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THE METAPHYSICS OF QUANTITY

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ABSTRACT. A formal theory of quantity T_Q is presented which is realist, Platonist, and syntactically second-order (while logically elementary), in contrast with the existing formal theories of quantity developed within the theory of measurement, which are empiricist, nominalist, and syntactically first-order (while logically non-elementary). T_Q is shown to be formally and empirically adequate as a theory of quantity, and is argued to be scientifically superior to the existing first-order theories of quantity in that it does not depend upon empirically unsupported assumptions concerning existence of physical objects (e.g. that any two actual objects have an actual sum). The theory T_Q supports and illustrates a form of *naturalistic Platonism*, for which claims concerning the existence and properties of universals form part of natural science, and the distinction between accidental generalizations and laws of nature has a basis in the second-order structure of the world.

1. INTRODUCTION

Quantitative theories generally assume the concept of a quantity as a primitive notion, and begin simply by postulating certain *quantities* to exist and to be capable of representation by real-number variables. (E.g. classical equilibrium thermodynamics assumes given certain "state variables" such as volume and energy.) The *laws* of the theory are then formulated as mathematical propositions relating the values of these quantities under certain circumstances. However, the very existence of quantities is itself an empirical matter, and quantitative theories thus presuppose a *theory of quantity*, i.e. an empirical theory which itself does not presuppose any quantitative concepts, but which yields as consequences the existence of the quantities taken as primitive in the theory as ordinarily presented, and justifies their representation by real numbers in the standard way. Only such a theory can provide the link between the numerical apparatus of the quantitative theory and the qualitative facts of observation.

Since ordinary quantitative theories do not make explicit their associated theory of quantity it becomes a task for foundational research to fill this gap. These theories of quantity are in fact primitive empirical theories in their own right, having their own laws and empirical foundation independently of the more sophisticated quantitative theories based upon them. For example, it was necessary to discover that length, mass or temperature existed as quantities which could be measured, before one could go on to formulate the specific quantitative theories of geometry, dynamics or thermodynamics involving these quantities. The present paper presents a general formalism for a theory of quantity in this sense which embodies a *Platonist metaphysics*, in contrast to the existing theories of quantity which have an empiricist and nominalist orientation. This difference is manifested in the use of a formal language with a *second-order syntax*, allowing quantifiers and second-order predicates applying to first-order quantitative properties. The semantics of this language is *logically elementary* (being based on general or Henkin models), so that the logic upon which the theory is based is complete and recursively axiomatizable.

The empirical interpretation of the theory is discussed, and its general adequacy as a theory of quantity is displayed. I argue that this theory has significant scientific advantages over the existing first-order theories of quantity developed within the theory of measurement, both in respect of internal or theoretical considerations and in respect of empirical support. The empirical difference stems from the fact that the first-order theories of quantity require certain existence assumptions which appear to be empirically false, while the second-order theory does not. We thus seem to derive direct empirical support for some form of *naturalistic Platonism*. These metaphysical implications are discussed in the final section.

2. FIRST-ORDER AND SECOND-ORDER THEORIES OF QUANTITY

In the present paper I shall be concerned only with *extensive* quantities, which possess a linear order and some form of addition or concatenation of magnitudes. Nothing essential turns upon this restriction. A formal theory of extensive quality must formalize the qualitative facts concerning the relations of order and addition which form the basis for the standard practice of representation of such quantities using real variables. (For the extensive quantity of *mass*, for example, these facts involve the order relation established by an equal-arm balance and the addition relation established by simultaneous placement of two masses on one pan of the balance.) Within the usual framework of formal logic, such a theory T will be determined by a set of axioms expressed as sentences of a definite formal language L, having specified rules of inference and an empirical interpretation which corresponds in some way to the qualitative facts in question. The formal adequacy of the theory is to be demonstrated by proving that every model M of the theory possesses a representation (unique up to scale transformations) in the additive ordered group of real numbers.

Within this framework an important question is still left open: are the entities connected by the relations of order and addition, and assigned numerical values on the resulting scales, to be taken as *particular physical objects*, or as *quantitative properties* (henceforth called *magnitudes*) belonging to particular material objects? In brief, does quantification consist in an assignment of numbers to *objects*, or to *properties* of objects? In the first case L will be an *atomically first-order* language having object variables x ranging over concrete objects, two first-order predicate constants $\leqslant$ and $*$, and atomic formulas of the forms $x = y$, $x \leqslant y$, and $x * y = z$. The predicates $\leqslant$ and $*$ will be interpreted as observable relations of the objects. In the second case L will be an *atomically second-order* language having object variables x ranging over concrete objects, first-order variables X ranging over properties of concrete objects, two second-order predicate constants $\leqslant$ and $*$, and atomic formulas of the forms $x=y$, $X=Y$, $X(y)$, $X \leqslant Y$, and $X * Y = Z$. (Here $X(y)$ of course means that the object y possesses the property X.) The first-order predicate variables X and second-order predicate constants $\leqslant$ and $*$ will be interpreted as theoretical entities, and additional first-order relations among concrete objects (corresponding to the actual measurement operations) will be introduced to provide the empirical basis for the theory.

Theories of quantity may accordingly be classified as *first-order* or as *second-order* according to whether the underlying language is atomically first-order or atomically second-order. The formal theories of quantity which have been developed within the context of modern logic (most notably within the *theory of measurement*, as in Krantz *et al.*, 1971 or Roberts, 1979) are first-order theories. Here I will present a comparable second-order theory.[1]

There are several motives for doing so. The most important one is that a second-order theory of quantity has definite *scientific* advantages over the existing first-order theories, considered simply as empirical theories. Specifically, the first-order theories all depend essentially upon at least one strong existence axiom asserting the *existence of sums*, e.g. the existence, for any two objects x and y, of an object $z = x * y$ whose magnitude is the sum of those of x and y. It is recognized in first-order measurement theory that this particular assumption is unrealistic because of practical limitations on the process of concatenation, but the only weakened first-order axiom system addressing this point known to me is that of Krantz *et al.*, 1971, pp. 81–85 (based on Luce and Marley, 1969), which replaces the assumption of universal existence of sums with the assumption that sums exist whenever they are not larger than a certain size, and which then yields existence of a scale only for objects not greater than that size.

