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Planar Graphs and Poset Dimension

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Abstract. We view the incidence relation of a graph $G = (V, E)$ as an order relation on its vertices and edges, i.e. $a <_G b$ if and only if a is a vertex and b is an edge incident on a. This leads to the definition of the *order-dimension* of G as the minimum number of total orders on $V \cup E$ whose intersection is $<_G$. Our main result is the characterization of planar graphs as the graphs whose order-dimension does not exceed three. Strong versions of several known properties of planar graphs are implied by this characterization. These properties include: each planar graph has arboricity at most three and each planar graph has a plane embedding whose edges are straight line segments. A nice feature of this embedding is that the coordinates of the vertices have a purely combinatorial meaning.

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1. Introduction

For a poset $(X, <)$ consisting of a set X and a (transitive, irreflexive) order-relation $<$ on X, a *realizer* is a nonempty set of total orders on X whose intersection is the relation $<$.

E. Szpilrajn [17] has proved that each poset has a realizer. B. Dushnik and E. W. Miller [3] have defined the *dimension* of a poset P (*dim P*) as the minimum cardinality of its realizers.

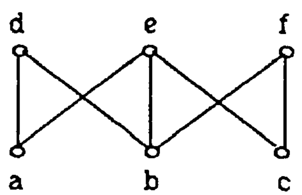

Fig. 1.

For example, the diagram shown in Figure 1 defines an order $<$ (of height 1) on the set $X = \{a, b, c, d, e, f\}$ by $x < y$ if and only if 'x is in the lower part, y is in the upper part and x, y are adjacent'. The dimension of $(X, <)$ cannot be 1 ($<$ is not total) and must therefore be 2 as $<$ is the intersection of the two total orders

$$<_1 : a\ b\ d\ c\ e\ f \quad \text{and} \quad <_2 : c\ b\ f\ a\ e\ d.$$

(Here as everywhere else in this paper, the order is from left to right.)

The importance of orders in mathematics has motivated numerous investigations of the dimension concept. For example, Komm [11] established very early that the dimension of the power-set of a set is the cardinality of this set and Hiraguchi [8] proved that, for $n \geqslant 4$, the dimension of a n-element poset is at most $n/2$. Later, Gysin [7] and Trotter, Moore, and Sumner [20] applied results of Gallai [5] to show that the dimension of a poset $(X, <)$ depends only of the underlying comparability graph (the graph where vertices $x, y \in X$ are adjacent if and only if $x < y$ or $y < x$), i.e. two posets with the same comparability graph have the same dimension. These are but a few of many results in this area. For more information we refer to the overviews in [18, 12, 22].

Although two-dimensional posets are well understood [3, 10, 19] and can be recognized in polynomial time [6], no simple characterization of d-dimensional posets is known for $d \geqslant 3$. Indeed, Yannakakis [23] has shown that deciding whether a poset has dimension d is a NP-complete problem, for all $d \geqslant 3$. Even the recognition of four-dimensional posets of height 1 is NP-complete. The complexity of the recognition of three-dimensional posets of height 1 remains an open problem.

One of the simplest examples of posets of height 1 is the poset $P(n, k)$ induced by the inclusion relation on the class of all 1- and k-element subsets of a n-element set. The study of this poset dates back to [3] proving $\dim P(n, n-1) = n$ and the lower bound $\dim P(n, 2) \geqslant \log_2 \log_2(n)$. These investigations were continued in [2] with a formula for $\dim P(n, k)$ when $2\lfloor\sqrt{n}\rfloor \leqslant k \leqslant n$. Eventually, Spencer [14] investigating the asymptotic behavior of $\dim P(n, k)$ has shown that $\dim P(n, 2)$ is asymptotically equal to $\log_2 \log_2(n)$.

In this paper, posets of height 1 are viewed as hypergraphs whose incidence relation is interpreted as order relation. For example, $P(n, k)$ is the complete k-uniform hypergraph on n vertices and $P(n, 2)$ is the complete graph K_n. We propose to extend the investigation of $P(n, 2)$ to general graphs, defining the *(order-)dimension* of a graph as the dimension of its incidence relation, and to try to relate the dimension of a graph to other graph-properties.

Our main result is that a graph has dimension at most three if and only if this graph is planar.

From this characterization we deduce that, given a maximal planar graph G on at least four vertices the poset consisting of the vertices, the edges and the faces of G ordered by the inclusion relation has dimension four. Removal of any face from this poset results in a three-dimensional poset. The hypergraph whose vertices are the vertices of G and whose hyperedges are the edges and faces of G has the same property.

Our result also implies strong versions of several known properties of planar graphs. For example, the property of having embeddings in the plane where edges are segments of straight lines [4, 16, 21]; here we can give a purely combinatorial meaning to the vertex coordinates of the embedding. Another property is the decomposition of maximal planar graphs in three edge disjoint trees [13, 9] that correspond to three-dimensional realizers.

Notice that the planarity of three-dimensional graphs ('only if' part of our characterization) has also been shown by Babai and Duffus [1] in a different context.

2. The Order-Dimension of a Graph

Throughout this paper, V is a finite nonempty set and $|V|$ denotes the cardinality of V. *Graphs* are finite simple graphs, their *edges* are identified with 2-element sets. A *triangle* of a graph G is a set of three pairwise adjacent vertices of G.

The symbols G and H are reserved for graphs, $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G; a formula $G=(V, E)$ means that $V=V(G)$ and $E=E(G)$. G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

If R is a binary relation on $V(G)$, an R-*path* in G is a path $v_0, v_1, \ldots, v_n$ of G such that $R(v_i, v_{i+1})$ holds for all i, $0 \leqslant i < n$. The *outdegree in* R of a vertex x of G is the number of neighbors y of x that satisfy $R(x, y)$.

DEFINITION. For a graph $G=(V, E)$ the partial order $<_G$ *on* $V \cup E$ is defined by $a <_G b \Leftrightarrow a \in V$ and $b \in E$ and $a \in b$.

The *(order-)dimension* $dim(G)$ of G is the dimension of the poset $(V \cup E, <_G)$.

Remark. If H is a subgraph of G, the relation $<_H$ is the restriction of $<_G$ to $V(H) \cup E(H)$. Therefore $\dim(H) \leqslant \dim(G)$, i.e., the dimension is a *monotone* function of graphs.

The definition of $\dim(G)$ relates $<_G$ to its realizers, thus to total orders on $V(G) \cup E(G)$. It will be useful to consider the restrictions of the latter orders to $V(G)$. Their essential properties are stated in the following lemma whose straightforward proof is left to the reader. An analogous statement for $P(n, k)$ can be found in [2].

