Euler's Invention of Integral Transforms

MICHAEL A. B. DEAKIN

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Abstract

EULER invented integral transforms in the context of second order differential equations. He used them in a fragment published in 1763 and in a chapter of *Institutiones Calculi Integralis* (1769). In introducing them he made use of earlier work in which a concept akin to the integral transform is implicit. It would, however, be reading too much into that earlier work to see it as contributing to the theory of the integral transform. Other work sometimes cited in this context in fact has different concerns.

1. Introduction

Integral transforms occur explicitly twice in EULER'S work. They are to be found in a fragment (EULER, 1763) devoted to a relatively specialised differential equation, and later in a chapter of *Institutiones Calculi Integralis* (EULER, 1769) where the treatment is more complete, more systematic and more general. That these papers involve integral transforms is quite clear and has long been recognised. (See, *e.g.*, DULAC, 1936.) However, claims have been made (DULAC, 1936; ENESTRÖM, 1914) that other papers of EULER'S, papers indeed predating those mentioned above, also involve integral transforms.

The purpose of the present paper is to dispute such claims. In three cases, we may (from our vantage point) see integral transforms as implicit in the analysis, and EULER by the time of his paper of 1763 seems also to have taken this view, but nowhere in these papers is the notion advanced as such, let alone studied in depth. In relation to other papers mentioned in this context, it is clear that no transform is involved at all.

Thus the two papers referred to above, rather than the earlier work, should be seen as the genesis of the concept of integral transform.

2. Integral Transforms

According to modern usage (see, *e.g.*, NEČAS, 1969), an integral transform is a map from one function, $f(t)$, to another, $F(p)$, by means of a definite integral

$$F(p) = \int_a^b K(t, p) f(t)\, dt, \tag{1}$$

where a, b are constants. Other authors, more generally, would consider contour integrals in the t-plane, but such usage clearly post-dates EULER.

Integral transforms are widely used in the solution of differential equations, although this does not exhaust their range of application. EULER, however, considers them only in this context. In the modern approach, a differential equation in an unknown function $f(t)$ would be transformed by the use of Equation (1) into another (to be hoped simpler) equation involving $F(p)$.

The earlier approach, initiated by EULER and surviving into this century, was different. This began with a differential equation in $F(p)$ and assumed a solution of the form given by Equation (1). This then led to a new (and, in useful cases, simpler) equation in $f(t)$. (For more detail on these two approaches, see my earlier studies: DEAKIN, 1980, 1981, 1982.)

One might view this older approach as constituting a special case of a more general one, described by DULAC as EULER'S, in which a differential equation in $F(p)$ is solved by seeking a solution of the form

$$F(p) = \int_a^b \Phi(p, t)\, dt \tag{2}$$

for some specified function Φ.

The problem of specifying Φ in Equation (2), as opposed to that of specifying K in Equation (1), has meant that this method is hardly, if ever, employed.

3. Bibliographic Details

The following papers have been cited (DULAC, 1936) as bearing on the subject. They will be referred to in the sequel by the ENESTRÖM numbers (*e.g.* E28) preceding them in the listing. Full details are given in the bibliography. The papers are: E11 (EULER, 1733), E28 (EULER, 1738a), E31 (EULER, 1738b), E44 (EULER, 1740a), E45 (EULER, 1740b), E70 (EULER, 1744), E274 (EULER, 1763), and Chapter 10 of Book 1, Part 2 of *Institutiones Calculi Integralis* (EULER, 1769).

Also relevant is a paper E49 (EULER, 1741) on the oscillations of a hanging chain, whose methodology has some kinship with E28 and E31, but which is not referenced by DULAC. This paper is summarised by TRUESDELL (1960), pp. 162–165.

This list is ordered according to the obvious chronological sequence. The studies fall naturally into two groups. Dates of initial presentation of each of the papers are as follows: E11, 1733; E28, 1732/3; E31, 1732/3; E44, 1734/5; E45, 1734/5; E49, 1736; E70, 1737; E274, 1760/1; *Institutiones*, 1769.