This aspect of first-order theories of quantity does not seem to be satisfactory. First, there may perfectly well exist magnitudes which are larger than the practical limit of concatenation but to which we nonetheless wish our scale to assign a value (astronomical distances, etc.). Second and more important, regardless of what bound is chosen the hypothesis that for any two actual objects whose sum does not exceed that bound there *actually exists* an object equal to that sum is an extremely implausible one, and certainly not well supported by empirical evidence if 'object' is understood normally (i.e. so as to include only actual separate objects, not mereological "subobjects" of actual objects and similar problematic entities). Furthermore it seems absurd to treat this claim as a *law of nature* even if by some chance it should happen to be true. Surely the whole system of physical quantities and quantitative laws would not collapse if through some cosmic accident all of the actual examples of objects precisely two meters long were to be destroyed while the standard meter itself remained intact. Clearly the only reasonable interpretation of these existence claims is not that all of these sums actually exist but rather than one *could con-*

struct such sums for any two objects (or of any two which are not too long).

However, this reading raises serious questions. In the first place, without the introduction of an explicit modal apparatus into the language L the theory T now becomes ambiguous, since we have no way of knowing which existential quantifiers are to be interpreted normally and which as asserting mere possible existence in this sense. Second and more important, the introduction of such a modal quantifier seems to completely undercut the *empirical* character of the theory, since we have no means whatever of determining the truth of falsehood of such claims of "possible existence". The only clear-cut empirical interpretation which we possess for the first-order language of a first-order theory is the standard one which takes the existential quantifier to assert actual existence of an actual object, and under that interpretation the existence assumptions of standard first-order extensive measurement theories seem likely to be straightforwardly *false*, and in any case not to be deserving of acceptance as universally true on the basis of evidence available to us.[2]

Second-order theories of quantity, by contrast, ae not liable to this empirical objection, since the assumption of existence of actual sums of actual objects will be replaced by an assumption of existence of sums $X^* Y$ of magnitudes X and Y, which is not subject to direct empirical test because the order and addition relations of magnitudes are theoretical facts. It will be shown below that a second-order theory of quantity may be constructed which satisfies the formal conditions on a theory of quantity as well as do the first-order theories, and which seems to includes among its empirical consequences all of the *true* universal conditionals derivable from the corresponding first-order theory without having any obvious false consequences. On ordinary empirical grounds such a second-order theory of quantity seems to be a better theory.

In addition to its empirical superiority, there are a number of intuitive and philosophical motives for the construction of a second-order theory of quantity. From an intuitive viewpoint it seems hard to deny that we think of the size or magnitude of a physical object as a quantitative property of the object, one which could be shared by other objects. At the same time, we think of the relations of greater or less, or of numerical relations such as double or half, as being relations of particular magnitudes *to* one another, as expressed in statements such as, 'the size of x is twice the size of y'. Here the natural reading is obviously that the 'the size of x' and 'the size of y' refer to certain first-order quantitative properties of the objects x and y, and that the statement ascribes a certain second-order relation to these two first-order properties. One may also construe the statement as expressing directly a first-order relation between x and y, with no reference to sizes as properties, but this does not seem to be the natural reading.

Most authors dealing with the concept of quantity prior to the advent of logical positivism seem to have taken a second-order interpretation to be at least as plausible as a first-order one. The mathematical analysis of the seventeeth century is concerned with the variation in time of certain entities called "quantities", which clearly are taken to include properties of objects, such as velocity (Boyer, 1949). Helmholtz, 1887, Section 9, describes magnitudes (the bearers of quantitative relations) as either objects or attributes of objects. Russell, 1903, Ch. XIX, clearly distinguishes the two views, calling the first-order approach the *relative* theory of magnitude since it bases quantitative facts in relations among objects, and the second-order approach the *absolute* theory, in analogy to absolute and relative theories of space. He advocates the absolute or second-order theory, appealing to his Principle of Abstraction to infer from the existence of a first-order equivalence relation of equality in magnitude the existence of magnitudes as first-order quantitative properties. Campbell, 1920, Ch. X, explicitly asserts that the objects of measurement are quantitative properties.

Thus there seems to be a long and well-established tradition of second-order interpretations of quantity, which in itself seems to provide a sufficient justification for developing a corresponding formal theory of quantity. For even if the actual views of scientists concerning quantity have changed radically in the twentieth century (which I think unlikely, at least for classical physics), it is important to have a precise formulation of the older second-order view for comparison with the newer first-order one.

Concerning philosophical motives, the most obvious remark is that the replacement of the second-order approach by the first-order one in the recent literature on quantity and measurement seems clearly to have been motivated by empiricist (and perhaps nominalist) philosophical considerations. The earliest explicit rejection of the second-order approach known to me is Nagel's 1932 article on measurement in *Erkenntnis*, which relies in part upon the empiricist criterion of meaning.[3] The 1951 article of Suppes commences with criticisms of the earlier axiomatic formulation of Hölder 1901; these criticisms essentially show that Hölder's theory is not naturally interpretable in the first-order manner.[4] The 1958 article of Scott and Suppes (which marks the beginning of the modern model-theoretic approach to measurement) adopts a first-order viewpoint on general empiricist principles.[5] Indeed, even the name 'theory of measurement' replacing the older 'theory of quantity' suggests an empiricist, reductionist or nominalist approach, focusing on the medium rather than the object of quantitative knowledge.[6] The first-order framework of measurement theory is obviously well suited to such an empiricist viewpoint, since the two types of term may be taken to correspond to observable objects and their observable features.

Of course I do not mean to suggest that empiricism or nominalism are worthless or untenable viewpoints, but simply to call attention to the obvious fact that they are not the *only* viewpoints. It must be regarded as something of an anomaly that the theory of quantity has been studied almost exclusively from those viewpoints.[7] The only exception known to me is an article of Byerly and Lazara, 1973 (cf. also Byerly, 1974), who observe that quantitative reasoning may equally well be interpreted from the contrary viewpoints of scientific realism as opposed to positivism and Platonism as opposed to nominalism, and who advocate such interpretations on philosophical grounds. However, they do not develop a corresponding formal theory of quantity as an alternative to those proposed in standard measurement theory.

