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PRESERVICE ELEMENTARY TEACHERS' EXPLICIT BELIEFS ABOUT MULTIPLICATION AND DIVISION

ABSTRACT. This study, conducted in the United States, was designed to assess the extent to which the beliefs, "multiplication always makes bigger" and "division always makes smaller," are explicitly held by preservice elementary teachers. Paper and pencil instruments were administered to 136 preservice elementary teachers in a large university in the southeastern United States. Responses on the instruments and in interviews provide insight into the sources of preservice teachers' beliefs and into the relationship among preservice teachers' beliefs, computational skills, and performance on word problems.

In a previous study conducted in the United States, the authors (Graeber, Tirosh, and Glover, 1989) found that a substantial number of preservice teachers had difficulty selecting the correct operation to solve multiplication and division word problems involving positive decimal factors less than one. Interviews indicated that some of the preservice teachers held explicit misbeliefs about the operations. Other preservice teachers apparently were influenced, as Fischbein et al. (1985) hypothesized, by "implicit, unconscious and primitive intuitive" models of the operations. One purpose of this study was to assess the extent to which two common misbeliefs about multiplication and division are explicitly held. The study was also designed to provide insight into the sources of the preservice teachers' beliefs about multiplication and division.

MISBELIEFS AND PRIMITIVE MODELS

Fischbein et al. (1985) hypothesized that the primitive model associated with multiplication is repeated addition. According to this model a (whole) number of collections of the same size are "put together." Multiplication is not seen as commutative in this model. One factor (the number of equivalent collections) is treated as the operator and the other (the magnitude of each collection) as an operand. When this concept of multiplication prevails, the operator "must" be a whole number, and, consequently, the product "must" be greater than the operand. In the domain of whole numbers, where instruction usually begins, possession of the primitive multiplication model can be a source of the belief that "multiplication always makes bigger."

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Fischbein et al. (1985) also describe two primitive models for division, a partitive model and a measurement model. In using the primitive partitive model of division, an object or collection of objects is divided into a given whole number of equal parts or subcollections. In using the primitive measurement model, one seeks to determine how many times a given quantity is contained in a larger quantity. Earlier work (Graeber, Tirosh, and Glover, 1986), suggests that American, preservice elementary teachers tend to think of division predominately in partitive terms. This primitive model, by its behavioral nature, imposes constraints on the operation of division. Two of these constraints are: the divisor "must" be a whole number and the quotient "must" be less than the dividend. These constraints can be the source of the belief that "division always makes smaller."

Fischbein et al. (1985) claimed that the "models become so deeply rooted in the learner's mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct" (p. 16). Although a number of researchers from different countries (Bell, 1982; Ekenstam and Greger, 1983; Greer, 1987; Hart, 1981; Owens, 1987; Sowder, 1986) have reported that children often explicitly express misbeliefs such as "multiplication always makes larger" and "division always makes smaller," less has been written about whether or not these two beliefs are explicitly held by adults or about adults' sources of support for the beliefs.

METHOD

Subjects

The subjects were college preservice teachers enrolled in the mathematics content or mathematics methods course for early elementary (kindergarten to grade five) education majors in a large university in the United States. Preservice teachers typically enter the methods courses in their third year of university study having completed at least the first of the two mathematics content courses in their second year at the university. The preservice teachers who were subjects in this study were enrolled in one of six classes in either the winter or spring of 1986.

Instruments

The preservice teachers were asked to respond to the following six statements related to the misbeliefs, "multiplication always makes larger" and "division always makes smaller."

A. In a multiplication problem, the product is greater than either factor.
B. The product of $.45 \times 90$ is less than 90.
C. In a division problem, the quotient must be less than the dividend.
D. In a division problem, the divisor must be a whole number.
E. The quotient for the problem $60/.65$ is greater than 60.
F. The quotient for the problem $70 \div 1/2$ is less than 70.