LEMMA 2.1. *A graph G with vertex set V has $dim(G) \leqslant d$ if and only if there exists a sequence $<_1, <_2, \ldots, <_d$ of total orders on V satisfying the following conditions:*

(1) *the intersection of $<_1, <_2, \ldots, <_d$ is empty,*
(2) *for each edge $\{x, y\}$ and each vertex $z \notin \{x, y\}$ of G, there is at least one order $<_i$ in the sequence such that $x <_i z$ and $y <_i z$.*

Condition (1) is a consequence of condition (2) when the minimum degree of

G is at least 2. In this case, already the existence of d *partial* orders satisfying condition (2) implies $\dim(G) \leqslant d$.

In the remainder of this paper, the names *vertex property* and *edge property* will refer to (1) and (2), respectively.

DEFINITION. A *d-dimensional representation* on a set V is a sequence $<_1$, $<_2, \ldots, <_d$ of total orders on V that has the vertex property. This sequence *represents* all graphs with vertex set V for which the edge property holds. Among these graphs, the graph G with maximal edge set is the graph *induced* by $<_1, <_2, \ldots, <_d$.

The following examples and Proposition 2.4 illustrate the use of Lemma 2.1 and show some elementary properties of the dimension of graphs.

EXAMPLE 2.2. *A graph has dimension less than 3 if and only if it is a subgraph of a path* (notice that the only 1-dimensional graph is the isolated vertex): a two-dimensional representation on an n-element set V has the form

$$<_1: v_1 v_2 \ldots v_{n-1} v_n, \qquad <_2: v_n v_{n-1} \ldots v_2 v_1$$

and represents a graph $G = (V, E)$ iff $E \subseteq \{\{v_i, v_{i+1}\} \mid 1 \leqslant i < n\}$.

EXAMPLE 2.3. A three-dimensional representation of the triangle with vertices a, b, c is any sequence $<_1, <_2, <_3$ of total orders on $\{a, b, c\}$ such that each of a, b and c is the maximum of one of these orders.

PROPOSITION 2.4. *Each 4-colorable graph has dimension at most 4.*

Proof. Let $G = (V, E)$ be a four-colorable graph and consider a partition X, Y, Z, W of V in four color classes. Let X^+, Y^+, Z^+, W^+ denote any total orderings of X, Y, Z, W and X^-, Y^-, Z^-, W^- denote the inverse orderings. It is easy to see that the following four orderings of V form a four-dimensional representation of G.

$$\begin{aligned} &<_1: X^+ \ Y^+ \ Z^+ \ W^+, \\ &<_2: Y^- \ X^- \ W^- \ Z^-, \\ &<_3: W^+ \ Z^+ \ X^- \ Y^-, \\ &<_4: Z^- \ W^- \ Y^+ \ X^+ \end{aligned}$$

(the superscripts chosen in column 1 are irrelevant).

The converse of Proposition 2.4 is false. For example, it can be shown that $\dim(K_{12}) = 4$. (K_{13} is the first complete five-dimensional graph).

It will be proved in Section 4 that every three-dimensional graph is planar. Together with Proposition 2.4, this implies the existence of *graphs of arbitrarily high genus in dimension 4*.

The following observations (Lemma 2.5 and Corollary 2.6) show that each d-dimensional graph has *standard* representations. This fact has also been used in [2].

LEMMA 2.5. *Let* $<_1, <_2, \ldots, <_d$ *be a d-dimensional representation of* $G = (V, E)$ *and x be the maximum of* $<_k$. *For* $i \neq k$, *let* $<_i'$ *be the total order on V where x precedes all elements of* $V - \{x\}$ *and the elements of* $V - \{x\}$ *are ordered by* $<_i$. *Then* $<_1', \ldots, <_{k-1}', <_k, <_{k+1}', \ldots, <_d'$ *is also a d-dimensional representation of G.*

DEFINITION. A d-dimensional representation $<_1, <_2, \ldots, <_d$ on a set V is *standard* if $|V| \geqslant d$ and, for all $i \neq j$, the maximum element of $<_i$ is one of the $d-1$ smallest elements of $<_j$. The maxima of the orders of a standard d-dimensional representation are the *exterior* elements of this representation. The other elements of V are the *interior* elements.

Notice that, for $d \geqslant 3$, any two exterior elements of a standard d-dimensional representation are adjacent in the graph induced by this representation.

COROLLARY 2.6. *Each Graph G with at least d vertices and* $\dim(G) \leqslant d$ *has a standard d-dimensional representation.*

Proof. By d-fold application of Lemma 2.5 to an initial d-dimensional representation of G.

3. Three Dimensional Representations

For a sequence R_1, R_2, R_3 of binary relations on a set V, we define the *dual* sequence R_1^*, R_2^*, R_3^* by

$$R_k^*(x, y) \Leftrightarrow R_k(x, y) \; \& \; [R_i(y, x) \text{ for } i \neq k].$$

In particular, if $<_1, <_2, <_3$ is a three-dimensional representation on V, the relations $<_1^*$, $<_2^*$, $<_3^*$ are partial orders on V and any two distinct elements x and y of V are comparable in exactly one of these orders. Each edge of the graph G induced by $<_1, <_2, <_3$ receives therefore a unique *label* and *direction*. This decomposes G in three arc disjoint digraphs whose arc-sets are denoted A_1, A_2, A_3:

$$A_k = \{(x, y) \mid \{x, y\} \in E(G) \; \& \; x <_k^* y\}$$

and $E(G)$ is the disjoint union of the three underlying edge sets E_1, E_2, E_3 defined by $E_k := \{\{x, y\} \mid (x, y) \in A_k\}$. An immediate consequence of the edge and vertex properties is then Lemma 3.1.

LEMMA 3.1. *For a 3-dimensional representation* $<_1, <_2, <_3$ *on the set V, there holds:* $A_k = \{(x, y) \mid y \text{ is the minimum in } <_k \text{ of } \{v \in V \mid x <_k^* v\}\}$.

EXAMPLE 3.2. The three-dimensional representation $<_1: b\,c\,x\,y\,z\,a$, $<_2: c\,a\,z\,x\,y\,b$, $<_3: a\,b\,y\,z\,x\,c$ on $V = \{a, b, c, x, y, z\}$ induces the graph shown as Figure 2, where the thick lines correspond to A_1.

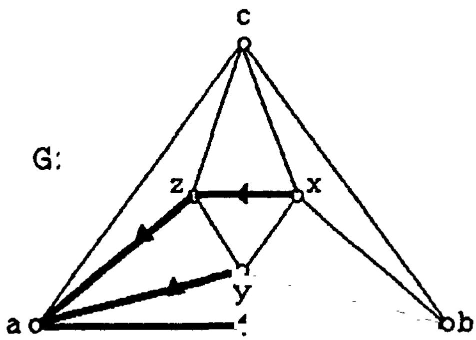

Fig. 2.