3. THE QUANTITATIVE THEORY T_Q

The language L_Q has the two non-logical second-order constants * and $\leqslant$, countably many first-order predicate variables X_i of degree 1, and countably many zero-order object variables x_i. (Some additional non-logical first-order predicate constants belonging to the observational part of the theory will be added later.) The identity relations for zero-order and for first-order terms are also included as logical constants. To increase readability I will use alphabetic variants such as x, y, z instead of x_1, x_2, x_3. The atomic formulas of L_Q thus have the forms $x=y$, $X=Y$, $X(y)$, $X \leqslant Y$ and $*(X, Y, Z)$. The logical connectives are the standard propositional connectives, first-order quantifiers binding the variables x_i, and second-order quantifiers binding the variables X_i. A *model* M for L_Q consists of a domain $M(0)$ of *objects*; a set $M(1)$ of *properties*; a function E carrying each element P of $M(1)$ to a subset of $M(0)$, where $E(P)$ is called the *extension* of P; a two-place relation $M(\leqslant)$ over $M(1)$, and a three-place relation $M(*)$ over $M(1)$. Identity of properties in M is not extensional; we may have $E(P)=E(Q)$ as subsets of $M(0)$ while $P \neq Q$ as elements of $M(1)$.

The free or bound object-variables x of L_Q range over the elements of $M(0)$, and the free or bound property-variables X range over the elements of $M(1)$. (For a *standard model* $M(1)$ would be the full power set of $M(0)$ and E would be identity; the present *general* or *Henkin* models are not subject to this restriction, and $M(1)$ may even be empty.) A valuation υ of L_Q in a model M satisfies $x=y$, $X=Y$, $X(y)$, $X \leqslant Y$ or $*(X, Y, Z)$ according as $\upsilon(x)=\upsilon(y)$, $\upsilon(X)=\upsilon(Y)$, $\upsilon(y)$ is in $E(\upsilon(X))$, $\langle \upsilon(X), \upsilon(Y)\rangle$ is in $M(\leqslant)$, or $\langle \upsilon(X), \upsilon(Y), \upsilon(Z)\rangle$ is in $M(*)$, respectively. Satisfaction for quantified formulas is defined in the standard way, with the condition that a universal (existential) quantification of a property variable X in a formula F is satisfied just in case $F(X)$ is satisfied by every (some) element P *of* $M(1)$. The proof rules for L_Q are exactly like those of standard predicate logic, except that the quantifier rules apply to both types of variable. The completeness and compactness theorems may be proved exactly as for standard predicate logic (Henkin 1950); provability in L_Q thus corresponds exactly to truth in all models M, for the present concept of a model. I shall sometimes use the primitive or defined terms of the language L_Q in contexts such as 'the relation *' to refer to the corresponding primitive or defined relations such as $M(*)$ in an unspecified model M for L_Q.

One further departure from standard measurement theory should be noted. First-order extensive measurement theory presents separate axioms for each quantity to be considered, based on different first-order relations of addition and order. This is obviously necessary on the empiricist viewpoint, since different quantities correspond to different observational first-order relations of objects. In the second-order context the basic quantitative relations $\leqslant$ and * are theoretical rather than observational, and thus nothing prevents us from taking these to be the same second-order relations for all extensive quantities. The relation $M(\leqslant)$ will then be a partial ordering rather than a full linear ordering, and particular quantities such as mass and length will be disjoint subsets of the field of $M(\leqslant)$ which are linearly ordered by it. (I will call these *rays* following Whitney 1968.) The relation $M(*)$ will hold only between elements of the same ray.

This yields a formal unification and simplification of the theory of quantity, since a single set of axioms stated for the second-order relations $M(\leqslant)$ and $M(*)$ will apply to all of the rays at once. Whether this unification is of any genuine scientific significance I shall not try to say. In support of such a claim one might argue that this theory of quantity explains why so many known physical quantities possess the same type of extensive structure, while on the first-order approach this remains an unexplained coincidence.

We begin with elementary second-order extensive measurement axioms on the relations $\leqslant$ and *, as restricted to rays in the $\leqslant$ ordering. These axioms are in a sense formally first-order, since they involve only the upper two type levels. The present extensive axioms are based on those of Mundy (b).

Define 'X *sim* Y' as '$X \leqslant Y$ or $Y \leqslant X$'. (Free variables are understood as universally quantified; propositional connectives are Anglicized.) In Axom 2c and thenceforth, '$X * Y$' refers to the unique element Z such that $*(X, Y, Z)$, whose existence and uniqueness is guaranteed by Axioms (2a) and (2b).

(1) *reflexive linear order on rays:*
- (a) If $X \leqslant Y$ and $Y \leqslant Z$ then $X \leqslant Z$.
- (b) $X \leqslant X$.
- (c) If $X \leqslant Y$ and $Y \leqslant X$ then $X = Y$.
- (d) If X sim Y and X sim Z then Y sim Z.

(2) *existence of* $X * Y$:
  (a) If $*(X, Y, Z)$ and $*(X, Y, Z')$ then $Z = Z'$.
  (b) X sim Y iff $(\exists Z)*(X, Y, Z)$.
  (c) X sim $X * Y$.

(3) *associativity of* $*$:
  $(X * Y) * Z = X * (Y * Z)$

(4) *monotonicity*:
  Let X sim Z; then $X \leqslant Y$ iff $X * Z \leqslant Y * Z$ iff $Z * X \leqslant Z * Y$.

Axiom 1 ensures that *sim* is an equivalence relation and that $\leqslant$ is a full linear ordering on each equivalence class (ray). Axiom (2) ensures that * is a function, is defined always and only on similar pairs, and yields a result similar to the arguments. Finally, Axioms (3) and (4) make each ray an ordered semigroup under *. Thus in each model M for T_Q each element P of the field of $M(\leqslant)$ will belong to a unique ray $Ra(P)$, and $M(*)$ will act as an ordered semigroup on $Ra(P)$. I will simply write '$P \leqslant Q$' to mean that $\langle P, Q\rangle$ is in $M(\leqslant)$, and write '$P * Q$' for the unique element R of $M(1)$ for which $\langle P, Q, R\rangle$ is in $M(*)$. Similarly, all other concepts defined for elements of an ordered semigroup will, when applied to elements P of the field of $M(\leqslant)$, refer to the semigroup defined by $M(*)$ on $Ra(P)$.