Thus the first seven all date from the 1730s and the final two are much later. The composition of E274 has been assigned various dates (see ENESTRÖM, 1913), but none earlier than 1758, so that it comes over twenty years after the presentation of the latest of the others, E70. It is argued that this separation in time is reflected in a separation of subject matter and of approach.

4. Duclac's Account

On DULAC'S account, E274 continues a scholarly tradition begun with E28 and continuing through the whole sequence (except that he omits reference to E49) to its culmination in *Institutiones*. He writes (my translation):

> A series of papers are devoted to the method EULER termed "Constructio per quadratura curvarum". Led by chance, as he mentioned in Paper 28, to the representation of the solution $y(x)$ of a differential equation in terms of a definite integral in which x appears as a parameter, EULER sought a systematic approach to this type of representation, of which it seems to give the first example, and of which well-known applications were made by, in particular, LAPLACE, GAUSS and KUMMER. EULER uses this method in two different ways. In Paper 31, and in Chapter XI of the 1st part of Volume 2 of *Calculus integralis,* he first obtains the solution in the form of a series and then evaluates the sum of this series by means of an integral of the type indicated. EULER later sought a more direct approach, deriving the differential equation satisfied by a given definite integral, in which x appears as a parameter. This method is described in Papers 44 and 45, and later applied in 70, 274 as well as in Chapter X of the 1st part of Volume 2 of *Calculus integralis.*

While this makes no claim about integral transforms (*i.e.* Equation (1)) as such, it dates the use of Equation (2) as early as 1732/3. It will be argued that, although a form of Equation (2) may be *inferred* from E28, and analogous equations from E31, it misrepresents EULER'S discussion to present these as the subject matter of these papers. A similar point may be made in respect of E49. It will further be argued that E44, E45 and E70 have quite different concerns and do not involve Equation (2) at all. E11, a brief summary of E31, will not be separately discussed.

E274 thus will emerge as the seminal paper and it will be argued that EULER saw it as such, although there is some evidence that he recognised the connection with the integrals implicit in the earlier works.

The separate papers will now be considered *seriatim.*

5. E28

E28 is the first of a number of papers to be concerned with the "construction" of differential equations. As ENGELSMAN (1982, p. 145) remarks, EULER was not initially concerned with solving differential equations so much as with arriving at them from geometrically posed problems, but later "the idea struck him that problem and solution should instead by reversed: the geometrical definition ... should be regarded [as] the solution rather than the problem, and the differential equation that is found should be regarded [as] the problem instead of the solution".

This statement is made in respect of a paper, *De differatione*, published only as Appendix 2 of ENGELMAN'S study and dated "from about 1830". E28, which would seem to be a somewhat later study, retains some of the earlier flavour in that it commences with a geometric problem—the determination of the perimeter of an ellipse—and results in the production of a RICCATI equation. It is, however, quite clear (see, in particular, Section 2 of the paper) that EULER is quite conscious that he is contributing principally to the *theory* of the RICCATI equation.

The point of departure is Figure 1, whose notation I have slightly modernised and simplified. M has the coordinates (x, y) and $AT = t$. $s = AM$, measured along the arc. Let $n = (a^2 - b^2)/b^2$ be a measure of the eccentricity of the ellipse.

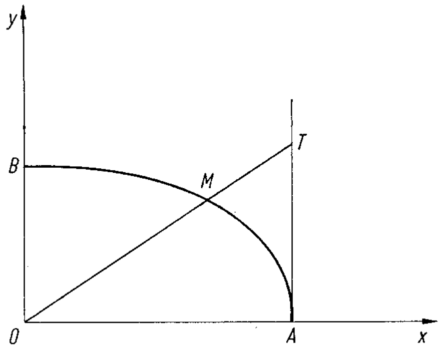

Fig. 1

Then it may readily be shown that

$$\frac{ds}{dt} = \frac{b^2 \sqrt{\{(b^2 + t^2) + nt^2\}}}{(b^2 + t^2)^{3/2}}. \tag{3}$$