THEOREM 3.3. *For $k = 1, 2, 3$, the digraph (V, A_k) determined by a three-dimensional representation $<_1, <_2, <_3$ on V is a rooted forest (whose arcs are directed toward the roots). If the representation is standard and a_1, a_2, a_3 denote the respective maxima of $<_1, <_2, <_3$, then the component T_k of a_k in (V, A_k) has a_k as root and includes at least all interior elements of the representation as further vertices. The other components, if any, are trivial.*

Proof. By Lemma 3.1, the outdegree in (V, A_k) of a vertex $x \in V$ is at most 1 (it is 1 iff there exists a vertex $y \in V$ such that $x <_k^* y$). Hence, to each undirected cycle of the underlying graph (V, E_k) would correspond a directed cycle of (V, A_k). Since $<_k^*$ is an (acyclic) order relation, (V, E_k) is a forest. The same outdegree property of (V, A_k) implies that each connected component of (V, E_k) contains exactly one vertex whose outdegree in (V, A_k) is 0. This vertex is the root of the given component.

If $<_1, <_2, <_3$ is standard, the only vertices that may have outdegree 0 in (V, A_k) are the respective maxima a_1, a_2, a_3 of $<_1, <_2, <_3$. Of these, only a_k can have indegree greater than 0 in (V, A_k). Hence, (V, A_k) has at most one nontrivial component (rooted in a_k). This tree includes all interior elements of the representation.

Remark. For $i \neq k$ the element a_i is a vertex of T_k (the notation is the same as in Theorem 3.3) if and only if the edge $\{a_i, a_k\}$ is directed from a_i to a_k. Therefore, if $|V| = n$, *either the trees T_1, T_2, T_3 have each exactly $n - 1$ vertices, or the cardinalities of their vertex sets form a permutation of the numbers n, $n - 1$ and $n - 2$.*

We observe that Theorems 1 and 3 of [9] are a consequence of the last remark (combined with the characterization of planar graphs as graphs of dimension at most three).

COROLLARY 3.4. *A three-dimensional representation $<_1$, $<_2$, $<_3$ on a set V of cardinality $n \geq 3$ is standard if and only if the induced graph G has exactly $3n - 6$ edges.*

Proof. (1) The 'only if' statement is a direct consequence of the last remark. (2) Conversely, let a_1, a_2, a_3 denote the respective maxima of $<_1, <_2, <_3$. Suppose that $<_1, <_2, <_3$ is not standard. For example, there exist vertices $x, y \in V$ such that $x <_2 y <_2 a_1$. The vertex condition implies then that $a_1 \neq a_3$. One of the vertices x, y must be different from a_3. This vertex is in no order greater than both a_1 and a_3. Consequently $\{a_1, a_3\}$ is not an edge of G. However, applications of Lemma 2.5 to $<_1$ and $<_3$ result in a supergraph $H = (V, E')$ of G such that $\{a_1, a_3\} \in E'$. Part (1) of this proof and Corollary 2.6 imply then that $|E(G)| < |E'| \leqslant 3n - 6$.

4. Three Dimensional Graphs

The following notation and terminology concerning planar embeddings will be used in this and the next sections.

A *triangular graph* G is a maximal planar graph on at least three vertices that is embedded in the plane (i.e. whose exterior face has been chosen). The triangle of G whose edges form the boundary of the exterior face of G is the *exterior triangle* of G; its vertices and edges are the *exterior vertices and edges* of G, the other vertices and edges are the *interior vertices and edges* of G. An *elementary triangle* of G is a triangle whose edges form the boundary of an interior face of G. Given a cycle Z of G, the *region with boundary* Z is the subgraph of G induced by the vertices of Z and the vertices lying in the interior of Z. Therefore 'face' and 'region' have different meanings.

The notation xy (for $x, y \in \mathbb{R}^3$) represents the straight line segment with endpoints x and y. A *straight line embedding* of a graph G in a plane is an injection f of $V(G)$ in this plane such that for any two distinct edges $\{x, y\}$ and $\{u, v\}$ of G: $f(x)f(y) \cap f(u)f(v) = f(\{x, y\} \cap \{u, v\})$.

THEOREM 4.1. *Each graph $G = (V, E)$ of dimension at most three is planar. Moreover, to each three-dimensional representation $<_1, <_2, <_3$ of G corresponds a straight-line embedding f: $v \in V \to (v_1, v_2) \in \mathbb{R}^2$ of G in the plane, such that for all $u, v \in V$: $u_i < v_i \Leftrightarrow u <_i v$ $(i = 1, 2)$.*

Proof. For $v \in V$ and $i \in \{1, 2\}$, let v_i be the power of 2 whose exponent is the ordinal of v with respect to $<_i$.

It suffices to verify that the so defined mapping $f: V \to \mathbb{R}^2$ is a straight line embedding of G in the plane. To simplify the notation, the same symbol will be used to denote a vertex of V and its image under f.

Note that, by the definition of f, if $x, y \in V$ satisfy $y <_1 x$ and $x <_2 y$, no vertex $z \neq x, y$ will be mapped by f in the (closed) triangle delimited by the points x, y and (x_1, y_2) shown in Figure 3, since each point z of this triangle satisfies

$$\frac{x_1}{2} < z_1 \leqslant x_1 \quad \text{or} \quad \frac{y_2}{2} < z_2 \leqslant y_2.$$

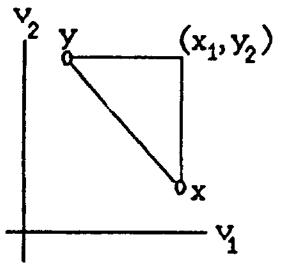

Fig. 3.

Assume, for contradiction, that there exist two disjoint edges $\{x, y\}$ and $\{a, b\}$ such that $xy \cap ab \neq \emptyset$. (The case of nondisjoint edges has a similar proof.)

Let, for example, x be the maximum of $\{x, y, a, b\}$ with respect to $<_1$. By the edge property, $y >_i a, b,\ a >_j x, y,\ b >_k x, y$ must hold for some choice of $i, j, k \in \{1, 2, 3\}$.

$xy \cap ab \neq \emptyset$ and the maximality of x in $<_1$ imply $i, j, k \geqslant 2$. Thus $j = k$ and therefore $j = k = 3$ and $i = 2$. Then $y >_2 a, b$ and $xy \cap ab \neq \emptyset$ imply $y >_2 x$.