An ordered semigroup is a special case of the type of structure called an *extensive semigroup* in Mundy (b). Following the terminology of that paper, the *sign* of an element P is *positive* if $P * P > P$, *zero* if $P * P = P$, *negative* if $P * P < P$; P is *proper* if it is either positive or negative. The *extended real number system* R' is $\langle R, \leqslant, +\rangle$ together with the elements $-\infty$ and $+\infty$, these being given the natural relations of order and addition to the finite elements of R.[8] A function f from an ordered semigroup $\langle S, \leqslant, *\rangle$ to R' is called *weakly faithful* if it satisfies:

(WF) (a) For all x and y in S, if $f(x) < f(y)$ then $x < y$.
  (b) For all x, y and z in S for which $f(x) + f(y)$ is defined, if $f(x) + f(y) < f(z)$ then $x * y$ is defined and $x * y < z$; if $f(z) < f(x) + f(y)$ then $x * y$ is defined and $z < x * y$.

From Theorem 5 of Mundy (b) we may now immediately conclude:

Representation Theorem: Let M be a model of T_Q.

(a) Let P be a proper element of the field of $M(\leqslant)$, and let j be a non-zero real number whose sign is the same as that of P. Then there is a unique weakly faithful function f from $Ra(P)$ into R' such that $f(P)=j$.

(b) Let $Ra(P)$ be a ray in the field of $M(\leqslant)$, let f be a weakly faithful function from $Ra(P)$ into R', and let g be a function from $Ra(P)$ into R' such that there is at least one element Q in $Ra(P)$ for which both $f(Q)$ and $g(Q)$ are finite and non-zero. Then g is weakly faithful if and only if there is a positive real number k such that $f(R)=k\cdot g(R)$ for all R in $Ra(P)$.

According to Mundy (a) the essential property of our actual numerical scales for physical quantities is to be weakly faithful functions from the ordered semigroup of magnitudes into R'. The above theorem then shows Axioms 1–4 to give a formally adequate theory of quantity, i.e. one sufficient to yield the existence of such a family of numerical scales, and thus to justify the assumptions about physical quantities made in ordinary quantitative theories. (Of course we must adjoin axioms specifying the existence of as many distinct rays in the field of $M(\leqslant)$ as there are distinct fundamental quantities employed in the theory.)

As in standard measurement theory, part (a) of the theorem expresses our freedom, in constructing an extensive scale for a given quantity, to choose any non-zero magnitude P of the quantity as our unit of measurement and to assign to it any finite non-zero numerical value j consistent with its sign. Part (b) expresses the fact that this is the only degree of freedom present for a weakly faithful scale, in that any other such scale will be related to the given one by a constant positive factor k. The present version of this uniqueness theorem is weaker than the standard one because of the absence from our theory of quantity of a logically non-elementary Archimedean axiom, thus allowing the possibility of different weakly faithful scales f and g for $Ra(P)$ which fail to be related by any finite factor k because the unit of one scale is infinite or infinitesimal relative to the unit of the other. This is why the uniqueness result (b) must include the condition that the two scales f and g both assign a finite value to some common magnitude Q.

It remains to make explicit the connection of the first-order structure of objects and their magnitudes with the theoretical second-order structure on magnitudes. It is here that the second-order character of L_Q plays an essential role, since this axiom contains terms of all three type levels simultaneously.

(5) *uniqueness of magnitudes:*
If $X(x)$ and $Y(y)$ and $X < Y$ then $x \neq y$.

Axiom 5 and the linear order on rays imply that a given object x can possess at most one magnitude X in any one ray of $M(\leqslant)$. Note that we have not yet stated any axioms requiring the field of $M(\leqslant)$ to be non-empty, and thus no rays may exist. Even if a ray does exist, we still do not want to assume that every object possesses a magnitude in that ray, since some quantities are only defined for certain types of object.

Axioms (1)–(5) suffice to define numerical scale values for all objects x in $M(0)$ which possess a magnitude in Ra, for any ray Ra of $M(\leqslant)$. The Representation Theorem allows us to choose a weakly faithful scale f for each ray $Ra(P)$ in $M(\leqslant)$, and Axiom (5) allows us to define a corresponding partial function g from $M(0)$ into R', by setting $g(x) = f(Q)$ for any x in $M(0)$ which possesses a magnitude Q in $Ra(P)$, i.e. satisfies Q sim P and $Q(x)$. The function g serves as a numerical scale for objects in $M(0)$ with respect to the quantity corresponding to the ray $Ra(P)$; we will ordinarily construct one such scale for each ray in $M(\leqslant)$.

4. EMPIRICAL BASIS OF T_Q

Axioms (1)–(5) give the *theoretical* portion of our second-order theory of quantity T_Q. This is true in two senses: first Axioms (1)–(5) suffice for the derivation of the *formal* results required of a theory of quantity (namely the existence and uniqueness of numerical scales for each quantity), and second that the theory based on Axioms (1)–(5) alone seems incapable of an appropriate empirical interpretation in the sense intended for a theory of quantity. This is because the atomic facts of the form $P(a)$, $P = Q$, $P \leqslant Q$ and $P * Q = R$ which determine the truth or falsehood of propositions of L_Q do not appear to be empirically accessible under the intended interpretation. (One may argue that some magnitudes P of certain quantities such as length should count as observable properties of objects, so that some atomic facts of the form $P(a)$ or $P \leqslant Q$ should count as observable, but I do not wish to rely upon this; one may equally well argue that what is observed is a relation between the magnitude of the object and a magnitude characteristic of the observer.) We therefore must add to L_Q some new non-logical terms corresponding to observational predicates of observable objects and add to T_Q some bridge axioms connecting these with the theoretical terms in such a way as to yield empirical propositions which may provide an empirical basis for the theory. In particular, to complete the basic task of a theory of quantity, we must establish the connection between actual measurement procedures and the abstract scale-functions $g(x)$ constructed using Axioms (1)–(5).