They were asked to label each statement "True" or "False" and to justify their response. Since our previous work indicated that some students were confused about the various terms used in division, the following reminder preceded the statements.

$$\text{divisor}\overline{\smash{)}\text{dividend}}^{\text{quotient}}$$

Data were also collected on the preservice teachers' computational skills and on their performance in writing expressions to solve word problems. Two of the computational exercises, $.38 \times 5.14$, and $.75\overline{\smash{)}3.75}$, provided counter examples to the beliefs under discussion. Preservice teachers responded to either 16 or 21 word problems (including 13 multiplication and division problems) depending on whether they were enrolled in the course in the winter or spring. The problems used in the winter and spring were similar but not always identical; both sets were similar to the problems reported by Graeber, Tirosh, and Glover (1986).

Interviews

About one-half of the preservice teachers were interviewed in order to obtain more information about their conceptions of multiplication and division. The preservice teachers were asked to write expressions that would solve multiplication or division word problems similar to those they had missed on the written word problem instrument. They were also asked to explain the logic they used in selecting the operation.

Procedures

The paper and pencil instruments were administered to the 135 female and 1 male students comprising six classes. The preservice teachers' justifications for each of the statements of belief were compiled and similar responses were grouped into categories. When a respondent gave no justification or simply restated the belief or its negation, the response was tallied in the "Said only true" or "Said only false" category. Any justification, whether logical or illogical, that referred to "moving the decimal point" was coded into a special category. Other justifications that were unrelated to the main issue, were uninterpretable, or supported a point of view contrary to the "True" or "False" claim a student made were coded in the "Illogical or irrelevant argument" category.

After the paper and pencil instruments had been administered, all 33 preservice teachers from one of the winter classes and all 38 from one of the spring classes were interviewed by the authors. Systematic data on the audiotaped interviews are not presented here; excerpts illustrating preservice teachers' reasoning are included.

RESULTS

Beliefs about Multiplication

Eighty-seven percent of the 130 preservice teachers who responded to both of the multiplication statements related to the misbelief "multiplication always makes bigger" responded correctly to both; only 3% of them responded incorrectly to both of the statements.

Table I shows that in responding to Statement A, "the product is always larger," 40% of the preservice teachers relied on specific examples from the domain of whole numbers, frequently citing products with zero or one as factors.

TABLE I

Distribution (in percent) of responses to Statement A: "In a multiplication problem, the product is greater than either factor."

Responded correctly (said False)	85
Example with either 0 or 1 as a factor	38
Rule about multiplication with fractions/decimals	25
Specific example using fractions/decimals	16
Is only true for whole numbers	2
Said only False	3
Illogical or irrelevant argument	1
Responded incorrectly (said True)	11
Multiplication is repeated addition	5
Specific example using whole numbers	2
Said only True	3
Illogical or irrelevant argument	1
Did not respond	4

($n = 136$).

TABLE II

Distribution (in percent) of responses to Statement B: "The product of .45 × 90 is less than 90."

Responded correctly (said True)	90
Computed the answer	32
Rule about multiplication with fractions/decimals	26
Move the decimal point	10
Estimated	9
Multiplication is repeated addition	2
Said only True	6
Illogical or irrelevant argument	5
Responded incorrectly (said False)	10
Multiplication always makes bigger	5
Move the decimal point	1
Illogical or irrelevant argument	4

($n = 136$).

The data in Table II show that the preservice teachers' performed somewhat better in responding to Statement B, .45 × 90 is less than 90, than to Statement A. The truth of the second statement could easily be tested by computation; and, indeed, about one-third of the respondents correctly computed the answer. Other responses to Statement B such as, "You are taking .45 sets of 90 which would be less than 90," seem to reflect an interpretation of multiplication as repeated addition. More commonly the use of this primitive model led to incorrect responses with justifications such as "multiplication is an increasing function, it is like addition many times over." Some preservice teachers used their procedural knowledge about the multiplication algorithm to provide justifications for both "True" and "False" responses to Statement B. Justifications such as "you have to move the decimal point over and this will make the answer less than 90" comprised the 10% in the "Move the decimal point" category. Justifications for a response of "False" included "The decimal has to be moved over twice, making the number larger than 90."