We may now expand the right-hand side of Equation (3) as a power series in n. Thus

$$\frac{ds}{dt} = \frac{b^2}{b^2 + t^2} + \frac{Ab^2nt^2}{(b^2 + t^2)^3} + \frac{Bb^2n^2t^4}{(b^2 + t^2)^3} + \dots, \tag{4}$$

where A, B, ... are numerical constants. If we write Equation (4) as

$$\frac{ds}{dt} = P(t) + AnP(t)Q(t) + Bn^2P(t)Q^2(t) + \dots, \tag{5}$$

where $P(t) = b^2/(b^2 + t^2)$ and $Q(t) = t^2/(b^2 + t^2)$, we see the feature that EULER noticed. We have

$$\begin{aligned} \int P(t)\,Q(t)\,dt &= \tfrac{1}{2}\int P(t)\,dt - \tfrac{1}{2}\cdot\frac{b^2t}{b^2+t^2}, \\ \int P(t)\,Q^2(t)\,dt &= \frac{1\cdot 3}{2\cdot 4}\int P(t)\,dt - \frac{1\cdot 3}{2\cdot 4}\frac{b^2t}{b^2+t^2} - \tfrac{1}{4}\frac{b^2t^3}{(b^2+t^2)^2}, \\ &\ldots, \end{aligned} \tag{6}$$

where each successive integral on the left is expressed in terms of the first of the integrated terms of Equation (5). This allows for ready term by term integration and so yields a power series for s.

In particular, if we now impose the limits $t = 0, \infty$ on the integrals, the series gives the quadrant of the ellipse:

$$AMB = \frac{b\pi}{2}\left(1 + \tfrac{1}{2}\cdot\frac{n}{2} - \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}\cdot\frac{n^2}{4} + \ldots\right). \tag{7}$$

Further, notice that the integrated terms of each of Equations (6) vanish at both of the limits $t = 0, \infty$.

EULER now alters his notation, setting $s = \frac{2}{b}(AMB)$ and $n = -x^2$. (That this x is actually an imaginary quantity if, as his diagram indicates, $b < a$, he allows to pass without comment.) Then from Equation (7) he deduces the equation

$$(x^2 - 1)\frac{d}{dx}\left(x\frac{ds}{dx}\right) = xs \tag{8}$$

(where again I have modernised the notation). The substitution

$$y = \frac{x}{s}\frac{ds}{dx} \tag{9}$$

now yields

$$\frac{dy}{dx} + \frac{y^2}{x} = \frac{x}{x^2 - 1}, \tag{10}$$

a RICCATI equation, which may also be written

$$2\frac{dy}{dn} + \frac{y^2}{n} = \frac{1}{n-1}, \tag{11}$$

also a RICCATI equation.

Equations (10), (11) have now been "constructed". They may, of course, be solved in the modern sense by reversing the procedure and obtaining the series solution (7). However, no integral transform is involved, and DULAC's claim that the procedure involves an integral of the form (2) rests on implied equalities.

We can, indeed, using EULER's work, express the solution of Equation (8) as a definite integral involving x as a parameter.

First note that Equation (8) has the series solution

$$s = 1 - \tfrac{1}{2}\cdot\frac{x^2}{2} - \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}\cdot\frac{x^4}{4} - \dots. \tag{12}$$

Next note that (with this meaning of s) Equation (3) may be written

$$s = \frac{2}{\pi}\int_0^\infty \frac{b\sqrt{(b^2+t^2) - x^2t^2}}{(b^2+t^2)^{3/2}}\,dt, \tag{13}$$

and setting $t = ub$, we find

$$s = \frac{2}{\pi}\int_0^\infty \sqrt{\frac{1+u^2-x^2u^2}{(1+u^2)^3}}\,du. \tag{14}$$

This is the basis of DULAC's claim that such an integral is involved, but nowhere does EULER make this point explicit and indeed his change of notation serves to distance Equation (3) from Equation (8), although as Equation (8) is linear, the scale factor $2/\pi b$ is largely irrelevant. However, the integral form of Equation (3) is in fact indefinite. It is only when the special case of a quadrant is considered that a definite integral emerges.