There are as many different ways of doing this as there are ways of measuring the quantities to which the theory refers. Indeed, one of the formal virtues of second-order theories of quantity is precisely the fact that they treat quantities and magnitudes as theoretical, and hence allow one to maintain consistently that the same quantity is capable of being measured in different ways, in contrast to the counter-intuitive consequence of simple empiricism or operationism that each measurement procedure defines a different quantity. (This theoretical conception of quantities is advocated for example in Carnap 1966, 102–104.) Here I shall simply introduce the formal apparatus corresponding to the simplest and most natural empirical procedure for measurement of extensive quantities, which is also the one forming the basis of the standard first-order theories of extensive quantity. This allows a simple comparison with the corresponding first-order theory.

Since each fundamental quantity is measured by different procedures, we must add a distinct observational component to T_Q for each such quantity q. Therefore we add to the language L_Q two non-logical first-order predicate constants for each fundamental quantity q: a two-place relation $\leqslant_q(x, y)$ and a three-place relation $*_q(x, y, z)$, which will be interpreted as the observable order and addition relations between the observable objects x, y, z in respect of the quantity q. (These are the first-order relations taken as primitive in the standard first-order theory of the single extensive quantity q.) The second-order models M for L_Q are thus expanded to include distinguished first-order predicates

$M(\leqslant_q) \subseteq M(0)^2$ and $M(*_q) \subseteq M(0)^3$ for each q. The empirical interpretation of these first-order primitives and the role which they play in practical measurement procedures has been thoroughly discussed in standard works on measurement such as Helmholtz, 1887 and Campbell, 1920, 1928.

In the extended language L_Q we now add the following axioms to T_Q (these are of course distinct axioms for each quantity q):

(6q) *bridge law for q:*
$(\exists V)\{V \leqslant V$ and $(\forall x, y, z)$
$[\leqslant_q(x, y)$ iff $(\exists X, Y)(X$ sim V and $X(x)$ and $Y(y)$ and $X \leqslant Y)]$ and
$[*_q(x, y, z)$ iff $(\exists X, Y, Z)(X$ sim V and $X(x)$ and $Y(y)$ and $Z(z)$ and $*(X, Y, Z))]\}$

The clause $V \leqslant V$ of this axiom asserts for the first time the existence of an element V in the field of $M(\leqslant)$. The ray $Ra(V)$ is then asserted to correspond to the observable first-order relation $\leqslant_q$, in the sense that two objects x and y bear that relation if and only if they possess magnitudes in the ray $Ra(V)$ which bear the second-order relation $\leqslant$; similarly the first-order relation $*_q$ corresponds to the second-order relation $*$ on the elements of the ray $Ra(V)$. The second-order predicate $\mathbf{Q}_q(X)$ will be defined as $(\exists Y)(\exists y)[\leqslant_q(y, y)$ and $Y(y)$ and X sim $Y]$; it means that X is a magnitude of the quantity q. The theory T_Q consists of all logical consequences of Axioms 1–6q under the elementary second-order logic of L_Q.

The bridge law 6q may be used to derive from Axioms (1)–(5) various empirical laws governing the first-order relations $\leqslant_q$ and $*_q$, including some which would be included among the axioms on $\leqslant_q$ and $*_q$ in a standard first-order theory of the extensive quantity q. Define the relation '$\sim_q(x, y)$' as '$\leqslant_q(x, y)$ and $\leqslant_q(y, x)$', and '$\mathrm{Q}_q(x)$' as '$\leqslant_q(x, x)$'; '$\mathrm{Q}_q(x)$' means that x is an object with a value of the quantity q. We easily derive the following theorems of T_Q:

(1) $\leqslant_q(x, y)$ and $\leqslant_q(y, z)$ imply $\leqslant_q(x, z)$.
(2) If $Q_q(x)$ and $Q_q(y)$ then $\leqslant_q(x, y)$ or $\leqslant_q(y, x)$.
(3) If $*_q(x, y, z)$ then $Q_q(z)$.

(4) If $*_q(x, y, z)$ and $\sim_q(x, x')$ and $\sim_q(y, y')$ and $*_q(x', y', z')$ then $\sim_q(z, z')$.

(5) If $*_q(x, y, u)$ and $*_q(u, z, v)$ and $*_q(y, z, t)$ and $*_q(x, t, w)$ then $\sim_q(v, w)$.

(6) If $*_q(x, z, u)$ and $*_q(y, z, v)$ then $\leqslant_q(x, y)$ iff $\leqslant_q(u, v)$.

(7) If $*_q(z, x, u)$ and $*_q(z, y, v)$ then $\leqslant_q(x, y)$ iff $\leqslant_q(u, v)$.

Here Ths. (1) and (2) show that $\leqslant_q$ is a reflexive weak ordering on the objects with values of q, (3) and (4) show that $*_q$ carries $\sim_q$-equivalent objects to $\sim_q$-equivalent objects, and (5)–(7) show that this action of $*_q$ is that of an ordered semigroup insofar as the sums are defined. Thus if we adjoined the further first-order axiom $(\exists z)*_q(x, y, z)$ asserting existence of sums we would have a full extensive semigroup in the sense of Mundy (b) at the first-order level. This is essentially the axiomatization which is used in standard first-order theories of quantity, with the addition of a non-elementary Archimedean axiom.

The theory T_Q however does not contain any such axioms asserting the existence of actual objects or their sums; the theorems of T_Q listed above are all hypothetical in form, and say that *if* the necessary sums exist then the relations of an ordered semigroup will hold among them. These theorems express empirical laws concerning the relations $\leqslant_q$ and $*_q$. Since these laws are empirically very well confirmed, they provide an empirical basis for T_Q.

Axiom ($6q$) also enables us to formulate the theory of the empirical procedures of measurement for the weakly faithful scales f which are abstractly defined in the preceding section. Let some object e with $Q_q(e)$ be selected as the unit of measurement for the quantity q, and assigned the numerical scale value j. Then the unique magnitude E with $\mathbf{Q}_q(E)$ and $E(e)$ is the magnitude of our unit, and in consequence of Axioms (1)–(5) there will exist a unique weakly faithful function f from the ray $\mathbf{Q}_q$ into R' with $f(E)=j$, and a unique scale g defined thereby on all x with $Q_q(x)$, by the condition that $g(x)=f(X)$ where $\mathbf{Q}_q(X)$ and $X(x)$. However, it remains to be seen what connection there may be between this abstract construction in the metalanguage, based on the theoretical part of T_Q alone, and actual concrete measurement procedures carried out with material objects, describable in the empirical part of L_Q.

