Cavalieri's Method of Indivisibles

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BONAVENTURA CAVALIERI (Archivio Fotografico dei Civici Musei, Milano)

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Introduction

CAVALIERI is well known for the method of indivisibles which he created during the third decade of the 17^{th} century. The ideas underlying this method, however, are generally little known. This almost paradoxical situation is mainly caused by the fact that authors dealing with the general development of analysis in the 17^{th} century take CAVALIERI as a natural starting point, but do not discuss his rather special method in detail, because their aim is to trace ideas about infinitesimals. There has even been a tendency to present the foundation of his method in a way which is too simplified to reflect CAVALIERI's original intentions.

The rather vast literature, mainly in Italian, explicitly devoted to CAVALIERI does not add much to a general understanding of CAVALIERI'S method of indivisibles, because most of these studies either presuppose a knowledge of the method or treat specific aspects of it.

Yet there is one apt presentation of CAVALIERI'S theory of indivisibles; it was published by ENRICO GIUSTI in 1980. Only when this paper was almost finished did I become aware of GIUSTI's study which is published as an introduction, *fuori commercio*, to a reprint of CAVALIERI'S *Exercitationes geometricae sex*; hence GIUSTI's and my work overlap. To keep my paper coherent I have not changed the sections which are similar to some in GIUSTI's book, but only added references to the latter. This procedure is further motivated by the fact that GIUSTI's approach and mine to CAVALIERI's method are rather different. He gives a background to the understanding of CAVALIERI'S theory and its weakness and does not treat many technical aspects. I intend to supplement the existing literature on CAVALIERI with a detailed presentation of his method, including its fundamental ideas, the concepts involved in it, its technique of proofs, and its applications. Further, I try to sketch how mathematicians have understood CAVALIERI'S ideas.

A complete study of the interpretations of CAVALIERI's theory would be very useful, but requires a paper of its own (a presentation of some of the interpretations can be found in GALUZZI & GUERRAGGIO 1983). Hence I here concentrate on the reactions of some of the 17th-century mathematicians, for example GALILEO, GULDIN, TORRICELLI, ROBERVAL, PASCAL and WALLIS, who were influental in creating an opinion on CAVALIERI's method; and I add a few examples of 18th century views on the method.

My wish to treat CAVALIERI's theory in one paper has resulted in another restriction: to leave out the otherwise interesting questions about similarities between CAVALIERI's ideas and those occurring in medieval and renaissance mathematics.

To avoid a confusion between modern concepts and the CAVALIERIan ones I have introduced an *ad hoc* notation; a list of the symbols employed can be found at the end of the paper.

I. The Life and Work of Cavalieri

I.1. Our knowledge of CAVALIERI's life stems mainly from his letters to GALILEO and other colleagues, from a few official documents, and from his first biographer and pupil URBANO DAVISO, who is not always to be trusted. Since complete biographies of CAVALIERI have been compiled several times from this material, I shall only give a few particulars about his life as background for his mathematical work. (For more biographical information, see PIOLA 1840, FAVARO 1888, MASOTTI 1948, CARRUCCIO 1971, and GIUSTI 1980.)

CAVALIERI was born about 1598 and as a boy in Milan he came in contact with the rather small order of Jesuats, the male section of which was dissolved in 1668.

In 1615 CAVALIERI entered this order and on that occasion he probably took the first name BONAVENTURA. The years 1616–1620, with an interruption of a one-year stay in Florence around 1617 (*cf.* GIUSTI 1980, p. 3), CAVALIERI spent at the Jesuati convent in Pisa, and he became a mathematical pupil of the Benedictine BENEDETTO CASTELLI. CASTELLI was so satisfied with this student that about 1617 he arranged a contact to his own teacher, GALILEO GALILEI (*cf.* GALILEI *Opere*, vol. 12, p. 318). This resulted in more than 100 letters from CAVALIERI to GALILEO in the period 1619–1641. GALILEO did not answer all of them, but sent an occasional letter to CAVALIERI; of these all but a very few have disappeared.

DAVISO's version of how CAVALIERI took up mathematics is more dramatic than the one presented here. DAVISO claimed that at the age of twenty-three CAVALIERI started an intense study of mathematics, having been told by CASTELLI that mathematics was an efficacious remedy against depression. However, the facts that in 1617 FEDERIGO BORROMEO asked GALILEO to support CAVALIERI, that in 1618 CAVALIERI temporarily took over CASTELLI's lectures on mathematics in Pisa, and that CAVALIERI in 1619 applied for a vacant professorship in mathematics destroy DAVISO's thesis of CAVALIERI's late start (GALILEI *Opere*, vol. 12, p. 320, GIUSTI 1980, pp. 3–4, FAVARO 1888, pp. 4 and 35).

CAVALIERI did not obtain the chair in Bologna in 1619, but he went on applying for a *lettura* in mathematics at various places while moving between Jesuati monasteries in Milan, Lodi, and Parma. In his letters he ventilated the idea that the reason why he was not appointed professor of mathematics was that the Jesuats were not very popular in Rome. Probably through GALILEO'S influence, CAVALIERI eventually obtained a professorship in mathematics at the university of Bologna in 1629. About the same time he also became prior at the Jesuati monastery there. The appointment to the chair of mathematics was only for a period of three years, but CAVALIERI had it renewed until his death in 1647.

In his teaching CAVALIERI seems to have followed a three years' cycle of lectures, consisting of comments on EUCLID, *Theorica planetarum*, and PTOLEMY'S astronomy (FAVARO 1888, p. 22). Besides his two major works, which will be presented shortly, CAVALIERI published eight books on mathematics and mathematical sciences and a table of logarithms. One of the books treating astrology was published under the pseudonym SILVIO FILOMANTIO. The other books were mainly textbooks, and although some of them contain references to results obtained by the method of indivisibles they do not deal with the method. Therefore I shall not discuss them further, but refer to the bibliography where their titles are listed.

I.2. The book which made CAVALIERI famous in mathematical circles was *Geometria indivisibilibus continuorum nova quadam ratione promota*, Bologna 1635 (Geometry, advanced in a new way by the indivisibles of the continua); I shall abbreviate this impressive title to *Geometria*. It is difficult to follow CAVALIERI through the almost 700 pages of this book, so difficult that MAXIMILIEN MARIE suggested that if a prize existed for the most unreadable book, it should be awarded to CAVALIERI for *Geometria* (MARIE 1883–1888, vol. 4, p. 90); further, the mathematical language CAVALIERI employed in *Geometria* was characterized by CARL B. BOYER as "confusingly obscure" (BOYER 1941, p. 85). Nevertheless, *Geometria* was in its time considered so important that it was reprinted in 1653 in an edition which, unlike the first, is paginated continuously.

The main reason why *Geometria* attracted attention was doubtless that most mathematicians of the 17th century were interested in its topic, quadratures and cubatures, and that the number of publications on this subject was small. The mathematicians who carefully studied *Geometria* were probably few, but nevertheless it remained a well known book. This is reflected in the treatment of *Geometria* in general works and articles on 17th-century mathematics: it is mentioned, but its content is not thoroughly described.

That *Geometria* is still considered an important contribution to mathematics can be seen from the fact that it has been translated into modern languages. Thus in 1940 parts of *Geometria* with elaborate comments appeared in a Russian edition made by S. J. LUR'E (my inability to read Russian has prevented me from consulting this edition). A complete translation of *Geometria* into Italian, containing many clarifying comments, was published in 1966 by LUCIO LOMBARDO-RADICE.

Shortly before his death CAVALIERI published another work on indivisibles, the *Exercitationes geometricae sex* (1647, 543 pp.); this book has received much less attention than *Geometria*, but is still mentioned in expositions of CAVALIERI'S contributions to mathematics. My presentation of CAVALIERI'S method of indivisibles is based on features both from *Geometria* and *Exercitationes*, and since I am not going to discuss all his results I shall briefly outline the contents of these two books.

I.3. *Geometria* consists of seven books. In the first, CAVALIERI clarifies some of his assumptions concerning plane and solid figures. In Book II he introduces the method of indivisibles, or rather his first method which I term the collective method, and proves some general theorems concerning collections of indivisibles. These theorems he applies in Books III, IV and V where he deals with quadratures and cubatures related to conic sections. The sixth book is mainly devoted to the quadrature of the spiral, but contains also some results concerning cylinders, spheres, paraboloids and spheroids. In the last book CAVALIERI presents a new approach to the method of indivisibles, which I am going to call the distributive method (*cf.* Section IX.1).

The content of the six geometrical *Exercitationes* is more varied: in the first CAVALIERI presents a revised version of the collective method of indivisibles from Book II of *Geometria* and suggests some simplifications. The second *exercitatio* similarly takes its starting point in Book VII of *Geometria* and develops a new presentation of the distributive method. The third book is evoked by PAUL GULDIN'S reaction to CAVALIERI'S method.

Throughout the last three of the four volumes of his *Centrobaryca* (1635–1641) GULDIN had commented upon CAVALIERI'S use of indivisibles and had particularly criticized it very outspokenly in Chapter 5 of the fourth volume. Shortly before GULDIN'S death CAVALIERI published a defense of his method in the section "Admonitio circa auctorem centrobaryacae" of his *Trigonometria* (1643, pp. 6–8). This is only a short and rather superficial reaction, not taking up the technical and philosophical aspects of the method criticized by GULDIN; it was only preliminary as CAVALIERI had plans about writing a more detailed answer. His letters to TORRICELLI from the period between September 1643 and September 1644 show that for a time he worked on presenting the answer as a dialogue in three parts with the participant USULPA GINULDUS—an anagram of PAULUS GULDINUS (TORRICELLI *Opere*, Vol. 3, pp. 145, 157, 159–160, 167, 170, 179–180, 226–227). The first dialogue was printed, but CAVALIERI gave up his plan, destroyed the printed copies and eventually formulated his defense as the third *exercitatio* (*cf.* GIUSTI 1980, pp. 56–58).

In the fourth book of *Exercitationes* CAVALIERI presents a generalization of the collective method of indivisibles, which enabled him to deal with algebraic curves of degrees higher than two. In the fifth *exercitatio* CAVALIERI turns to determinations of centres of gravity, partly based on concepts related to his method of indivisibles. The last *exercitatio* contains miscellaneous material (*cf.* GIUSTI 1980, pp. 85–90).

I.4. The publication of *Geometria* in 1635 was the climax of a longer process whose history will briefly be indicated here because it reveals the interesting emergence of the method of indivisibles. (For more details see the *Introduzione* in LOMBARDO-RADICE 1966.)

It was in the early 1620's that CAVALIERI got the idea of using indivisibles in comparisons of two areas and two volumes. In letters to GALILEO he described how he worked on this idea and how his thoughts gradually took form and resulted in a finished version of *Geometria* consisting of six books (*cf.* CAVALIERI's letters from the period between December 15, 1621 and December 17, 1627 in GALILEI *Opere*, Vol. 13). In November 1627 CAVALIERI sent this version to GIOVANNI CIAMPOLI to whom he also later dedicated *Geometria* (*cf. ibid.*, Vol. 13, p. 381). The content of the version of 1627 of *Geometria* was unknown until recently when GINO ARRIGHI became aware of a manuscript in the *Bibliotheca Cortona* (Codice (211) 292) which most probably is a copy of it (ARRIGHI 1973). Being unable to get a reproduction of the Cortona manuscript, I have had no occasion to study it carefully. A cursory perusal, however, gave me the impression that before the text was printed, CAVALIERI changed several things in Book I, made only linguistic changes in the Books II–V, and changed Book VI essentially (the last change will be described in Section VIII.2).

Thus in 1627 CAVALIERI had a manuscript almost ready for the press, and yet it took another eight years before *Geometria* appeared. This long period cannot be explained by the changes CAVALIERI made, so there must be other reasons for the delay. CAVALIERI himself maintained that the delay was caused by his teaching duties as a professor, by his desire to publish textbooks, and by failing health (*cf.* letters from CAVALIERI to GALILEO within the period from February 24, 1629, to September 18, 1635, GALILEI *Opere*, Vols. 14, 15 and 16, esp. Vol. 14, p. 171). Still, there may have been another reason also, which I shall discuss in Section IX.4, namely that CAVALIERI was waiting for GALILEO's approval of his method.

In 1634 when the printing of the planned *Geometria* was almost finished a further delay was introduced, because CAVALIERI decided to add a seventh book, as we shall see in Section IX.1.

II. Figures

II.1. In this chapter I am going to deal with some of the explicit and implicit assumptions that CAVALIERI made concerning figures and which he used in connection with his method of indivisibles.

CAVALIERI himself devoted the whole of *Geometria*, Book I, to introductory theorems. His main concern there was to study solids of rotation, general cylinders and cones (having an arbitrary closed curve as generating curve), and sections of these solids, in particular those which are similar. Interesting for our purpose, however, is only the first part of Book I dealing with tangents to plane and solid figures. As there is no essential difference between the plane and the solid case, I shall deal only with the first one.

In Book I CAVALIERI takes it for granted that his readers understand what he means by a tangent, whereas in Book VII of *Geometria* he defines it. This definition is fortunate because it clears up misunderstandings which a reader with a traditional concept of a tangent may have had:

> I say that a straight line touches a curve situated in the same plane as the line when it meets the curve either in a point or along a line and when the curve is either completely to the one side of the meeting line [in the case when the meeting is a point] or has no parts to the other side of it [in the case when the meeting is a line segment].[1]

This definition is in accordance with the classical concept of tangents in the sense that it excludes tangents as t_1 and t_2 in Figure II.1, unless limited parts of the curves are considered. However, it differs from both the ancient and modern

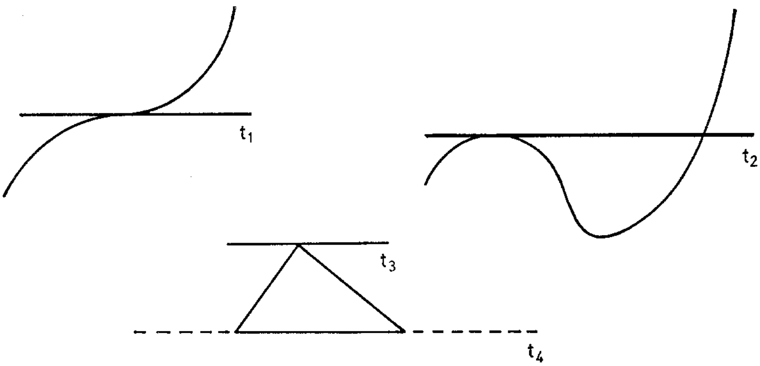

Fig. II.1

concept by considering t_3 as touching; further it stresses that t_4 is a tangent. CAVALIERI included lines like t_3 as tangents because he had a definite aim with his study of tangents, namely to ensure the general validity of the following theorem:

Given a closed plane figure, ABCD (*cf.* Figure II.2) and a direction RS, called *regula*; the figure will then have two tangents, AE and CG, parallel to the

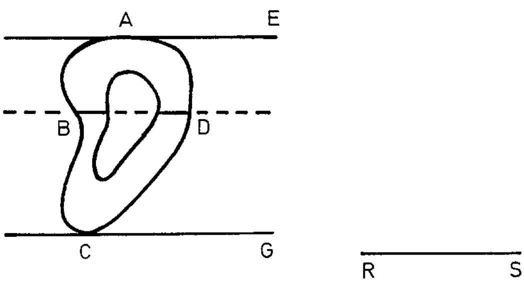

Fig. II.2. A redrawing with altered letters of a figure in *Geometria*, page 8.

[1] *Geometria*, p. 492: Tangere autem dico rectam lineam aliam quamcunque curvam totam in eodem plano cum ea existantem, cum ipsa recta linea sive in puncto, sive in recta linea, curvae, occurrente, eadem curva vel tota est ad eandem partem, vel illius nihil est ad alteram partem illi occurrentis rectae lineae.

regula; moreover any line parallel to the *regula* situated between the two tangents, for example BD, will intersect the figure in line segments, whereas any line parallel to the *regula* outside the tangents will have no points in common with the figure (*Geometria*, pp. 14, 17–18). CAVALIERI called the two tangents *opposite* tangents.

He needed such a result for his definition of the concept of "all the lines", which will be presented in Section III.1. Most of the figures to which CAVALIERI applied his method of indivisibles do have two opposite "traditional" tangents parallel to a chosen *regula*. However, triangles play an important role in his theory and the fact that they have no "traditional" opposite tangents apparently motivated CAVALIERI to introduce a line through a vertex of a triangle like t_3 as a tangent. CAVALIERI'S definitions and theorems concerning opposite tangents and "all the lines" show a striking generality, very advanced for his time. Thus in building up his theory he had in mind figures as complicated as the one of Figure II.2.

This generality made some of CAVALIERI'S demonstrations concerning curves erroneous, which is no wonder, since he did not have at his disposal concepts like differentiability which could have enabled him to analyse the assumed properties of curves. He implicitly supposed that his curves behaved 'nicely', more 'nicely', in fact, than some of his drawn curves actually do.

To illustrate his style of argument I shall present one of CAVALIERI'S means for obtaining the main result, the existence of two opposite tangents. It is a part of Theorem 1 of the first book of *Geometria* and is repeated separately as Lemma 3 in Book VII in the following form:

> If a curved line is situated in one plane and if a straight line meets it in either two points, two line segments, or in a line segment and a point, then we can draw another straight line parallel to the previous line which touches the part of the curve situated between the two mentioned meetings.[2]

In Figure II.3 let BAC be the curve and BC the meeting line. CAVALIERI'S proof is based on the intuitive idea, that if a line KV parallel to BC is moved

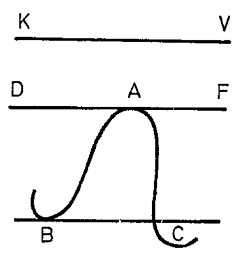

Fig. II.3. The figure except the line KV is on page 492 of *Geometria.*

[2] *Ibid.*: Si curva linea quaecunque tota sit in eodem plano, cui occurrat recta in duobus punctis, aut rectis lineis, vel in recta, & puncto, poterimus aliam rectam lineam praefatae aequidistantem ducere, quae tangat portionem curvae lineae inter duos predictos occursus continuatam.

either towards BC or away from BC remaining parallel to it, then in a certain position it will become tangent to ABC (*Geometria* pp. 14–15).

Besides being an essential tool for CAVALIERI, Lemma 3 is an interesting example of a precursor of an important theorem in the calculus: *The mean value theorem*, which states that if the function f(x) is continuous in the closed interval [a, b] and differentiable at every point of the open interval, then there is an intermediate value ξ such that $(f(b) - f(a))/(b - a) = f'(\xi)$. Or, translated into geometrical language (*cf.* Figure II.4): there is at least one point on the graph of f(x) at which the tangent is parallel to the secant between the points (a, f(a)) and (b, f(b)).

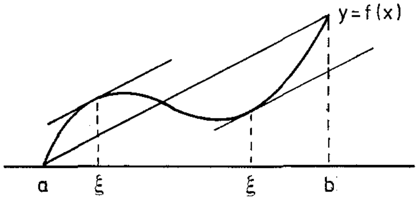

Fig. II.4

It is exactly the last property CAVALIERI stated in Lemma 3; nevertheless I only consider it a precursor and not an early version of the mean value theorem, because it is based on geometrical ideas very different from the concepts underlying the mean value theorem (*cf.* LOMBARDO-RADICE 1966, pp. 81–82, where a different opinion is expressed).

II.2. The purpose of CAVALIERI'S method of indivisibles was to provide a means for quadratures and cubatures of figures. His method was new, but his ideas of what should be understood by a quadrature or a cubature were entirely based on the Greek theory of magnitudes; therefore I shall briefly sketch it.[3]

Greek mathematicians divided mathematical objects into different categories. Of particular interest for CAVALIERI'S work are the category containing the natural numbers and the three categories containing one-dimensional, two-dimensional, and three-dimensional geometrical figures respectively. Two objects belonging to the same category were said to be of the same kind and could be combined or related in various ways. The compositions and relations considered were not defined, but some of their properties were postulated in the common notions in EUCLID'S *Elements*, Book I. There the objects are described only as "they", τά, whereas in Book V of the *Elements*, the word magnitude, μέγεθος, is introduced

[3] My understanding of the Greek theory of magnitudes originates from OLAF SCHMIDT'S lectures which unfortunately have not been published. In this brief outline of the theory I have avoided special problems connected with the theory, as *e.g.* the fact that not all Greek mathematicians considered a curve and a straight line as magnitudes of the same kind. For more information on magnitudes see HJELMSLEV 1950 and ITARD 1953.

to characterize an object. It is difficult to define a magnitude precisely, but for our purpose it is enough to note that in the category of natural numbers the numbers themselves were magnitudes, and in the categories of figures, the figures when considered movable were magnitudes (or more precisely, the magnitudes were the equivalence-classes determined by the relation of congruence).

The Greek mathematicians' assumptions concerning magnitudes, relations and compositions implied that when any two magnitudes, A and B, of the same kind, are given then

1. A and B can be ordered so that precisely one of the following three possibilities holds:
$$A > B \quad A = B \quad A < B.$$
2. A and B can be added; the result, which will be denoted by $A + B$, is a magnitude of the same kind as A and B.
3. If $A > B$ B can be subtracted from A, forming the magnitude, $A - B$, of the same kind as A and B.[4]
4. A and B can form a ratio [A:B].

Moreover, according to EUDOXUS's theory of proportions, set forth in Book V of EUCLID'S *Elements*, ratios between magnitudes can be ordered.

To understand CAVALIERI'S treatment of geometrical figures it is very important to keep in mind that the described calculations concern the proper magnitudes. Given two geometrical figures, A and B, we can interpret the relations = and > by saying that A is equal to or greater than B if the measure of A is equal to or greater than the measure of B. However, the Greeks' concept of numbers did not allow measures like length, area and volume to be ascribed to figures, and therefore they calculated directly with the figures or magnitudes. Thus to effect a quadrature or a cubature meant for Greek mathematicians to find the ratio between the figure to be determined and a "known" figure, *e.g.* the ratio between a segment of a parabola and its circumscribed parallelogram.

CAVALIERI took over all the above mentioned assumptions concerning magnitudes and even attempted, as we shall see in the next sections, to extend the set of magnitudes.

III. "All the lines"

III.1. The first definition in Book II of *Geometria* introduces the concept of "all the lines" (*omnes lineae*):

> If through opposite tangents to a given plane figure two parallel and indefinitely produced planes are drawn either perpendicular or inclined to the plane of

[4] EUCLID only subtracted B from A if—set-theoretically speaking—$A \supset B$, whereas the subtraction $A - B$ is allowed in ARCHIMEDES' lemma when $A > B$.

the given figure, and if one of the parallel planes is moved toward the other, still remaining parallel to it, until it coincides with it; then the single lines which during the motion form the intersections between the moving plane and the given figure, collected together, are called all the lines of the figure taken with one of them as *regula*; this when the planes are perpendicular to the given figure. When, however, the planes are inclined to the figure the lines are called all the lines of the same given figure with respect to an oblique passage (*obliqui transitus*), the *regula* being likewise one of them.[5]

CAVALIERI added that when the moving plane is perpendicular to the given figure "all the lines" can be characterized as *recti transitus*.

The important concept in CAVALIERI'S theory about plane figures is exactly that of "all the lines" *recti transitus*, which he also called the indivisibles of a given figure (*Geometria*, p. 114: "indivisibilia. s. omnes lineas figurae"). He did not make extensive use of "all the lines" *obliqui transitus*; but he had a motive for introducing them, to which I shall return in Section III.7. Unless otherwise stated "all the lines" should be thought of as *recti transitus*.

Since the concept of "all the lines" is crucial in CAVALIERI'S collective method of indivisibles, I shall discuss it in some detail in this and the following sections.

Let us consider a plane figure like F = ABC in Figure III.1 and let the line BC be the *regula* determining a direction in the plane of F (*cf.* Section II.1) "All the lines" belonging to F, taken with BC as *regula*, constitute the set of chords in F parallel to BC. It is useful to have a notation for this set, and I have chosen the symbol $\mathcal{O}_F(l)_{BC}$ ($\mathcal{O}$ from *omnes*). If there is no doubt about the *regula*, the symbol $\mathcal{O}_F(l)$ will be used. Further, I often find it convenient to employ the term *the collection of lines of F* instead of "all the lines".

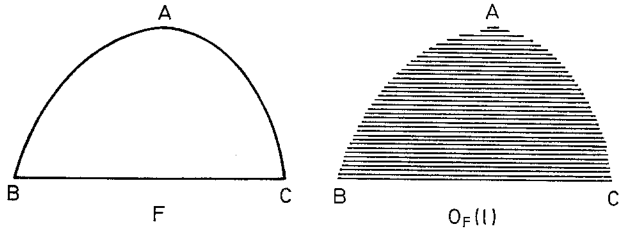

Fig. III.1

[5] *Geometria*, p. 99: Si per oppositas tangentes cuiuscunque datae planae figurae ducantur duo plana invicem parallela, recta, sive inclinata ad planum datae figurae, hinc inde indefinitè producta; quorum alterum moveatur versus reliquum eidem semper aequidistans donec illi congruerit: singulae rectae lineae, quae in toto motu fiunt communes sectiones plani moti, & datae figurae, simul collectae vocentur: Omnes lineae talis figurae, sumptae regula una earundem; & hoc cum plana sunt recta ad datam figuram: Cum verò ad illam sunt inclinata vocentur. Omnes lineae eiusdem obliqui transitus datae figurae, regula pariter earundem una.