6. E11 and E31

We need not consider E11 separately, for, as DULAC remarks, it is merely a summary of the subsequent E31. It is referenced in E28, which shows that EULER saw it as part of the same general investigation. We thus turn to E31.

This deals with the RICCATI equation

$$ax^n\,dx = dy + y^2\,dx. \tag{15}$$

(This is the original RICCATI equation, subsequently generalised to

$$\frac{dy}{dx} + Q(x)\,y + R(x)\,y^2 = P(x) \tag{16}$$

a different special case of which is "constructed" in E28.)

EULER'S concern here is with series of the form (5) which will yield equations of the form (6). In particular, he considers the case

$$P = \frac{1}{(1+bz^\mu)^\nu}, \quad Q = \frac{z^\mu}{1+bz^\mu} \tag{17}$$

which has a reduction formula

$$\int PQ^\theta\,dz = \frac{(\theta-1)\,\mu+1}{b\mu(\nu+\theta-1)}\int PQ^{\theta-1}\,dz - \frac{1}{b\mu(\nu+\theta-1)}\,\frac{z^{(\theta-1)\mu+1}}{(1+bz^\mu)^{\nu+\theta-1}}, \tag{18}$$

leading to equations analogous to Equations (6). (Here θ is a positive integer). If μ, ν satisfy $\mu\nu > 1$, the integrated terms of Equation (18), and thus of all equations based on it, vanish not only when $z = 0$, but also at $z = \infty$.

Thus it is possible to proceed in a fashion analogous to that used in E28. On the one hand, we have the substitution $y = t^{-1}\, dt/dx$ which converts Equation (15) into the linear second order differential equation

$$\frac{d^2t}{dx^2} = ax^n t \tag{19}$$

for which a series solution is readily found. On the other, we seek to sum this series in the form of a term by term integration of the form

$$\int_0^\infty P(1 + AgQ + ABg^2Q^2 + \ldots)\, dz, \tag{20}$$

where $A, B, \ldots$ are constants, g is a parameter and z a variable of integration. Intricate sequences of successive substitutions relate g to x in various cases, so that the form (20) is used in a way similar to that in which Equation (4) was used, towards the solution, in this case, of Equation (11).

If now $Z = Ht$, H being a constant, Z is given by

$$Z = \int_0^\infty \phi(x, z)\, dz \tag{21}$$

where $\phi(x, z)$ is an elaborate function of x, z given correctly by DULAC in a footnote in the *Opera Omnia* version. y then is given by $y = dZ/Z\, dx$.

EULER is here quite explicit in his use of an equation of the form (2), whereas, in E28, although Equation (14) is implied by the work, it is never exhibited explicitly.

7. E44 and E45

E44 is a seminal paper in the theory of partial derivatives and, for its contribution to that field, the reader is referred to ENGELSMAN'S (1982) study for a fuller account.

The starting point (Par. 3 of the paper) is the equation, in modern notation,

$$z = \int P(a, z, x)\, dx, \tag{22}$$

which DULAC in a footnote (Footnote 2 to Par. 3) interprets as (in essence)

$$z = \int_{x_0}^{x} P(a, z, t)\, dt \tag{23}$$

where x_0 is independent of a.

During the integration, a is held constant, but it is subsequently allowed to vary and EULER sets

$$dz = P\, dx + Q\, da, \tag{24}$$

the second term on the right being due to the variation of a. He now has

$$dP = A\, dx + B\, da \tag{25}$$

(say) and

$$dQ = C\,dx + D\,da \tag{26}$$

(say). But then $B = C$, and thus

$$Q = \int B\,dx. \tag{27}$$

In other words, Q may be found by differentiating under the integral sign.

The rest of the paper is occupied with examples and extensions, being concerned with the now defunct theory of *modular equations*. (Equation (24) is a modular equation if Q can be determined algebraically in terms of x, z, a.) For an account of EULER's theory of modular equations see ENGELSMAN (1982).