The connection is easily made. The actual procedure for extensive measurement of an object x involves the construction of a *standard sequence* in the sense of Krantz *et al.*, i.e. a sequence of multiples of the unit, or of known fractions of the unit (so many meters, so many centimeters, etc). Then the scale value of x is determined up to the unit or the chosen fraction $1/m$ of the unit, by the information as to which successive pair of multiples it lies between (unless x is infinite or infinitesimal relative to the unit e). Any such standard sequence is a sequence of actual objects, but the bridge axioms $6q$ will enable us to draw corresponding conclusions about the magnitudes of those objects, and hence to infer a relation between the magnitude X of the measured object x and the magnitude E of the unit e. In this way our physical construction of a standard sequence will yield information about the numerical value $g(x) = f(X)$ of the abstractly-defined scale g.

Specifically, suppose that we have divided our unit into m parts, i.e. we have a set of actual objects s_i (of which there are at least m) with the properties that:

(1) $\sim_q(s_i, s_j)$ for all i, j.
(2) The sum of any m of the s_i exists and is $\sim_q$-equivalent to e.

(1) and (2) represent certain formulas of L_Q which may easily be constructed for each value of m. An empirical application of the method of standard sequences yields the information:

(3) The sum of n of the s_i exists and is $\leqslant_q$ than x.
(4) The sum of $n+1$ of the s_i exists and is $\geqslant_q$ than x.

These also represent definite formulas of L_Q for each value of n. These formulas of L_Q may be accepted as direct reports of what is observed in a standard-sequence construction with respect to an object x.

The basis for the theory of the empirical measurement of values of the abstract scale $g(x)$ is then the fact that in T_Q we can derive from (1), (2) and (3) the conclusion '$nE \leqslant mX$', where 'nE' means as usual the n-fold sum $E*E*...*E$. From the mode of definition of the abstract scale f on the ray $\mathbf{Q}_q$ as described in Mundy (b) (following the method of Hölder 1901) it follows that $nE \leqslant mX$ if and only if $jn/m \leqslant f(X)$ (where $j = f(E)$). Similarly from (1), (2) and (4) we can derive '$mX \leqslant (n+1)E$' in T_Q, which is equivalent to $j(n+1)/m \geqslant f(X)$. Thus every concrete application of the empirical method of standard sequences to measure an object x determines the value of the abstract function $g(x)$ to within j/m, and conversely for every question of the form 'Does $g(x)$ lie within the interval $[a, b]$?' we can determine values of m and n which characterize a concrete application of the method of standard sequences which may be carried out in order to answer this question. Thus the abstract scales $g(x)$ of the preceding section acquire a definite empirical basis through the bridge law ($6q$), and the theory T_Q allows us to draw the appropriate conclusion concerning the scale value $q(x) = f(X)$ from any empirical application of the method of standard sequences, as expressed in the language L_Q.

5. CONCLUSION: NATURALISTIC PLATONISM

The second-order theory T_Q has been shown to be formally adequate as a theory of quantity, and to possess an empirical interpretation in terms of which the well-known empirical laws governing the basic processes of measurement appear as theorems of T_Q. Thus T_Q possesses considerable empirical support, and, unlike the corresponding first-order theories of quantity, does not appear to have obvious false consequences. Thus it seems that T_Q or some formal equivalent is worthy of provisional acceptance as the correct theory of actual physical extensive quantities, at its level of detail. It remains to be considered what wider significance this may have.

In the first place, T_Q is an interesting example of definite *empirical* superiority of a realist theory over a corresponding empiricist or phenomenal one. Usually we think of such comparisons as being between empirically *equivalent* theories, the central issue being the choice between the parsimony of empiricism and the formal advantages (generality, unification) of realism. Here, by contrast, the result of a serious attempt to explicitly formulate a workable empiricist theory of quantity (rather than a mere artifice such as "the set of empirical consequences of the realist theory") yields a theory which seems to be empirically inferior to the corresponding realist theory.

In the second place, the particular character of the theoretical entities and propositions introduced in the realist theory T_Q is of considerable metaphysical interest. There is an obvious formal sense in which the first-order variables X of L_Q should be thought of as ranging over certain *properties* of objects, namely magnitudes, so that by Quine's criterion of ontological committment T_Q appears to be reasonably characterized as a *Platonist* theory. However, unlike the traditional forms of Platonism which assert that universals and facts about them are to be discovered *a priori*, we have here a form of *a posteriori* or *naturalistic* Platonism, according to which propositions about universals and their second-order properties and relations are supported by reference to their observable consequences, using the hypothetico-deductive method of natural science. Such a view has been maintained for example by Armstrong (1978) concerning properties and relations, and by Quine concerning sets. In the remainder of the paper I will discuss such naturalistic Platonist programs from the present viewpoint.

The first point is that the empirical superiority of T_Q provides *prima facie* support for a (naturalistic) Platonist *ontology*. Of course it is possible that T_Q may ultimately be reduced to some deeper non-Platonist theory, but we have no idea what such a theory would look like. Current fundamental physical theories are all quantitative, and hence should be assumed (on the present grounds) to include T_Q as a part. Thus on the basis of current theory T_Q should be taken not only as true but as fundamentally true, and its ontolology taken as part of basic physical ontology.

The inclusion of the second-order or Platonist apparatus within the deductive structure of science of course resolves the special *epistemological* problems which beset *a priori* Platonism. The status of universals is no different from that of any other theoretical entities within natural science: they are postulated as part of a theory (e.g. T_Q), and the empirical confirmation of the predictions of the theory (e.g. Ths. (1)–(7)) provides grounds for belief in their existence. The empirical superiority of T_Q over the corresponding first-order theories shows that the postulation of universals is by no means devoid of significance, as is often asserted. This example of empirical support for an assertion of existence of universals fills a gap in Armstrong, 1978, noted e.g. by Sanford 1980.