Although Tables I and II indicate that only about 10% of the preservice teachers held the misbelief explicitly, the data on preservice teachers' performance in writing expressions for word problems suggest that many of the preservice teachers were implicitly influenced by this misbelief. When the operator in a word problem was a decimal less than 1, about 50% of the preservice teachers responded with a division expression. However, when the operator was a whole number, 90–95% of the preservice teachers wrote correct expressions. This high rate of correct response for whole number operators held for both whole number and decimal operands.

The influence of the misbelief was also evident in interviews. The following excerpt is from an interview in which the student is attempting to solve the problem. "The price of one bolt of silk fabric is $12,000. What is the cost of .55 of the bolt?"

Student: ... You want to find out what is the price of just this portion of the bolt. So you will have to divide .55 into that amount to get the portion.
Interviewer: Can you explain it again?
Student: OK. This [points to $12,000] is the price of the bolt of fabric. And you want to know the price of this part, a part, of the bolt. So you are going to divide .55 into 12,000 to find out what that part is.

This excerpt is typical of the explanations offered by preservice teachers acting with the related beliefs that "division always makes smaller, and multiplication always makes bigger."

Beliefs about Division

Four statements related to the misbelief "division always makes smaller" were included on the paper and pencil instrument. Of the 129 preservice teachers who responded to all four of these statements, 28% responded correctly to all four of the statements and 3% responded incorrectly to all four. Table III indicates that the majority of the preservice teachers responded incorrectly to Statement C, the statement that most closely parallels this misbelief. This is the only one of the six belief statements that the majority of the preservice teachers answered incorrectly. Table III shows that the justification given by more than half of the preservice teachers who responded incorrectly are valid if one is restricted to the domain of whole numbers. The majority of those who responded correctly either cited a specific example, or appealed to special cases within the domain of whole numbers, e.g., divisor of 1, or dividend of 0. Nine percent of the preservice teachers argued that Statement C was true using a specific example from the domain of whole numbers to "justify" their answer.

The pervasive nature of this misbelief is also evidenced by the fact that about 45% of the preservice teachers wrote multiplication expressions for the division word problems with decimal divisors less than one. A word problem of this type was, "Girls club cookies are packed .65 pounds to a box. How many boxes can be filled with 5 pounds of cookies?" Eighteen of the 40 preservice teachers who responded to this problem did so incorrectly:

TABLE III

Distribution (in percent) of responses to Statement C: "In division problems, the quotient must be less than the dividend."

Responded correctly (said False)	45
Examples with specific decimal or fractional divisors	17
If the divisor is 1, the quotient can equal the dividend	7
Quotient can be greater if the divisor is less than 1	7
Not in the case when the divisor is a decimal or a fraction	4
The divisor could be negative or a high fraction – meaning the dividend would be lower	1
The dividend can equal 0	1
Can be any number	1
Not really because the decimal is moved	1
Said only False	4
Illogical or irrelevant argument	2
Responded incorrectly (said True)	52
You are taking a part of or reducing the dividend	14
Specific example such as $2\overline{)6}$	9
You must be able to multiply the quotient times the divisor and get the dividend	5
You are finding out how many times a number goes into the dividend	5
Move the decimal point	1
Said only True or merely repeated the premise	17
Illogical or irrelevant argument	1
Did not respond	3

($n = 136$).

14 of those 18 wrote .65 × 5 or 5 × .65. During the interviews, a number of individuals explained that they rejected division as the appropriate operation for the problem because they knew that the answer ought to be larger than 5. Since division makes smaller, they chose multiplication.

Although the responses to Statement C, "the quotient less than dividend," suggest that a great number of the preservice teachers tacitly assumed that divisors were whole numbers, the responses to the "divisor must be a whole number," Statement D, indicate that 61% of the preservice teachers explicitly acknowledged the possibility of decimal or fraction divisors (see Table IV). This discrepancy may indicate that the preservice teachers acknowledge the existence of forms with decimal divisors but do not link such forms with a resulting quotient greater than the divisor.

Another apparent contradiction arises when the data from the computation exercises are compared with the written justifications. All of the

TABLE IV

Distribution (in percent) of responses to Statement D: "In a division problem, the divisor must be a whole number."