We may note, however, that even with DULAC's amendment (Equation (23)), Equation (22) is concerned with *indefinite* rather than with definite integrals, and thus no true transform is involved.

E45 is an addendum, notable, for our purposes, only in that it explicitly omits z from the right-hand side of Equation (22). Again no transform is involved.

8. E49

EULER's work on the oscillations of a flexible hanging chain derives from discussion of the problem with DANIEL BERNOULLI. The mathematical statement of the problem reduces to the solution of the equation

$$\frac{x}{n+1}\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{y}{\alpha} = 0, \tag{28}$$

where n, α are constants. This equation reduces, after the substitution $y' = yz$, to the RICCATI equation

$$\frac{x}{n+1}\left(\frac{dz}{dx} + z^2\right) + z + \frac{1}{\alpha} = 0. \tag{29}$$

Furthermore, EULER derives a series solution and is able to express this (it represents a modified Bessel function) in terms of a definite integral. The calculation is discussed in detail by TRUESDELL (1960).

This paper concludes: "And so, likewise, there follows from this the construction of the Riccati equation, which I gave some years ago". (My translation.) The reference is presumably to E31, which was three or four years prior to E49.

9. E70

E70 considers the equation

$$z = \int P(a, x)\,dx,$$

where again no limits are imposed on the integration. In particular the form

$$z = \int e^{ax} X(x)\,dx \tag{30}$$

is studied. This equation has a form somewhat reminiscent of the LAPLACE Transform and this may have led ENESTRÖM (1914) to see it as an early form of that, whereas, in fact, as I have argued elsewhere (DEAKIN, 1981), it is nothing of the kind, as an indefinite, not a definite, integral is involved.

Again EULER's main concern is the production of modular equations. One example will suffice to show his approach. If

$$z = \int e^{ax}xX(x)\,dx = \alpha \int e^{ax}X(x)\,dx + me^{ax}X(x) - m$$

where m, α are constants, then the modular equation

$$dz = (b - ma)\,z\,da - m\,da + e^{\frac{x^2-2bx-2max}{2m}}(dx + m\,da) \tag{31}$$

results, b being equal to $\alpha + ma$.

The focus of attention is far removed from the use of equations like (1) and (2) to solve differential equations.

10. E274

The fragment E274 considers the differential equation

$$(Fu^2 + Eu + D)\frac{d^2y}{du^2} + (Cu + B)\frac{dy}{du} + Ay = 0, \tag{32}$$

and does so by explicitly considering y to be a definite integral, not only of the form (2) but indeed of the form (1). EULER sets

$$y = \int P\,dx(u + x)^n, \tag{33}$$

"where P, which we need to define later, denotes some function of x, but not u. When this function becomes known, the integration is performed, at least by quadrature, for each value of u, which during the integration is like a constant. The integral is then so taken that it vanishes for some assigned value of x, and is evaluated for some fixed constant value of x (quite independent of u); when this is done, y is equal to some determined function of u, with the property that it satisfies the given equation."
(Translation by DEAKIN & ROMANO, 1983.)

This is a very clear statement of the principle behind the use of the integral transform, and indeed, EULER gives a more general statement, which may be taken to apply to the form (2):

"I now suppose y to be given by some integral form involving, as well as u, a new variable x, such that, during the integration, only x is treated as a variable, u being regarded as a constant. But when the integration is completed, whether by analytic means or by quadrature, the quantity x will be assigned some given constant value, and, as a result, the integral appears as some function of u, which is to be precisely that which satisfies the equation."
(Translation by DEAKIN & ROMANO, 1983.)

This is clear and explicit and occurs at the beginning of the study. While E31 does contain explicit accounts of Equation (2), they are not central to the concern of that paper, and E28 contains only implicit integral forms. The approach of E274 is

> "... the author puts forward a new and quite remarkable technique for dealing with these equations, and produces, even at this early stage, some remarkable examples of its use ...".
> (Translation by DEAKIN & ROMANO, 1983.)