Finally, from a *logical* viewpoint T_Q shows how a naturalistic Platonist theory should be formalized: namely, by means of *elementary* second-order logic. The use of general or Henkin models corresponds to the naturalistic Platonist thesis that existence of universals is a contingent matter, in that propositions asserting existence of universals (which often express important scientific discoveries, e.g. of new physical properties such as radioactivity) are not treated as logical truths.[9, 10]

There is one important respect in which the naturalistic Platonism suggested by T_Q differs from that of Armstrong, 1978, and related views such as those of Swoyer, 1982, 1983. These authors maintain that universals should be postulated to exist only if they are *exemplified* by actual objects. The basis for this view seems to be the belief that science can only be concerned with what in some sense exists in space and time. I think that this is mistaken: it is indeed a contingent fact that the bulk of science has been concerned with spatio-temporal objects, but there is no *a priori* limitation of scientific knowledge to such objects. The scope of science is limited only by our ingenuity in the construction of theories and the derivation from them of observational consequences, and it is entirely possible that certain facts within space and time might turn out to be best explained by a theory which postulates the existence of certain entities which are not in space and time.

Indeed, I think that this is not only possible but is actually the case. This is illustrated by Axiom (2b) of T_Q, which asserts the existence of a sum of any two similar magnitudes. This axiom is empirically confirmed by our observation of a large number of instances in which, for two objects x and y bearing the first-order similarity relation for some quantity q, it is found that there exists a third similar object z satisfying the first-order addition relation $*_q(x, y, z)$. Applying the theoretical laws of T_Q (as followed for example by Glymour's bootstrap principles of confirmation), we may prove in L_Q that each such observation provides an instance of Axiom (2b). Axiom (2b) is therefore well supported, and may be applied to infer the existence of magnitudes which are not known to be possessed by any particular objects.

The present type of naturalistic Platonism may also be contrasted with the *set-theoretic* Platonism of Quine (1960, Ch. 7; or 1976). Quine also takes a Platonist ontology to be based on the empirical success of science, but sees the ontology as one of sets rather than of properties and relations, where sets are understood in the sense of axiomatic set theory. The present Platonist viewpoint seems to have several advantages over that of Quine. In the first place, the role of set theory in natural science has never been explicitly demonstrated; Quine's argument is roughly that science depends upon mathematics and mathematics upon set theory. A plausible reply (which I endorse) is that science may not depend upon *all* of mathematics, and that many of the most implausible, unempirical and logically objectionable features of set theory (e.g. the strong set-existence axioms such as the comprehension and power-set axioms) may be needed only for the derivation of parts of mathematics which do *not* play any essential role in physical science. The empirical basis of set theory thus remains somewhat tenuous, while the empirical basis for Platonism with regard to properties and relations may be presented in a straightforward hypothetico-deductive manner, as here for T_Q.

A further advantage of the present view over set-theoretic Platonism is that it is possible to identify particular empirical facts and laws as constituting the empirical basis for particular Platonist existence propositions, in keeping with the anti-holist confirmation theory of Glymour, 1980, and in contrast with the reliance of set-theoretic Platonism upon an amorphous holism. This is evident, for example, in the fact that the evidence for the existence of each separate ray Q_q in the field of $M(\leqslant)$ consists of distinct empirical laws such as those expressed in Ths. (1)–(7) of T_Q, involving the distinct empirical primitives $\leqslant_q$ and $*_q$. Moreover, distinct empirical laws provide evidence for distinct axioms of T_Q.

Finally, from a purely formal viewpoint, elementary second-order Platonist theories such as T_Q are mathematically much weaker than any standard axiomatic set theory, hence more parsimonious and more likely to be consistent.

One of the most interesting features of naturalistic Platonism is that it may provide the basis for an objective distinction between accidental generalizations and *laws of nature*. The idea that laws of nature are second-order relations among first-order universals has formed the basis for several recent analyses of laws of nature (Armstrong 1983, Dretske 1977, Tooley 1977, Swoyer 1982). I believe that while this basic approach is sound, the existing proposals suffer from two general defects.

First, it seems that these analyses do not go far enough, in that they appear to retain certain modal concepts as primitive *in addition to* the second-order structure which they introduce. The fundamental problem (cf. Salmon, 1976) is to provide the concepts of natural law of physical necessity with an *empirical basis*, not merely to give some philosophically satisfying analysis of them in terms of possible worlds, second-order necessitation relations, or any other concepts as devoid of empirical basis as that of physical necessity itself. The crucial problem is to find a way of breaking *out* of this circle of interdefinable but empirically empty concepts to establish a link to what is actual and observable. The presence of second-order structure in the language of science may provide the necessary link. To solve the problem of physical modality using this link, it is necessary to *define* the physical modalities completely in terms of second-order aspects of scientific theory.

Second, I think that the existing proposals do not sufficiently address the issue of the *empirical foundation* for the second-order structure which they attribute to the world. They seem to simply *attribute* to scientific theory or to the world such second-order features as will suffice to yield a philosophically satisfactory analysis of physical modality, without offering any justification for this attribution beyond the fact that it does yield such an analysis. This seems to bypass the essential point: to solve the problem of physical modality in second-order terms we must not only *define* 'natural law' in second-order terms, but must also show how the resulting second-order propositions expressing the desired theory of natural laws have an *empirical foundation* in the actual facts of the actual world, independent of any philosophical theory. To ground physical modality in second-order structure is of no value unless the second-order structure can itself be grounded in actual observable facts, as the theory T_Q is grounded.

The successful implementation of this program therefore depends essentially upon what type of second-order structure the world can be shown to possess. For this reason I think that it is premature to try to formulate specific second-order definitions of 'natural law', in terms for example of a second-order necessitation relation (as proposed by most of the cited authors). Only through careful analysis of existing scientific theory can we arrive at any well-founded views as to the actual second-order structure of the world, in terms of which the analysis of physical modality may be carried out. Here I will merely mention one general line of analysis suggested by T_Q.

Recall the discussion in Section 2 of the existence assumptions of a first-order theory of quantity. We noted a natural sense in which the assumption of the existence of sums which a first-order theory expresses using existential quantification over objects ("for any two objects there exists an object equal to their sum") is better understood as having a *modal* content, as asserting the *physical possibility* of the existence of such an object, not the actual fact. At the same time, we saw that the deductive role played by these tacitly modal propositions in the first-order theory of quantity (and perhaps their intuitive content as well) can be fully taken over by the *non-modal* second-order Axiom (2b) of T_Q asserting the *actual existence* of the first-order magnitudes corresponding to such sums.