In the quadratures which CAVALIERI actually carried out the "lines" l in $\mathcal{O}_F(l)$ are line segments, but, as we saw in Chapter II, when dealing with general statements he considered figures like the one depicted in Figure II.2; hence it can happen that some of the l's in $\mathcal{O}_F(l)$ consist in two or more line segments.

CAVALIERI introduced "all the lines" as a tool for quadratures. How exactly he used this tool will be illustrated later, but for the further discussion it is adequate to be aware of the connection between quadratures and collections of lines. This is expressed in the fundamental Theorem II.3 of *Geometria* which states that the ratio between two figures equals the ratio between their collections of lines taken with respect to the same *regula*: *i.e.*

$$F_1 : F_2 = \mathcal{O}_{F_1}(l) : \mathcal{O}_{F_2}(l). \tag{III.1}$$

CAVALIERI'S demonstration of this relation will be indicated in Section V.4; here I shall return to the investigation of the very concept of "all the lines" and particularly I will be concerned with the two following questions: How did CAVALIERI get the idea of using "all the lines" to calculate the left-hand side in (III.1)? And, how did he conceive of "all the lines"?

III.2. An explicit answer to the first question cannot be found in CAVALIERI'S writings. However, I find it likely that he was inspired by the intuitive idea of considering a plane or solid figure as composed of infinitesimals. Many mathematicians since DEMOCRITUS have used such an idea as a starting point for their approach to quadratures and cubatures, and with JOHANN KEPLER'S *Stereometria* (1615) it became the basis of a method of integration.

Probably CAVALIERI only became acquinted with *Stereometria* about 1626, that is after he had invented his own method (LOMBARDO-RADICE 1966, pp. 51–52). But even if he had read the book earlier, he would not have met the ideas underlying his own method. CAVALIERI could agree with KEPLER that a new method should unify heuristics and proof which in the Greek method of exhaustion were kept separate. In other words a new method should provide the results and the proofs at the same time. However, CAVALIERI did not find KEPLER'S new method satisfactory. Although he was impressed by the number of problems KEPLER dealt with, he thought that KEPLER had built his method on a too weak a foundation which led to errors (*Geometria*, the Introduction).

CAVALIERI believed that a sound foundation could be obtained only by keeping the Greek tradition of not using infinitesimals in proofs. Hence he had to suppress all intuitive ideas about how a plane figure is composed and which role "all the lines" played in the composition.

This leads us back to the question: How did CAVALIERI actually conceive of "all the lines". The answer to this will be divided into two parts. First I shall discuss his mathematical treatment of "all the lines", and then I am going to follow his more philosophical ideas (for other accounts of CAVALIERI'S ideas concerning indivisibles see WALLNER 1903 and CELLINI 1966_1).

III.3. The mathematical role of collections of lines can best be characterized by comparing them with the categories of the Greek magnitudes, because CAVALIERI treated collections of lines in a way suggesting he considered them a new category of magnitudes to which EUDOXUS's theory of magnitudes could be applied. Seeing "all the lines" under this aspect brings an important clue to the understanding of CAVALIERI's ideas and proofs. In the following I shall justify the interpretation of collections of lines as a category of magnitudes by some examples (see also GIUSTI 1980, p. 28).

A *sine qua non* of the relation (III.1) is that the ratio between two collections of lines exists. Since each collection of lines consists of indefinitely many lines, the existence of the ratio is not obvious. From his early letters to GALILEO it can be seen that CAVALIERI was aware of this problem from the very start of his work on indivisibles. Thus on December 15, 1621, he asked GALILEO for his opinion on the dilemma, that on the one hand

> it seems that "all the lines" of a given figure are infinite [in number] and hence not covered by the definition of magnitudes which have ratios,[6]

and on the other

> for the reason that if the figure is made larger also the lines become larger ... it seems that they ["all the lines"] are covered by the mentioned definition.[7]

His later letters to GALILEO and passages in *Geometria* and *Exercitationes* show that CAVALIERI frequently returned to the problem for the rest of his life. In his continuous struggle during the period 1621–1647 CAVALIERI mainly used two mathematical arguments to prove the existence of the ratio between two collections of lines. The first is found in a letter from March 22, 1622 to GALILEO stating that the ratio does exist, because any collection of lines can be multiplied to exceed another collection of lines (GALILEO *Opere*, vol. 13, p. 86). Thus CAVALIERI referred to the fundamental property of Greek magnitudes described in EUCLID's *Elements*, Definition V.4:

> Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another.

This definition, which among other things had the function of excluding infinitesimals and infinite magnitudes, apparently prescribed a method to test whether two magnitudes have a ratio. However, in the surviving Greek mathematical texts it has not been used to prove the existence of a ratio. It is the other implication of the definition, also called EUDOXUS's axiom, that was used: namely that two magnitudes of the same kind can be multiplied to exceed each other. The Greek mathematicians seem to have considered the existence of a ratio between two

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[6] GALILEI *Opere*, vol. 13, p. 81: pare che tutte le linee d'una data figura sieno infinite, e però fuor della diffinitione delle grandezze che hano proportione.

[7] *Ibid.*: ma perchè poi, se si aggrandisse la figura, anco le linee si fano maggiori, essendovi quelle della prima et anco quelle di più che sono nell'eccesso della figura fatta maggiore sopra la data, però pare che non sieno fuora di quella diffinitione.

magnitudes of the same kind a consequence of the fact that the magnitudes can be added and ordered. CAVALIERI must have had similar ideas, because as his second argument he applied the following: Since two collections of lines have the property that they can be added and subtracted, it is obvious that they can be compared (see *e.g. Geometria*, p. 111, and *Exercitationes*, pp. 202–203). And by comparing CAVALIERI meant to form a ratio. (CAVALIERI never commented upon an ordering of collections of lines, but it is an implicit assumption in his theory, as we shall see in Section V.2.)

CAVALIERI did not trust this argument of additivity sufficiently to let it be conclusive. Thus he provided the first theorem of *Geometria*, Book II,

> "all the lines" of plane figures ... are magnitudes which have a ratio to each other,[8]

with a proof which was an attempt to verify his thesis that collections of lines can be multiplied to exceed each other. (His proof will be taken up in Section V.3.)

Although CAVALIERI's argumentation for the existence of a ratio between two collections of lines strikes more chords, his whole approach to the problem shows that he had the Greek magnitudes in mind when he created and developed his theory, and that he used the word *magnitudes* for "all the lines" in the Greek mathematical sense.

In *Exercitationes*, a further support for regarding "all the lines" as an abstract magnitude can be found. Here CAVALIERI drew a parallel between collections of lines and algebraic magnitudes like roots, with which one may calculate although their nature is unknown:

> This [to obtain a result by using "all the lines"] is just like what happens when algebraists who, although they are ignorant of the nature of what they call a root, a side or a thing, and undetermined roots, yet by multiplying, dividing *etc.* these, at the end are led to a result of a problem by these obscure roundabout methods.[9]

Apart from GIUSTI's book the literature on CAVALIERI does not clearly suggest an interpretation of collections of lines as a new category of magnitudes. (BOYER stressed that CAVALIERI built his methods on conceptions of EUCLIDean geometry but did not explain how (BOYER 1941, p. 83).) This may partly be due to the fact that more attention has been paid to CAVALIERI's comments on "all the lines", put forward in scholia and annotations, than to how he actually used the concept in mathematical arguments. CAVALIERI directed his comments particularly toward

[8] *Geometria*, p. 108: Quarumlibet planarum figurarum omnes lineae ... sunt magnitudines inter se rationem habentes.

[9] *Exercitationes*, pp. 202–203: Hic enim perinde sit ac apud Algebricos, qui nescientes; quae sit, quam dicunt Radicem, Latus, aut Cossam, seu quales ineffabiles radices, tamen easdem multiplicantes, dividentes &c denique in quaesiti inventionem quasi per has obscuras ambages manuducuntur.

two problems; the first, already touched upon in this section, was the problem of avoiding absurdities in dealing with collections which consist in infinitely many lines. The second problem concerned the relation between "all the lines" and the composition of a continuum.

The purpose of CAVALIERI's elaborations of the concept of "all the lines" was undoubtedly to clarify his method of indivisibles and to anticipate scepticism, but they probably had the effect of creating more confusion than clarity. This confusion will be illustrated in the following two sections.

III.4. In his comments on collections of lines CAVALIERI tried to explain how it was possible to work with a magnitude containing indefinitely many lines. Thus in a scholium to *Geometria*, Theorem II.1 (collections of lines are magnitudes which have a ratio to each other), he explained that it is not the number of lines in a collection which is used in a comparison, but

> the magnitude which is equal to being, congruent with it, the space occupied by these lines.[10]

In his very critical analysis of CAVALIERI's method of indivisibles GULDIN bluntly denied the existence of a ratio between two collections of lines with the argument that

> between one infinity and another there is never a proportion or ratio.[11]

Further, GULDIN speculated about what CAVALIERI meant by the magnitude which is equal to the space occupied by "all the lines". He came to the conclusion that there were only two possibilities, and that neither was of any help to quadratures. The one was that CAVALIERI mixed up "all the lines" of a given plane figure with the space inside the figure described by the moving tangent plane, *i.e.* the figure itself. The other possibility was that the magnitude equal to the space occupied by "all the lines" was a length consisting in an indefinite number of lines.

In answering GULDIN's point of the illegitimate comparison of two collections of lines CAVALIERI emphasized that although a collection of lines is infinite with respect to the number of lines it is finite with respect to extension (*in spatio*). Further he tried to make the ratio between two collections of lines comprehensible by posing the question: Is it not obvious that two collections of lines belonging to two congruent squares are the double of each of the collections? (*Exercitationes*, p. 202.)

CAVALIERI also answered GULDIN's remarks on the space occupied by "all the lines". CAVALIERI's approach to this problem was much related to the question

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[10] *Geometria*, p. 111: magnitudinem, quae adaequatur spatio ab eisdem lineis occupato, cum illi congruat.

[11] GULDIN 1635–1641, vol. 4, p. 342 quoted in *Exercitationes*, p. 201: sed infiniti ad infinitum nulla est proportio, sive ratio.

of the composition of the continuum and the role of indivisibles in this; hence I shall first present his ideas about this question before presenting his answer to GULDIN.

III.5. The ancient discussion about how to conceive of a continuum was still considered very important, partly because of GALILEO'S work, when CAVALIERI wrote his *Geometria*. The questions with which CAVALIERI was confronted can, slightly simplified, be described in the following way (*cf.* STAMM, 1936).

What happens if a continuum, as for example a line segment, is divided indefinitely often? Would it be, as stated by ARISTOTLE, that at each step one obtains parts which can again be divided, being of the same kind as the continuum. Or would it be, as maintained by others, that ultimately indivisible parts (atoms) are obtained? And would they then have the same dimensions as the continuum or a lower one? (*E.g.* if the continuum is a line segment: would the indivisibles be line segments or points?) And if ARISTOTLE was right, how then could the points of a line segment or parallel chords in a plane figure be conceived? Were they linked to the continuum as a kind of indivisible which, although they did not compose it, was related to its continuous nature? And if so, could this property be explained by letting a moving point describe a line segment, or by letting a line segment describe a rectangle?

Although these questions were closely related to his method of indivisibles, CAVALIERI decided not to take part in the discussion, or at least not to reveal his opinion. Thus, if we return to his reply to GULDIN'S interpretations of the magnitude equal to the space occupied by "all the lines" we see an approach to the problem of the continuum which is very typical of CAVALIERI. He answered that if one conceives the continuum to be composed of indivisibles, then a given plane figure and the "magnitude of all the lines" will be one and the same thing. Further, he said that if one assumes a continuous divisibility, then it can be maintained that this magnitude consists only in lengths, but because "all the lines" ought to be considered at their actual position the magnitude is limited by the same limits as those of the given figure (*Exercitationes*, p. 203).

CAVALIERI'S inclination to leave two possibilities open in the case of the composition of the continuum is often reflected in his writings (besides the reference above see *e.g. Geometria*, pp. 111, 113–114 and *Exercitationes*, p. 199). He did not state exactly how the space occupied by "all the lines" should be understood if continuous divisibility was assumed, but he argued for the existence of the ratio between two collections of lines even in this case. By following his argument, which should not be confused with his mathematical proof (*cf.* Sections III.3 and V.3), we can get a vague idea about CAVALIERI'S conception of "all the lines". He claimed that if the indivisibles do not make up a continuum, then a given plane figure consists of "all the lines" and something else (*aliquid aliud, Geometria*, p. 111). From this he concluded that the space occupied by "all the lines" is limited; and that made him deduce that collections of lines can be added and subtracted. As we saw in the last section, this last property he considered to be significant for the existence of a ratio between two collections of lines.

This argument is previous to the answer to GULDIN, presented above, but it discloses the same idea: In his attempt to imagine "all the lines" of a given figure

CAVALIERI saw them either as making up the figure or as a part of it – which somehow could be considered as having the same properties as two-dimensional figures. One gets the same impression from the famous comparison CAVALIERI made between a plane figure containing "all its lines" and a piece of cloth woven of parallel threads deprived of their thickness (*Exercitationes*, pp. 4, 239–240).

Although CAVALIERI did not make it clear what he thought of the composition of the continuum, one may wonder whether he did not incline more to the one possibility than to the other. The following remark from CAVALIERI's letter to GALILEO dated June 28, 1639,

> I have not dared to say that the continuum was composed of these [the indivisibles] ... Had I dared ...[12]

could give the impression that most likely CAVALIERI conceived of the continuum as composed of indivisibles. LUR'E seems to have maintained that this was indeed CAVALIERI'S opinion not expressed too explicitly because he feared an opposition from the Catholic church holding the ARISTOTELian view (*cf.* LOMBARDO-RADICE 1966, p. 206).

However, I think that CAVALIERI never took a definite point of view on the composition of the continuum and that the could be trusted when on October 2, 1634, he wrote to GALILEO

> I absolutely do not declare to compose the continuum by indivisibles.[13]

Moreover I find it likely that CAVALIERI'S apparent ambivalence should be ascribed to the circumstance that he was not genuinely interested in the philosophical aspects of the composition of the continuum. The function of "all the lines" was first of all, as CAVALIERI himself stated in the introduction to *Exercitationes* (p. 3), to be an instrument for quadratures; and his mathematical treatment of them was independent of any conception of the continuum. (This point of view is also expressed in LOMBARDO-RADICE 1966, *e.g.* p. 206, and in CELLINI 1966_1, p. 9.)

III.6. In Sections X.2–5 I shall discuss some of the misunderstandings connected with CAVALIERI's method, but I find it appropriate even here to touch upon later interpretations of CAVALIERI'S concept of *omnes lineae*. Let me start by repeating that CAVALIERI developed his method of indivisibles under the assumption that there is a difference between a plane figure and its collection of lines. In modern terms we can describe his procedure as a construction of a map

$$F \to \mathcal{O}_F(l)$$

which to each 'nice' plane figure, F, assigns a "magnitude" $\mathcal{O}_F(l)$. Let us introduce a time parameter, t; this is given implicitly by the motion of the plane which defines

[12] GALILEI *Opere*, vol. 18, p. 67: Io non ardii di dire che il continuo fosse composto di quelli [gli indivisibili] ... s'io havessi havuto tanto ardire ...

[13] GALILEI *Opere*, vol. 16, p. 138: ... assolutamente io non mi dichiaro di componere il continuo d'indivisibili ...

"all the lines". When choosing the time unit so that the time used to traverse a plane figure F, *recti transitus*, equals the altitude, a, of F, we can characterize the collections of lines of F, taken with respect to a given *regula* AB, as the set

$$\mathcal{O}_F(l)_{AB} = \{l(t) \mid l(t) \text{ is a chord in F parallel to AB and } t \in [0, a]\}.$$

This set was considered a magnitude by CAVALIERI. Most historians of mathematics have chosen to describe CAVALIERI'S *omnes lineae* as a sum of line segments (the only exceptions I am aware of being LOMBARDO-RADICE and GIUSTI). This transcription is unfortunate, because neither CAVALIERI'S definition nor his applications of "all the lines" imply the concept of a sum. In Section X.5. I shall return to this matter and argue that the impulse to regard collections of lines as sums originates in the 17th century.

In short outlines of the method of indivisibles CAVALIERI'S results are often described by means of definite integrals. I trust that by now it should be evident that CAVALIERI'S abstract magnitude, "all the lines", is conceptually very different from a definite integral. Still it is sometimes convenient to explain CAVALIERI'S geometrical results by a modern concept; and occasionally I am going to use integrals for that purpose. Thus $\mathcal{O}_F(l)$ will be 'transcribed' as $\int_0^a l(t)\,dt$ when F has the altitude a.

III.7. This last section on "all the lines" will treat those *obliqui transitus*. CAVALIERI introduced this concept in his definition of "all the lines" (*cf.* Section III.1), but he did not employ it much in *Geometria*. Hence one could wonder why CAVALIERI did not restrict his considerations to "all the lines", *recti transitus*, and why he did not define them by letting a line move instead of a plane.

Some explanation of this can be found in CAVALIERI'S letter of October 2, 1634, to GALILEO (GALILEI *Opere*, vol. 16, pp. 137–138). However, his ideas and motivation are more clearly exposed in *Exercitationes*, through comments on the concept of "all the lines" *obliqui transitus*.

In elaborating on the definition CAVALIERI considered two parallel planes α and β (Figure III.2) and two parallelograms IKLM and IKNO. The common side IK is situated in the plane α and the sides LM and NO are situated in β; moreover IKLM is perpendicular to α and β (*Exercitationes*, pp. 15–16). He then

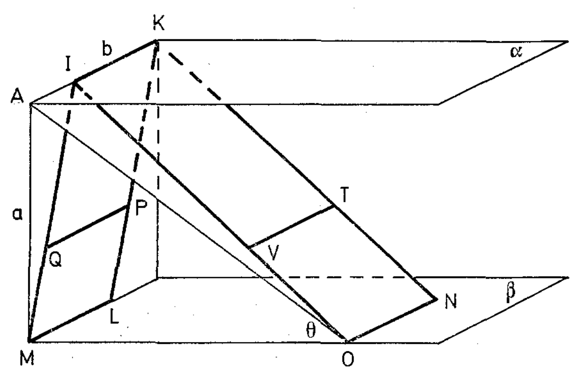

Fig. III.2

imagined that α moves toward β while staying parallel to it. When the line IK is taken as *regula*, the intersections between the moving plane and the two parallelograms make up "all the lines" of IKLM, *recti transitus* and "all the lines" of IKNO, *obliqui transitus*. CAVALIERI maintained that these two collections are equal, *i.e.*

$$\mathcal{O}_{\mathrm{IKLM}}(\mathrm{l})_{recti\ transitus} = \mathcal{O}_{\mathrm{IKNO}}(\mathrm{l})_{oblique\ transitus}\,. \qquad \text{(III.2)}$$

His argument was that at each position of the moving plane, the intersections, *e.g.* QP and VT, are equal.

CAVALIERI further emphasized that the two parallelograms IKLM and IKNO are not equal (*i.e.* they do not have the same area), and explained that the reason that two unequal parallelograms can have equal collections of lines in the sense of (III.2), is that the lines in $\mathcal{O}_{\mathrm{IKLM}}(\mathrm{l})_{recti\ transitus}$ lie more closely than the lines in $\mathcal{O}_{\mathrm{IKNO}}(\mathrm{l})_{obliqui\ transitus}$.

Thus it seems that CAVALIERI'S purpose of creating the concept of "all the lines" *obliqui transitus* was to obtain a means enabling him to distinguish between different distributions of lines. What purpose would this serve? Apparently he wished to avoid some of the paradoxes which may be the consequence of using arguments about infinitely many elements. That is, at least, the reason he displayed in the letter mentioned above to GALILEO, and in *Exercitationes* he illustrated this further (pp. 238–240).

Here he presented an objection to his method which he had already dealt with in a letter to TORRICELLI written April 5, 1644 (TORRICELLI *Opere*, vol. 3, pp. 170–171). Also TORRICELLI used it in a slightly different form as an example in *De indivisibilibum doctrina perperam usurpata* (*ibid.*, vol. 1, part 2, pp. 417–418). At no place does it become clear from whom the objection came the only information being that it was sent to CAVALIERI from an anonymous person (*ibid.*, vol. 3, p. 170).

The aim of this person was to show the weakness of the method of indivisibles by demonstrating that it could lead to the result that all triangles are equal, or equivalently that all triangles having the same altitude are equal. His argument was this: Let ADH and DGH have the same altitude HD (Figure III.3), if it can be proved, taking HD as *regula*, that

$$\mathcal{O}_{\mathrm{ADH}}(\mathrm{l})_{\mathrm{HD}} = \mathcal{O}_{\mathrm{DGH}}(\mathrm{l})_{\mathrm{HD}}; \qquad \text{(III.3)}$$

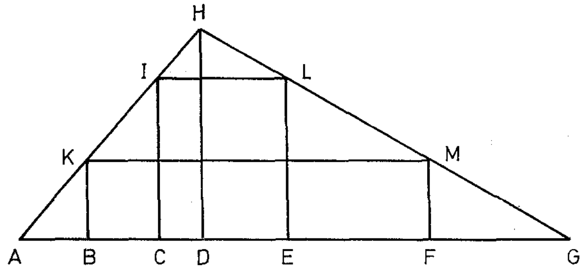

Fig. III.3

then it follows from *Geometria*, Theorem II.3 (*cf.* the relation (III.1)) that ΔADH $= \Delta$DGH. But to obtain (III.3), he said, is easy, because to every line segment BK in $\mathcal{O}_{\mathrm{ADH}}(\mathrm{l})$, by drawing KM parallel to AG one can find a line segment FM in $\mathcal{O}_{\mathrm{DGH}}(\mathrm{l})$ which is equal to it (and vice versa).

CAVALIERI refuted the objection by pointing out that Theorem II.3 applies only to two collections of lines generated by the same *transitus* (*Exercitationes* p. 239). Such is obviously not the case in the above example, because the distance between two line segments as KB and IC in $\mathcal{O}_{\mathrm{ADH}}(\mathrm{l})$ is not the same as the distance between their two corresponding line segments MF and LE in $\mathcal{O}_{\mathrm{DGH}}(\mathrm{l})$.

CAVALIERI'S concepts of "all the lines", *recti transitus* and *obliqui transitus*, and his distinction between different distributions of "all the lines" can be illustrated by 'transcribing' the concepts to integrals: Let us again consider the two parallelograms IKLM and IKNO in Figure III.2. Let the angle between the plane of the parallelogram IKNO and β be Θ, let IK $=$ b, AM $=$ a; and let us, as earlier, also assume that a is the time used for generating the two collections of lines $\mathcal{O}_{\mathrm{IKLM}}(\mathrm{l})_{\textit{recti transitus}}$ and $\mathcal{O}_{\mathrm{IKNO}}(\mathrm{l})_{\textit{obliqui transitus}}$.

Further, letting AO $=$ c, we can then transcribe

$$\mathcal{O}_{\mathrm{IKLM}}(\mathrm{l})_{\textit{recti transitus}} \quad \text{as} \quad \int_0^a \mathrm{b\,dt} = \text{area IKLM}$$

and

$$\mathcal{O}_{\mathrm{IKNO}}(\mathrm{l})_{\textit{recti transitus}} \quad \text{as} \quad \int_0^c \mathrm{b\,dt} = \text{area IKNO}.$$

The equality (III.2) then leads to transcribing

$$\mathcal{O}_{\mathrm{IKNO}}(\mathrm{l})_{\textit{obliqui transitus}} \quad \text{as} \quad \int_0^a \mathrm{b\,dt} = \sin\Theta \text{ area IKNO} = \int_0^c \mathrm{b\,d(t}\sin\Theta).$$

This enables us to interpret CAVALIERI'S considerations about different distributions of line segments in a *rectus transitus* $\left(\int_0^c \mathrm{b\,dt}\right)$ and in an *obliquus transitus* $\left(\int_0^c \mathrm{b\,d(t}\sin\Theta)\right)$ as the geometrical equivalence of integrating with respect to different variables, the one being a constant times the other (the argument is also valid if, instead of the constant line segment b, we consider line segments l(t)).

IV. Other omnes-concepts

IV.1. In *Geometria* CAVALIERI introduced a variety of *ad hoc* concepts to be used for quadratures and cubatures. In this and the following sections I shall present some of them and mainly those which are necessary for understanding the text of the more important theorems in *Geometria*.

To deal with solid figures CAVALIERI introduced collections of planes. This concept is defined in *Geometria*, Definition II.2 for the kind of figures CAVALIERI considered, that is, solids which have two opposite tangent planes (Figure IV.1). CAVALIERI imagined the one tangent plane to move toward the other, remaining parallel to it. "All the planes" of the solid figure, taken with one of the tangent planes as *regula*, consist in the intersections between the figure and the moving plane. Such a collection of planes belonging to a solid figure, S, and with *regula* BCD will be denoted by $\mathcal{O}_S(p)_{BCD}$ or simply by $\mathcal{O}_S(p)$. (Despite the fact that the p's in $\mathcal{O}_S(p)$ are plane figures, like CAVALIERI I shall call them planes.)

