(y - 2)(x + 2)(x - 2) = 0$$

which yields a quadratic equation

$$x^2 + y^2 = 5.$$

Conversely, this means that if we are to find all Pappus polynomials of degree n, we may not restrict attention to all $P_n(v)$, because such polynomials may also arise as $P_m(v)$ for some $m > n$. This possibility will be important in interpreting and checking Descartes' assertion. In particular we shall need the following theorem about the degree of $P_n(v)$:

Theorem 2. Let $v \in V_n$ such that all factors $a_i x + b_i y + c_i$ in $P_n(v)$ are non-constant and such that not all the lines L_i are parallel. Then

$$\frac{n}{2} \leqq \text{degree } P_n(v) \leqq n.$$

Proof. It is obvious that degree $P_n(v) \leqq n$. To prove the other inequality I call $L_1, \ldots, L_n$ the first set of lines and $L_{n+1}, \ldots, L_{2n}$ the second set of lines, and I consider first the case where the two sets have no line in common. If there is a line in the first set which is not parallel to any of the lines in the second set, I take that line as L_1 and proceed by taking $k = 0$ in the argument below. If there is no such line in the first set, I choose (renumbering if necessary) $L_1, \ldots, L_k$ from the first set of lines and $L_{n+1}, \ldots, L_{n+k}$ from the second set such that

$$L_1 \,//\, L_2 \,//\, \ldots \,//\, L_k \,//\, L_{n+1} \,//\, \ldots \,//\, L_{n+k},$$

and such that either in the first set of lines or in the second set there are no more lines parallel to L_1. I assume that in the second set of of lines there are no more than k lines parallel to L_1 (in the other case I can switch the sets). Because not all the lines are parallel there is such a set of lines $L_1, \ldots, L_k, L_{n+1}, \ldots, L_{n+k}$ with

$$0 \leqq k \leqq \frac{n}{2},$$

because if k were $> \frac{n}{2}$, a smaller set with the same properties could be chosen from the remaining lines. If $b_1 \neq 0$, I substitute

$$y = \frac{-a_1 x - c_1}{b_1}$$

in

$$P_n(v)\,(x, y) = \prod_{i=1}^{n} (a_i x + b_i y + c_i) - \alpha \prod_{i=n+1}^{2n} (a_i x + b_i y + c_i).$$

The first product then becomes 0 (because $a_1 x + b_1 y + c_1 = 0$), and in the second product the factors become

$$\frac{a_i b_1 - a_1 b_i}{b_1} x + \frac{b_1 c_i - b_i c_1}{b_1}.$$

For $n + 1 \leqq i \leqq n + k$, the lines L_i are parallel to L_1 but not equal to L_1; hence in that case

$$a_i b_1 - a_1 b_i = 0 \quad \text{and} \quad b_1 c_i - b_i c_1 \neq 0,$$

so the first k factors of the second product become constants $e_i \neq 0$. The lines

$L_{n+k+1}, \ldots, L_{2n}$ are not parallel to L_1, hence for $n+k+1 \leqq i \leqq 2n$

$$a_i b_1 - a_1 b_i \neq 0,$$

so the last $n-k$ factors of the second product can be written as

$$f_i x + g_i, \quad f_i \neq 0.$$

Hence

$$P_n(v)\left(x, \frac{-a_1 x - c_1}{b_1}\right) = -\alpha \prod_{i=n+1}^{n+k} e_i \prod_{i=n+k+1}^{2n} (f_i x + g_i),$$

a polynomial in x of degree $n-k$. But this means that

$$\text{degree } P_n(v)\,(x, y) \geqq n - k.$$

Also

$$n - k \geqq \frac{n}{2},$$

so

$$\text{degree } P_n(v)\,(x, y) \geqq \frac{n}{2}.$$

If $b_1 = 0$, it can be proved analogously $\left(\text{substitution } x = \frac{-c_1}{a_1}\right)$ that $P_n(v)\left(\frac{-c_1}{b_1}, y\right)$ is a polynomial in y of degree $n-k$, so that also in that case

$$\text{degree } P_n(v)\,(x, y) \geqq n - k \geqq \frac{n}{2}$$

Finally, if some of the lines of the first and second sets coincide, let these lines be $L_1, \ldots, L_d$, that is, $L_1 = L_{n+1}$, $L_2 = L_{n+2}, \ldots, L_d = L_{n+d}$. Then, for $1 \leqq i \leqq d$,

$$a_i x + b_i y + c_i = \delta_i (a_{n+i} x + b_{n+i} y + c_{n+i}), \quad \delta_i \neq 0.$$