Consequently $x >_1 y$, $y >_2 x$ and $y_2 > a_2, b_2$, $x_1 > a_1, b_1$. Therefore, a and b are both on the same side (below) of the straight line through x and y. Hence, $xy \cap ab = \emptyset$ in contradiction to the original assumption.

By Corollary 3.4 and Theorem 4.1, a three-dimensional representation $<_1, <_2, <_3$ on a set V of cardinality $|V| \geqslant 3$ induces a maximal planar graph if and only if this representation is standard. In this case, we refer to the graph G induced by $<_1, <_2, <_3$ together with its planar embedding defined in the proof of Theorem 4.1 as the *triangular graph induced by* $<_1, <_2, <_3$. Elementary geometric considerations show that the exterior vertices of G are precisely the exterior elements of the representation. This justifies the terminology introduced in the definition of standard representations.

As Lemma 4.2, we now prove the converse of Theorem 3.3. For later use, it is convenient to formulate this lemma in terms of acyclic relations.

LEMMA 4.2. *Let G be a triangular graph whose vertex set V has cardinality $|V| \geqslant 4$ and $<_1, <_2, <_3$ be three acyclic relations on V such that every interior vertex of G has outdegree exactly one in each of $<_1^*$, $<_2^*$, $<_3^*$. Then for each edge $\{x, y\}$ of G and each vertex $z \notin \{x, y\}$ there exist, for some $k \in \{1, 2, 3\}$, a $<_k$-path from x to z and a $<_k$-path from y to z. Therefore, any three total orders extending $<_1, <_2, <_3$ on V form a three-dimensional representation of G.*

Proof. Notice that for $i \neq j$, a $<_i^*$-path and a $<_j^*$-path starting in the same vertex v have no common vertex except v as the existence of such a common vertex u would imply a $<_i$-cycle from v to v via u.

Let v be any interior vertex of G. By the above, the (unique longest) $<_1^*$, $<_2^*$ and $<_3^*$-paths starting at v end in distinct exterior vertices that will be denoted respectively by a_1, a_2, a_3. In particular, this implies the existence of some $<_i$-path from a_j to a_i for any choice of $j \neq i$.

If $u \neq v$ is any further interior vertex of G, the (unique longest) $<_i^*$-path starting at u must also end at a_i, as an ending in a_j for $j \neq i$ would imply the existence of some $<_k^*$-path $(k \neq i)$ from u to a_i and therefore the existence of a $<_i$-cycle from a_i to a_i via u, a_j and v (Figure 4).

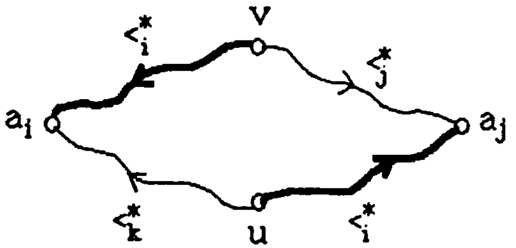

Fig. 4.

Therefore there exists a $<_i$-path from each vertex $u \neq a_i$ of G to a_i. This proves the lemma for the case where $z \in \{a_1, a_2, a_3\}$.

If $z \notin \{a_1, a_2, a_3\}$, the $<_1^*$, $<_2^*$ and $<_3^*$-paths from z to a_1, a_2 and a_3 divide G in three regions R_1, R_2 and R_3 (where R_k denotes the region opposite a_k, including its boundary) (Figure 5).

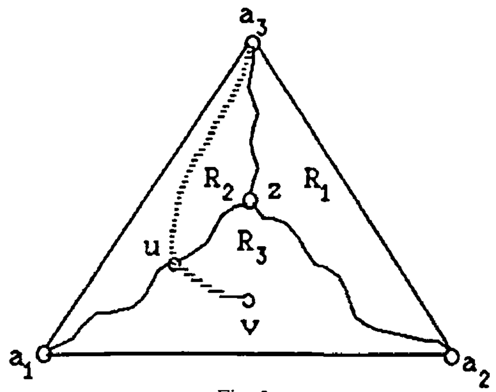

Fig. 5.

As the end-vertices x and y of each edge $\{x, y\}$ of G are both in the same region, it suffices to show that there exists for each vertex v of G, $v \in R_k$, a $<_k$-path from v to z.

Let, for example, $v \in R_3$ and consider some $<_3$-path from v to a_3. This path must intersect the $<_1^*$-path from z to a_1 or the $<_2^*$-path from z to a_2 in a vertex u. Suppose, for example, that u belongs to the $<_1^*$-path from z to a_1. Combining the two paths, there results a $<_3$-path from v to z via u.

5. Barycentric Representations

It has been shown in Section 4 that the graph G induced by a three-dimensional representation $<_1, <_2, <_3$ is always planar. It is therefore tempting to look for an embedding of G in a plane (or sphere) in $\mathbb{R}^3$ such that $<_1, <_2$ and $<_3$ are precisely the orders defined on the vertices by their three coordinates. Such an embedding however, does not exist in general, as illustrated (for planes) by Example 5.1.

EXAMPLE 5.1. Let $V = \{a, b, c, x, y, z\}$ and $<_1$, $<_2$, $<_3$ be the three-dimensional representation defined on V by

$<_1: a\ x\ b\ z\ c\ y,$
$<_2: b\ y\ c\ x\ a\ z,$
$<_3: c\ z\ a\ y\ b\ x.$

There exists no mapping $f: v \in V \to (v_1, v_2, v_3) \in \mathbb{R}^3$ such that

(1) $f(V)$ is contained in a plane,
(2) for all $u, v \in V$: $u_i < v_i \Leftrightarrow u <_i v \quad (i = 1, 2, 3)$.

Since for $i = 1$, 2, 3 there exist elements $u, v \in V$ with $u <_i^* v$, the coefficients α_1, α_2, α_3 of the equation $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \delta$ of such a plane would have to be of the same sign. Replacing in each row $<_i$ of the above matrix every element v by the product $\alpha_i v_i$ and comparing the sums of columns 1, 3, 5 and columns 2, 4, 6 would then lead to the contradiction $3\delta \neq 3\delta$.

Nevertheless, Lemma 4.2 implies that the edge and vertex properties of a standard three-dimensional representation are essentially determined by the restrictions of its orders to the edges of the induced graph. This suggests a relaxation of condition (2) of the last example and will result (Theorem 5.3 and Corollary 5.4) in the correct interpretation of three-dimensional representations by coordinates.

DEFINITION. A *barycentric embedding* of a graph G is an injective function $v \in V(G) \to (v_1, v_2, v_3) \in \mathbb{R}^3$ that satisfies the conditions:

(1) $v_1 + v_2 + v_3 = 1$ for all vertices v,
(2) For each edge $\{x, y\}$ and each vertex $z \notin \{x, y\}$, there is some $k \in \{1, 2, 3\}$ such that $x_k < z_k$ and $y_k < z_k$.