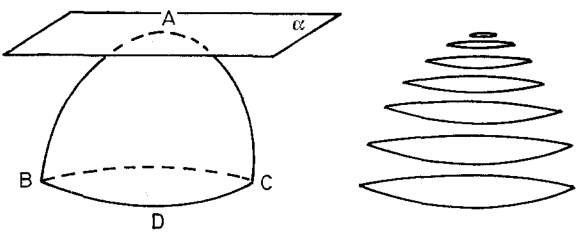

Fig. IV.1. Inspired by CAVALIERI's figure on page 105 of *Geometria.* To the left the solid S = ABCD having the opposite tangent planes α and BCD. To the right some of the planes of $\mathcal{O}_S(p)$.

CAVALIERI also assigned collections of plane figures to plane figures (*Geometria*, p. 103): For a given plane figure, F, the collection of "all similar plane figures of the given figure" is obtained by describing a plane figure on each l in $\mathcal{O}_F(l)$ in such a way that the figures are similar and situated in parallel planes determined by the transit of F (Figure IV.2). I shall denote a plane figure on l by A(l), and the collection of similar and parallel figures by $\mathcal{O}_F(A(l))$.

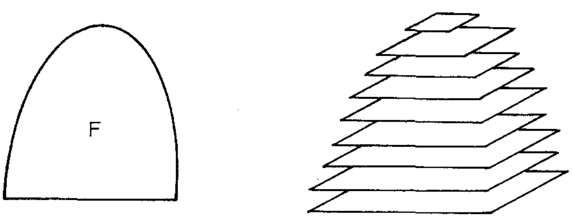

Fig. IV.2. A given figure F and some of the similar plane figures of the collection $\mathcal{O}_F(A(l))$ when A(l) is a rectangle.

CAVALIERI was particularly interested in the case when all the A(l)'s are squares and called the corresponding collection "all the squares" of the given figure; introducing the symbol $\square\, l$ for the square on l, I am going to denote this collection by $\mathcal{O}_F(\square l)$. The requirement that all figures in $\mathcal{O}_F(A(l))$ should be similar implies that for all l_1 and l_2 in $\mathcal{O}_F(A(l))$

$$A(l_1) : A(l_2) = \square\, l_1 : \square\, l_2, \qquad \text{(IV.1)}$$

a relation CAVALIERI proved and to which we shall return (Section VI.4).

It is natural to consider the solid whose surface is determined by "all the similar plane figures of a given figure"; the collection of planes of this solid is exactly the collection of similar plane figures. This result is central in several of CAVALIERI'S proofs and is stated as a separate postulate, the second and last in Book II (*Geometria*, p. 138). That CAVALIERI found this postulate necessary is rather surprising, because the identity between the two collections of planes seems clear. Probably, CAVALIERI'S real problem was not to establish the identity, but to come from an intuitive understanding of the solid determined by the similar figures to a real definition of it, and this problem he then hid in an apparently obvious postulate.

IV.2. A third kind of collection of plane figures belonged to CAVALIERI'S tools, namely some emerging from a pair of plane figures (*Geometria*, p. 104): Let F_1 and F_2 be two plane figures having the same altitude and the common *regula* AB (Figure IV.3). If from each pair of corresponding l_1 and l_2 in $\mathcal{O}_{F_1}(l)$ and $\mathcal{O}_{F_2}(l)$ the rectangle $l_1 \times l_2$ is formed, we obtain "all the rectangles" of the two figures. (I call l_1 and l_2 corresponding when they have the same distance to AB.) Such a collection of rectangles will be denoted by $\mathcal{O}_{F_1,F_2}(l \times l)_{AB}$ or by $\mathcal{O}_{F_1,F_2}(l \times l)$.

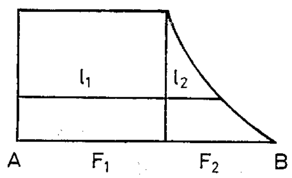

Fig. IV.3. In the figure CAVALIERI used to illustrate the concept $\mathcal{O}_{F_1,F_2}(l \times l)$. F_1 and F_2 have a straight line in common (*Geometria*, page 107), but the definition applies for all figures F_1 and F_2 having a common *regula* and the same altitude.

CAVALIERI'S creation of a collection of rectangles has a certain resemblance to the construction of a solid figure from two figures standing on the same base and situated in perpendicular planes which was carried out by, among others, GILLES P. ROBERVAL and GRÉGOIRE DE SAINT VINCENT and called by the latter *ductus plani ad planum*. ROBERVAL, GRÉGOIRE, and later PASCAL used these solids to perform geometric transformations for problems of integration. CAVALIERI'S purpose in introducing "all the rectangles" was a different one, namely to obtain a concept he could use in his geometrical calculations in the cases where a modern mathematician would work with integrals of the form $\int_0^a l_1(t) \cdot l_2(t)\, dt$. Thus in connection with "all the rectangles" CAVALIERI did not consider the pertaining solid, but let the concept remain an abstract one. In his second method of integration – involving no *omnes*-concepts – CAVALIERI actually introduced solids similar to *ductus* but still not to achieve integral transformations. In Section IX.2 I shall return to these solids.

IV.3. The last *omnes*-concepts to be presented here relate to a line segment. The basis of these concepts is "all the points" of a given line segment: Let the two line segments ON and EM have their endpoints in two parallel planes, ON being perpendicular to the planes but EM not perpendicular (*cf.* Figure IV.4). We then imagine that the one plane moves toward the other, remaining parallel to it. The points of intersection between the moving plane and ON are called "all the points" of ON *recti transitus*, whereas those between the moving plane and EM are called "all the points" of EML, *obliqui transitus* (*Geometria*, p. 100).

From these collections of points CAVALIERI derived a spectrum of abscissae concepts which he repeatedly applied. He defined the concepts both *recti transitus* and *obliqui transitus*. The latter are related to the line segment EM, and since they can easily be imagined when the first ones, related to the line segment ON, are known, I am not going to deal with them.

The collection of abscissae itself, or "all the abscissae" of the line segment ON arises from its collection of points by assigning to each point H the line segment OH (*cf.* Figure IV.4). If instead of taking all the line segments OH we take the line segments NH, we obtain the collection which CAVALIERI called *residuæ omnium abscissarum*, "the residues of all the abscissae". Further, if, to each H in ON's collection of points, we assign the whole line segment ON, we get "the maxima of all the abscissae" of ON.

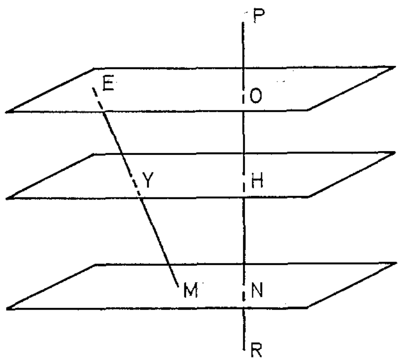

Fig. IV.4

"All the abscissae" and "the residues of all the abscissae" of ON differ only with respect to the order in which the line segments are considered. I shall let the order of the letters in the symbol for a line segment indicate this order and denote the collection of abscissae belonging to ON by $\mathcal{O}_{ON}(\alpha)$ and the collection of residues by $\mathcal{O}_{ON}(ON - \alpha)$; this latter collection is the same as $\mathcal{O}_{NO}(\alpha)$. The symbol $\mathcal{O}_{ON}(ON)$ will be used for "the maxima of all the abscissae".

The role of the abscissae concepts in CAVALIERI's geometrical calculations can be compared with the modern use of integrals of linear functions. If we set $|ON| = a$, we can make the following transcriptions:

$$\mathcal{O}_{ON}(\alpha) \quad \text{to} \quad \int_0^a t\,dt,$$

$$\mathcal{O}_{ON}(ON - \alpha) \quad \text{to} \quad \int_0^a (a - t)\,dt,$$

$$\mathcal{O}_{ON}(ON) \quad \text{to} \quad \int_0^a a\,dt.$$

By adding a fixed line segment, PO or NR, say of length b, to each abscissa in the collections of abscissae CAVALIERI achieved geometrical concepts which can be transcribed as

$$\int_0^a (t + b)\,dt, \qquad \int_0^a (a - t + b)\,dt \quad \text{and} \quad \int_0^a (a + b)\,dt.$$

Actually, CAVALIERI not only used abscissae concepts in connection with linear expressions; in his calculations he also introduced *ad hoc* collections of rectangles of abscissae such as the one, $\mathcal{O}_{ON}((ON - \alpha)\times(PO + \alpha))$, which is obtained by assigning to each H of the collection of points of ON the rectangle $NH\times(PO + OH)$ (*Geometria*, p. 172). This collection can be transcribed as $\int_0^a (a - t)(b + t)\,dt$. The abscissae concepts *obliqui transitus*, like "all the lines" *obliqui transitus*, correspond to integrals where the variable of integration has been multiplied by a constant (*cf.* Section III.7).

IV.4. Throughout *Geometria* the abscissae concepts occur very often and results concerning other collections are translated into the terminology of collections of abscissae. The idea behind the translation can be illustrated by the following example.

Let us consider the parallelogram AFDC, where $CD = DF$ (Figure IV.5). The collection of lines of the triangle CDF, $\mathcal{O}_{CDF}(l)_{CD}$, consists of all line segments HE parallel to CD, whereas the collection of abscissae of the line segment FD, $\mathcal{O}_{FD}(\alpha)$, *recti transitus* if the angle CDF is right, *obliqui transitus* otherwise, consists of all line segments FE. The equality $CD = DF$ implies that $HE = FE$ for each HE in $\mathcal{O}_{CDF}(l)_{CD}$ and each FE in $\mathcal{O}_{FD}(\alpha)$ *recti* or *obliqui transitus*. This made CAVALIERI state that the two collections are equal (*Geometria*, p. 148). Further he set $\mathcal{O}_{AFCD}(l)_{CD}$ equal to "the maxima of all the abscissae" of FD, $\mathcal{O}_{FD}(FD)$ *recti* or *obliqui transitus*, and similarly collections of of lines belonging to trapezia equal to other collections of abscissae.

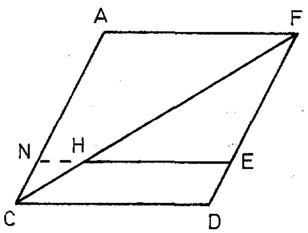

Fig. IV.5

CAVALIERI'S reason for translating his results concerning collections of lines, rectangles, *etc.*, to results concerning collections of abscissae was clearly related to his work on conic sections; this can be explained by a simple example: Let ABC be a segment of a parabola (Figure IV.6) having the property that

$$\square\, EF : \square\, GH = BE : GB. \qquad \text{(IV.2)}$$

In some theorems CAVALIERI needed a result concerning all the squares of the parabola segment, *i.e.* $\mathcal{O}_{ABC}(\square\, l)_{DC}$. He could then use the relation (IV.2), in a way which will become clearer later on, to transform the problem concerning $\mathcal{O}_{ABC}(\square\, l)_{DC}$ into an investigation of $\mathcal{O}_{BD}(\alpha)$ *recti transitus.*

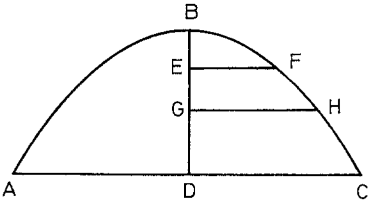

Fig. IV.6

In *Exercitationes* (p. 17) CAVALIERI remarked that the abscissae concepts were not really necessary in his theory. This statement indicates that CAVALIERI had the idea that his theory would have been simpler if he had omitted the abscissae concepts. Such an omission would certainly have meant fewer concepts, but with his way of calculating not much would have been gained by so doing, because instead of the abscissae concepts he would have had to introduce extra figures into the calculations. Thus in the above example he would need a triangle whose collection of lines could be used instead of $\mathcal{O}_{BD}(\alpha)$.

V. The foundation of Cavalieri's collective method of indivisibles

V.1. This and the following sections will concentrate on the two fundamental theorems of CAVALIERI'S collective method of indivisibles and on the question of which assumptions he used to establish them. (For another analysis of the foundation of CAVALIERI'S theory of indivisibles, see GIUSTI 1980, pp. 34–40.) Since there is no essential difference between his arguments concerning plane figures and solid ones, only the first will be treated. Further, I shall leave out the *regulae*, assuming that collections of lines have the same *regula* when they are compared.

The first of the fundamental theorems, already touched upon in Section III.1, is Proposition II.3 of *Geometria* establishing the connection between areas and collections of lines and between volumes and collections of planes:

$$F_1 : F_2 = \mathcal{O}_{F_1}(l) : \mathcal{O}_{F_2}(l) \tag{V.1}$$

and

$$S_1 : S_2 = \mathcal{O}_{S_1}(p) : \mathcal{O}_{S_2}(p). \tag{V.2}$$

The second is the succeeding Proposition II.4 containing a result which is often called *Cavalieri's principle.* For reasons of clarity I present it in a free translation (*cf.* BOYER 1968, p. 362):

If two plane (or solid) figures have equal altitudes, and if sections made by lines (or planes) parallel to the bases and at equal distances from them are always in the same ratio, then the plane (or solid) figures also are in this ratio.[14]

Thus if two plane figures, like $F_1 = ACM$ and $F_2 = MCE$ in Figure V.1, have the property that for each line BD parallel to the base AE the sections BR and RD, in F_1 and F_2 respectively, satisfy the relation

$$BR : RD = AM : ME, \tag{V.3}$$

then

$$ACM : MCE = AM : ME. \tag{V.4}$$

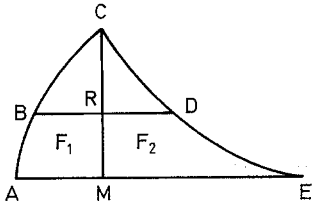

Fig. V.1. A reproduction of CAVALIERI's figure illustrating his principle (*Geometria* page 115). In making the figure CAVALIERI was not aware that the assumption (V.3) implies that the curves ABC and ECD have similar curvature at corresponding points.

After an investigation of CAVALIERI's assumptions concerning collections of lines I shall return to his proofs of these two theorems, *viz.* the proofs of (V.1) and (V.4).

CAVALIERI stated only one of his assumptions explicitly, namely the one contained in Postulate II.1:

"All the lines" of congruent plane figures ... are congruent.[15]

That is

$$F_1 \simeq F_2 \Rightarrow \mathcal{O}_{F_1}(l) \simeq \mathcal{O}_{F_2}(l). \tag{V.5}$$

In *Geometria* CAVALIERI took it for granted that his readers would understand what he meant by congruent collections of lines, but in *Exercitationes* (pp. 200–201) he added an explanation: When two congruent figures, F_1 and F_2, are placed so that they coincide, then each line in $\mathcal{O}_{F_1}(l)$ will coincide with exactly one line in $\mathcal{O}_{F_2}(l)$ (and vice versa), and the collections of lines are called congruent.

[14] *Geometria*, p. 115: Si duae figurae planae, vel solidae, in eadem altitudine fuerint constitutae, ductis autem in planis rectis lineis, & in figuris solidis ductis planis utcumque inter se parallelis, quorum respectu praedicta sumpta sit altitudo, repertum fuerit ductarum linearum portiones figuris planis interceptas, seu ductorum planorum portiones figuris solidis interceptas, esse magnitudines proportionales, homologis in eadem figura semper existentibus, dictae figurae erunt inter se, ut unum quodlibet eorum antecedentium, ad suum consequens in alia figura eidem correspondens.

[15] *Geometria*, p. 108: Congruentium planarum figurarum omnes linae, sumptae una earundem ut regula communi, sunt congruentes.

Although the concept of congruence between collections of lines is explicitly introduced, it did not play a great role in CAVALIERI's theory; thus in applying Postulate II.1 CAVALIERI used only the implication

$$F_1 \simeq F_2 \Rightarrow \mathcal{O}_{F_1}(l) = \mathcal{O}_{F_2}(l). \qquad (V.6)$$

He did not explain what it means for two collections of lines to be equal, but assumed that there is a relation of the kind we now call an equivalence relation, involving more than just identity or congruence.

V.2. The existence of an equality relation belongs to CAVALIERI's implicit assumptions originating from his idea that collections of lines constitute a category of magnitudes (or more precisely that the equivalence classes of collections of lines defined by congruence are magnitudes). On the whole he presupposed that collections of lines have the Properties 1, 2 and 3 listed in Section II.2 for magnitudes.

CAVALIERI further assumed an additive property of his *omnes*-concept, namely that

$$F \simeq F_1 + F_2 \Rightarrow \mathcal{O}_F(l) = \mathcal{O}_{F_1}(l) + \mathcal{O}_{F_2}(l). \qquad (V.7)$$

It is used, for instance, in his proofs of the first two theorems of *Geometria*, Book II.

Moreover he often employed the following implication without further comment:

$$F_1 > F_2 \Rightarrow \mathcal{O}_{F_1}(l) > \mathcal{O}_{F_2}(l). \qquad (V.8)$$

After the presentation of Theorem II.2 I shall reconstruct an argument on which CAVALIERI could have based this assumption.

Apart from assumptions concerning additivity, equality, and ordering, CAVALIERI employed a strong requirement on $\mathcal{O}$'s properties which I shall call the *ut-unum principle*:

> As one antecedent is to one consequent so are all the antecedents to all the consequents.[16]

CAVALIERI formulated this principle as a corollary to Theorem II.4 of *Geometria* employing the word corollary in a rather unusual sense, because the principle is not a consequence of Theorem II.4, but is used to prove it.

For plane figures the content of the principle is the following: If two figures, F_1 and F_2, have their bases situated on the same line, have equal altitudes, and if each pair of corresponding l_1 and l_2, in $\mathcal{O}_{F_1}(l)$ and $\mathcal{O}_{F_2}(l)$ respectively, are in the same ratio, then $\mathcal{O}_{F_1}(l)$ and $\mathcal{O}_{F_2}(l)$ are also in that ratio. (As earlier, I call l_1 and l_2 corresponding when they are at equal distance from the bases.) Thus, if in Figure V.1

$$l_1 : l_2 = AM : ME \text{ for all corresponding } l_1 \text{ and } l_2 \qquad (V.9)$$

[16] *Geometria*, p. 116: ut unum antecedentium ad unum consequentium, ita esse omnia antecedentia ad omnia consequentia.

then

$$\mathcal{O}_{F_1}(l) : \mathcal{O}_{F_2}(l) = AM : ME. \qquad (V.10)$$

In Sections VI.3 and 4 we are going to see that while developing his theory CAVALIERI also generalized the *ut-unum* principle, so that it came to mean that $\mathcal{O}$ can be applied to relations between line segments in which there is a certain constancy.

CAVALIERI did not reveal his thoughts concerning the use of the *ut-unum* principle, but in *Exercitationes* (p. 30) he emphasized that he was not the only one employing this principle, but that also TORRICELLI had used it in *De dimensione parabolae*; how TORRICELLI used the principle we shall see in Section X.2.

V.3. It would serve no purpose – indeed it would also be quite impossible within the compass of one paper – to present all of CAVALIERI'S theorems and proofs. Nevertheless, to facilitate the understanding of CAVALIERI'S theory and the reaction to it, I shall present in some detail the theorems on which the theory is based.

Although CAVALIERI implicitly assumed that collections of lines have Properties 1–3 of Section II.2 he found it necessary, as we saw in Section III.3, to prove that they also have the last Property 4. Thus he stated in *Geometria*, Theorem II.1, that collections of lines are magnitudes which have a ratio to each other.

As explained in Section II.3 CAVALIERI'S aim was to show that collections of lines fulfill Definition V.4 of EUCLID'S *Elements*. Let $\mathcal{O}_{F_1}(l)$ and $\mathcal{O}_{F_2}(l)$ be the collections of two plane figures, $F_1 = GOQ$ and $F_2 = EAG$ (Figure V.2). CAVALIERI then had to show that they can be multiplied to exceed each other. This means that he would for example have to show that a multiple

$$\mathcal{O}_{F_1}(l) + \mathcal{O}_{F_1}(l) + \ldots + \mathcal{O}_{F_1}(l) \text{ of } \mathcal{O}_{F_1}(l) \text{ can be found such that}$$

$$\mathcal{O}_{F_1}(l) + \mathcal{O}_{F_1}(l) + \ldots + \mathcal{O}_{F_1}(l) > \mathcal{O}_{F_2}(l).$$

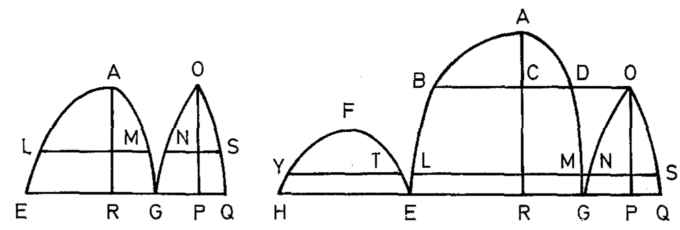

Fig. V.2. A reproduction of CAVALIERI'S figure on page 100 of *Geometria*, the left for the case AR = OP, and the right for the case AR > OP.

His proof is based on rather loose considerations: First he supposed the altitudes AR and OP of the two figures to be equal. He then argued that each $l_1 = NS$ in $\mathcal{O}_{F_1}(l)$ can be multiplied to exceed the corresponding $l_2 = LM$ in $\mathcal{O}_{F_2}(l)$, and concluded that a multiple of $\mathcal{O}_{F_1}(l)$ greater than $\mathcal{O}_{F_2}(l)$ exists. Thus he did not perceive the problem of finding a maximum of those infinitely many n's for which $nl_1 > l_2$, l_1 belonging to $\mathcal{O}_{F_1}(l)$ and l_2 to $\mathcal{O}_{F_2}(l)$; even if he had seen it, he would have had no way to establish the existence of that maximum.

In the case where $AR \neq OP$, *e.g.* $AR > OP$, CAVALIERI split AR into parts equal to OP and a remaining part which was not greater than OP. For the sake of simplicity he assumed that $AR = CR + AC$, where $CR = OP$ and $AC < OP$. Through C he drew the line CO parallel to EG and moved the figure BAD into the figure HFE. As in the first part of the proof, to conclude the existence of a multiple of $\mathcal{O}_{GOQ}(l)$ which is greater than $\mathcal{O}_{EBDG}(l) + \mathcal{O}_{HFE}(l)$ and hence greater than $\mathcal{O}_{EAG}(l)$ he used the fact that each $l_1 = NS$ in $\mathcal{O}_{GOQ}(l)$ can be multiplied to exceed the sum of the corresponding $l_2 = LM$ in $\mathcal{O}_{EBDG}(l)$ and $l_3 = YT$ in $\mathcal{O}_{HFE}(l)$ (in the last step (V.7) is used).

The next theorem, II.2 (*Geometria*, p. 122) states that

$$F_1 = F_2 \Rightarrow \mathcal{O}_{F_1}(l) = \mathcal{O}_{F_2}(l). \tag{V.11}$$

This implication constitutes a very important element of the foundation of CAVALIERI'S method; it is stronger than most of his implicit and explicit assumptions and cannot be derived from those without a further assumption. Before saying more about the fundamental problem involved in (V.11), I shall sketch CAVALIERI'S proof of it.

Let it be given that $AEB = ADC$ (Figure V.3), the aim is to show that $\mathcal{O}_{AEB}(l) = \mathcal{O}_{ADC}(l)$. For this CAVALIERI used a method of superposition and first placed the figures so that they had the area ADB in common; then he placed one residuum over the other and continued the process "until all the residual parts have been placed over each other".[17]

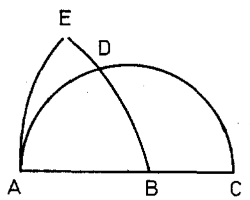

Fig. V.3

He then concluded that, since the two figures are split up into congruent parts and these, by Postulate II.1 (*cf.* (V.5) and (V.6)), have equal collections of lines, the figures also have equal collections of lines (in the last step (V.7) is employed).

The problematic part of the proof is the sentence "until all residual parts have been placed over each other", because this process may turn out to be infinite (AEB and ACD could *e.g.* be a triangle and a semicircle), and would then introduce infinite sums in CAVALIERI'S argument. This would be unfortunate since one of the main points of CAVALIERI'S method is to avoid infinite sums. In *Geometria* CAVALIERI did not comment upon the aspect of an infinite process, but he took it up later in a letter to TORRICELLI, dated March 10, 1643 (TORRICELLI *Opere*, vol. 3, p. 114) and in *Exercitationes* (p. 212). He seems to have thought that an argument in the style of the method of exhaustion could save his proof in the case of an infinite process, but he did not go into detail about this.

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[17] *Geometriae*, p. 112: donec omnes residuae partes ad invicem superpositae fuerint.