Responded correctly (said False),	80
Fractions or decimals can be divisors	33
Specific (e.g., 5, 1.5) possible decimal divisors; (all were either .5 or mixed decimals)	14
Decimals, in this event the decimal must move	14
It can be any number, or can be an integer	8
Said only False	7
Illogical or irrelevant argument	4
Responded incorrectly (said True)	19
Even if you have a decimal for a divisor, it must be changed to a whole number	12
If you have a fraction you invert and multiply	1
You cannot divide negative numbers into positive numbers	1
Said only True	3
Illogical or irrelevant argument	2
Did not respond	1

($n = 136$).

preservice teachers attempted a computation exercise with a decimal divisor, $.75\overline{)5}$ and 87% completed it correctly. However 19% of these preservice teachers argued that a divisor must be a whole number. As shown in Table IV, most of these preservice teachers' written justifications referred to algorithmic procedures. Their reliance on procedure is also evidenced in the interviews. Some interviewees felt division by a decimal was clearly impossible, because "you must change the decimal to a whole number" or, "you can start with a decimal but it changes to a whole number." Division by a fraction was rejected because "with fractions, you invert and multiply." Even preservice teachers who disagreed with the statement "the divisor must be a whole number" made written reference to the algorithmic procedures involved in dividing by a decimal or a fraction.

The justifications for refuting the statement included responses such as "You can divide by natural numbers, too." Such statements, which ignored the set/subset relationships between the whole numbers and the natural numbers were included in the "Illogical or irrelevant argument" category.

Tables V and VI, present preservice teachers' reactions to two specific examples with divisors less than one and quotients greater than the dividends. The justification of many of those who responded incorrectly

TABLE V

Distribution (in percent) of responses to Statement E: "The quotient for 60/.65 is greater than 60."

Responded correctly (said True)	62
The divisor is less than 1	26
Computed the answer	20
Move the decimal point	6
You are dividing by a fraction (decimal)	3
Said only True	6
Illogical or irrelevant argument	1
Responded incorrectly (said False)	34
Any division of 60 will make the quotient smaller	13
Refer to 60 divided by 65	4
The quotient is less than 60 because of the decimal places	1
The problem means to take .65 of 60	1
Calculated incorrectly	1
Said only False or simply negated the premise	4
Illogical or irrelevant argument	10
Did not respond	4

(n = 136).

included a statement to the effect that division makes smaller. For example, one preservice teacher wrote "False, because 60 is being divided. Therefore it is unlikely that the answer is greater than 60." Some preservice teachers responded to the statement, 70 ÷ 1/2 is less than 70, with arguments similar to the following: "Because 70 is being divided into parts, the quotient (the answer) will be less than 70." These examples suggest that the preservice teachers are defining division in terms of the primitive partitive model. This dominance of the primitive partitive model is consistent with earlier findings (Graeber, Tirosh, and Glover, 1986).

It is interesting to recall that 87% of the preservice teachers correctly completed the computation of $.75\overline{)5}$; however only 62% correctly claimed that 60/.65 is greater than 60. This discrepancy may be explained by the fact that the computational exercise did not require any interpretation of the relation of the calculated quotient, $75\overline{)500}$, to the original problem $.75\overline{)5}$. In fact, interviews indicated that some preservice teachers were not able to interpret the results of the algorithmic procedure for simple exercises such as $.5\overline{)4}$ or for more difficult ones such as $.55\overline{)12{,}000}$. For example, the student who claimed that to find .55 of $12,000 you must

TABLE VI

Distribution (in percent) of responses to Statement F: "The quotient for the problem $70 \div 1/2$ is less than 70."

Responded correctly (said False)	60
Computed the answer	25
The divisor is less than 1	11
1/2 will go into 70 more than 70 times	10
Move the decimal point (wrote 1/2 as .5)	5
Invert and multiply and this gives a bigger result	4
If divisor was 1, the answer would be 70. Since divisor is less than 1, answer is more than 70	2
The divisor is 1/2	1
70 divided by .5 is equal to 700 divided by 5.	1
Said only False	1
Responded incorrectly (said True)	40
In dividing any number, one will come up with a number less than itself	16
Problem is 1/2 of 70	14
Calculated incorrectly, answers $\neq 35$	5
Divide 70 by half, or in half, answer is less	1
Said only True	1
Illogical or irrelevant argument	3

($n = 136$).

divide by .55, carried out her division. The following is an excerpt from the interview:

Interviewer: Work it out.
Student: Ah. Try to divide this into 12,000. I'm not sure I can. I move this [decimal point in the divisor] over here and put a decimal here [indicates correct placement of decimal in dividend].
Interviewer: That's right.
Student: [Uses standard algorithm and writes 21,818 as the quotient.]
Interviewer: What is the answer, about?
Student: I moved it [the decimal point] over here [in dividend], so I move it back two places [writes 218.18]. About two hundred eighteen.