LEMMA 5.2. *Each barycentric embedding of a graph is a straight line embedding of this graph in the plane* $v_1 + v_2 + v_3 = 1$.

Proof. By applying the proof of Theorem 4.1 to the 'projection' $f: v \in V(G) \to (v_1, v_2) \in \mathbb{R}^2$ of an initial barycentric embedding of the graph G. Observe that if $\{x, y\}$ is an edge of G and $y_1 \leqslant x_1$, $x_2 \leqslant y_2$, no vertex z of G will be mapped by f in the convex hull of the points (x_1, x_2), (y_1, y_2), (x_1, y_2).

THEOREM 5.3. *Let G be the triangular graph induced by a standard three-*

dimensional representation $<_1, <_2, <_3$ *on a set* V *and let* a^1, a^2, a^3 *be the respective maxima of* $<_1, <_2, <_3$. *Then there exists a straight line embedding of* G *such that* v_1, v_2, v_3 *denoting the barycentric coordinates of vertices* $v \in V$ *relative to* a^1, a^2, a^3, *there holds for each interior edge* $\{u, v\}$ *of* G: $u <_i v \Leftrightarrow u_i < v_i$ $(i = 1, 2, 3)$.

Proof. Let n denote the cardinality of V. It suffices to define a mapping f: $v \in V \rightarrow (v_1, v_2, v_3) \in \mathbb{R}^3$ with the properties:

(0) $f(a^1) = (2n-5, 0, 0), f(a^2) = (0, 2n-5, 0), f(a^3) = (0, 0, 2n-5)$,
(1) $v_1, v_2, v_3 \geqslant 0$ and $v_1 + v_2 + v_3 = 2n - 5$ for all vertices v,
(2) $u <_i v \Rightarrow u_i < v_i$ for all interior edges $\{u, v\}$ of G and $i = 1, 2, 3$.

By Lemmas 4.2 and 5.2, the mapping resulting from f upon division by $2n - 5$ satisfies the statement of the proposition.

Using the notation of the proof of Lemma 4.2, recall that each interior vertex v divides G in three regions $R_1(v)$, $R_2(v)$, $R_3(v)$ and that each vertex $u \neq v$ of region $R_k(v)$ satisfies $u <_k v$ $(k = 1, 2, 3)$.

$f(a^1)$, $f(a^2)$, $f(a^3)$ are defined by (0) above. For an interior vertex v of G, we let v_i $(i = 1, 2, 3)$ be the number of elementary triangles in region $R_i(v)$. This clearly satisfies condition (1).

To verify condition (2), consider an interior edge $\{u, v\}$ of G and assume, for example, that $u <_3^* v$. To show: $u_3 < v_3$, $v_1 < u_1$, $v_2 < u_2$. This is trivial if $v = a^3$. Suppose, therefore, that $v \neq a^3$. Then v is an interior vertex of G and u must lie in the interior of region $R_3(v)$ (Figure 6).

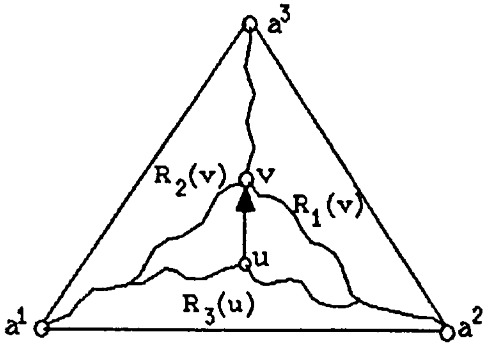

Fig. 6.

For $i = 1$ and $j = 2$ (or $i = 2$ and $j = 1$), the $<_i^*$-path from u to a^i and the $<_j^*$-path from v to a^j have no common vertex, as the existence of such a vertex x would imply $u \leqslant_i x \leqslant_i v$, contradicting $u <_3^* v$. The outdegree (at most 1) properties of $<_1^*$ and $<_2^*$ imply thus: $R_3(u) \subseteq R_3(v)$, $R_1(v) \subseteq R_1(u)$, and $R_2(v) \subseteq R_2(u)$. These inclusions being proper, there follows $u_3 < v_3$, $v_1 < u_1$, $v_2 < u_2$.

As each three-dimensional representation of a graph G can be extended, by adjunction of three new exterior elements, to a standard representation of a triangular supergraph of G, we obtain

COROLLARY 5.4. *To each three-dimensional representation* $<_1, <_2, <_3$ *of a graph G corresponds a barycentric embedding* $v \in V(G) \rightarrow (v_1, v_2, v_3) \in \mathbb{R}^3$ *of G such that* $u <_i v \Leftrightarrow u_i < v_i$ *for all edges* $\{u, v\}$ *of G and* $i = 1, 2, 3$.

6. Planar Graphs have Dimension at Most Three

We now show that each planar graph has dimension at most three. By Theorem 4.1 or Corollary 5.4, this fact implies, in particular, that each planar graph has straight line embeddings, a result of Fáry [4], Stein [16] and Wagner [21]. Our proof of three-dimensionality is based on their methods, as analyzed by Kampen [9].

Two independent versions are given: construction of a straight line embedding in this section and a coordinate-free proof in Section 7.

We first review the method of edge contraction. For a vertex x of a graph G, $N(x)$ denotes the set of neighbors of x in G. If $\{x, y\}$ is an edge of G, the *contracted graph* $G/(x, y)$ is obtained from G by removal of the vertex y and the edges incident on y and by introduction of an edge $\{x, z\}$ for each vertex $z \in N(y) - N(x)$. The edge $\{x, y\}$ is *contractible* if x and y have exactly two common neighbors. If G is a maximal planar graph on at least four vertices and $\{x, y\}$ is a contractible edge of G, then $G/(x, y)$ is a maximal planar graph.

LEMMA 6.1 [9]. *Let G be a triangular graph on* $n \geqslant 4$ *vertices. If a, b and c denote the exterior vertices of G, then there exists a neighbor* $x \neq a, b$ *of c such that the edge* $\{c, x\}$ *is contractible.*

Thus to each triangular graph corresponds a sequence of 'allowed contractions' transforming this graph into a triangle.

THEOREM 6.2. *Each planar graph has dimension at most three.*

Proof. By the monotonicity of the dimension, it suffices to prove the theorem for triangular graphs G. Let V be the vertex set of G and let a, b, c denote the exterior vertices of G in counterclockwise order. We show the existence of an embedding of V in the plane such that (identifying the vertices and their images):

(a) a, b, c are affinely independent and each vertex $v \in V$ lies in the triangle delimited by a, b and c.
(b) The partial orders $<_1, <_2, <_3$ defined on V by the barycentric coordinates v_1, v_2, v_3 of the vertices $v \in V$ with respect to a, b, c ($u <_i v \Leftrightarrow u_i < v_i$) satisfy the hypothesis of Lemma 4.2 (if $|V| \geqslant 4$).