As pointed out by GIUSTI, the problem with which CAVALIERI was confronted stems from the fact that in the theory of magnitudes the equality $F_1 = F_2$ is mainly obtained as exclusions of $F_1 > F_2$ and $F_1 < F_2$, and that a direct definition of $F_1 = F_2$, when F_1 and F_2 are not polygons, leads to an infinite process (GIUSTI 1980, p. 34).

Theorem II.2 enables us to see how CAVALIERI could have concluded the implication (V.8): He might have argued that $F_1 < F_2$ implies that there are figures F_3 and F_4 such that $F_2 = F_3$ and $F_1 \simeq F_3 + F_4$; from this (V.7) and (V.11) it then follows that

$$\mathcal{O}_{F_1}(l) = \mathcal{O}_{F_3}(l) + \mathcal{O}_{F_4}(l) = \mathcal{O}_{F_2}(l) + \mathcal{O}_{F_4}(l)$$

and hence that

$$\mathcal{O}_{F_1}(l) > \mathcal{O}_{F_2}(l).$$

V.4. With the tools presented in the last three sections at his disposal CAVALIERI was able to achieve the two fundamental results (V.1) and (V.4) in the best EUCLIDean style. To obtain

$$F_1 : F_2 = \mathcal{O}_{F_1}(l) : \mathcal{O}_{F_2}(l), \tag{V.1}$$

CAVALIERI had to prove (according to Definition V.5 of the *Elements*) that

$$\underbrace{F_1 + F_1 + \dots + F_1}_{\text{n times}} > \underbrace{F_2 + F_2 + \dots + F_2}_{\text{m times}}$$

implies that

$$\underbrace{\mathcal{O}_{F_1}(l) + \mathcal{O}_{F_1}(l) + \dots + \mathcal{O}_{F_1}(l)}_{\text{n times}} > \underbrace{\mathcal{O}_{F_2}(l) + \mathcal{O}_{F_2}(l) + \dots + \mathcal{O}_{F_2}(l)}_{\text{m times}},$$

and similarly for $=$ and $<$. The cases involving $>$ and $<$ follow directly from (V.7) and (V.8); and the equality case is a consequence of (V.7) and (V.11).

The results (V.1), (V.2) and their generalizations are really the central idea of CAVALIERI'S theory: to transform a determination of, for instance, a ratio between two areas into a calculation of the ratio between their collections of lines. In later sections I shall illustrate how CAVALIERI managed to calculate such a ratio for some figures. But even now it can be noticed that one of his useful tools was what is called CAVALIERI'S principle ((V.3) and (V.4)). The principle itself is an immediate consequence of the *ut-unum* principle and (V.1):

The assumption (Figure V.1) $BR : RD = AM : ME$ (V.3) is equivalent to $l_1 : l_2 = AM : ME$ (V.9); hence (V.10) implies that

$$\mathcal{O}_{ACM}(l) : \mathcal{O}_{MCE}(l) = AM : ME$$

and the result, $ACM : MCE = AM : ME$ (V.4) is obtained by (V.1).

In the sections dealing with CAVALIERI'S foundation of his method, I have concentrated on collections of lines; CAVALIERI himself also treated collections of planes. However, in working out the more complicated parts of his theory, he let some of his fundamental results and assumptions, including the implicit ones, cover every *omnes* concept, as for instance collections of squares.

The presentation of his foundation has revealed that CAVALIERI could have been more careful in making his assumptions clear, and that Theorem II.2 ((V.11)) contains a really weak point. There are some problems involved also in the proofs of his later theorems, but after Theorem II.4 of *Geometria* CAVALIERI could by and large proceed in the style he aimed at, namely one coming near to Greek mathematics, although it implied other tools.

VI. The basic theorems of Book II of Geometria

VI.1. Although CAVALIERI's motive for writing *Geometria* clearly was to create a new method of quadratures and cubatures, he was not first of all concerned with finding new results, but in showing that his method worked, and he did this by using his method to obtain well known theorems. Some of these are to be found in Book II; the book also contains preparatory theorems for Books III–V.

In this and the following sections I shall give a survey of the topics of Book II and illustrate some of them by examples. The first four theorems of Book II have already been presented in Sections V.1–4, and the last eight are purely algebraic and have no direct connection with CAVALIERI's method; the remaining theorems, II.5–34, contain:

A. Some elementary statements about the ratio between two parallelograms.
B. Calculations of ratios between collections of lines, between collections of squares, *etc.* These theorems are CAVALIERI's geometrical equivalents for calculating the integrals of polynomials of first and second degree. His technique did not permit him to determine the geometrical equivalent of $\int_0^a (\alpha t^2 + \beta t + \gamma)\,dt$ at one stroke. Instead he considered cases corresponding to

$$\int_0^a t\,dt, \quad \int_0^a (a - t)\,dt, \quad \int_0^a (t + b)\,dt, \quad \int_0^a (a - t + b)\,dt, \qquad \text{(VI.1)}$$

$$\int_0^a t^2\,dt, \quad \int_0^a (a - t)^2\,dt, \quad \int_0^a (t + b)^2\,dt, \quad \int_0^a (t + b)\,t\,dt,$$
$$\int_0^a (t + b)(a - t)\,dt, \quad \int_0^a (t + b)(t + b + c), \quad \int_0^a (t + b)(a - t + c)\,dt \qquad \text{(VI.2)}$$

where a, b and c are positive constants.
C. Generalized forms of the *ut-unum* principle.
D. Two general theorems stating that two similar plane figures are in the duplicate ratio of their linear ratio, and that two similar solids are in the triplicate ratio of their linear ratio.
E. Results concerning the ratios between pairs of the following solids: cylinders, cones (including truncated ones), prisms and pyramids (including truncated ones).

VI.2. CAVALIERI's method was well fitted to achieve results concerning parallelograms: If two parallelograms have equal altitudes it follows directly from CAVALIERI's principle (*cf.* (V.3) and (V.4)) that they are in the ratio of their bases. And from this CAVALIERI easily concluded that two parallelograms are in a ratio composed by the ratio of their altitudes and the ratio of their bases.

The group of theorems mentioned in B contains the most important results in Book II and constitutes the base of the calculations in Books III–V concerning conic sections. Therefore I shall give some detailed examples of their formulation and proofs. As previously mentioned CAVALIERI introduced many *ad hoc* concepts to obtain the geometrical equivalents of the integrals occurring in (VI.1) and (VI.2); I shall restrict the presentation to the theorems involving the least complicated concepts. The first one to be considered is the result corresponding to $\int_0^a t\,dt = \frac{1}{2}a^2$, *Geometria,* Theorem II.19:

> If a diagonal is drawn in a parallelogram, the parallelogram is the double of each of the triangles determined by the diagonal.[18]

Let ACDF be a parallelogram and CF one of its diagonals (Figure VI.1); Theorem II.19 then states that

$$ACDF = 2\Delta FAC = 2\Delta CDF. \tag{VI.3}$$

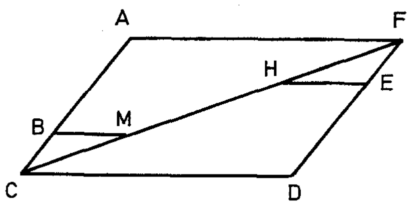

Fig. VI.1

To prove this CAVALIERI considered an arbitrary pair of corresponding line segments BM and HE in $\mathcal{O}_{FAC}(l)_{CD}$ and $\mathcal{O}_{CDF}(l)_{CD}$ respectively; that BM and HE are corresponding means that CB = FE. By using Theorem I.26 of the *Elements* CAVALIERI concluded that

$$\Delta CBM \cong \Delta FEH, \tag{VI.4}$$

and hence

$$BM = HE.$$

Since this is valid for all corresponding line segments in $\mathcal{O}_{FAC}(l)$ and $\mathcal{O}_{CDF}(l)$, CAVALIERI deduced that

$$\mathcal{O}_{FAC}(l) = \mathcal{O}_{CDF}(l). \tag{VI.5}$$

(In relating this deduction to the question of the foundation of CAVALIERI'S method, we can instead of triangle FCD consider a triangle congruent to it having its base on the line AF and the same altitude as triangle CAF; (VI.5) then follows

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[18] *Geometria,* p. 146: Si in parallelogrammo diameter ducta fuerit, parallelogrammum duplum est cuiusvis triangulorum per ipsam diametrum constitutorum.

from the *ut-unum* principle.) The relation (VI.5) together with Theorem II.3 (*cf.* (V.1)) implies that

$$\Delta FAC = \Delta CDF, \qquad (VI.6)$$

from which the result (VI.3) is obtained.

In this proof CAVALIERI seems to have taken unnecessary trouble, since instead of employing Theorem I.26 of the *Elements* to achieve (VI.4) he could have used it to deduce that the triangles FAC and CDF are congruent, which means that he could have obtained (VI.6) directly without using "all the lines".

The reason why CAVALIERI used collections of lines in the proof may of course be that he wanted to base most of the proofs in *Geometria* on *omnes*-concepts; but it is also possible that he made this detour because his main concern was not to prove (VI.3) but to obtain a result concerning collections of lines or abscissae. Thus in the second corollary to Theorem II.19, he returned to equality (VI.5) and deduced that

$$\mathcal{O}_{ACDF}(l) = 2\mathcal{O}_{FAC}(l). \qquad (VI.7)$$

Considering a parallelogram in which CD = FD, CAVALIERI could translate this result, as mentioned in Section IV.4, into terms of "all the abscissae" (*Geometria*, p. 148):

$$\mathcal{O}_{FD}(FD) = 2\mathcal{O}_{FD}(\alpha), \qquad (VI.8)$$

these abscissae being *recti transitus* if the angle CDF is right, and otherwise *obliqui transitus*. In the first case (VI.8) is CAVALIERI's geometrical analogy of

$$a^2 = 2\int_0^a t\,dt.$$

Similarly he found the geometrical results corresponding to the other integrations listed in (VI.1).

VI.3. The next step was to find a geometrical equivalent of integrating t^2; CAVALIERI did so in *Geometria*, Theorem II.24:

> Let there be given a parallelogram in which a diagonal is drawn; then "all the squares" of the parallelogram will be the triple of "all the squares" of any one of the triangles determined by the diagonal, when one of the sides of the parallelogram is taken as common *regula*.[19]

CAVALIERI built his proofs of this theorem on results obtained in the preceding theorems. Before presenting his proof I shall outline three of these results; first Theorem II.11 which, in my notation, states that when P_1 and P_2 are two parallelo-

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[19] *Geometria*, p. 159: Exposito parallelogrammo quocunque in eoque ducta diametro; omnia quadrata parallelogrammi ad omnia quadrata cuiusvis triangulorum per dictam diametrum constitutorum erunt in ratione tripla, uno laterum parallelogrammi communi regula existente.

grams (Figure VI.2) with altitudes h_1 and h_2 and bases b_1 and b_2, then (*Geometria*, p. 123)

$$\mathcal{O}_{P_1}(\square\, l) : \mathcal{O}_{P_2}(\square\, l) = (\square\, b_1 : \square\, b_2) \cdot (h_1 : h_2), \tag{VI.9}$$

the *regula* being parallel to the bases.

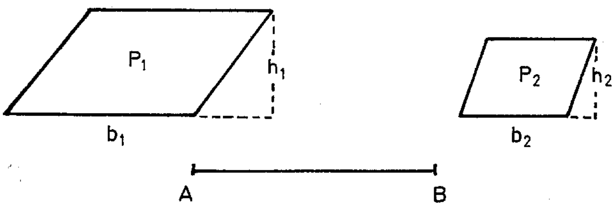

Fig. VI.2

To achieve this result CAVALIERI first considered the situation where $h_1 = h_2$. All corresponding squares in the collections of squares of the two parallelograms are in the constant ratio $\square\, b_1 : \square\, b_2$, and since the altitudes are equal, CAVALIERI concluded, through the *ut-unum* principle, that

$$\mathcal{O}_{P_1}(\square\, l)_{AB} : \mathcal{O}_{P_2}(\square\, l)_{AB} = \square\, b_1 : \square\, b_2 . \tag{VI.10}$$

He then considered the case where $h_1 \neq h_2$ and $b_1 = b_2$ and claimed that according to the *Elements*' Definition V.5 of equal ratio and (VI.10), it can be seen that

$$\mathcal{O}_{P_1}(\square\, l)_{AB} : \mathcal{O}_{P_2}(\square\, l)_{AB} = h_1 : h_2 . \tag{VI.11}$$

(In this deduction he implicitly presupposed that $nh_1 > mh_2$ implies that $\mathcal{O}_{nP_1}(\square\, l) > \mathcal{O}_{mP_2}(\square\, l)$, that is, a generalized form of (V.8).)

The required relation (VI.9) then follows from a combination of (VI.10) and (VI.11).

Another result which CAVALIERI used in his proof of Theorem II.24 is formulated in Theorem II.22. It states that if Δ_1 and Δ_2 are two triangles determined by diagonals in the parallelograms P_1 and P_2, respectively, then

$$\mathcal{O}_{P_1}(\square\, l) : \mathcal{O}_{\Delta_1}(\square\, l) = \mathcal{O}_{P_2}(\square\, l) : \mathcal{O}_{\Delta_2}(\square\, l); \tag{VI.12}$$

which means that the ratio between the collection of squares of a parallelogram and the collection of squares in a triangle determined by the diagonal is constant.

CAVALIERI established this result by a thorough *reductio ad absurdum* proof. He supposed that (VI.12) is not valid, and arrived at a contradiction by letting the triangles be circumscribed and inscribed by figures consisting of parallelograms of equal and sufficiently small altitudes (Figure VI.3) and by using (VI.9).

The remaining problem in Theorem II.24 is to prove that the ratio between the collections of squares of the parallelogram and the triangle is 3.

Before showing this, CAVALIERI introduced in Theorem II.23 a rule which might be called a generalized *ut-unum* principle (*Geometria*, pp. 155–158): He considered

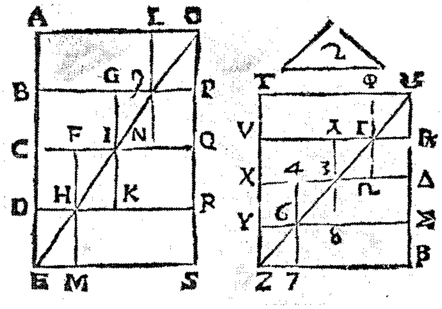

Fig. VI.3. From *Geometria*, page 151.

a plane figure ABCD divided by some curves like AC and AI in Figure VI.4 and imagined that for each $l = BD$ in $\mathcal{O}_{ABCD}(l)$ there is the same relation between the parts BE, EF, and FD into which BD is divided by the curves AC and AI (endpoints of the curves may be excluded). For example in the case where the curve AC bisects all BD so that $BE = ED$ we have, according to Theorem II.9 of the *Elements*, that

$$\square\, BF + \square\, FD = 2(\square\, BE + \square\, EF) \tag{VI.13}$$

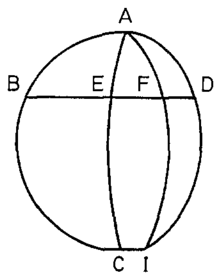

Fig. VI.4. A reproduction of CAVALIERI's figure in *Geometria*, page 156.

for all BD. The generalized *ut-unum* principle, Theorem II.23, then states that $\mathcal{O}$ can be applied to this relation, resulting in

$$\mathcal{O}_{ABIA}(\square\, l) + \mathcal{O}_{AIDA}(\square\, l) = 2(\mathcal{O}_{ABCA}(\square\, l) + \mathcal{O}_{ACIA}(\square\, l)). \tag{VI.14}$$

CAVALIERI'S proof of this theorem consists of a reference to the *ut-unum* principle.

Now let ACGE be the parallelogram and CGE the triangle of Theorem II.24 (Figure VI.5); it then has to be proved that

$$\mathcal{O}_{ACGE}(\square\, l)_{EG} = 3\mathcal{O}_{CGE}(\square\, l)_{EG}. \tag{VI.15}$$

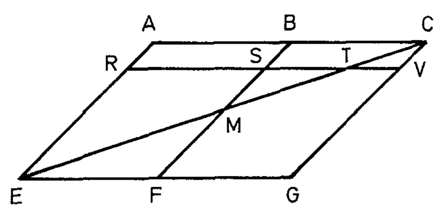

Fig. VI.5

CAVALIERI drew the line BF bisecting the parallelogram and let the lines BF and CE have the role of the curves in the generalized *ut-unum* principle. When RV is an arbitrary line parallel to EG, intersecting BF in S and EC in T, a relation analogous to (VI.13) holds, namely

$$\Box\, RT + \Box\, TV = 2\Box\, RS + 2\Box\, ST.$$

From this CAVALIERI concluded as in the case of (VI.14), by letting M be the midpoint of the parallelogram, that

$$\mathcal{O}_{ACE}(\Box\, l) + \mathcal{O}_{CEG}(\Box\, l) = 2\mathcal{O}_{ABFE}(\Box\, l) + 2(\mathcal{O}_{BCM}(\Box\, l) + \mathcal{O}_{MEF}(\Box\, l)).$$

By using the result that congruent figures have equal collections of squares CAVALIERI could reduce this equality to

$$\mathcal{O}_{CEG}(\Box\, l) = \mathcal{O}_{ABFE}(\Box\, l) + 2\mathcal{O}_{MEF}(\Box\, l). \qquad \text{(VI.16)}$$

A combination of (VI.9) and (VI.12) leads to

$$\mathcal{O}_{CEG}(\Box\, l) : \mathcal{O}_{MEF}(\Box\, l) = (\Box\, EG : \Box\, EF) \cdot (CG : MF) = 8:1. \qquad \text{(VI.17)}$$

Moreover (VI.9) implies that

$$\mathcal{O}_{ACGE}(\Box\, l) : \mathcal{O}_{ABFE}(\Box\, l) = \Box\, EG : \Box\, EF = 4:1. \qquad \text{(VI.18)}$$

The relations (VI.16), (VI.17) and (VI.18) lead to (VI.15), that is, to the statement of Theorem II.24. This result has as an immediate consequence that a pyramid with a square base is one third of its corresponding prism. In Section VI.6 more results based on Theorem II.24 will be presented.

Having obtained the relation (VI.15) CAVALIERI introduced new *ad hoc* concepts related to collections of squares of abscissae, and he could then give a geometrical formulation of the relation

$$\int_0^a t^2\, dt = \tfrac{1}{3}a^3.$$

Similarly he could by considering collections of rectangles of trapezia achieve results which were geometrical equivalents to a calculation of the final integrals in (VI.2).

VI.4. We saw in the last section that CAVALIERI generalized the *ut-unum* principle so that $\mathcal{O}$ could be applied to relations of the form (VI.13); Book II of *Geometria* contains another generalization of the *ut-unum* principle put forward in the Propositions 25 and 26. In these CAVALIERI considered two figures like BEC and BFE in Figure VI.6 having bases situated on the same line CF, which is taken as *regula*, and having the same altitude BE; further he looked at their circumscribed parallelograms ABEC and BDFE. He imagined that for each line MQ parallel to the *regula* cutting AC in M, CB in I, BE in O, BF in P, and DF in Q there is given a relation, for instance, like

$$MO : IO = OQ : OP, \qquad \text{(VI.19)}$$

or like

$$MO : IO = \Box\, OQ : \Box\, OP. \qquad \text{(VI.20)}$$

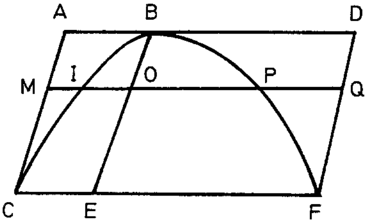

Fig. VI.6

The statement of Propositions 25 and 26 is that $\mathcal{O}$ can be applied to such relations; this is not an immediate consequence of the *ut-unum* principle, because there is no constant ratio in the proportions.

In the case of a homogeneous relation like (VI.19) the result

$$\mathcal{O}_{ABEC}(l) : \mathcal{O}_{BEC}(l) = \mathcal{O}_{BDFE}(l) : \mathcal{O}_{BFE}(l)$$

is easily obtained by interchanging MO and OP in (VI.19), using the *ut-unum* principle twice to conclude $\mathcal{O}_{BFE}(l) : \mathcal{O}_{BEC}(l) = OQ : MO = \mathcal{O}_{BDEF}(l) : \mathcal{O}_{ABEC}(l)$, and making a second interchange. In the case when the proportion is inhomogeneous, such as (VI.20), CAVALIERI could not use this procedure because only magnitudes of the same kind have a ratio. He solved that problem by multiplying the terms on the left-hand side of (VI.20) by a constant and by introducing auxiliary "cylinders" having the rectangles obtained by the multiplication as collections of planes and the figures BEC and BFE as bases. Since by that device CAVALIERI had achieved a situation where interchanging of terms is possible, he was able, after some calculation, to reach the result

$$\mathcal{O}_{ABEC}(l) : \mathcal{O}_{BEC}(l) = \mathcal{O}_{BDFE}(\square\, l) : \mathcal{O}_{BEF}(\square\, l).$$

Thence he concluded that in general $\mathcal{O}$ can be applied to both sides of a proportion where either the antecedents or the consequents are constants.

VI.5. The results mentioned under D, concerning similar plane and solid figures are contained in Theorems II.15 and II.17 of *Geometria*. In this section I shall paraphrase CAVALIERI's proof of the first result, stating that when F_1 and F_2 are similar plane figures with altitudes h_1 and h_2 and "bases" b_1 and b_2 ("bases" here mean "horizontal altitudes", Figure VI.7), then

$$F_1 : F_2 = (h_1 : h_2) \cdot (h_1 : h_2) = (h_1 : h_2) \cdot (b_1 : b_2). \qquad \text{(VI.21)}$$

CAVALIERI's idea was to use the *ut-unum* principle, and for that he needed an auxiliary figure with base b_1 and altitude h_2. To obtain this he transformed the figures F_1 and F_2 in two steps into figures H_1 and H_2 situated around two axes and limited by curves which, in modern terms, are graphs of monotone functions. (For a brief account of CAVALIERI's proof, using modern concepts and terminology, see ZEUTHEN 1903, pp. 257–258.) First he constructed the figure G_i with the same altitudes as F_i, $i = 1, 2$, such that each l in $\mathcal{O}_{G_i}(l)_{AC}$ was defined by having its one endpoint at B_iC_i and by being equal to the line segment or to the sum of the line segments of the corresponding l in $\mathcal{O}_{F_i}(l)_{AC}$; *e.g.* $T_iU_i = P_iQ_i + R_iS_i$.

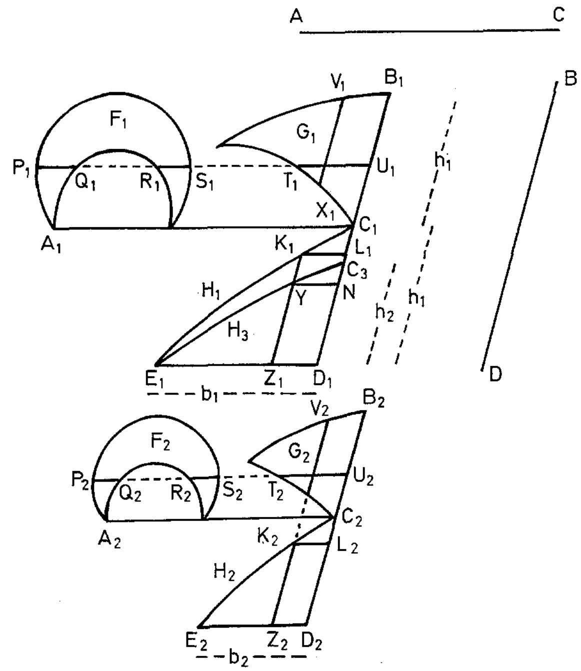

Fig. VI.7. The figure occurs in *Geometria*, page 128, but with different letters.