Some preservice teachers' confusion about such quotients may stem from the contradiction between the answer derived from the procedure and their belief that the quotient (e.g., 21,818) must be less than the dividend (e.g., 12,000). Further their reliance on procedural knowledge of the algorithm may support their misbelief about the relative size of the quotient and the dividend. A preservice teachers' inability to access the measurement model

TABLE VII

Percent of preservice teachers by response pattern to statements of belief about division

C $Q < DD$	D *WND*	E 60/.65	F $70 \div 1/2$	%
+	+	+	+	28
+	+	+	−	2
+	+	−	+	5
+	+	−	−	4
+	−	+	+	4
+	−	+	−	0
+	−	−	+	1
+	−	−	−	1
−	+	+	+	16
−	+	+	−	8
−	+	−	+	1
−	+	−	−	16
−	−	+	+	5
−	−	+	−	3
−	−	−	+	3
−	−	−	−	3

Lack of response to any one item resulted in exclusion, $n = 129$.
+ indicates correct response.
− indicates an incorrect response.

of division would also hamper efforts to determine what would be a reasonable answer.

All of the division beliefs statements discussed above are logically connected with the notion that division always makes smaller. However, as shown in Table VII, only about one-third of the preservice teachers gave answers that were consistent across the four statements. Thirty-six percent agreed that the quotient must be less than the dividend but responded correctly to at least one of the statements $60/.65 > 60$ and $70 \div 1/2 < 70$. The preservice teachers' responses suggest that they successfully used procedural knowledge (in the sense suggested by Hiebert, 1986) when justifying specific examples, and they used the primitive models of the operations or their generalizations about procedures when responding to general statements.

Relating Misconceptions about Multiplication and Division

If one accepts multiplication and division as inverse operations, the two statements "multiplication always makes bigger, and "division always

TABLE VIII

Percent of preservice teachers by response pattern to statements of belief about division

Product greater than factor(s)	Quotient smaller than dividend	%
+	+	46
+	–	45
–	+	4
–	–	5

Lack of response on any one item resulted in exclusion, $n = 129$.
+ indicates correct response.
– indicates an incorrect response.

makes smaller" are logically equivalent. The preservice teachers' written justifications, their comments during interviews, and their performance in writing expressions to solve word problems suggest that these beliefs are strongly tied – perhaps to the extent of being one belief. However as shown in Table VIII, forty-five percent of the preservice teachers successfully refuted the statement of misbelief about multiplication but did not refute the misbelief about division.

In the domain of nonnegative numbers, one can refute the multiplication misbelief by using either (1) a factor of one to obtain a product equal to the other factor, (2) a factor of zero to obtain a product less or equal to one of the factors, or (3) a factor less than one. If the factor less than one is considered as the operand and the operator is a whole number, the statement can be refuted within the framework of the primitive, repeated addition, model of multiplication. In the case of division, one can either (1) show that the quotient may equal the dividend by using the special case of one as a divisor, (2) show the two can be equal by using the case of zero as a dividend, or (3) show that the quotient is less than the dividend by using a rational divisor less than one. In the third case one must go outside the domain of whole numbers and use a model other than the primitive, partitive model. One possible explanation for the preservice teachers' higher success with the multiplication statement than with the division statement is that for multiplication refuting the statement on the basis of inequality can be done within the primitive model. For division one cannot refute the statement on the basis of an inequality within the framework of the dominant primitive partitive model.