The proof proceeds by induction on the number $n = |V|$. The claim is trivial if $n = 3$. Let $n \geqslant 4$ and assume that the claim is true for all triangular graphs with less than n vertices.

Consider some neighbor $x \neq a, b$ of c such that the edge $\{c, x\}$ is contractible and let $p = v^1, v^2, \ldots, v^r = q$ be the neighbors of x, distinct from c, in counter-clockwise order. The vertices p and q are the two common neighbors of c and x (Figure 7).

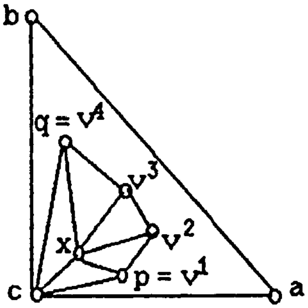

Fig. 7.

By induction hypothesis, there is an embedding f of $V - \{x\}$ in the plane with the above properties (a) and (b). Lemma 4.2 implies that f is a barycentric embedding. By Lemma 5.2, this embedding is therefore a straight line embedding of $G/(c, x)$ (Figure 8).

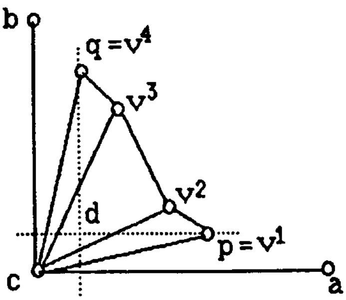

Fig. 8.

Notice that as $c_3 > u_3$ for all $u \in V - \{c\}$ and f is a straight line embedding of G, condition (2) of the definition of barycentric embeddings applied to the wheel of c implies that

$$p_2 = v_2^1 < v_2^2 < \cdots < v_2^r = q_2 \quad \text{and} \quad q_1 = v_1^r < v_1^{r-1} < \cdots < v_1^1 = p_1.$$

Let d denote the point with barycentric coordinates $(q_1, p_2, 1 - q_1 - p_2)$. By the above inequalities, there is a choice of x sufficiently close from d in the quarter-plane $u_1 > q_1$, $u_2 > p_2$ that will satisfy the relations: $x_3 > v_3^i$ $(i = 1, 2, \ldots, r)$, $x_1 > q_1, x_2 > p_2, x_1 < v_1^i$ and $x_2 < v_2^i$ $(i = 2, \ldots, r - 1)$.

For $i = 1, 2, 3$ let $<_i$ denote the partial order defined on V by $u <_i v \Leftrightarrow u_i < v_i$. By the choice of x, we have $x <_1^* p$, $x <_2^* q$, $x <_3^* c$ and $v^i <_3^* x$ for all $i \neq 1, r$. Therefore, the outdegree of x in each of the relations $<_1^*$, $<_2^*$, $<_3^*$ is exactly one. Using the induction hypothesis, it is easy to verify that the outdegrees of each interior vertex $y \neq x$ of G in the relations $<_1^*$, $<_2^*$, $<_3^*$ are exactly one (notice that for $i = 2, \ldots, r-1$ the edge $\{v^i, c\}$ of $G/(c, x)$ has been replaced in G by the edge $\{v^i, x\}$ and that there holds $v^i <_3^* x$).

Observe that we placed x *close* to d and *far* from c. The standard proof that G has a straight line embedding would position x arbitrarily close to the vertex c.

Remark. If $<_1$, $<_2$, $<_3$ is a standard three-dimensional representation on a set V of cardinality $|V| \geqslant 4$ and c is the maximum of $<_3$, then the maximum x of $V - \{c\}$ in $<_3$ is a neighbor of c and the edge $\{c, x\}$ is contractible. This fact could have been used to prove Theorem 5.3 by suitably modifying the proof of Theorem 6.2. However, the resulting proof would not provide the interpretation of barycentric coordinates as counts of elementary triangles.

Theorem 3.3 has shown that each three-dimensional representation of a triangular graph decomposes this graph in three edge-disjoint directed trees whose vertices include all interior vertices of the graph. However, not every such tree-decomposition corresponds to a three-dimensional representation as the acyclicity of the induced relations $<_1$, $<_2$, $<_3$ is not guaranteed. This acyclicity was ensured in the proof of Theorem 6.2 by the use of coordinates. In the next section, we present the topological properties of the tree-decomposition that correspond to the above acyclicity. Rather than using trees, however, it will be convenient to apply the equivalent notion of labelings of angles.

7. Labeling the Angles of a Triangular Graph

The *angles* of a triangular graph G are the angles of its elementary triangles. If G is induced by a standard three-dimensional representation $<_1$, $<_2$, $<_3$, then each angle $\angle(xy, xz)$ of G determines a unique *label* $k \in \{1, 2, 3\}$ such that $x >_k y, z$ (since $\{x, y, z\}$ is a triangle and the inequalities $y >_j x, z$ and $z >_i x, y$ must also be satisfied).

The representation $<_1$, $<_2$, $<_3$ *induces* in this way a labeling of the angles of G with the labels 1, 2, 3.

EXAMPLE 7.1. The representation $<_1$: $b\ c\ x\ y\ z\ a$, $<_2$: $c\ a\ z\ x\ y\ b$, $<_3$: $a\ b\ y\ z\ x\ c$ on $V = \{a, b, c, x, y, z\}$ induces the labeled graph given as Figure 9.

The essential properties of this labeling are summarized in the following definition.

DEFINITION. A *normal labeling* of a triangular graph G is a labeling of the angles of G with the labels 1, 2, 3 that satisfies the conditions:

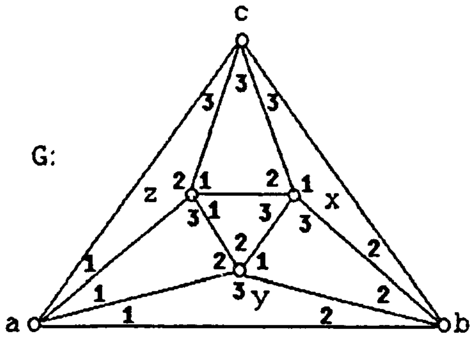

Fig. 9.

(1) Each elementary triangle of G has an angle labeled 1, an angle labeled 2 and an angle labeled 3. The corresponding vertices appear in counterclockwise order.
(2) The labels of the angles of an interior vertex x of G form, in counterclockwise order, an interval of 1's followed by an interval of 2's followed by an interval of 3's.