Since corresponding l's in $\mathcal{O}_{F_i}(l)$ and $\mathcal{O}_{G_i}(l)$ are equal, it follows from the principle named after CAVALIERI (*cf.* (V.4)) that

$$F_1 = G_1 \quad \text{and} \quad F_2 = G_2. \tag{VI.22}$$

CAVALIERI transformed the figure G_i into an H_i with altitude h_i, $i = 1, 2$, by the same process, now considering the collections of lines with respect to the *regula* BD; thus K_iZ_i in $\mathcal{O}_{H_i}(l)_{BD}$ was defined by having the endpoint Z_i on E_iD_i and by $Z_iK_i = X_iV_i$. Again it follows from the construction that $H_i = G_i$, and hence by (VI.22) that

$$F_1 = H_1 \quad \text{and} \quad F_2 = H_2. \tag{VI.23}$$

The advantage of dealing with figures H_1 and H_2 instead of F_1 and F_2 is that each "line" of $\mathcal{O}_{H_i}(l)$, both with respect to the *regula* AC and the *regula* BC, is a line segment having one endpoint on an axis, and not a sum of arbitrarily situated line segments. Having introduced the figures H_1 and H_2, CAVALIERI was able to construct an auxiliary figure H_3 with base $E_1D_1 = b_1$ and altitude $C_3D_1 = h_2$, namely by defining each $\bar{l}_3$ like YZ_1 in $\mathcal{O}_{H_3}(l)_{BD}$ by the corresponding $\bar{l}_1$ in $\mathcal{O}_{H_1}(l)_{BD}$ through the relation

$$\bar{l}_1 : \bar{l}_3 = h_1 : h_2; \tag{VI.24}$$

thus $K_1Z_1 : YZ_1 = C_1D_1 : C_2D_2 = h_1 : h_2$. By applying his principle, CAVALIERI concluded from (VI.24) that

$$H_1 : H_3 = h_1 : h_2. \tag{VI.25}$$

CAVALIERI'S next step was to prove that for each pair of corresponding l_3 in $\mathcal{O}_{H_3}(l)_{AC}$ and l_2 in $\mathcal{O}_{H_2}(l)_{AC}$ the following relation holds:

$$l_3 : l_2 = h_1 : h_2. \qquad \text{(VI.26)}$$

To obtain it, CAVALIERI first noticed that the similarity of F_1 and F_2 implies that H_1 and H_2 are similar. He then let $ND_1 = L_2D_2$ and considered the corresponding line segments $YN (= l_3)$ and $K_2L_2 (= l_2)$. According to the construction of H_3 the point Y is determined by the relations: $YN = K_1L_1$ and $K_1Z_1 : YZ_1 = h_1 : h_2$; since $YZ_1 = ND_1 = L_2D_2$ and $K_1Z_1 = L_1D_1$ we can conclude that $L_1D_1 : L_2D_2 = h_1 : h_2$. Thus the line segments K_1L_1 and K_2L_2 are similarly situated in the similar figures H_1 and H_2 which means that $K_1L_1 : K_2L_2 = h_1 : h_2$. This is equivalent to (VI.26) because $l_3 = YN = K_1L_1$ and $l_2 = K_2L_2$.

Applying his principle to (VI.26), CAVALIERI got the relation

$$H_3 : H_2 = h_1 : h_2. \qquad \text{(VI.27)}$$

The required result (VI.21) is then obtained by a combination of (VI.23), (VI.25) and (VI.27).

VI.6. Having proved that similar plane figures are in a ratio which is the square of their linear ratio CAVALIERI was able to deduce results concerning collections of similar plane figures. As we saw in Section IV.1, a collection of similar plane figures, $\mathcal{O}_F(A(l))$, belonging to a given figure F is obtained by constructing similar and parallel plane figures, A(l), on the l's in $\mathcal{O}_F(l)$.

We also saw that together with $\mathcal{O}_F(A(l))$ CAVALIERI considered the solid whose collection of planes is $\mathcal{O}_F(A(l))$; I shall denote this solid by S(A(l), F).

CAVALIERI was particularly interested in comparing two solids, $S(A(l), F_1)$ and $S(A(l), F_2)$, which have the property that the plane figures of their collections of planes, $\mathcal{O}_{F_1}(A(l))$ and $\mathcal{O}_{F_2}(A(l))$, are mutually similar. By combining the result concerning similar plane figures (VI.21), with (V.2) and the *ut-unum* principle he found in Theorem II.33 of *Geometria* that

$$S(A(l), F_1) : S(A(l), F_2) = \mathcal{O}_{F_1}(\Box\, l) : \mathcal{O}_{F_2}(\Box\, l). \qquad \text{(VI.28)}$$

This result is essential in CAVALIERI'S theory and was employed to obtain several relations between solid figures. In Book II of *Geometria* CAVALIERI especially used it for comparing cylinders, cones, truncated cones, prisms, pyramids, and truncated pyramids. These solids can be obtained by letting the given plane figures be parallelograms, triangles or trapezia, and by letting the similar figures be either circles, parallelograms or triangles. Thus, if the given plane figures are chosen as two parallelograms, P_1 and P_2, and the similar figures as circles, two cylinders C_1 and C_2 are generated. The relation (VI.28) provides the first step in finding the ratio between the cylinders (*Geometria*, p. 184):

$$C_1 : C_2 = \mathcal{O}_{P_1}(\Box\, l) : \mathcal{O}_{P_2}(\Box\, l). \qquad \text{(VI.29)}$$

The second step is to employ the result (VI.9) concerning the ratio between collections of squares belonging to parallelograms.

To have another illustration of CAVALIERI'S use of (VI.28) let us consider a cone and its corresponding cylinder (*Geometria*, p. 185). These solids can be obtained by assigning a collection of circles to a triangle, Δ, and a parallelogram, P. Therefore (VI.28) gives the result

$$\text{cone} : \text{cylinder} = \mathcal{O}_{\Delta}(\square\ l) : \mathcal{O}_{P}(\square\ l), \qquad \text{(VI.30)}$$

and the right-hand side CAVALIERI had already calculated (*cf.* (VI.15)) to be 1:3.

Proceeding along this line, CAVALIERI was able to find relations between all the solids mentioned under E. When the considered cylinders and cones have circular bases the results CAVALIERI got from (VI.29) and (VI.30) were not new; but he was aware that (VI.30) also applied for any cone having a closed curve as base, and that (VI.29) can be used for two cylinders having similar closed curves as bases (*Geometria*, pp. 184–185). Thus CAVALIERI showed in the second book of *Geometria* that his method led to results which were otherwise difficult to demonstrate.

VII. Application of the theory to conic sections

VII.1. In Books III–V of *Geometria* CAVALIERI showed how his new theory could be used to obtain results concerning conic sections. As the pattern in these books is the same, I shall only deal with Book III. It consists of one section with 33 theorems and one with 29 corollaries which are applications of the theorems. In addition to a determination of the ratio between two ellipses (*i.e.* the ratio between their areas), the theorems mainly contain calculations of relations between collections of squares and rectangles belonging to figures defined by an ellipse. These relations are applied to effect cubatures of various solids in the corollaries.

To give an impression of how this worked I shall present one theorem, its proof and its applications in detail. Further, I shall indicate the content of the most important of the remaining theorems. First, however, I am going to give a slightly simplified description of how CAVALIERI found the ratio between two ellipses.

In dealing with ellipses CAVALIERI made drawings where their properties were related to the axes and where the *regula* was one of the axes. Actually most of his theorems are formulated so that they also cover the situation where two conjugate diameters are used instead of the axes. For simplicity I shall mainly refer to the axes.

To find the ratio between two ellipses (*Geometria*, pp. 211–214) CAVALIERI first considered two quarter-ellipses having one axis in common, like Q_1 = AFTB, Q_2 = AFVC in Figure VII.1, and their circumscribed rectangles R_1 = AFGB

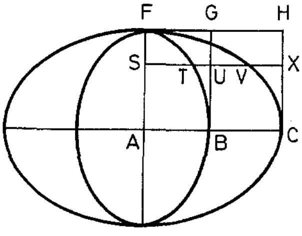

Fig. VII.1

and $R_2 = AFHC$; he let the *regula* be parallel to AC. For any line SX parallel to AC cutting the axis at S, the first ellipse at T, the line BG at U, the second ellipse at V, and the line CH at X, he found from the properties of the ellipses that

$$ST : SV = SU : SX.$$

An application of $\mathcal{O}$ to this relation leads to

$$\mathcal{O}_{Q_1}(l) : \mathcal{O}_{Q_2}(l) = \mathcal{O}_{R_1}(l) : \mathcal{O}_{R_2}(l); \tag{VII.1}$$

since areas have the same ratio as their collections of lines (*cf.* (V.1)) and since R_1 and R_2 have the same altitude, the relation (VII.1) implies that

$$Q_1 : Q_2 = R_1 : R_2 = AB : AC.$$

Using this relation twice, CAVALIERI found that the ratio between two ellipses is equal to the ratio between the rectangles of their axes or to the ratio between rectangles of corresponding conjugate diameters. In modern terms we would say that the area of the ellipse is proportional to the product of the lengths of its axes; that the factor of proportionality is $\pi/4$ CAVALIERI expressed by considering the ratio between an ellipse and its inscribed circle.

VII.2. The other results of Book III are – as mentioned – related to two-dimensional collections. As an example of these I present Proposition III.1 of *Geometria*. Let DEPR be an ellipse having ER as the one axis and the *regula* parallel to the other axis, and let DEP be a segment of the ellipse where DP is parallel to the *regula* and cuts the axis at the point B (Figure VII.2). Further, let DFHP be the rectangle having the same base and altitude as the segment and ΔDEP be the inscribed triangle of the segment. The theorem then states that

$$\mathcal{O}_{DEP}(\square\, l) : \mathcal{O}_{DFHP}(\square\, l) = (\tfrac{1}{6}BE + \tfrac{1}{2}RB) : RB \tag{VII.2}$$

and

$$\mathcal{O}_{DEP}(\square\, l) : \mathcal{O}_{\Delta DEP}(\square\, l) = (\tfrac{1}{2}RE + RB) : RB \tag{VII.3}$$

or, in CAVALIERI's words,

> When the base is taken as regula, all the squares of a segment of a circle or an ellipse will be to all the squares of the parallelogram, having the same base and altitude as the segment, as the composition of a sixth part of the axis or diameter of the same [segment] and a half of the remaining part is to the axis or diameter of the remaining part. And the same will be to all the squares of the triangle with the same [base and altitude] as the composition of a half of the whole axis or diameter and the remaining part is to the axis or diameter of the remaining part.[20]

[20] *Geometria*, p. 197: Omnia quadrata portionis circuli, vel Ellipsis, ad omnia quadrata parallelogrammi in eadem basi, & altitudine cum portione constituti, regula basi, erunt, ut composita ex sexta parte axis, vel diametri eiusdem, & dimidia reliquae portionis, ad axim, vel diametrum reliquae portionis: Eadem verò ad omnia quadrata trianguli in iisdem existentis erunt, ut composita ex dimidia totius, & reliquae portionis axi, vel diametro, ad axim, vel diametrum reliquae portionis.

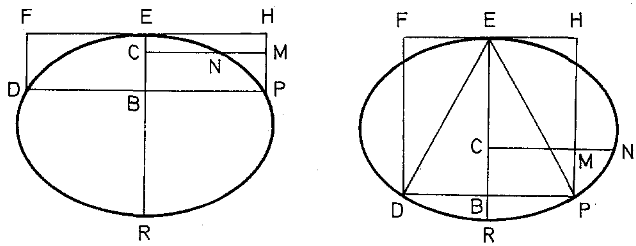

Fig. VII.2. The ellipse segment DEP and the rectangle having the same base and altitude as the segment in the cases where DEP is less than or greater than half an ellipse.

To prove this, CAVALIERI considered a line CM parallel to DP cutting the ellipse in N and the line HP in M, and observed that the properties of the ellipse imply that $\square\, \mathrm{BP} : \square\, \mathrm{CN} = (\mathrm{RB} \times \mathrm{BE}) : (\mathrm{RC} \times \mathrm{CE})$, or since $\mathrm{BP} = \mathrm{CM}$, that

$$\square\, \mathrm{CM} : \square\, \mathrm{CN} = (\mathrm{RB} \times \mathrm{BE}) : (\mathrm{RC} \times \mathrm{CE}).$$

Since the antecedents, $\square\, \mathrm{CM}$ and $\mathrm{RB} \times \mathrm{BE}$, are the same for all lines CM, CAVALIERI concluded that $\mathcal{O}$ can be applied to this proportion (*cf.* Section VI.4), and in this way he was led to the following relation between collections of squares and collections of rectangles of abscissae (*cf.* Section IV.3):

$$\mathcal{O}_{\mathrm{BEHP}}(\square\, \mathrm{l}) : \mathcal{O}_{\mathrm{BENP}}(\square\, \mathrm{l}) = \mathcal{O}_{\mathrm{BE}}(\mathrm{RB} \times \mathrm{BE}) : \mathcal{O}_{\mathrm{BE}}((\mathrm{RB} + \alpha) \times (\mathrm{BE} - \alpha)). \quad (\mathrm{VII.4})$$

In a corollary to Proposition II.30 of *Geometria* CAVALIERI had calculated that

$$\mathcal{O}_{\mathrm{BE}}(\mathrm{RB} \times \mathrm{BE}) : \mathcal{O}_{\mathrm{BE}}((\mathrm{RB} + \alpha) \times (\mathrm{BE} - \alpha)) = \mathrm{RB} : (\tfrac{1}{6}\mathrm{BE} + \tfrac{1}{2}\mathrm{RB}) \quad (\mathrm{VII.5})$$

which is his geometrical result corresponding to

$$\left(\int_0^a b\, a\, dt\right) : \left(\int_0^a (b + t)\,(a - t)\, dt\right) = b : (\tfrac{1}{6}\, a + \tfrac{1}{2} b).$$

Using $\mathcal{O}_{\mathrm{DEP}}(\square\, \mathrm{l}) : \mathcal{O}_{\mathrm{DFHP}}(\square\, \mathrm{l}) = \mathcal{O}_{\mathrm{BENP}}(\square\, \mathrm{l}) : \mathcal{O}_{\mathrm{BEHP}}(\square\, \mathrm{l})$, CAVALIERI obtained the required relation (VII.2) from (VII.4) and (VII.5). The second result, (VI.6), then follows from the theorem that $\mathcal{O}_{\mathrm{DFHP}}(\square\, \mathrm{l}) = 3\mathcal{O}_{\Delta\mathrm{DEP}}(\square\, \mathrm{l})$ (*cf.* (VI.15)).

The relation (VII.2) gives the ratio between the collections of squares belonging to a segment of an ellipse and its circumscribed rectangle in the case where the segment DEP is less than or equal to half the ellipse. When DEP is greater than half the ellipse, DFHP is not the circumscribed rectangle of the segment, but since CAVALIERI disposed of a result concerning the ratio between collections of squares belonging to two rectangles (*cf.* (VI.9)), he was able to use (VII.2) to find the ratio between "all the squares" of DEP and "all the squares" of its circumscribed rectangle for both cases (*Geometria*, p. 200). Thus, although the ratios do not have the same form in the two cases, there is in principle no difference between them, wherefore I, unlike CAVALIERI, shall deal only with the case where DEP is less than or equal to half an ellipse.

In Corollary one of the third book of *Geometria* CAVALIERI drew conclusions from (VII.2) by considering collections of similar plane figures belonging to the segment DEP and the rectangle DFHP, $\mathcal{O}_{\mathrm{DEP}}(\mathrm{A(l)})$ and $\mathcal{O}_{\mathrm{DFHP}}(\mathrm{A(l)})$, and the solids S(A(l), DEP) and S(A(l), DFHP), generated from these (*cf.* Section IV.1). By a combination of the relations (VI.28) and (VII.2) he obtained

$$S(A(l), DEP) : S(A(l), DFHP) = (\tfrac{1}{6}BE + \tfrac{1}{2}RB) : RB. \qquad (VII.6)$$

When the A(l)'s are circles, S(A(l), DEP) is a section of the ellipsoid of revolution, or spheroid as CAVALIERI called it, obtained by rotating the ellipse around the axis ER, and S(A(l), DFHP) is its circumscribed cylinder. If we set the altitude, BE, of these solids equal to h and $RE = 2a$, we get from (VII.6) that

$$\text{section of ellipsoid} : \text{circumscribed cylinder} = (a - \tfrac{1}{3}h) : (2a - h).$$

Setting $h = a$, CAVALIERI realized that the ratio between half the spheroid and its circumscribed cylinder or the ratio between the whole spheroid and its circumscribed cylinder is 2:3 (*Geometria*, p. 259).

CAVALIERI also considered the situation where a plane cuts a spheroid in an ellipse DQPS having its center on the diameter ER (Figure VII.3), and applied relation (VII.6) to find the ratio between the section EDQPS and its circumscribed cylinder with elliptic base (*Geometria*, p. 259). He maintained, rightly, that the collection of planes belonging to EDQPS consists of similar ellipses, whence (VII.6) gives the ratio between EDQPS and its circumscribed cylinder (in the case that EDQPS is less than half the spheroid).

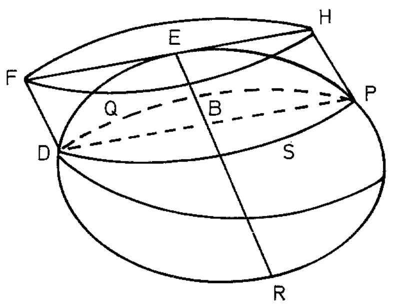

Fig. VII.3. Illustrating a section of an ellipsoid and its circumscribed cylinder with elliptic base.

Further, CAVALIERI applied the relation (VII.3) and obtained results which were similar to the one just described, involving inscribed cones instead of cylinders. These last results were in accord with theorems ARCHIMEDES had proved in *On Conoids and Spheroids*. It was doubtlessly rather important for CAVALIERI to show that his new theory led to the results concerning conic sections which had been found by the method of exhaustion or had been published by KEPLER in *Stereometria*. The next step was to gain new insight. Actually he used (VII.6) for obtaining a new cubature: he considered the situation where the A(l)'s are squares and thus found the ratio between a solid like the one in Figure VII.4 and its circumscribed parallelepiped. On the whole, CAVALIERI reached new relations by transforming the results of Book II, particularly those which correspond to calculating the integrals listed in (VI.2), to conic sections.

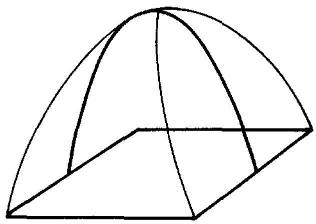

Fig. VII.4. A solid generated by a section of an ellipse and squares.

VII.3. To give an idea of the content of the remaining theorems of Book III, I have shown in Figure VII.5 some figures determined by ellipses; for each of them CAVALIERI found the ratio between its collection of squares and the collection of squares belonging to its circumscribed rectangle (or an inscribed triangle), all the *regulae* being horizontal. CAVALIERI applied these results in the same way as described in the last section. He found it particularly important to point out how each result implied that the ratio between a solid of revolution and its circumscribed cylinder was determined.

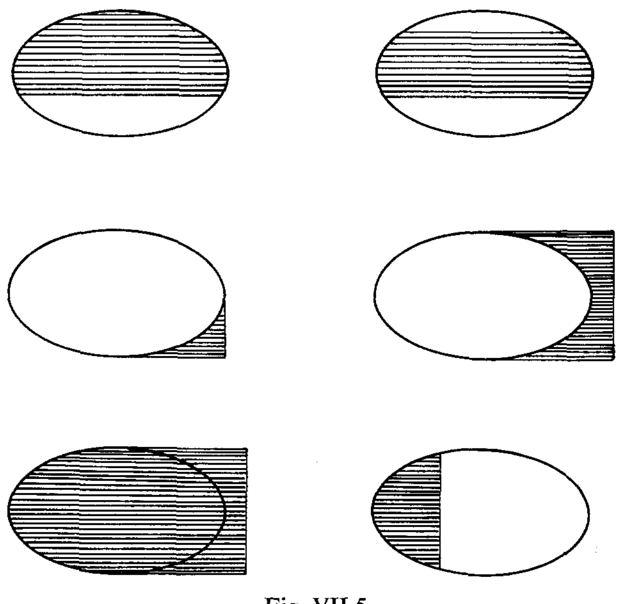

Fig. VII.5

CAVALIERI carried his calculation of ratios between solids of revolution further than determining those equal to a ratio between the collections of squares belonging to the generating figures. As an illustration of this I shall outline how he found the ratio between an "elliptic ring" and its circumscribed "cylindrical ring"; exactly what these solids are will be explained in the following. Let R_1 = CDGH (Figure VII.6) be the circumscribed rectangle of an ellipse E, R_2 = GHIJ be a rectangle between the same parallels as R_1, and having a side in common with it, and F be the "complement" of the ellipse, the hatched figure in Figure VII.6. In Proposition III.30 CAVALIERI proved that

$$(\mathcal{O}_{R_1}(\Box\, l) + 2\mathcal{O}_{R_1,R_2}(l \times l)) : (\mathcal{O}_E(\Box\, l) + 2\mathcal{O}_{F,E}(l \times l)) = R_1 : E. \quad \text{(VII.7)}$$

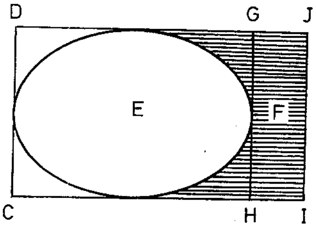

Fig. VII.6

From this relation he obtained the ratio between the ring, A, obtained by rotating E around IJ, and its circumscribed cylindrical ring, B (*Geometria*, p. 279): An elaboration of relation (VI.28) leads to

$$B : A = [\mathcal{O}_{R_1+R_2}(\Box\, l) - \mathcal{O}_{R_2}(\Box\, l)] : [\mathcal{O}_{E+F}(\Box\, l) - \mathcal{O}_F(\Box\, l)]; \quad \text{(VII.8)}$$

since by the generalized *ut-unum* principle (*cf.* Section VI.3) $\mathcal{O}_{R_1+R_2}(\Box\, l) - \mathcal{O}_{R_2}(\Box\, l) = \mathcal{O}_{R_1}(\Box\, l) + 2\mathcal{O}_{R_1,R_2}(l \times l)$ and $\mathcal{O}_{E+F}(\Box\, l) - \mathcal{O}_F(\Box\, l) = \mathcal{O}_E(\Box\, l) + 2\mathcal{O}_{E,F}(l \times l)$ the relations (VII.7) and (VII.8) imply that

$$B : A = R_1 : E.$$

In other words the ratio between the two rings is equal to the ratio between the rectangle circumscribing the ellipse and the ellipse, and thus independent of the distance between the ellipse and the axis of rotation.

Book III ends with corollaries drawn from a theorem similar to Proposition III.30 just described but more complicated because instead of circumscribed rectangles or parallelograms determined by the directions of a set of conjugate diameters, CAVALIERI considered an arbitrary circumscribed parallelogram.

The purpose of the presentation of CAVALIERI's achievements concerning ellipses has been to show that CAVALIERI's method led to impressive results; however, the reader should not be misguided by the compactness of my description; employing hardly any symbols, CAVALIERI needed almost 90 pages to prove the results sketched here.

VIII. Generalizations of the omnes-concept

VIII.1. Even while developing his method of indivisibles CAVALIERI was interested in applying it to quadratures of figures found by a straight line and a part of the ARCHIMEDEAN spiral. It is rather obvious that collections of lines cannot be of much help for this, and CAVALIERI therefore introduced the concept of "all the circumferences" which he presented in Book VI of *Geometria* and which will soon be explained.

After having published *Geometria* CAVALIERI naturally could turn to other problems to try out his method. During the 1630's and 1640's the activity in mathematical analysis was concentrated on the following topics: quadratures of the new curves which emerged in that period, cubatures of solids of revolution, determination of centres of gravity and of tangents. CAVALIERI's method was particularly applicable to investigations of centres of gravity and the new algebraic curves; hence he took up these problems and presented his results in *Exercitationes*. In adapting his method to the determination of centres of gravity he created some more *omnes*-concepts. In dealing with new quadratures and cubatures he invented an almost algebraic *omnes*-concept which enabled him to perform a geometrical integration of t^n for $n > 2$.

The following sections are devoted to a discussion of the three different generalizations CAVALIERI made of the *omnes*-concept in connection with his work on the spiral, on the geometrical integration of t^n, $n > 2$, and on centres of gravity.

VIII.2. CAVALIERI's work on spirals dates back to the early 1620's when he got the idea of comparing a spiral with a parabola. On the 9th of April 1623 he sent a manuscript to GALILEO, containing some of his results (GALILEI *Opere*, vol. 13, p. 114). ANTONIO FAVARO has traced this manuscript among GALILEO's papers and in 1905 he gave a description of it (*cf.* FAVARO 1905). In Book Six of the manuscript from 1627 of *Geometria* (*cf.* Section I.4) CAVALIERI presented another version of his work on spirals, and the printed *Geometria* from 1635 contains a third version.

The three different versions indicate that CAVALIERI was in doubt about the extent to which he should base his proofs on the concept of "all the circumferences". These are introduced in the manuscript of 1623, whereas the proofs in the manuscript of 1627 are in the style of the Greek method of exhaustion without indivisibles, and then again the printed *Geometria* contains proofs using the indivisibles of a spiral figure. In my presentation of these indivisibles "all the circumferences" will be based on the latter version of CAVALIERI's work.

CAVALIERI first defined *omnes circumferentiae* of a given circle (*Geometria*, p. 427). Let a circle ABD (Figure VIII.1) with radius ED be given, and consider the collection of points of ED (*cf.* Section IV.3). If one imagines that through each of the points of this collection a circle is drawn with centre E, one obtains the collection of concentric circles which CAVALIERI called "all the circumferences" of the given circle. Further, CAVALIERI defined "all the circumferences" of a

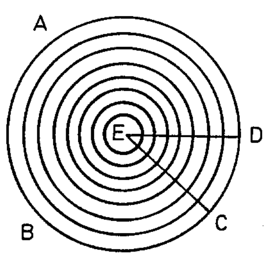

Fig. VIII.1

figure F inside the circle ABD as all the parts of "all the circumferences" of ABD cut off by the figure (*Geometria*, p. 427). Thus the segment ECD (Figure VIII.1) has a collection of circumferences consisting of the arcs of "all the circumferences" of ABD cut off by ED and EC. I shall use the notation $\mathcal{O}_F(c)$ for the collection of circumferences of F. In general $\mathcal{O}_F(c)$ depends on the circle used to define the circumferences, but when the figure F is limited by a straight and a spiral line, CAVALIERI lets the circumferences be determined by having the origin of the spiral as their centre.