This is not to say, however, that it is not implicit in the earlier work. EULER recognises that it is.

> "... I demonstrate the construction of this equation in the same way as I earlier set out for the RICCATI equation ..."
> (Translation by DEAKIN & ROMANO, 1983.)

The reference is to E31 and possibly also to E28. DULAC (1936) in a footnote includes E70, but we have seen that the procedure there is different. We may, however, view E274 as a new departure because of its clear and explicit account of the method, and, indeed, also because of its use of Equation (1), as opposed to Equation (2). This latter feature is not made explicit by EULER, but it is this that allows for the simple forms obtained by differentiating under the integral sign.

11. Institutiones Calculi Integralis

These matters are taken up more generally in Chapter X of the first part of Volume 2 of *Institutiones Calculi Integralis.* This begins with the question of differentiating Equation (2) under the integral sign, which quite clearly now denotes definite integration. Solutions to second order differential equations are then envisaged as having the form (2).

The integrands of Equation (2) are first posited and the differential equations whose solutions they give are then derived, so that, in a sense, EULER is still "constructing" the differential equations. He does, however, see solution of differential equations as the problem and his construction as contributing to the exploration of the solvability of different equations. Thus the substitution (in modern notation)

$$y = \int_0^c x^n \sqrt{\frac{u^2 + x^2}{c^2 - x^2}}\, dx \tag{34}$$

is shown to solve

$$\frac{d^2y}{du^2} - \frac{(n + 1)}{u}\frac{dy}{du} + \frac{(n + 1)\,y}{c^2 + u^2} = 0, \tag{35}$$

and a generalisation

$$y = \int_0^a x^{n-1}(u^2 + x^2)^{\mu}(c^2 - x^2)^{\nu}\,dx \tag{36}$$

to solve a more complicated differential equation.

Equation (34) may be thought of as an integral transform of x^n applied to the solution of Equation (35). The kernel of the transform is thus

$$\sqrt{\frac{u^2 + x^2}{c^2 - x^2}}.$$

A number of other transforms are adduced with kernels which relate to modern transforms. In Section 1036, for example, we have e^{ux} appearing for the first time as a kernel in the integral

$$u = \int_0^a e^{ux}x^n(a - x)^{\nu}\,dx \tag{37}$$

satisfying the differential equation

$$u\frac{d^2y}{du^2} + (n + \nu + 2 - au)\frac{dy}{du} - (n + 1)\,ay = 0. \tag{38}$$

This form and others like it are, as I have pointed out elsewhere (DEAKIN, 1980, 1981, 1982), versions of the LAPLACE Transform akin to that which preceded our modern version, which became standard during the early years of this century. In particular, EULER points out that it may be used to solve what we now term the LAPLACE Equation

$$(A + \alpha u)\frac{d^2y}{du^2} + (B + \beta u)\frac{dy}{du} + (C + \gamma u)\,y = 0. \tag{39}$$

More generally, he considers the transform

$$y = \int_b^a e^{K(u)Q(x)}P(x)\,dx, \tag{40}$$

whose properties are discussed in a previous paper (DEAKIN, 1980).

The transform considered in E274 is also generalised to

$$y = \int_b^a P(x)\,(K(u) + Q(x))^n\,dx. \tag{41}$$

All of these endeavours are aimed at exploring the scope of solution techniques for second order differential equations.

> "So while even now we are a long way from a solution of the problem, that is of finding a formula which provides the integral of any given second order differential equation (whether the solution of this problem will ever be discovered seems quite uncertain), we nonetheless apply ourselves to it, so that,

at least in special cases, we try to base the inquiry into the form of the integral on the nature of the given equation, and so to some extent we progress towards an explicit solution."
(Translation by DEAKIN & ROMANO, 1983).

This material continues into Chapter XI, some of which is reminiscent of E31, but this takes us away from a concern with integral transforms proper.

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Department of Mathematics
Monash University
Clayton, Australia

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