We then have

$$P_n(v)\,(x, y) = \prod_{i=1}^{d} (a_i x + b_i y + c_i) \cdot$$

$$\cdot \left\{ \prod_{i=d+1}^{n} (a_i x + b_i y + c_i) - \alpha \prod_{i=1}^{d} \delta_i^{-1} \prod_{i=n+d+1}^{2n} (a_i x + b_i y + c_i) \right\}.$$

The factor between brackets is a PAPPUS polynomial $P_{n-d}(w)$ for which the first and second sets of lines have no lines in common. Hence its degree is $\geqq \frac{n-d}{2}$, so

$$\text{degree } P_n(v) \geqq d + \frac{n-d}{2} \geqq \frac{n}{2}.$$

This completes the proof of Theorem 2.

DESCARTES noted that a case where all lines are parallel is exceptional. Theorems 1 and 2 do not apply in that case. I shall therefore discuss it separately.

Let $v \in V_n$ be such that none of the factors $a_i x + b_i y + c_i$ is constant and that the $2n$ lines L_i are all parallel. Then the equations of the lines L_i can all be written as

$$\alpha_i z + \beta_i = 0$$

with

$$z = a_1 x + b_1 y.$$

Hence

$$P_n(v)\,(x, y) = \prod_{i=1}^{n} (\alpha_i z + \beta_i) - \alpha \prod_{i=n+1}^{2n} (\alpha_i z + \beta_i) = F(z),$$

where $F(z)$ is a polynomial in z formed as the difference between two n^{th} degree polynomials in z, each having n real roots. But it may happen that $F(z)$ itself has no real roots or that its degree is $< \frac{n}{2}$. Hence the analogues of Theorems 1 and 2 do not apply in this case. We note that in this case of $2n$ parallel lines, the surface in R^3

$$w = P_n(v)\,(x, y) = F(z)$$

is a ruled surface.

As to the degree of F we prove:

Theorem 3. Let $v \in V_n$ be such that none of the factors $a_i x + b_i y + c_i$ is constant and such that the $2n$ lines L_i are all parallel. If $P_n(v)$ has degree $m < n$, then there is a $w \in V_m$, such that

$$P_n(v) = P_m(w)$$

and the lines corresponding to w are all parallel.

(In other words, if a problem of PAPPUS in $2n$ parallel lines yields an equation of degree m, then there is a problem of PAPPUS in $2m$ parallel lines which yields the same equation.)

Proof. Because

$$P_n(v)\,(x, y) = \prod_{i=1}^{n} (\alpha_i z + \beta_i) - \alpha \prod_{i=n+1}^{2n} (\alpha_i z + \beta_i) = F(z),$$

we have to prove that every polynomial $F(z)$ of m^{th} degree in one variable can be written as the difference between two polynomials $G(z)$ and $H(z)$ of m^{th} degree, each having only real roots. Consider a segment $[a, b]$, and let $|F(z)| < K$ for $z \in [a, b]$. Choose points in $[a, b]$ as follows:

$$a = d_0 < c_1 < d_1 < c_2 < d_2 < \ldots < c_{m-1} < d_{m-1} < c_m < d_m = b$$

and consider

$$G(z) = \prod_{i=1}^{m} (z - c_i).$$

Clearly $G(d_i) \neq 0$, and between each pair d_i, d_{i+1}, G has a change of sign. Choose α such that $|\alpha G(d_i)| > K$ for all d_i, and consider the polynomial of m^{th} degree

$$H(z) = F(z) + \alpha G(z).$$

Because $|F(d_i)| < K$ and $|\alpha G(d_i)| > K$, $H(d_i)$ has the same sign as $G(d_i)$; thus H has a change of sign between each pair d_i, d_{i+1}. Therefore all the roots of H are real. The required decomposition is therefore

$$F(z) = H(z) - \alpha G(z).$$

Corollary of Theorems 2 and 3. If $F(x, y)$ is a PAPPUS polynomial and degree $F = k$, then there is an $m \leqq 2k$ and a $v \in V_m$ such that $F = P_m(v)$.