Let $S = ABDE$ be the first turn of the spiral with origin A and first distance AE; we shall now see how CAVALIERI applied the concept of "all the circumferences" and his idea of comparing spirals and parabolas to obtain the quadrature of the sector S. First he let the spiral define a parabola OTQ, by setting (*cf.* Figure VIII.2)

$$RT = \text{arc } GHD, \quad \text{where} \quad OR = AD = AG. \qquad \text{(VIII.1)}$$

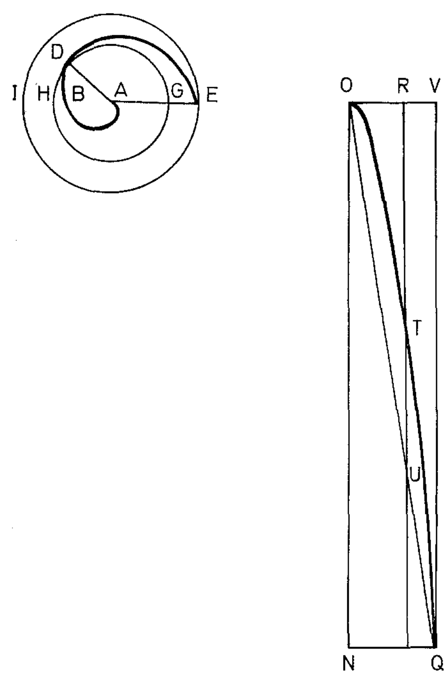

Fig. VIII.2

(That this really defines a parabola can be seen in the following way: let the equation of the spiral in polar coordinates be $r = a\Theta$; let $\angle GAD = \Theta$, and suppose $x = OR = AD = a\Theta$; then $y = RT = \text{arc } GHD = a\Theta^2$ and hence $T(x, y)$ is a point of the parabola $ay = x^2$.)

Let Q be the point on the parabola whose abscissa OV is equal to AE; CAVALIERI introduced the straight line OQ and investigated the properties of its points. Thus from the fact that the parabola has been constructed so that VQ is equal to the circumference of the circle having radius AE, and from

RU : VQ = OR : OV = AG : AE, he concluded that

$$RU = \text{circumference of circle with radius AG.} \qquad \text{(VIII.2)}$$

Instead of determining the sector S = ABDE directly CAVALIERI considered the difference between the circle, C, circumscribing the sector, and S, that is the figure AEIEDA = C − S; he compared this figure with the figure X = OTQVO cut off by the parabola. I shall not repeat CAVALIERI's actual deduction because it is only the underlying idea which is important for understanding his work on collections of circumferences. His idea was to verify that for each pair of corresponding points G on AE and R on OV (*i.e.* AG = OR) the c = arc GHD in $\mathcal{O}_{C-S}(c)$ and l = RT in $\mathcal{O}_X(l)_{ON}$ are equal (*cf.* VIII.1); he used this to realize the result

$$\mathcal{O}_{C-S}(c) = \mathcal{O}_X(l)_{ON}. \qquad \text{(VIII.3)}$$

Moreover CAVALIERI compared the triangle Δ = OQV with the circle C. Using the relation (VIII.2), he intuitively concluded that

$$\mathcal{O}_C(c) = \mathcal{O}_\Delta(l)_{ON}. \qquad \text{(VIII.4)}$$

The two last results combined with an intuitive interpretation of the indivisibles lead to the following relations:

$$C : (C - S) = \mathcal{O}_C(c) : \mathcal{O}_{C-S}(c) = \mathcal{O}_\Delta(l) : \mathcal{O}_X(l) = \Delta : X. \qquad \text{(VIII.5)}$$

The last ratio is known from the quadrature of the parabola to be 3 : 2; hence (VIII.5) gives ARCHIMEDES'S result that the first turn of the spiral, S, is one-third of the circumscribed circle, C. In a similar way that ratio between other sectors of spirals and sectors of circles can be found.

CAVALIERI formulated some theorems in an attempt to legitimize the above form of deduction, as for instance the theorem that if each of two figures has the property that it contains "all the radii" from a given point to the points of the perimeter, then the ratio between the figures will be the same as the ratio between their collections of circumferences (*Geometria*, p. 433). This guarantees the first equality in (VIII.5); but CAVALIERI did not give any arguments for the second equality or for (VIII.3) and (VIII.4). Thus CAVALIERI seems to have found it too laborious to establish a foundation of his technique based on "all the circumferences". Instead of working out a careful theory he supplemented his proofs in the printed *Geometria* with exhaustion proofs like those he had used in the version of 1627 of *Geometria*. The function of collections of circumferences in the printed *Geometria* therefore appears as being mainly heuristic: to show how results concerning the spiral can be realized. What mattered most to CAVALIERI in this connection was not the actual results since all of them could be found in ARCHIMEDES'S *On Spirals*, apart from one concerning sectors of spirals of more than one revolution. The important thing for him was to show that the very idea of comparing a spiral and a parabola could lead to these results. This turned out to be very inspiring to other mathematicians working on quadratures and rectifications (*cf.* PEDERSEN 1970 and KRIEGER 1971).

VIII.4. CAVALIERI's second generalization of the *omnes*-concept, "all the powers", relates more directly to the concepts of *Geometria*, Book II. We have seen that in this book he managed to find geometrical results which correspond to integrations of polynomials of second degree. His results could not, however, solve all existing problems, as for example the one KEPLER had posed in *Stereometria* (1615): The cubature of a parabolic cask, *i.e.* a solid obtained by resolving a segment of a parabola around its base (KEPLER *Opera*, vol. 4, p. 601, and Section VIII.8).

CAVALIERI found this problem fascinating and in the late 1630's he started to look for a generalization of his method of indivisibles which would enable him to perform geometrical integrations of t^n for $n > 2$. In the cases $n = 1$ and $n = 2$ CAVALIERI had transformed such integrations to a determination of $\mathcal{O}_P(l) : \mathcal{O}_\Delta(l)$ and $\mathcal{O}_P(\square\, l) : \mathcal{O}_\Delta(l)$ where Δ is a triangle and P its corresponding parallelogram. By extrapolating from this CAVALIERI got the idea of introducing collections of powers of line segments belonging to P and Δ (in the next section it will be explained exactly what these collections are). He soon guessed that the required ratios between the collections of powers were equal to $(n + 1) : 1$, for $n = 3, 4$ *etc.* (CAVALIERI 1639_1, p. 523 and *Exercitationes*, p. 243); his problem was to prove these results.

From the beginning of his investigations CAVALIERI was aware that results concerning the ratio between collections of powers belonging to a triangle and a parallelogram had more applications than solving KEPLER'S problem, and that it for instance could be used for effecting quadratures of the parabolas $y = x^n$. Often he even characterized his investigations by referring to those quadratures. Thus on December 29, 1637 CAVALIERI wrote to GALILEO that he had found *questa bella cosa*, namely the quadratures of $y = x^n$; he added that he was going to publish his discovery in his forthcoming book (GALILEO *Opere*, vol. 17, pp. 243–244). The book he meant was *Centuria*, a kind of handbook for solving various problems by means of logarithms. It appeared in 1639 and had a postscript containing CAVALIERI'S results formulated as relations between collections of powers. Here the results were stated generally, but CAVALIERI very honestly admitted that he had been able to prove them only for $n \leqq 4$. An analysis of the technique CAVALIERI had used in *Geometria* for $n = 2$ and of the one he later employed in *Exercitationes* shows that his main difficulty was to express $(a + b)^n + (a - b)^n$ in terms of powers of a and b, and that this difficulty was caused partly by his wish for an entirely geometrical formulation.

In the preface to *Exercitatio* IV (p. 243) CAVALIERI related that he had discussed his problem with J. F. NICERON in 1640, asking him to present it to J. BEAUGRAND in Paris. On that occasion NICERON may have told CAVALIERI, that FERMAT, DESCARTES and ROBERVAL had already achieved the quadratures of the parabolas $y = x^n$ for $n > 2$; but anyway he complied with CAVALIERI'S wish (for more details see MERSENNE *Correspondance*, vol. 9, p. 221). On the 8^{th} of November 1640, shortly before he died, BEAUGRAND wrote a reply to CAVALIERI, who received it only after more than six months (*cf. Exercitationes*, p. 245 and the letter from CAVALIERI to GALILEO, dated August 20, 1641, GALILEI *Opere*, vol. 18, p. 346). BEAUGRAND'S letter is lost but the way CAVALIERI referred to it in *Exercitationes* shows that BEAUGRAND among other things supplied CAVALIERI with the needed expression for $(a + b)^n + (a - b)^n$ (p. 245 and p. 283).

CAVALIERI seems to have searched for a proof of his own, probably not wanting to use as much algebra as BEAUGRAND had done. Thus on September 22, 1643, he wrote to TORRICELLI that he was working on a proof of the geometrical integration of t^n, which had already been found by his pupil GIANNANTONIO ROCCA for $n = 5$ and generally by BEAUGRAND (TORRICELLI *Opere*, vol. 3, p. 144). In the following sections we shall see how CAVALIERI eventually decided to present his solutions in *Exercitationes.*

VIII.5. The results CAVALIERI proved in book four of *Exercitationes* were the geometrical equivalences of integrating t^n for $n = 3, 4, 5, 6$ and 9, but he stated the general result. CAVALIERI's reason for limiting the n's in the proofs is mainly due to his lack of a notation for expressing general statements. In particular he had no way of expressing the coefficients of $a^{n-2i} b^{2i}$ in the relation we would write as

$$(a + b)^n + (a - b)^n = 2 \sum_{i=0}^{[\frac{1}{2}n]} \binom{n}{2i} a^{n-2i}b^{2i} \qquad \text{(VIII.6)}$$

unless n was a specified number. With a reference to BEAUGRAND he calculated $(a + b)^n + (a - b)^n$ explicitly for $n = 5$, 6 and 9 – one of the few instances where CAVALIERI applied an algebraic notation (*Exercitationes*, pp. 283–285). For $n = 3$ and 4 he employed his usual verbal style to obtain (VIII.6), formulated as a result concerning the third and fourth powers of line segments produced by dividing a given line segment into equal and unequal parts (*Exercitationes*, pp. 269–271).

A general proof of the result of "integrating" t^n, based on CAVALIERI's idea, also requires the use of complete induction. CAVALIERI was unable to do so; yet he was aware that an induction argument was necessary, and that his proof for $n = 9$ assumed that the result had been proved for $n = 7$ and 8. Having pointed out the limitations in CAVALIERI's proofs, I shall for simplicity introduce a general n in the presentation of his arguments.

CAVALIERI'S first step was to introduce the concept which has been mentioned several times in the preceding section: powers, *potestates,* of a line segment, l. These he denoted by l.q., l.c., l.qq., l.q.c., *etc.* (the letters after the l meaning *quadratum, cubum, quadrato-quadratum, quadrato cubum*) and, when the power was unspecified, l.p. (*potestas*); I shall use the notation l^n. The second power l^2 is the same as the square $\square$ l considered in *Geometria*; the third power l^3 also has a geometrical meaning as a cube with the line segment l as side; beyond $n = 3$ the powers are abstract or imaginary (*imaginariae, Exercitationes,* p. 247), but still they are geometrical magnitudes, not numbers. CAVALIERI assumed that powers of line segments can be added and multiplied, and that they obey the usual algebraic rules like

$$l^n \cdot l^m = l^{n+m} \qquad \text{(VIII.7)}$$

(*Exercitationes*, p. 247). For specified n's he also used

$$(l_1 + l_2)^n = l_1^n + nl_1^{n-1} l_2 + \frac{n(n-1)}{2} l_1^{n-2}l_2^2 + \ldots + l_2^n. \qquad \text{(VIII.8)}$$

CAVALIERI then generalized some of the concepts which had been very instrumental in his first theory of indivisibles. Thus he introduced the concepts *omnes* *cubi, omnia quadrato-quadrata, etc.*, or, in general, *omnes potestates* of a given figure. These collections of powers belonging to a figure are produced similarly for "all the squares", *i.e.* by assigning l^n to each l in $\mathcal{O}_F(l)$ for $n = 3, 4,$ *etc.* CAVALIERI sometimes used the notation *o.p.* for "all the powers"; I shall use the notation $\mathcal{O}_F(l^n)$. CAVALIERI further employed a generalization of "all the rectangles" of two figures F and G having the same altitude and common *regula.* He described this new concept as the product (*factum*) of "all the powers" of figure F and "all the powers" of G (*Exercitationes*, p. 249). This expression may lead the reader to think of a kind of product: $\mathcal{O}_F(l^n) \times \mathcal{O}_G(l^m)$; however, the way CAVALIERI applied the concept clearly shows that he had something different in mind: the magnitude emerging when the product $l_1^n \times l_2^m$ is formed for each corresponding l_1 and l_2 in F and G respectively. This collection of "rectangles" will be denoted by $\mathcal{O}_{F,G}(l^n \times l^m)$.

VIII.6. CAVALIERI treated his new *omnes*-concepts as he had treated the earlier ones, and among other things stated a generalized *ut-unum*-principle (*cf.* Section VI.3) for "all the powers" (*Exercitationes*, p. 249, p. 265). Hence if F_1 and F_2 have the same altitude, CAVALIERI allowed an application of $\mathcal{O}$ to (VIII.8) and found that

$$\mathcal{O}_{F_2+F_2}(l^n) = \mathcal{O}_{F_1}(l^n) + n\,\mathcal{O}_{F_1,F_2}(l^{n-1} \times l) + \frac{n(n-1)}{2}\mathcal{O}_{F_1,F_2}(l^{n-2} \times l^2) + \ldots + \mathcal{O}_{F_2}(l^n), \tag{VIII.9}$$

again for specified numbers *n*.

Further, CAVALIERI maintained (*Exercitationes*, p. 247) that it is clear that

$$\mathcal{O}_{F,F}(l^n \times l^m) = \mathcal{O}_F(l^{n+m}). \tag{VIII.10}$$

Moreover he stated that when a figure, F, and a parallelogram, P, have the same altitude and *regula*, then

$$\mathcal{O}_P(l^{n+m}) : \mathcal{O}_{P,F}(l^n \times l^m) = \mathcal{O}_P(l^m) : \mathcal{O}_F(l^m). \tag{VIII.11}$$

To obtain this result (*Exercitationes*, pp. 260–263) CAVALIERI first noticed that for corresponding l_1's in P and l_2's in F, we have that

$$(l_1^n \times l_1^m) : (l_1^n \times l_2^m) = l_1^m : l_2^m; \tag{VIII.12}$$

from which (VIII.11) would follow by an application of $\mathcal{O}$ (*cf.* (VIII.10)). However, CAVALIERI could not apply $\mathcal{O}$ directly to this proportion since none of the ratios is a constant (l_2 varies with its distance from the *regula*). Neither could he use his technique for proving a generalized *ut-unum*-principle for collections of lines and squares (*cf.* Section VI.4), because there are no geometrical figures connected with collections of powers of higher degrees. The only solution CAVALIERI saw to this problematic situation was to prove that (VIII.12) implies that

$$(\Sigma l_1^n \times l_1^m) : (\Sigma l_1^n \times l_2^m) = (\Sigma l_1^m) : (\Sigma l_2^m)$$

where Σ is a *finite* sum, and to assert that TORRICELLI had employed a result which corresponds to deriving (VIII.11) from (VIII.12) (*ibid.*, p. 261; *cf.* TORRICELLI *Opere* vol. 1, part 1, pp. 123 and 153–154). CAVALIERI's "proof" did not only cover the particular case of (VIII.12), but all cases where the antecedents in a proportion are constants, and he concluded that $\mathcal{O}$ can be applied to both sides of such a proportion.

It is noticeable that CAVALIERI here tried to argue for a result concerning $\mathcal{O}$ by showing that a similar result is valid for finite sums. Such an argument may very well have opened the way for the interpretation of CAVALIERI'S *omnes*-concepts as infinite sums (*cf.* GIUSTI 1980, pp. 38–39).

In terms of integrals the three relations (VIII.9–11) can be understood as the geometrical equivalents of

$$\int_0^a (l_1(t) + l_2(t))^n \, dt = \int_0^a (l_1(t))^n \, dt + n \int_0^a (l_1(t))^{n-1} \cdot l_2(t) \, dt + \dots + \int_0^a (l_2(t))^n \, dt, \tag{VIII.9'}$$

$$\int_0^a (l(t))^n \times (l(t))^m \, dt = \int_0^a (l(t))^{n+m} \, dt, \tag{VIII.10'}$$

and

$$\int_0^a b^n (l(t))^m \, dt = b^n \int_0^a (l(t))^m \, dt. \tag{VIII.11'}$$

VIII.7. The presentation of CAVALIERI'S concepts and assumptions has now come so far that we can follow his proofs of his most noteworthy achievement, the geometrical equivalent of the result $\int_0^a t^n \, dt = \frac{a^{n+1}}{n+1}$, that is, the relation

$$\mathcal{O}_P(l^n) : \mathcal{O}_\Delta(l^n) = (n+1) : 1, \tag{VIII.13}$$

where P is a parallelogram and Δ one of the triangles determined by a diagonal in P.

CAVALIERI'S proof, or rather his separate proofs for $n = 4$, 5, 6 and 9, employ the same idea as his proof for $n = 2$ in *Geometria* (*cf.* Section VI.3)[21]: Let ABCD

[21] For $n = 3$ CAVALIERI gives an elegant proof based on an idea which "transcribed" to integrals looks like

$$\int_0^a a^3 \, dt = \int_0^a (a - t + t)^3 \, dt =$$

$$\int_0^a (a-t)^3 \, dt + 3 \int_0^a (a-t)^2 \cdot t \, dt + 3 \int_0^a (a-t) \cdot t^2 \, dt + \int_0^a t^3 \, dt.$$

From symmetry we get

$$\int_0^a a^3 \, dt = 2 \int_0^a t^3 \, dt + 6 \int_0^a (a-t) \cdot t^2 \, dt. \tag{1}$$

On the other hand it follows from $\int_0^a a^2 \, dt = 3 \int_0^a t^2 \, dt$ that

$$\int_0^a a^3 \, dt = 3 \int_0^a at^2 \, dt = 3 \int_0^a (a - t + t) \, t^2 \, dt;$$

be the parallelogram, AC one of its diagonals, and suppose that ABCD is bisected vertically and horizontally by EF and LN (Figure VIII.3). Further let RV be an arbitrary line parallel to AD which intersects AB in R, EF in S, AC in T, and CD in V.

Owing to the fact that RS = SV, the following relation holds:

$$(RT)^n + (TV)^n = (RS + ST)^n + (RS - ST)^n.$$

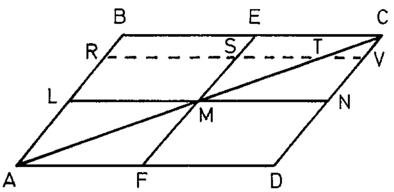

Fig. VIII.3

Hence by using the result (VIII.6), which CAVALIERI called BEAUGRAND'S lemma, he obtained

$$(RT)^{2k+1} + (TV)^{2k+1} = 2\sum_{i=0}^{k} \binom{2k+1}{2i} (RS)^{2k+1-2i} (ST)^{2i},$$

$$(RT)^{2k} + (TV)^{2k} = 2\sum_{i=0}^{k} \binom{2k}{2i} (RS)^{2k-2i} (ST)^{2i}.$$

An application of $\mathcal{O}$ to these relations led CAVALIERI to

$$\mathcal{O}_{ABC}(l^{2k+1}) + \mathcal{O}_{ACD}(l^{2k+1}) = 2\mathcal{O}_{ABEF}(l^{2k+1}) + 2\sum_{i=1}^{k} \binom{2k+1}{2i} [\mathcal{O}_{FMND,AMF}(l^{2k+1-2i} \times l^{2i}) + \mathcal{O}_{LBEM,MEC}(l^{2k+1-2i} \times l^{2i})] \qquad \text{(VIII.14)}$$

and

$$\mathcal{O}_{ABC}(l^{2k}) + \mathcal{O}_{ACD}(l^{2k}) = 2\mathcal{O}_{ABEF}(l^{2k}) + 2(\mathcal{O}_{AMF}(l^{2k}) + \mathcal{O}_{MEC}(l^{2k})) + 2\sum_{i=1}^{k-1} \binom{2k}{2i} [\mathcal{O}_{FMND,AMF}(l^{2k-2i} \times l^{2i}) + \mathcal{O}_{LBEM,MEC}(l^{2k-2i} \times l^{2i})]. \qquad \text{(VIII.15)}$$

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hence

$$\int_0^a a^3\, dt = 3\int_0^a (a - t)\, t^2\, dt + 3\int_0^a t^3\, dt. \qquad (2)$$

By comparing (1) and (2) we see that $3\int (a - t) \cdot t^2\, dt = \int_0^a t^3\, dt$, which inserted in (1) or (2) gives

$$\int_0^a a^3\, dt = 4\int_0^a t^3\, dt.$$

This idea cannot be used for even numbers, and for odd n's higher than 3 the calculations become rather complicated, involving $\frac{1}{2}(n + 1)$ equations.

By use of the equalities $\mathcal{O}_{ABC}(l^n) = \mathcal{O}_{ACD}(l^n)$, $\mathcal{O}_{AMF}(l^n) = \mathcal{O}_{MEC}(l^n)$, and $\mathcal{O}_{FMND,AMF}(l^n \times l^m) = \mathcal{O}_{LBEM,MEC}(l^n \times l^m)$ the relations (VIII.14) and (VIII.15) can be reduced to

$$\mathcal{O}_{ABC}(l^{2k+1}) = \mathcal{O}_{ABEF}(l^{2k+1}) + 2\sum_{i=1}^{k} \binom{2k+1}{2i} \mathcal{O}_{LBEM,MEC}(l^{2k+1-2i} \times l^{2i}) \quad \text{(VIII.16)}$$

and

$$\mathcal{O}_{ABC}(l^{2k}) = \mathcal{O}_{ABEF}(l^{2k}) + 2\mathcal{O}_{MEC}(l^{2k}) + 2\sum_{i=1}^{k-i} \binom{2k}{2i} \mathcal{O}_{LBEM,MEC}(l^{2k-2i} \times l^{2i}). \quad \text{(VIII.17)}$$

In the required relation (VIII.13) the collection $\mathcal{O}_{ABC}(l^n)$ is compared with $\mathcal{O}_{ABCD}(l^n)$; hence CAVALIERI compared the magnitudes on the right-hand side of (VIII.16) and (VIII.17) with $\mathcal{O}_{ABCD}(l^n)$. Before proving (VIII.13) CAVALIERI had calculated that (*Exercitationes*, p. 255)

$$\mathcal{O}_{ABEF}(l^n) : \mathcal{O}_{ABCD}(l^n) = (AF)^n : (AD)^n = 1 : 2^n \quad \text{(VIII.18)}$$

and

$$\mathcal{O}_{LBEM}(l^n) : \mathcal{O}_{ABCD}(l^n) = (LB : AB) \cdot [(LM)^n : (AD)^n] = 1 : 2^{n+1}. \quad \text{(VIII.19)}$$

To compare $\mathcal{O}_{LBEM,MEC}(l^{n-j} \times l^j)$ with $\mathcal{O}_{ABCD}(l^n)$, CAVALIERI put in the term $\mathcal{O}_{LBEM}(l^n)$ and used (VIII.11) and (VIII.19) to get

$$\begin{aligned}\mathcal{O}_{LBEM,MEC}(l^{n-j} \times l^j) : \mathcal{O}_{ABCD}(l^n) \\ &= [\mathcal{O}_{LBEM,MEC}(l^{n-j} \times l^j) : \mathcal{O}_{LBEM}(l^n)] \cdot [\mathcal{O}_{LBEM}(l^n) : \mathcal{O}_{ABCD}(l^n)] \\ &= [\mathcal{O}_{MEC}(l^j) : \mathcal{O}_{MECN}(l^j)] \cdot [1 : 2^{n+1}].\end{aligned} \quad \text{(VIII.20)}$$

For an odd number, 2k + 1, the relation (VIII.13) can now be obtained if it is assumed that the relation has been proved for numbers less than 2k + 1, because then (VIII.18) and (VIII.20) inserted in (VIII.16) give

$$\begin{aligned}\mathcal{O}_{ABC}(l^{2k+1}) : \mathcal{O}_{ABCD}(l^{2k+1}) &= \frac{1}{2^{2k+1}} + 2\sum_{i=1}^{k} \frac{1}{2^{2k+2}} \binom{2k+1}{2i} [\mathcal{O}_{MEC}(l^{2i}) : \mathcal{O}_{MECN}(l^{2i})] \\ &= \frac{1}{2^{2k+1}} \left[1 + \sum_{i=1}^{k} \binom{2k+1}{2i} \frac{1}{2i+1}\right] \\ &= \frac{1}{2^{2k+1}} \left[1 + \frac{1}{2k+2} \sum_{i=1}^{k} \binom{2k+2}{2i+1}\right] \\ &= \frac{1}{2^{2k+1}} \cdot \frac{1}{2k+2} \sum_{i=0}^{k} \binom{2k+2}{2i+1} = \frac{1}{2k+2}.\end{aligned} \quad \text{(VIII.21)}$$

This last calculation CAVALIERI could carry out only for specified numbers.