Briefly then, we could say that the *possible* existence of an *object* may be analyzed as the *actual* existence of the first-order *property* characteristic of that object. The present analysis of the theory T_Q shows that such second-order existence propositions play a perfectly definite and legitimate deductive role in second-order scientific theories, and may be regarded as quite solidly based upon empirical evidence provided by empirical confirmation, in actual instances, of their first-order deductive consequences. In other words, a full and complete second-order theory of what first-order *properties* exist may be founded upon observations of the *actual* world, together with ordinary scientific principles governing the construction of simple and general theories to unify and deductively explain what is observed. 'Existence' here means simply what is expressed by the ordinary non-modal existential quantifier of elementary second-order logic, for which we can formulate a complete set of inference rules and whose scientific significance is arguable fully expressed by those inference rules; no modal concepts are involved. And yet these second-order existence propositions seem in some way to correspond to what in first-order terms appears as an assertion of *physical possibility*. This observation seems to provide substantial support for the view that the presence of second-order structure in scientific theories provides the empirical basis for propositions involving physical modality.

NOTES

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[1] The present distinction between first- and second-order languages is a purely syntactical one, referring only to the primitive predicates and atomic formulas of the language L, and should not be confused with the distinction between elementary and non-elementary *logics* which is often also expressed using the terms 'first-order' and 'second-order'. An *elementary logic* is one whose logical connectives are just those of standard predicate logic, i.e. truth-functional propositional connectives and the standard universal and existential quantifiers. (If the language is atomically second-order or higher-order the models and semantics of the quantifiers are assumed to be those of general or Henkin models, as outlined below for the particular second-order language L used here.) The Gödel completeness theorem and the compactness theorem hold for all elementary logics, including atomically higher-order ones. (See Van Benthem and Doets, 1983).

The existing first-order formal theories of quantity are also *non-elementary*: they employ non-elementary logical connectives in order to state a single axiom (usually a form of Archimedean axiom) which cannot be expressed in logically elementary terms. This axiom is necessary in order to derive the existence of a numerical scale which is both an additive homomorphism and an order *iso*morphism from the model to the real number system. I have argued in Mundy (a) and (b) that a theory of quantity does not require this axiom, because it is not necessary or appropriate to assume our actual numerical scales to be order isomorphisms. The argument involves an analysis of the concept of representation, leading to a weaker set of conditions on a numerical scale. (These conditions are equivalent to the concept of *weak extensive measurement* introduced in Holman, 1969 and discussed in Colonius, 1978, in which a scale is required only to be an order homomorphism, not an order isomorphism.) The existence of a real-number representation in this weaker sense may be derived from logically elementary axioms. This method of arriving at a logically elementary theory of quantity will be used here; it may be employed equally well in either the atomically first-order framework of standard measurement theory or the present atomically second-order framework.

[2] The assumption of closure under addition may be replaced in the standard representation theorems by assumptions of closure under other operations such as subtraction. However, these assumptions do not appear to be any more likely to be empirically correct when the variables are taken as ranging only over actual objects than does the assumption of additive closure. (In the case of mass, for example, there are elementary particles of differing masses for which the mass difference is much smaller than the smallest known non-zero mass, that of the electron.) The situation may change if one allows a non-standard ontology. For example the theory of quantity in Field, 1980, is based on an ontology which rejects ordinary objects altogether and uses only quantitative properties of space-time points or regions, interpreted as values of fields at those points or in those regions. In this context the assumption of closure under subtraction is a plausible consequence of the continuity of the field, which will result in pairs of nearby points for which the difference in field values is as small as desired. However, this method of resolving the problem depends crucially upon the acceptance of this special ontology of classical field theory and its associated strong existence assumptions. This ontology is at least as complicated formally as the second-order one proposed here, and in certain cases (such as that of the mass spectrum in elementary particle physics) appears to be unsuitable for current physical theory.

[3] "So when magnitudes, which are always found to be relations exhibited in the physical operations of things, are invoked as the locus of those operations, it seems legitimate to

ask what empirical difference their existence or nonexistence as 'common essences' would make." (Danto and Morgenbesser, reprint, p. 132).

4 Hölder's system is criticized (reprint, pp. 36–37) for treating equality of magnitudes as strict identity and for presenting a categorical axiomatization, both of which are appropriate on a second-order view which takes the axioms to describe the second-order structure of a system of first-order quantitative properties.

5 "...the point of a theory of measurement is to lay bare the structure of a collection of empirical relations which may be used to measure the characteristic of empirical phenomena corresponding to the concept. Why a collection of relations? From an abstract standpoint, a set of empirical data consists of a collection of relations between specified objects." (Reprint, p. 46).

6 It may also be significant that this body of literature has been developed mainly by psychologists concerned to arrive at a firm basis for the use of quantitative methods in psychology, not by authors primarily concerned with the foundations of physical science. The status of psychology as a last stronghold of positivism has often been noted.

7 Ellis, 1966, while not lying within the mainstream tradition of representational measurement theory, explicitly aims to give "a consistent positivist account of the nature of measurement" (p. 3). Field, 1980, p. 55 mentions briefly the possibility of a second-order theory of quantity but does not pursue it.

8 For x in R we have $-\infty < x < +\infty$. Addition R' is defined so that $x + \pm\infty = \pm\infty$, where x is any finite element or is the element $\pm\infty$; only the sum of $+\infty$ and $-\infty$ fails to be defined.

9 Some logicians seem to regard non-elementary "standard" second-order logic as the most natural and important form of second-order logic, with Henkin's elementary version being of little intrinsic interest except for the convenient metatheoretic properties which it possesses. From a naturalistic Platonist viewpoint the reverse is true.

10 The formalization of naturalistic Platonist theories within elementary second-order logic seems to me to be a useful means of clarifying their content. For example, Armstrong 1978 contains extensive discussion of various infinite regresses supposedly involved in the postulation of universals, brought about by asking what relation the universals bear to the particulars to which they apply. I think that these regresses disappear when the theory is formalized in a second-order language, since we see that this "relation of instantiation" plays a quite different formal role in the theory than do the universals (properties and relations) which are designated by the variables and constants of the second-order language. It is only the latter which are assumed to *exist* by the theory; the former corresponds syntactically to a formation-rule rather than a term, and cannot coherently be construed as one of the universals to which the theory itself applies.

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