In the remainder of this section, the expression *labeled triangular graph* will be used to denote a triangular graph together with a normal labeling of its angles.

PROPOSITION 7.2. *The labeling induced by a standard three-dimensional representation is a normal labeling.*

Proof. Let G be the triangular graph induced by a standard three-dimensional representation $<_1, <_2, <_3$ and consider a straight line embedding of G with the properties shown in Proposition 5.3.

Define the counterclockwise orientation of the plane by the cycle a^1, a^2, a^3 (in this order). Property (1) of normal labelings is then trivially satisfied.

Let x be an interior vertex of G and let u^1, u^2, u^3 denote the (unique) neighbors of x with $x <_i^* u^i$. Notice that the ordering u^1, u^2, u^3 is counterclockwise. It is easy to see that all angles at x in a sector (xu^i, xu^{i+1}) have the same label $i+2$ (indices and labels are modulo 3). This shows that property (2) of normal labelings is also satisfied.

A vertex v belonging to a cycle Z in a labeled triangular graph will be said to be of *type* i ($i = 1, 2, 3$) with respect to Z if all angles at v, interior to Z, have the label i. The interpretation of three-dimensional representations by barycentric coordinates motivates the following lemma.

LEMMA 7.3. *Each cycle Z in a labeled triangular graph G has a vertex of type 1, a vertex of type 2 and a vertex of type 3.*

Proof. Each interior edge $\{u, v\}$ of G belongs to two elementary triangles

$\{x, u, v\}$ and $\{y, u, v\}$. The labels of the angles at u and v in these triangles will be called *labels of u and v along* $\{u, v\}$. The definition of a normal labeling implies that these labels have (up to a renaming of u and v) the form given in Figure 10, where i, j, k is a cyclic permutation of 1, 2, 3. Thus, the labels of u along $\{u, v\}$ are distinct from the labels of v along $\{u, v\}$.

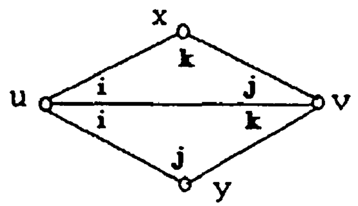

Fig. 10.

Assume that there exist cycles for which the statement of the lemma is false. Among these cycles, consider a cycle Z enclosing the minimum number n of elementary triangles. By definition $n \geqslant 2$. Suppose, for example that Z has no vertex of type 1.

Case 1 (Figure 11): There is an edge $\{u, v\}$ interior to Z whose vertices u and v are nonconsecutive vertices of Z. The edge $\{u, v\}$ divides then Z in two cycles Z_1 and Z_2, each containing less than n elementary triangles of G.

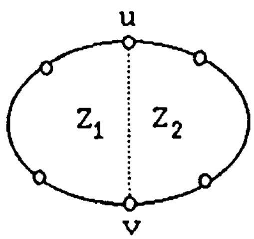

Fig 11.

Z_1 and Z_2 have vertices of type 1. As Z does not have such a vertex, one of the vertices u, v has type 1 in Z_1 and the other has type 1 in Z_2. Therefore the labels of u along $\{u, v\}$ and the labels of v along $\{u, v\}$ are not distinct, in contradiction to the preliminary observation.

Case 2 (Figure 12): No two nonconsecutive vertices of Z are joined by an edge interior to Z. Let u, v be counterclockwise consecutive vertices of Z and let x be the vertex in the interior of Z such that $\{x, u, v\}$ is an elementary triangle (observe that x is not a vertex of Z). Let the angles at u, v and x in this triangle be labeled i, j and k, respectively.

Then Z has a vertex of type k. Indeed, let Z' denote the cycle obtained from Z through replacement of the edge $\{u, v\}$ by the two edges $\{u, x\}$ and $\{x, v\}$. Z' encloses $n-1$ elementary triangles and has, therefore, a vertex w of type k. The remark preceding case 1 implies that $w \neq u, v$. The definition

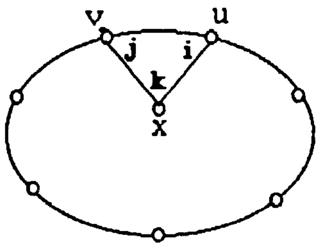

Fig. 12

of normal labelings implies that $w \neq x$. Therefore, w is a vertex of type k in the cycle Z.

Applying this argument to $i = 2$ and $j = 3$, it follows that Z has no vertex of type 2 or 3. Therefore Z has no vertex of type 1, 2, 3; again a contradiction to the above argument.

In particular, the exterior cycle of a labeled triangular graph G has a vertex of type 1, a vertex of type 2 and a vertex of type 3: the exterior vertices of G. It can easily be seen that these vertices appear in counterclockwise order. (For an illustration, see Example 7.1.)

PROPOSITION 7.4. *Let G be a labeled triangular graph and $<_1, <_2, <_3$ be the binary relations defined on $V(G)$ by: $x <_k y \Leftrightarrow$ there exists an elementary triangle $\{x, y, z\}$ of G such that the label of $\angle(yx, yz)$ is k. Then the relations $<_1, <_2, <_3$ are acyclic and any three total orders extending $<_1, <_2, <_3$ on $V(G)$ form a three-dimensional representation of G.*

Proof. Suppose that there exists a cycle $x = x_0 <_k x_1 <_k \cdots <_k x_n = x$ $(n \geqslant 1)$. The underlying cycle Z of G does not have any vertex of type k. This contradicts Lemma 7.3.

Condition (2) of the definition of a normal labeling implies that each interior vertex of G has outdegree exactly one in $<_k^*$ $(k = 1, 2, 3)$. Together with Lemma 4.2, this implies the second statement of Proposition 7.4.

We now give the second (coordinate-independent) version of the proof of Theorem 6.2.

THEOREM 7.5. *Each triangular graph has a normal labeling. Therefore, each planar graph has dimension at most three.*

Proof. The second statement of the theorem follows from the first combined with Proposition 7.4 and the monotonicity of the dimension.

We prove the first statement by induction on the number n of vertices of a triangular graph G. The case $n = 3$ is trivial. Let $n \geqslant 4$ and assume that the theorem is true for all triangular graphs having less than n vertices.

Let a, b, c denote the exterior vertices of G and $x \neq b, c$ be a neighbor of a such that a and x have exactly two common neighbors. Let $a, v_1, v_2, \ldots, v_l$ be the vertices of the wheel of x, listed in counterclockwise order (Figure 13).