Further, an even number requires the calculation of $\mathcal{O}_{MEC}(l^{2k}) : \mathcal{O}_{ABCD}(l^{2k})$ (*cf.* (VIII.17)). Maintaining that Theorem 22 of *Geometria*, Book Two (*cf.* (VI.12)) is valid for all powers (*Exercitationes*, p. 277), CAVALIERI first deduced that

$$\mathcal{O}_{MEC}(l^{2k}) : \mathcal{O}_{ABC}(l^{2k}) = \mathcal{O}_{MECN}(l^{2k}) : \mathcal{O}_{ABCD}(l^{2k});$$

since $\mathcal{O}_{MECN}(l^{2k}) = \mathcal{O}_{LBEM}(l^{2k})$, the last ratio is known from (VIII.19) to be $1 : 2^{2k+1}$. From this he obtains

$$\mathcal{O}_{MEC}(l^{2k}) : \mathcal{O}_{ABCD}(l^{2k}) = (1 : 2^{2k+1}) \cdot (\mathcal{O}_{ABC}(l^{2k}) : \mathcal{O}_{ABCD}(l^{2k})). \quad \text{(VIII.22)}$$

A combination of (VIII.17), (VIII.18), (VIII.22) and (VIII.20) together with the assumption that (VIII.13) is valid for numbers less than 2k leads to the result

$$\mathcal{O}_{ABC}(l^{2k}) : \mathcal{O}_{ABCD}(l^{2k}) = \frac{1}{2k+1}. \quad \text{(VIII.23)}$$

Since the relation (VIII.13) has been proved in *Geometria* for $n = 1$ and 2, (VIII.21) and (VIII.23) show that it is true for all natural numbers n (*Exercitationes*, pp. 273–287, pp. 286–291).

After having obtained this geometrical equivalent of $\int_0^a t^n \, dt = \frac{a^{n+1}}{n+1}$, CAVALIERI turned to geometrical integrations of other polynomials of degree higher than two using collections of "rectangles of powers" belonging to trapezia and triangles. Building on BEAUGRAND'S achievements, he presented a result which corresponds to calculating $\int_0^a (a + b - t)^n t^m \, dt$ (*Exercitationes*, pp. 296–299).

VIII.8. CAVALIERI'S results concerning collections of powers served, as has already been noticed, for quadratures of the parabolas $y = x^n$, which CAVALIERI in *Exercitationes* (p. 279) called diagonals of higher degree. In a parallelogram or rectangle like ABCD in Figure VIII.4 he defined a series of diagonals in the following way: the first diagonal is a usual diagonal, namely the straight line AC, and the second diagonal is the curve AHC given by

$$\square\, DA : \square\, AF = FE : FH$$

which means that AHC is a parabola with axis AB, as CAVALIERI also notices (*Exercitationes*, p. 281). Similarly, the third diagonal AJC is defined by

$$(DA)^3 : (AF)^3 = FE : FJ$$

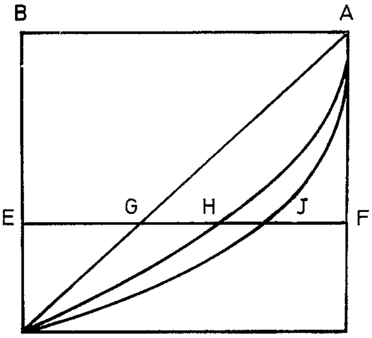

Fig. VIII.4

and is therefore a cubical parabola for which CAVALIERI gave no name but the third diagonal; continuing in this way, he obtained diagonals of arbitrary orders.

From the result $\mathcal{O}_{ABCD}(l^n):\mathcal{O}_{ACD}(l^n) = (n+1):1$ CAVALIERI obtained the quadratures of these diagonals or parabolas in the following way: let H_n be a point on the n^{th} diagonal; then by definition we have

$$FE:FH_n = (DA)^n:(AF)^n; \qquad (VIII.24)$$

further, from the similarity of the triangles AGF and ACD and from FE = DC it follows that

$$(FE)^n:(FG)^n = (DA)^n:(AF)^n.$$

A combination of the last two equations implies

$$FE:FH_n = (FE)^n:(FG)^n. \qquad (VIII.25)$$

Taking CD as *regula* CAVALIERI applied $\mathcal{O}$ to this relation (*cf.* his deduction of (VIII.11)), this—together with (V.1)—leads to (*Exercitationes*, pp. 279–280)

$$ABCD:AH_nCDA = \mathcal{O}_{ABCD}(l):\mathcal{O}_{AH_nCDA}(l)$$
$$= \mathcal{O}_{ABCD}(l^n):\mathcal{O}_{ACD}(l^n) = (n+1):1.$$

To demonstrate that his theory led to this result, which was a 17^{th} century discovery, must have been rather satisfactory to CAVALIERI. It was also natural for him to show that he could determine KEPLER'S *fusus parabolicus,* since the desire to solve that problem had motivated his creation of the concept of collections of powers. KEPLER'S parabolic cask is obtained if a segment of a parabola, AHCB (*cf.* Figure VIII.4), having the axis AB, is rotated about BC. The ratio between the paraboloid and the cylinder formed by rotating ABCD around BC is equal to (*cf.* (VI.28))

$$\mathcal{O}_{AHCB}(l^2)_{CD}:\mathcal{O}_{ABCD}(l^2)_{CD}.$$

To calculate this ratio CAVALIERI proceeded as follows (*Exercitationes*, pp. 281–282; I have made small simplifications in the presentation by introducing subtractions, by avoiding some of CAVALIERI'S manipulations with proportions, and by letting AB be an axis and ABCD a rectangle rather than a diameter and a parallelogram). CAVALIERI used the relation

$$\mathcal{O}_{AHCB}(l^2) = \mathcal{O}_{ABCD}(l^2) - 2\mathcal{O}_{AHCB,AHCD}(l\times l) - \mathcal{O}_{AHCD}(l^2) \qquad (VIII.26)$$

and the fact that the ratios between the magnitudes on the right-hand side and $\mathcal{O}_{ABCD}(l^2)$ can be determined. First he found $\mathcal{O}_{AHCD}(l^2):\mathcal{O}_{ABCD}(l^2)$: The parabola AHC is given by (VIII.24) for $n = 2$; hence a squaring of (VIII.25) implies that

$$(EF)^2:(HF)^2 = (EF)^4:(GF)^4.$$

The antecedents in this proportion are constants; hence CAVALIERI concluded, with a reference to the generalized *ut-unum*-principle, that

$$\mathcal{O}_{ABCD}(l^2):\mathcal{O}_{AHCD}(l^2) = \mathcal{O}_{ABCD}(l^4):\mathcal{O}_{\Delta ACD}(l^4) = 5:1, \qquad (VIII.27)$$

where the last equality is obtained by (VIII.13). To find $\mathcal{O}_{AHCB,AHCD}(l \times l)$: $\mathcal{O}_{ABCD}(l^2)$, CAVALIERI observed that

$$\mathcal{O}_{AHCB,AHCD}(l \times l) = \mathcal{O}_{ABCD,AHCD}(l \times l) - \mathcal{O}_{AHCD}(l^2); \qquad \text{(VIII.28)}$$

by using (VIII.11) and the quadrature of the parabola he obtained

$$\mathcal{O}_{ABCD,AHCD}(l \times l) : \mathcal{O}_{ABCD}(l^2) = \mathcal{O}_{AHCD}(l) : \mathcal{O}_{ABCD}(l) = AHCD : ABCD = 1 : 3. \qquad \text{(VIII.29)}$$

A combination of (VIII.26)–(VIII.29) gives the required ratio

$$\mathcal{O}_{AHCB}(l^2) : \mathcal{O}_{ABCD}(l^2) = 8 : 15, \qquad \text{(VIII.30)}$$

and thus the cubature of KEPLER's cask.

CAVALIERI did not stop at this but went on to calculate generalizations of the last results. Thus if in Figure VIII.4 we imagine that AJC is the "n^{th} diagonal", similarly to (VIII.27) he found that

$$\mathcal{O}_{ABCD}(l^m) : \mathcal{O}_{AJCD}(l^m) = (m \cdot n + 1) : 1. \qquad \text{(VIII.31)}$$

Moreover, he tabulated – in a complicated way – a generalization of (VIII.30) (*Exercitationes*, pp. 307–300) corresponding to

$$\mathcal{O}_{ABCD}(l^2) : \mathcal{O}_{AJCB}(l^2) = [(n + 1)(2n + 1)] : [2n^2]. \qquad \text{(VIII.32)}$$

Continuing this line of thought he described a procedure for finding the ratio $\mathcal{O}_{ABCD}(l^3) : \mathcal{O}_{AJCB}(l^3)$ (*Exercitationes*, pp. 309–311).

CAVALIERI did not list all possible applications of his results concerning collections of powers, but he showed how (VIII.31), for m = 2, and (VIII.32) could be used to find solids of revolutions of the parabolas $y = x^n$. One of his pupils, STEFANO ANGELI, later wrote a whole book, *De infinitis parabolis* (1659), on conclusions drawn from CAVALIERI's results in *Exercitationes*, Book IV. CAVALIERI himself ended this book by showing how cubatures of solids obtained by rotating hyperbolas could be reduced to the quadrature of the hyperbola (pp. 314–319).

VIII.9. The last *omnes*-concepts to be discussed here occur in *Exercitatio* V in connection with centres of gravity. This book opens with a careful list of definitions and postulates indicating that CAVALIERI originally intended to compose a complete theory of centres of gravity based on indivisibles. If he ever had this idea, he gave it up, maybe because he started his work on centres of gravity shortly before *Exercitationes* was printed, in a period when he was seriously ill (*cf.* GIUSTI 1980, pp. 80–81). Instead of working out a full theory he decided to take over some results from previous works on the subject; in particular, he referred to GUIDOBALDO DEL MONTE's edition of 1588 of ARCHIMEDES and LUCA VALERIO's *De centro gravitatis solidorum* (1604). In his own proofs he often avoided the use of indivisibles and instead used the "ARCHIMEDEan style" (*Exercitationes*, p. 322).

In a way, it is natural that the topic of centres of gravity invited CAVALIERI to employ indivisibles, especially because he considered both homogeneous and inhomogeneous distributions of mass, the latter being uniformly difform. In the theory of indivisibles a uniformly difform mass distribution of a figure has an obvious interpretation, namely that the masses of equal indivisibles of the figure, taken with respect to a *regula*, are proportional to their distances to a given line or plane parallel to the *regula*.

On the other hand the use of indivisibles for centres of gravity in the CAVALERIan style required a new system of *omnes*-concepts which was difficult to handle in a mathematically satisfactory way. CAVALIERI solved this dilemma by employing indivisibles without letting them be fundamental in his theory. I shall therefore not present the content of *Exercitatio* V, but only give an example of how CAVALIERI generalized the *omnes*-concept for the purpose of finding centres of gravity.

Let F be a plane figure and $\mathcal{O}_F(l)_{AB}$ its collection of lines. To each l in $\mathcal{O}_F(l)$ CAVALIERI assigned among other things a mass (*gravitas*) and a moment with respect to a given point o; let us denote these concepts by m(l) and $M_o(l)$ respectively. For CAVALIERI m(l) and $M_o(l)$, like areas and volumes, were magnitudes and not real numbers. That means that all calculations with them were to be based on ratios between masses or between moments. For example, if $m(l_1) = m(l_2)$, and d_1 and d_2 are the distances between l_1 and l_2 and the line through o parallel to the *regula* AB, we have

$$M_o(l_1) : M_o(l_2) = d_1 : d_2,$$

and if $d_1 = d_2$, then

$$M_o(l_1) : M_o(l_2) = m(l_1) : m(l_2).$$

When dealing with such or more complicated relations CAVALIERI applied $\mathcal{O}$, and in this way introduced the concept of "all the moments" (*Exercitationes* p. 340) and "all the masses". The latter he set, without comment, equal to the mass of "all the lines" (*ibid.*, p. 343).

Apart from formulating the postulate that the centre of gravity of "all the lines" of a given figure is the same as the centre of gravity of the figure (*ibid.*, p. 330), CAVALIERI was not at all careful in specifying the assumptions he made in dealing with masses, moment and other concepts in connection with "all the lines". His attitude towards magnitudes as collections of moments seems to have been that they, like collections of circumferences, were instruments for finding results which should be proved in the classical way.

In concluding the sections on CAVALIERI's three generalizations of the *omnes*-concept, it may be noticed that only one of them, namely "all the powers", was completely incorporated in CAVALIERI's theory, whereas the others mainly were used intuitively for making discoveries.

IX. The distributive method

IX.1. On July 22nd, 1634, CAVALIERI wrote to GALILEO:

> As the printing of the first five books of my *Geometria* is already finished I wanted to send them to you so that you can have a look at them when it is convenient for you; that would do me a great favor and greatest if you would tell me what you think of my foundation of the indivisibles. Suspecting that this concept of infinitely many lines or planes may cause difficulties for many,

I have later decided to compose the seventh book in which I show the same things in a different way—also different from Archimedes's.[22]

From this we gather that it was only at the eleventh hour CAVALIERI decided to add the seventh book to *Geometria.* Considering that the five books already printed were 541 pages long, that Book VI would add another 71 pages, that the first five books had been ready for printing since 1627 and the sixth since 1629, and that the printing process had been long and difficult, it was quite a decision to make (*cf.* LOMBARDO-RADICE 1966, p. 19). In the letter to GALILEO CAVALIERI touched upon the reason for this decision, and in the Introduction to Book VII he specified his motive: He feared that certain mathematicians and philosophers would doubt the validity of his arguments. This doubt could arise, he said, despite the fact that he had treated collections of lines and planes according to their finite nature, which is reflected in the property that they can be augmented and diminished (*Geometria,* p. 483).

In my opinion it was not so much a fear of *certain* mathematicians' opposition as a wish to convince GALILEO thath motivated CAVALIERI to create a second method. Although GALILEO did not receive a complete presentation of CAVALIERI's method of indivisibles until the summer of 1634, through CAVALIERI's letters he had got an impression of the idea it was based on, and he had put forward rather sceptical remarks concerning the *omnes*-concepts. In replying to these CAVALIERI strongly defended his concepts and at the same time expressed the hope of having *il maestro* accept his method. It is very likely that his waiting for such approval was one of the factors delaying the appearance of the *Geometria.* However, CAVALIERI did not get the acceptance he wished, and it may very well have made him so uneasy that he decided to add another approach.

IX.2. The best desription of the difference between CAVALIERI'S first and second method is found in *Exercitationes* Book One (p. 4). CAVALIERI there considered two figures ABCD and EFGH with the same altitude and the common *regula* LM (Figure IX.1). He supposed all lines parallel to LM to be drawn in both figures.

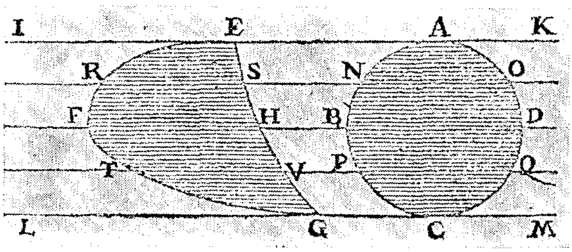

Fig. IX.1

[22] GALILEI *Opere*, vol. 16, p. 113: essendosi già finita la stampa de' primi cinque libri della mia Geometria, gliel'ho voluti inviare, acciò, havende agio, gli dia un puoco di un'occhiata, che mi sarà di molto favore, e massime se mi dirà quale gli riesca il mio fondamento delli indivisibili. E perchè dubito che a molti sia forsi per dar fastidio quel concetto delle infinite linee o piani, perciò ho poi volsuto fare il settimo libro, nel quale dimostro per altra via, differente anco da Archimede, le medesime cose.

These lines can be compared in two ways, either collectively (*collective*), that is the whole collection of lines of ABCD is compared with the whole collection of lines of EFGH, as it was done in the first books of *Geometria*, or distributively (*distributive*), which means that each pair of corresponding lines, like BD and FH, is compared separately.

In *Exercitationes* CAVALIERI named his two procedures the first and the second method of indivisibles[23]; in distinguishing between them I shall call them the collective and the distributive method. In a way it is strange that CAVALIERI attached the word indivisibles to the second method, because he had created it exactly to avoid the use of the indivisibles, "all the lines" and "all the planes". Nevertheless, when he gave the two methods similar names, the reason could be that he treated the distributive approach as supplementary to the collective method, and not as an independent theory. Thus his two presentations of the distributive method, Book VII of *Geometria*, and Book II of *Exercitationes*, have the character of appendices in which it is shown, or at times merely hinted, how results obtained by the use of an *omnes*-concept can be achieved in another way.

Since the main topic of this paper is CAVALIERI'S *omnes*-concepts, I shall restrict the treatment of the distributive method to a brief description of the means employed in Book Seven of *Geometria* to avoid the *omnes*-concepts.

The most important tool is the theorem called CAVALIERI'S principle. We saw that it had occurred as a proposition in Book II (*cf.* Sections V.I and V.4), but since it was there formulated in terms of "all the planes" and "all the lines", it had to be reestablished in the distributive method by a new proof. Because there are translations of this proof into English and analyses of it (EVANS 1917, pp. 447–451, SMITH 1929, pp. 605–609, STRUIK 1969, pp. 210–214, CELLINI 1966_2, CARRUCCIO 1971), I will only indicate some of CAVALIERI'S steps.

He started in the first theorem of *Geometria*, Book VII, by imagining that two figures like F = BZV, G = CRT in Figure IX.2 have equal altitudes with respect to the *regula*, YH, and that corresponding chords (or sum of chords), like MN + OP and SX, parallel to YH, are equal. The theorem then states that

$$F = G. \qquad \text{(IX.1)}$$

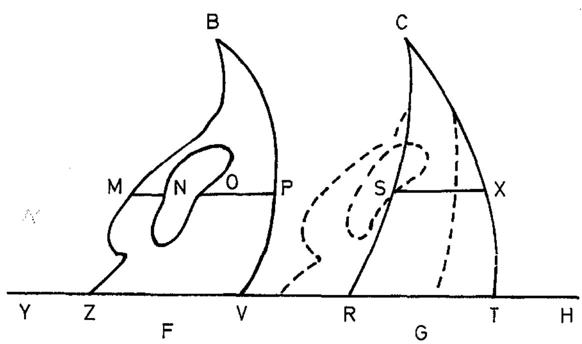

Fig. IX.2. Part of CAVALIERI'S figure in *Geometria*, page 485.

[23] *Exercitationes*, p. 3: Ultramque igitur non incongruè methodum Indivisibilium appellabimus, nempè illam priorem, posteriorem alteram.

To prove this CAVALIERI used superposition. He put F over G so that the points B and C coincided and corresponding "chords" were on the same line. He then considered the part, F_1, of figure F, which does not cover parts of figure G and the part, G_1, of figure G which is not covered. F_1 and G_1 have equal altitudes, and their corresponding "chords" parallel to YH will be equal. CAVALIERI maintained that continuing this process would at the end split F and G into parts which would cover each other and hence be equal. As in the proof of Theorem II.2 he did not state whether he had a finite or an infinite process in mind (*cf.* Section V.3). But anyway the proof is problematic. CAVALIERI may have been aware of this; at least he provided also a longer proof in which he transformed the figures, used an argument of exhaustion with quite some display of intuition. For the content of this proof I refer to the above-mentioned literature.

Turning to the general situation CAVALIERI looked at two figures F and G with equal altitude and the property that for all corresponding "chords" l_1 and l_2 in F and G:

$$l_1 : l_2 = a : b. \qquad \text{(IX.2)}$$

His principle then states that

$$F : G = a : b.$$

CAVALIERI's idea was to prove this by using Definition V.5 of EUCLID'S *Elements*. Hence he had to prove that

$$\underbrace{a + a + \ldots + a}_{n \text{ times}} > \underbrace{b + b + \ldots + b}_{m \text{ times}} \qquad \text{(IX.3)}$$

implies that

$$\underbrace{F + F + \ldots + F}_{n \text{ times}} > \underbrace{G + G + \ldots + G}_{m \text{ times}}, \qquad \text{(IX.4)}$$

for arbitrary multiples, and similarly for $=$ and $<$. In the case where there is an $=$ in (IX.3) we have the situation that corresponding "chords" in the multiples of the figures are equal, and (IX.4) with an $=$ then follows from (IX.1). CAVALIERI finished the proof by claiming that it is obvious that (IX.4) follows from (IX.3) in the cases involving $>$ or $<$ (*Geometria*, pp. 498–499). He progressed analogously in making his principle for solid figures independent of "all the planes".

IX.3. One of CAVALIERI'S ways of obtaining results in the distributive approach was a more extensive use of similar figures than in the previous books. That similar plane (solid) figures are in the duplicate (triplicate) ratio of their linear ratio, he took over from Book II (*cf.* Section VI.5). In his proofs of these theorems CAVALIERI had used "all the lines" and "all the planes", but these proofs, as he pointed out, can be reconstructed by using the CAVALIERIan principle instead (*Geometria*, pp. 504–505).

A large and important group of theorems in Books II–V were dependent on geometrical equivalents of integrating a polynomial of second degree. To obtain the less complicated of these theorems CAVALIERI took over from EUCLID'S *Elements* XII.7 the fact that a pyramid is a third of its corresponding prism, a theorem which is not based on a proof by exhaustion. Combining this result with one concerning similar figures and his principle, he was able to deduce theorems concerning pyramids, prisms, cones, and cylinders. Thus that the ratio between a cone and its corresponding cylinder is $\frac{1}{3}$ can be seen by applying CAVALIERI's principle in comparing the cone with a pyramid and the cylinder with a prism.

In deducing the more complicated results from the above mentioned group CAVALIERI introduced a substitute for the concepts of "all the rectangles" and "all the squares" which had played a considerable role in the collective method. This new concept, *solidum rectangulum*, is similar to GRÉGOIRE's *ductus* (*cf.* Section IV.2) and is defined in the following way (Figure IX.3):

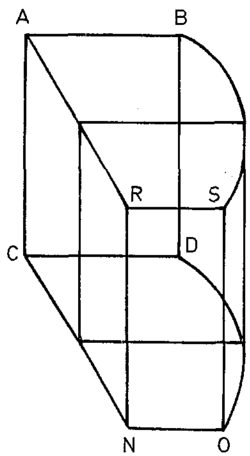

Fig. IX.3

The solid figure ABSRCDON is called *solidum rectangulum sub* the surfaces CNOD and ODBS if all sections made by planes parallel to a given *regula* (*in casu* RNOS) are rectangles having their perpendicular sides in CNOD and ODBS respectively (*Geometria*, p. 514). In case the rectangles are squares the solid is called *solidum quadratum*.

Applying his principle to these solids, CAVALIERI obtained new geometrical results which correspond to integrations of second degree polynomials; and on the whole he achieved the goal of Book Seven: to obtain the important theorems of the first books of *Geometria* without employing the *omnes*-concepts.

IX.4. One may naturally wonder whether CAVALIERI's creation of the distributive method meant that he dissociated himself from the collective approach. The answer is no. He continued to value the collective method, which had been on his mind for many years. This can be seen from the way he treated the two methods in *Exercitationes* in 1647. While the story of the printing of *Geometria* had made it natural to present the distributive method as a supplement to the collective method, in writing *Exercitationes* CAVALIERI had the chance to elevate the distributive method to the main theory, had he wished to, but he did not. He did revise his presentation of the distributive method, but not to such an extent, that it could be studied independently of the collective method.

In describing his views on the relative merits of the two methods, CAVALIERI in effect remarked only that the collective method had the advantage of allowing figures which have unequal altitudes to be compared, while the distribution method had the advantage of avoiding infinities (*cf.* for instance *Exercitationes*, pp. 5 and 30). In other words, he seems to have thought that the one method had its strength in its generality, the other in possessing a more acceptable foundation.

CAVALIERI clearly valued the first aspect above the second. He proceeded with his investigations of the applicability of the collective method and found, as we have seen, that it could be used for geometrical integrations of t^n and for determinations of centres of gravity. Thus the collective approach remained the most important part of CAVALIERI'S method of indivisibles while the distributive approach never became more than an appendix to the former.

X. Reaction to Cavalieri's method and understanding of it

X.1. Although CAVALIERI was convinced that his collective method was a useful new approach to quadratures and cubatures, he was probably also aware that he had not solved all the fundamental problems connected with it; in remarks in *Geometria* and in his letters to GALILEO he revealed at least that he foresaw some opposition to his method (*cf.* Section IX.1). He was right,; his method came to meet some opposition, but it also inspired other mathematicians to continue the study of geometrical quadratures along a new line. In this and the following sections I shall outline the main reactions to CAVALIERI'S method (this topic is also treated in GIUSTI 1980, pp. 40–65). Further I am going to show that most mathematicians did not make themselves acquainted with CAVALIERI'S own work but based their knowledge of his method on a study of EVANGELISTA TORRICELLI'S presentation of the method of indivisibles.