Another possible explanation may be found by considering only procedural knowledge about the operations. In performing the standard multiplication algorithm with one or more decimals, the form of the two factors remains constant and in the final step of the algorithm the decimal point is placed in the answer. However, in performing the standard division algorithm with a decimal in the divisor, the form of the dividend and the divisor are changed (as the decimal points are "moved"). The final written form does not juxtapose the original divisor and dividend with the derived quotient. If the preservice teachers' knowledge is limited to or dominated by procedural knowledge, they are more apt to recall the form of the completed multiplication algorithm with a product less than a factor, then to recall the form of a completed division algorithm with a quotient larger than the dividend.

DISCUSSION

Data from previous studies indicate that approximately 25–50% of adolescent or preservice teachers attempting to solve word problems with decimals less than one elect multiplication to find larger answers and elect division to find smaller answers (Bell, Fischbein, and Greer, 1984; Fischbein et al., 1985; Greer and Mangan, 1986; Tirosh, Graeber, and Glover, 1986). Fischbein and others have assumed that performance on word problems reflects the influence of implicit misbeliefs about these operations. This study showed that the percent of the American preservice teachers whose performance in solving word problems appears to be influenced by the misbeliefs is larger than the percent who explicitly hold these misbeliefs.

The responses that the American preservice teachers made to the statements of belief about the operations indicate that their conceptual understanding of multiplication is frequently expressed in terms of the repeated addition model and their understanding of division in terms of the primitive partitive model. The dominance of these primitive models is consistent with the preservice teachers' reliance on examples from the domain of whole numbers. The preservice teachers' heavy dependence on the domain of whole numbers is consistent with the reported tendencies of young adolescents (Bell, Fischbein, and Greer, 1984; Hart, 1981; Sowder, 1986).

Written justifications for statements about the operations indicate that only 10% of the preservice teachers explicitly hold the misconception that "multiplication always makes bigger." However, a majority of them agreed with the statement, "In a division problem, the quotient must be less than the dividend." Although these two beliefs are logically equivalent, nearly 50% of preservice teachers explicitly accepted one of the misbeliefs and refuted the other.

We believe that the discrepancies found among the preservice teachers' performance on different belief statements, and between their performance on computational examples and the related belief statements, may be explained by their reliance on procedural knowledge that dominates, or at least is not linked to, correct conceptual knowledge. For preservice teachers who draw only on the primitive model of multiplication and the primitive, partitive model of division, problems such as $.6 \times .5$ or $.5\overline{)4}$ become extremely difficult to endow with meaning, and the preservice teachers seem to rely on their procedural knowledge. Indeed, some of the preservice teachers appear to have used procedural knowledge to make generalizations which they then apply to the concept of division. For example, their procedural knowledge (e.g., change the decimal divisor to a whole number) seems to be translated into beliefs about operations (you can't divide by a decimal) and is used like conceptual knowledge to justify responses to statements such as "The divisor must be a whole number." The authors accept Hiebert's contention that "Students are not fully competent in mathematics if either kind of knowledge is deficient or if they have both been acquired but remain separate entities" (1986, p. 9). Hiebert and Wearne recently (1987) observed that once students acquire procedural knowledge "it is difficult for students to penetrate their own routinized procedures with meaningful information" (p. 397). Among their justifications, one can find instances of preservice teachers applying statements about procedures to the concept. One might say that a kind of "conceptual knowledge," albeit faulty, is developed from the procedural knowledge. This may be a source of difficulty when procedural knowledge is developed ahead of conceptual knowledge. From a constructivist point of view, further meaning may be unnecessary to a learner who has already constructed his or her own "conceptual knowledge."

Whether the source of the misconceptions is primarily the primitive models or primarily the dominance of procedural knowledge seems difficult to sort out and may be a moot point. A person limited to the primitive model of multiplication, will have difficulty ascribing meaning to the product of two decimals less than one. In this case, the strategy of learning the procedure and not linking it to any concept seems likely. In the case of division by a decimal less than one, constraints of the primitive partitive model (the divisor must be a whole number, the quotient must be less than the dividend) may also be reinforced by the procedures and form of the algorithm, or vice-versa. Overcoming or monitoring the misconceptions appears to require acquisition of less primitive conceptual models of the operations which may make possible a better linkage between concept and procedure. Nevertheless as Hiebert and others argue (Hiebert, 1986), the linkages, then theoretically possible, still need to be made.