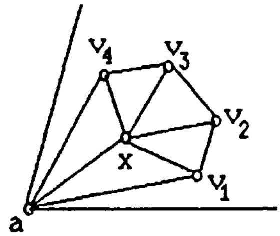

Fig. 13

By induction hypothesis, the graph $G/(a, x)$ has a normal labeling and we may assume that all angles at a have the label 1 (Figure 14).

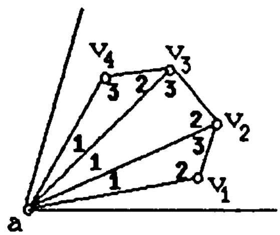

Fig. 14.

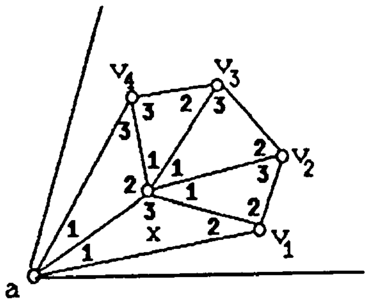

Fig. 15.

This labeling can trivially be transformed in a normal labeling of G by labeling the angles $\angle(xv_i, xv_{i+1})$ and $\angle(av_1, ax)$, $\angle(ax, av_r)$ with 1, as shown in Figure 15.

Notice that all normal labelings of a triangular graph can be obtained by the above method.

8. Adding the Faces: Hypergraphs

In this section, we consider (simple) *hypergraphs* $H=(V, E)$ consisting of a nonempty set V of *vertices* and a set E of *hyperedges* that are subsets of V having cardinality at least two. We call H a *closed* hypergraph if the implication $X \subseteq Y \Rightarrow |Y - X| = 1$ or $Y - X \in E$ holds for any two distinct hyperedges X and Y of H.

Each hypergraph $H = (V, E)$ determines two partial orders $<_H^\infty$ and $<_H^1$ on $V \cup E$. The order $<_H^\infty$ corresponds to the set inclusion (vertices being identified with singleton sets). The order $<_H^1$ corresponds to the incidence relation and is defined as for graphs. The superscripts ∞ and 1 indicate the upper bounds on the heights.

We therefore have two notions of order-dimension for a hypergraph H:

$$\dim_\infty(H) = \dim(V \cup E, <_H^\infty) \quad \text{and} \quad \dim_1(H) = \dim(V \cup E, <_H^1).$$

As a variant of Lemma 2.1, we obtain Lemma 8.1.

LEMMA 8.1. *A closed hypergraph H with vertex set V has $\dim_1(H) \leqslant d$ if and only if there exists a sequence $<_1, <_2, \ldots, <_d$ of total orders on V satisfying the following conditions:*

(1) *the intersection of $<_1, <_2, \ldots, <_d$ is empty,*
(2) *for each hyperedge X and each vertex $y \notin X$ of H, there is at least one order $<_i$ in the sequence such that $x <_i y$ for all $x \in X$.*

Proof. The 'only if' part is trivial. The 'if' part is proved by extending a sequence $<_1, <_2, \ldots, <_d$ of total orders on V that satisfies conditions (1) and (2) to a sequence $<_1', <_2', \ldots, <_d'$ of total orders on $V \cup E$ whose intersection is $<_H^1$.

The order $<_i'$ is obtained by insertion in $<_i$ of each hyperedge X just after its maximum (with respect to $<_i$). Hyperedges having the same maximum are inserted in order of decreasing cardinalities.

We show, for example, that distinct hyperedges are incomparable in the intersection of $<_1', <_2', \ldots, <_d'$. That is, given any two distinct hyperedges X and Y, there must exist an order $<_i'$ such that $X <_i' Y$.

Case 1: $Y - X \neq \emptyset$. Then there exists a vertex $y \in Y$ with $y \notin X$. By (2) there is an order $<_i$ such that $x <_i y$ for all $x \in X$. Hence, $X <_i' Y$.

Case 2: $Y - X = \emptyset$. Therefore $Y \subset X$. Since H is a closed hypergraph X is a disjoint union $X = Y \cup Z$, where $Z \in E$ or $Z = \{z\}$ for some $z \in V$. Let y be any element of Y. By (2) or (1) there is an order $<_i$ such that $z <_i y$ for all $z \in Z$. Hence, X and Y have the same maximum in this order $<_i$. Since $|Y| < |X|$, there follows $X <_i' Y$.

Remark. This lemma remains true if $\dim_1(H)$ is replaced by $\dim_\infty(H)$ (even when H is not closed). Therefore, if H is a closed hypergraph there holds $\dim_1(H) = \dim_\infty(H)$. We denote this common value with $\dim(H)$.

Let now G be a maximal planar graph on at least four vertices and let $H(G)$ be the hypergraph whose vertex set is $V(G)$ and whose hyperedges are the edges and faces of G. With $H^-(G)$ we denote the hypergraph obtained from

$H(G)$ by elimination of one of the faces of G. Then $H(G)$ and $H^-(G)$ are closed hypergraphs and dim $H(G)$, dim $H^-(G)$ are well defined.

THEOREM 8.2. *If G be a maximal planar graph on at least four vertices, then dim* $H(G) = 4$ *and dim* $H^-(G) = 3$.

Proof. View G as a triangular graph whose exterior face is the face missing in $H^-(G)$. Let $<_1$, $<_2$, $<_3$ be a three-dimensional representation of G. From the proof of Lemma 4.2, it is easy to see that for each elementary triangle $\{x, y, z\}$ and each vertex $v \notin \{x, y, z\}$ of G there is one order $<_i$ of the representation satisfying the inequalities $x <_i v$, $y <_i v$ and $z <_i v$.

Thus the sequence $<_1$, $<_2$, $<_3$ satisfies conditions (1) and (2) of Lemma 8.1 with respect to $H^-(G)$. As dim $H^-(G) > 2$, there follows dim $H^-(G) = 3$.

Let $<_4$ be any total order on the vertex set V of G whose three smallest elements are the exterior elements of the representation $<_1$, $<_2$, $<_3$. Clearly, the sequence $<_1$, $<_2$, $<_3$, $<_4$ satisfies both conditions of Lemma 8.1 with respect to $H(G)$. Therefore, dim $H(G) \leqslant 4$.

Suppose that dim $H(G) \leqslant 3$. Then there exists a sequence $<_1'$, $<_2'$, $<_3'$ of total orders on V that satisfies both conditions of Lemma 8.1. This sequence is, therefore, a three-dimensional representation of G in the sense of Section 2, thus standard as G is maximal planar. Hence, no vertex of G is greater than the three exterior elements of $<_1'$, $<_2'$, $<_3'$ in any of these orders. As these three elements form a face of G this contradicts condition (2) of Lemma 8.1.

Remark. The inequality $\dim_\infty H(G) \geqslant 4$ holds for all polyhedra [15].

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