I now return to the interpretation of DESCARTES' assertion. If it is taken to mean that every polynomial in two variables is a PAPPUS polynomial (and the first lines of DESCARTES' assertion could be read thus), then the assertion is quite easily proved wrong. A counter-example is

$$F(x, y) = x^2 + y^2 + 1,$$

because there are no real points (x, y) satisfying $F(x, y) = 0$; also $w = F(x, y)$ is not a ruled surface in R^3, so by Theorem 1 and the remark preceding Theorem 3, $F(x, y)$ cannot occur as $P_n(v)$ for any n and $v \in V_n$. There is no evidence that DESCARTES was aware of this complication in connection with polynomials without real roots. But, in view of the later part of the passage we are discussing, where DESCARTES speaks about curves, we must conclude that he did not have polynomials in mind but rather polynomial equations of curves, *i.e.* polynomials $F(x,y)$ such that the set $\{(x, y) \mid F(x, y) = 0\}$ is not empty.

Even with this restriction the assertion is wrong. I shall prove this with a dimensional argument. In particular:

Theorem 4. If $n > 21$, there are polynomials $F(x, y)$ of degree n, such that $F(x, y) = 0$ is a non-empty real curve and $F(x, y) = 0$ does not occur as the locus of a problem of PAPPUS.

Proof. Let Q_n be the space of polynomials in two variables of degree $\leqq n$. The dimension of Q_n is $\frac{(n+1)(n+2)}{2}$. Let $O_n \subset Q_n$ be an open subset of Q_n consisting of polynomials $F(x, y)$ such that $F(x, y) = 0$ is a non-empty real curve; the dimension of O_n is also $\frac{(n+1)(n+2)}{2}$. [It can be seen from the following that such a subset O_n exists. Let $O^{+}_{(1,0)}$ be the set of polynomials F with $F(1, 0) > 0$ and let $O^{-}_{(-1,0)}$ be the set of polynomials F with $F(-1, 0) < 0$. Both sets are open subsets of Q_n and their intersection O is not empty ($F(x, y) = x$ belongs to it). If $F \in 0$, then $F(1, 0) > 0$ and $F(-1, 0) < 0$ so that $F(x, y) = 0$ must be a non-empty real curve.] Suppose that all $F \in O_n$ are PAPPUS polynomials. Then, according to the corollary to Theorems 2 and 3, each $F \in O_n$ must occur as $P_{2n}(v)$ for some $v \in V_{2n} \cdot O_n \subset Q_n \subset Q_{2n}$, so O_n must be a subset of the image of P_{2n} in Q_{2n}. Hence

$$\dim O_n \leqq \dim \text{image } P_{2n} \leqq \dim V_{2n},$$

that is

$$\frac{(n+1)(n+2)}{2} \leqq 12n + 1.$$

This condition is not satisfied if $n > 21$, so in that case there must be polynomials in O_n which are not PAPPUS polynomials. (The estimate of dimension can easily be sharpened *e.g.* to $n > 13$.) This completes the proof that DESCARTES' assertion is wrong.

Apparently NEWTON was the first to question DESCARTES' assertion that all curves occur as loci of problems of PAPPUS and to give a proof that it is wrong (*cf. The mathematical papers of Isaac Newton* (ed. D. T. WHITESIDE, Cambridge (1967–) vol. 4 (1971), pp. 340–344). The basic idea in this proof is also a dimensional argument; NEWTON compared the number of free coefficients in the choice of n lines and the number of coefficients in a polynomial of degree n, and found that for $n \geqq 5$ the latter is larger than the former, so, NEWTON concluded, DESCARTES was wrong. NEWTON did not mention the occurrence of polynomials F such that $F(x, y) = 0$ is empty, nor did he consider the possibility that an equation $F(x, y) = 0$ of n^{th} degree might occur as the locus of a problem of PAPPUS in more than $2n$ lines. As we have seen, these two aspects make the question much more complicated.

H. G. ZEUTHEN states, in his *Geschichte der Mathematik im 16. und 17. Jahrhundert* (1966, p. 210), that DESCARTES' assertion is wrong, but he does not provide a proof.

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