IMPLICATIONS

Data indicate that a substantial percent of the preservice teachers involved in this study were influenced by misconceptions about multiplication and division. Although this study was conducted in the United States, there is some evidence that preservice teachers in Ireland (Greer and Mangan, 1986) and Israel (D. Tirosh, preliminary data) are also influenced by these misbeliefs. It is essential that teacher training programs in these countries help prospective teachers overcome or at least monitor the effects of these misbeliefs. Past studies suggest that this is not easily accomplished. Our work with preservice teachers in the United States provides some insights into the status, sources, and supports for their misbeliefs and suggests some instructional strategies that may prove useful.

Preservice teachers ought to be proficient at selecting the correct operation in solving relatively simple word problems. Those working with preservice teachers need to be aware of the possibility that preservice teachers may not have this skill, and instructors may find that they need to provide experiences to reverse the situation. Most theorists and researchers agree that awareness of misbeliefs is a necessary step in overcoming them (Flavell, 1977; Piaget, 1962; Pines and West, 1986; Schoenfeld, 1983; Shaughnessy, 1985; Silver, 1985; Strauss, 1987; Vygotsky, 1962). One strategy that first makes students aware of their misconceptions and shows some promise is the conflict teaching method. (Swan, 1983; Graeber, Jones, and Tirosh, 1987). The present study suggests that for many preservice students the misbeliefs about multiplication are likely to be implicit, while the misbelief about division is more apt to be explicit. The instructional path used to employ conflict teaching may differ with the implicit/explicit nature of the misbeliefs. For about 50% of these preservice teachers the misbelief, the quotient must be less than the dividend, is explicit. Since almost all of the preservice teachers correctly calculated quotients for examples that violated this misbelief, the explicit statements and the computational results can be used to penetrate a conflict and raise awareness of faulty beliefs. However for 90% of the preservice teachers, the multiplication misbelief acts only implicitly, and the preservice teachers must first become aware of the fact that they behave as if they hold the misbelief. Preservice teachers might be asked to respond to word problems and then to analyze their errors noting the extent to which the errors reflect a belief that multiplication makes larger.

The interviews indicate that the inverse nature of the operations is familiar to and used by many of the preservice teachers. Thus conflict with the belief "division always make smaller" might also be reached by using examples where multiplication results in a product smaller than a factor (e.g., $6 \times 1/2$) to generate the corresponding division sentence $3 \div 1/2 = 6$.

Another area of concern is the fact that a substantial number of preservice teachers apparently have only a procedural understanding of division by a decimal less than one. A possible instructional strategy is to review the models for division and require students to estimate or obtain answers without the use of an algorithm. For example, preservice teachers can be asked to provide solutions to "compatible number" problems such as $.25\overline{)2}$ without using the written algorithm. Such solutions and estimates should be required both before and after the standard algorithm has been taught.

Since a procedural understanding of the division algorithm seems to support the logic of the primitive partitive model (e.g., you can't divide by a decimal) and perhaps strengthen that model, instruction of preservice teachers probably needs to stress the measurement model. This seems especially appropriate as many preservice teachers access only the partitive model. Certainly preservice teachers should know what types of expressions can be interpreted as measurement and/or partitive type problems and should be able to demonstrate solutions in a manner consistent with the problem type.

The preservice teachers tended to restrict their justifications to examples from the domain of whole numbers – and their conceptions are likewise limited to this domain. One strategy for making them aware of the differences between the domain of whole numbers and the domain of rational numbers is to refer to the truth of the statements within the various domains – that is to explicitly contrast results in the domain of whole numbers with results of operations involving rational numbers less than one. This instruction can also focus students' attention on those instances in which calculations produce results that are contrary to the consequences of the constraints of the primitive models.

Such class discussions may also open the way for discussion of and practice with valid means of proof. The justifications that the preservice teachers gave to the statements of belief and statements that they made during interviews revealed confusion about what constitutes proof and the validity of generalizations for sets and subsets. The percent of illogical answers and use of specific examples to support a generalization (see for example Table III) suggests that this is a necessary activity.

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