CAVALIERI was, as mentioned in Section IX.1, very eager to get a positive reaction from GALILEO; however, his hope was not fulfilled. GALIELO'S own statements concerning *Geometria* are lost, but that he was not enthusiastic about it we learn from a letter CAVALIERI wrote to him October 23, 1635:

> I am sorry that my *Geometria* turns out to be as difficult and laborious as you say. This is my fault, because I have explained myself badly, but the subject is in itself also very difficult ... I therefore think that you ought to be indulgent towards me; that I have had nobody here with whom I could discuss similar matters is the reason why until now I found that easy which a dicussion would have taught me was difficult.[24]

What GALILEO himself thought of indivisibles as mathematical entities is not clear; thus his treatment of indivisibles and infinities in the *Discorsi* (1639) leaves many

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[24] GALILEI *Opere*, vol. 16, pp. 327–328: Mi dispiace che la Geometria mia riesca così difficile e laboriosa come dice: sarà colpa mia, che malamente mi sarò saputo esplicare, ma ad ogni modo la materia per sè stessa è anco molto difficile ... Mi dovrà però, credo, compatire V. S., che non havendo qua con chi conferire di simili materie, è cagione che mi sia tal hora parso facile quello che la conferenza mi harebbe fatto conoscer per difficile.

open questions. However, GALILEO's ideas clearly did not accord with CAVALIERI'S (*cf.* GIUSTI 1980, pp. 40–44). As LOMBARDO-RADICE has pointed out (LOMBARDO-RADICE 1966, pp. 18 and 767–768), it must have been a disappointment to CAVALIERI that GALILEO made a flattering remark on CAVALIERI'S *Lo specchio ustorio* (1632) in the *Discorsi*, where the discussion is about burning glasses but kept silent upon the *Geometria* earlier in the long discussion about indivisibles (GALILEI 1914, pp. 41–42).

Others, and particularly non-Italian mathematicians, expressed their scepticism and doubts more explicitly than GALILEO did. Among these GULDIN is the most noticeable; he virtually tried to tear CAVALIERI'S method of indivisibles completely apart, first of all by finding all the weak points in the foundation of the method. GULDIN did not stop at criticizing the method but maintained in addition that CAVALIERI had taken over all his ideas from others, especially KEPLER and BARTHOLOMEUS SOVER (GULDIN 1635–1641, vol. 2, pp. 3–4, or *Exercitationes*, pp. 179–182).

CAVALIERI must have found it rather insulting to learn both that this method had no value and that it was based upon plagiarism. His reaction to GULDIN'S attacks on the content of the method has already been discussed (*cf.* Section III, 3–5; see also LOMBARDO-RADICE 1966, pp. 773–777, and GIUSTI 1980, pp. 55–65), so here I will only deal with CAVALIERI'S reply to the accusation of plagiarism. He easily established his independence of KEPLER by a reference to the fact that KEPLER'S theory of quadratures and cubatures as presented in *Stereometria* is very different from his own theory of indivisibles (*Exercitationes*, p. 180, see also Section III.2). In showing that he was not influenced by SOVER and his book *Curvi ac recti proportio promota*, CAVALIERI gave a chronological argument; he had sent a copy of the manuscript of the seven books of *Geometria* to OTTAVIANO ZAMBECCARI in 1629, and SOVER'S book was not published until 1630 (*Exercitationes*, pp. 182–183).

That is not quite true, since only the first six books of *Geometria* were finished in 1629 (*cf.* Section I.4). The seven instead of the six may very well be a slip of the pen; but the argument is a shade suspicious, because CAVALIERI was of the opinion that GULDIN particularly compared SOVER'S work with CAVALIERI'S second method, the distributive one presented in Book VII. Thus CAVALIERI'S argument invites investigation of SOVER'S concepts.

In Book Five of *Curvi ac recti* we find the clue to GULDIN'S accusation. Here SOVER introduced the concept of *analogous* figures; in the simplest case (which is sufficient for the present purpose) SOVER considered two figures ABCD and KLMN (Figure X.1) having the "axes" AC and KM and situated between the

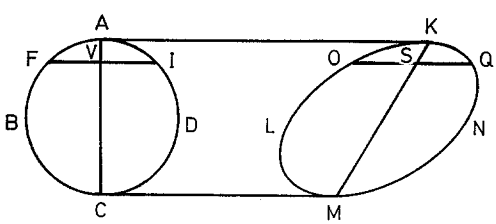

Fig. X.1

parallel lines AK and CM. He then imagined that the line AK is displaced parallel until it reaches CM, and defined KLMN to be *analogous* to ABCD, if at any moment of the motion the line segments cut off between the "axes" and the perimeters of the figures satisfy the relation

$$FV : VI = OS : SQ \tag{X.1}$$

(SOVER 1630, pp. 278–279). SOVER used this concept partly to define new curves and partly for quadratures which he obtained from relation (X.1) by exhaustion proofs.

GULDIN did not explicitly state which of CAVALIERI'S two methods he imagined to be inspired by SOVER. If he focused on the idea of using motion to define continuous relations between corresponding line segments, which by the way was in the air at the time, he might have thought of the collective method and would then have been wrong, because this method had been worked out long before CAVALIERI had any chance of knowing SOVER'S book. If, on the other hand, GULDIN concentrated on SOVER'S use of relations like (X.1) in a manner which CAVALIERI would call distributive, he would, as CAVALIERI thought, have had the second method in mind. The possibility that this method was inspired by SOVER cannot be completely excluded. CAVALIERI might have got the idea of simplifying SOVER'S work by replacing the latter's exhaustion proofs by a use of the principle later named after CAVALIERI. It is however more likely that CAVALIERI, independently of SOVER, got the idea to base a method on this principle which was already present in the collective theory.

The attacks by other mathematicians on CAVALIERI'S method were not so strong as GULDIN'S, but certain circles opposed the method. In the preface of *De infinitis parabolis* (1659) STEFANO ANGELI, who was a Jesuat like CAVALIERI, remarked that these circles mainly contained Jesuit mathematicians. ANGELI mentions not only GULDIN but also MARIO BETTINI and ANDREAS TACQUET.

X.2. ANGELI belonged to the group of Italian mathematicians who evaluated CAVALIERI'S method favorably. Among the other members were DAVISO, PIETRO MENGOLI, TORRICELLI, HONORATUS FABRY (*cf.* FELLMANN 1959), and C. RENALDINUS. The latter gave a rather clear exposition of CAVALIERI'S two methods of indivisibles in *De resolutione et compositione mathematica* (1668).

Also outside Italy the method of indivisibles became gradually acknowledged, and by the 1650's it was widely accepted. However, what outside Italy was called the method of indivisibles and attributed to CAVALIERI had in general very little in common with his elaborate theory of collections of lines and planes, which aroused the interest of only a few mathematicians. The method of indivisibles was thought to be based either on the idea that a plane figure is a sum of line segments, or on the idea that it is a sum of infinitesimals. In this section I will deal with the first idea, for which TORRICELLI is the main source.

During the first years after the publication of *Geometria* (1635) TORRICELLI took a rather sceptical attitude toward CAVALIERI'S method (*cf.* LOMBARDO-RADICE 1966, pp. 21–22). But about 1641 he changed his mind and found that it opened a "royal road" to quadratures (TORRICELLI *Opere*, vol. 1, part 1, p. 140), and he gave examples of the use of the method in his *Opera geometrica*, published in 1644. This book was well received by European mathematicians and became influential in spreading knowledge of the method of indivisibles; for several mathematicians it remained the only origin of this knowledge.

Since TORRICELLI is such an important link between CAVALIERI'S method and the general understanding of it, I shall present an example of how he used indivisibles. The example is from the part "De dimensione parabolae" of *Opera geometrica*, in which TORRICELLI provided 21 different examples of the quadrature of the parabola. After having given ten examples based on classical means TORRICELLI declared that he would show how the method of indivisibles, invented by CAVALIERI, could be used. He then considered a segment ABC of a parabola, whose tangent at C is CD, and whose diameter through A is AD (Figure X.2).

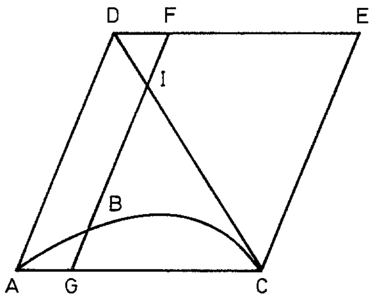

Fig. X.2

He completed the parallelogram ACED, and searched for the ratio between ABCD and ACED (TORRICELLI *Opere*, vol. 1, part 1, pp. 140–141).

To find this he drew an arbitrary diameter GF cutting the parabola in B and the line CD in I, and he proved that

$$FG : IB = (\text{circle with diameter } FG) : (\text{circle with diameter } IG).$$

For all FGs parallel to DA, the first term in this proportion is equal to DA, and the third term is equal to the circle with diameter DA. From this TORRICELLI concluded that

$$(\text{all } FG) : (\text{all } IB) = (\text{all circles with diameter } FG) :$$
$$(\text{all circles with diameter } IG);$$

or in his own words:

> All the first ones together, that is the parallelogram AE, will be to all the second ones together, that is to the trilineum ABCD, as all the third ones together, that is the cylinder AE, to all the fourth ones together, that is to the cone ACD.[25]

Thus, without further comment TORRICELLI set "all the lines" of a plane figure and "all the planes" of a solid figure equal to the figures themselves. Using the

[25] TORRICELLI *Opere*, vol. 1, part 1, p. 140: erunt omnes primae simul, nempe parallelogrammum AE, ad omnes secundas simul, nempe ad trilineum ABCD, ut sunt omnes tertia simul, nempe cylindrus AE, ad omnes quartas simul hoc est ad conum ACD.

fact that the cone ACD (which has the circle with diameter AD as base and vertex at C) is one third of its corresponding cylinder, TORRICELLI found that the acquired ratio between ABCD and ACED is $\frac{1}{3}$; from which he was able to deduce that the parabolic segment is $\frac{4}{3}$ of its inscribed triangle.

When CAVALIERI's elaborate system of concepts and theorems is compared with TORRICELLI's example, it becomes clear why mathematicians preferred the latter's exposition of the method of indivisibles. It should, be noticed however, that TORRICELLI obtained a short cut in the calculations by not bothering about the foundation of the collective method. CAVALIERI had taken great trouble to establish the existence of the ratio $\mathcal{O}_{F_1}(l) : \mathcal{O}_{F_2}(l)$ for two plane figures and to verify the relation

$$F_1 : F_2 = \mathcal{O}_{F_1}(l) : \mathcal{O}_{F_2}(l) \tag{X.2}$$

whereas TORRICELLI presented the method, as we saw in the quotation, in a way implying that

$$F = \mathcal{O}_F(l). \tag{X.3}$$

In *De dimensione parabolae* TORRICELLI did not comment on this aspect of the method; neither did he explain what should be understood by "all the lines". Yet he used *omnes* in other connections in the treatise, where it quite clearly meant a sum of a finite or infinite number of elements (TORRICELLI *Opere*, vol. 1, part 1, pp. 132–133, 148–149). Thus TORRICELLI opened the way to the interpretation

$$\mathcal{O}_F(l) = \sum_F l. \tag{X.4}$$

Combining (X.3) and (X.4), we get

$$F = \sum_F l \tag{X.5}$$

as the understanding of the collective method of indivisibles TORRICELLI offered. Indeed in this manner TORRICELLI created his own method of indivisibles, but by referring to CAVALIERI he gave the impression that he had employed CAVALIERI's method; that had the effect, as mentioned earlier, that many mathematicians identified the method of indivisibles with TORRICELLI's version of it.

TORRICELLI's way of employing indivisibles was accepted by many of his colleagues; their confidence in his work is illustrated in a letter FRANS VAN SCHOOTEN wrote to CHRISTIAAN HUYGENS September 27, 1650 (HUYGENS *Oeuvres*, vol. 1, pp. 130–132). In this letter VAN SCHOOTEN commented upon some examples HUYGENS had composed to warn against the use of "CAVALIERI's principles" (*ibid.*, p. 131). VAN SCHOOTEN found that HUYGENS was too sceptical and that one should not be afraid of building something on these principles as long as it was done as TORRICELLI had done in his demonstrations.

It is natural to wonder how CAVALIERI reacted to the fact that TORRICELLI presented the method of indivisibles as being based on the idea $F = \sum_F l$. In view of CAVALIERI's great concern about foundations, it could be expected that he refused to acknowledge TORRICELLI's treatment. That did not happen; on the contrary he took advantage of the respect TORRICELLI's work had gained. Thus a few times in *Exercitationes* CAVALIERI tried to make a problematic argument more convincing by referring to a similar argument in TORRICELLI'S *De dimensione parabolae* (*cf.* Section VIII.6).

On the whole CAVALIERI seems to have been delighted that TORRICELLI took interest in the method of indivisibles, implying among other things that after GALILEO's death in 1642 he still had a skilful mathematician with whom he could discuss the method. It may have been disappointing to CAVALIERI that TORRICELLI did not take up the points in his letters concerning the foundation of the method (*cf.* GIUSTI 1980, p. 45), but yet TORRICELLI was an inspiring correspondent. The two Italian mathematicians came so close scientifically in their last years that in his will TORRICELLI appointed CAVALIERI to be his scientific executor, an undertaking CAVALIERI could not carry out because he died a few months after TORRICELLI.

X.3. All of TORRICELLI'S examples of quadratures based on indivisibles consisted in a transformation of the ratio between a segment of a parabola and its inscribed triangle into a known ratio between two other geometrical figures, as in the example presented in the last section. Thus TORRICELLI'S technique did not lead to an approximation or to an arithmetic determination of $\Sigma 1$. Such approaches to quadratures were worked out by the French mathematicians PIERRE FERMAT and GILLES P. ROBERVAL at the same time as CAVALIERI'S *Geometria* was in print, and are quite often mentioned in the correspondence of the MERSENNE circle from 1636 and onwards.

In the literature the arithmetical methods have often been mixed up with the method of indivisibles. I shall therefore discuss some of the arithmetical methods, particularly under the aspect of how their users related them to CAVALIERI'S method; and I shall argue that some 17^{th}-century mathematicians provoked confusion of arithmetical methods and the method of indivisibles, and also the idea that CAVALIERI'S method employed infinitesimals.

FERMAT never disclosed his ideas about the foundation of arithmetical integrations, whereas ROBERVAL revealed his thoughts in his famous *Epistola an Torricellium* written in 1647 (printed in several books, for instance ROBERVAL 1730 and TORRICELLI *Opere*, vol. 3) and in "Traité des indivisibles", composed at an unknown time and printed posthumously in 1693 (reprinted in ROBERVAL 1730, pp. 207–290). When ROBERVAL first worked on his method he called it "a method of infinities". Later, however, he became so influenced by the fame of CAVALIERI'S indivisibles that he took over the word indivisibles. Thus "Traité des indivisibles" opens with the phrase "To draw conclusions by means of indivisibles ...".[26] This does not mean that ROBERVAL had adopted CAVALIERI'S method of indivisibles, or TORRICELLI'S version of it. ROBERVAL'S collection of "indivisibles" emerged by a continuous subdivision of the surface and were, as he stressed, "an infinity of small surfaces" (ROBERVAL 1730, p. 209); in other words they were infinitesimals.

Nevertheless the concept of *toutes les lignes* often occurs in connection with ROBERVAL'S quadratures. To illustrate how he employed this concept I shall

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[26] ROBERVAL 1730, p. 207: Pour tirer des conclusions par le moyen des indivisibles.

paraphrase his quadrature of the parabola. Let ABC be a segment of a parabola whose vertex is A and whose axis is AB (Figure X.3). ROBERVAL imagined that the tangent AD has been divided in "an infinity of equal parts", AE, EF, *etc.*, and that lines EL, FM, *etc.*, have been drawn through the division points parallel to AB (*ibid.*, p. 214). He then stated that

$$\text{(area ACD)} : \text{(rectangle ABCD)} = (\textit{toutes les lignes} \text{ of ACD}) : (\textit{toutes les lignes} \text{ of ABCD}). \qquad \text{(X.6)}$$

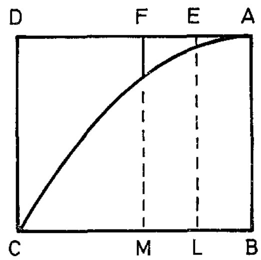

Fig. X.3

This relation is similar to CAVALIERI'S fundamental relation (X.2) concerning quadratures. ROBERVAL'S justification and employment of it is, however, quite different from CAVALIERI'S. ROBERVAL first maintained that

$$\text{(area ACD)} : \text{(rectangle ABCD)} = \text{(all infinitesimal rectangles of ACD)} : \text{(all infinitesimal rectangles of ABCD)}, \qquad \text{(X.7)}$$

where the infinitesimal rectangles are determined by the division of AD. Since the rectangles all have a base equal to AE, this line segment can be cancelled on the right-hand side of (X.7), whereby (X.6) is obtained. Thus in (X.6) *toutes les lignes* means the sum of the ordinates.

In modern terms ROBERVAL'S use of (X.6) can be explained in the following way: Let F_1 and F_2 be two plane figures with rectilinear base AD (Figure X.4) and further limited by the graphs of the functions f_k, $k = 1, 2$, and the lines AB,

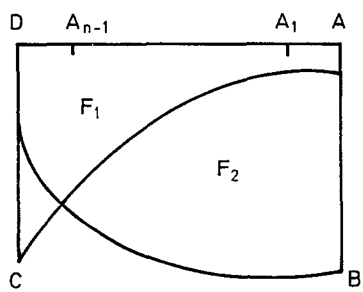

Fig. X.4

AD and DC. ROBERVAL then determined the ratio $F_1 : F_2$ from the relation

$$F_1 : F_2 = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} f_1\left(\frac{i}{n}AD\right)}{\sum_{i=1}^{n} f_2\left(\frac{i}{n}AD\right)}.$$

When F_1 is a segment of a parabola and F_2 a rectangle the functions f_1 and f_2 are characterized by $f_1\left(\frac{i}{n}AD\right) \sim \frac{i^2}{n^2}, f_2\left(\frac{i}{n}AD\right) \sim 1$. Hence ROBERVAL transformed the quadrature of the parabola to a determination of

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{i^2}{n^2}}{\sum_{i=1}^{n} 1} = \lim_{n\to\infty} \frac{\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n}{n^3}.$$

ROVERVAL argued that the quantity on the right-hand side is equal to $\frac{1}{3}$, because $\frac{\frac{1}{2}n^2 + \frac{1}{6}n}{n^3}$ can be neglected for large n's.

Thus it is obvious that ROBERVAL's procedure leading to a determination of a limit (which in the 17th century was done by omitting certain terms) was an approach to quadrature quite different from CAVALIERI's calculations with collections of lines. However, ROBERVAL himself did not seem to have considered his method very different from CAVALIERI's. This is illustrated in the above-mentioned letter he wrote to TORRICELLI in 1647. Besides disputing with TORRICELLI about the priority of the kinematic method of tangents, one of ROBERVAL'S concerns in this letter was to prove that he had discovered his method of quadrature independently of CAVALIERI. The point that the two methods were based on different foundations and required different procedures of calculations, could have provided the necessary argument. But that was not how ROBERVAL argued; he talked only about a "small difference" between the two methods.[27]

Further he stated that some mathematicians who were envious of CAVALIERI had wrongly claimed that CAVALIERI really meant that surfaces were composed of lines (TORRICELLI *Opere*, vol. 3, p. 489; *cf.* also WALKER 1932, pp. 15–16 and 35). This indicates that ROBERVAL thought that CAVALIERI's indivisibles of a surface were like his own small rectangles. On the whole ROBERVAL'S attitude toward CAVALIERI's theory was to project his own ideas into it, instead of concentrating on the differences between the two methods. Thereby he introduced new misunderstandings concerning CAVALIERI'S method of indivisibles, namely that it was based on infinitesimals and was related to arithmetical methods. Further he contributed to the idea, introduced earlier by TORRICELLI, that "all the lines" should be understood as a sum.

ROBERVAL'S views on indivisibles influenced the skilful architect of a well developed theory of arithmetical integration, BLAISE PASCAL. The similarities

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[27] TORRICELLI *Opere*, vol. 3, p. 489: exigua differentia.

between ROBERVAL'S and PASCAL'S ideas can be seen in PASCAL'S *Lettre de A. Dettonville à Monsieur Carcavy* (1658). In this treatise he explained that the indivisibles of a plane figure constitute an infinity of infinitesimal rectangles, whose sum differs from the figure only by a quantity smaller than any given one (PASCAL *Oeuvres*, vol. 8, p. 352). He further asserted that it was only in manner of speaking that the method of indivisibles was different from the ancient method of exhaustion.

In his other treatises on arithmetical integration PASCAL took the position that the method of indivisibles was so well established that he did not need to explain its foundation. Instead he concerned himself with finding the arithmetical sums applicable for quadratures, cubatures and determinations of centres of gravity. His expression for a sum used for quadratures was *summa linearum*, *la somme des lignes* or *la somme des ordonnées*.

These few remarks on PASCAL'S method should be sufficient to make it clear that those who became acquainted with "the method of indivisibles" through a study of PASCAL'S ingenious works would learn a method much different from CAVALIERI'S.

The other great contributor to arithmetical integrations was JOHN WALLIS who worked out an infinitesimal method independently of the French school and presented it in *Arithmetica infinitorum* (1655). In its preface WALLIS stated that while creating his own method, the arithmetics of infinities, he knew the method of indivisibles only through a study of TORRICELLI'S work, because he had not been able to see CAVALIERI'S *Geometria*, but still he found that there were similarities between the two methods (WALLIS 1972, p. 357).

Thus we have another example of how the method of indivisibles became known through TORRICELLI'S work and of how 17^{th}-century mathematicians did not pay much attention to the difference between an arithmetical integration and the geometrical method of indivisibles. Indeed we can conclude that the mathematicians of the 17^{th}-century tended to call all methods of integration emerging during the period 1635–1665 methods of indivisibles. Therefore they made no sharp distinction between CAVALIERI'S use of the concept of *omnes lineae* and *e.g.* PASCAL'S use of *summa linearum*. In particular, it was not recognized that CAVALIERI'S collection of lines belonging to a given figure was a magnitude, which was neither the area of the figure nor an approximation to it.

While working on quadratures and cubatures in the early 1670's G. W. LEIBNIZ took over the mixture of names and concepts connected with quadratures and used the abbreviation *omn.* for *omnes* to signify a sum. His first introduction of the symbol $\int l$ in October 1675 was only meant to be a further abbreviation of *omn.* l making it clear that *omn.* prefixed, for instance, to a line segment yields a two dimensional magnitude (LEIBNIZ 1820, p. 90). However, by that step he was inspired to investigate the operational rules for $\int$ and was led to the creation of the calculus.

X.4. The creation of the calculus did not imply that the method of indivisibles fell into complete oblivion. In the 18^{th} century the knowledge of it was kept alive through textbooks of infinitesimal calculus (especially their prefaces), through the encyclopedias of sciences, and through J. E. MONTUCLA'S *Histoire des mathématiques* (1758). Doubtless the 18^{th}-century views of the method of indivisibles played a great role for the later understanding of it and will therefore be described briefly in this section.

While the 17^{th}-century mathematicians had been at variance as to whether the method of indivisibles was based on the idea that a plane figure is composed by line segments or by infinitesimal rectangles, most 18^{th}-century writers agreed on the first opinion. Also they often focused on the central place CAVALIERI's principle had in the theory. Several characterizations of the method of indivisibles were indeed very close to the following one given by J. C. STURM:

> The Method of indivisibles ... goes to work after a way which seems to be more natural than any other, by supposing plane Figures to consist of innumerable lines, and solids of innumerable Plans (called their indivisible Parts or Elements because the Lines are conceived without latitude, and the Plans without any thicknes, and relying on this self-evident Axiom, That if all the Indivisibles of one Magnitude collectively taken, be equal or proportional to all the correspondent Indivisibles of another, or taken separately each to each, then also those Magnitudes will be equal or proportional among themselves (STURM 1700, Preface § IX).

STURM himself was influenced by reading RENALDINUS's treatment of the method of indivisibles (*cf.* Section X.2) and thus remarked that CAVALIERI had both a collective and a distributive method. MONTUCLA also made this point, but it was ignored by many writers.

In tracing the history of the method of indivisibles most 18^{th}-century mathematicians agreed that this method was the one which CAVALIERI and TORRICELLI had employed. Thus a distinction between the method of indivisibles and methods of infinitesimals was introduced. Although this did not add to the understanding of CAVALIERI's proper method, it could have been a means to improve on the 17^{th}-century confusion about methods of integration. However, this confusion survived because the important writer on history of mathematics, MONTUCLA, did not follow his contemporaries' general description of CAVALIERI's methods. MONTUCLA thought that in *Geometria* CAVALIERI had given the impression that he considered a plane figure as composed of line segments. But he was also of the opinion that in his answers to GULDIN, CAVALIERI had clearly shown "that his method is nothing but a simplified method of exhaustion" based on infinitesimals (MONTUCLA 1758, p. 27).

The evaluations of the method of indivisibles made in the 18^{th}-century were rather diverse. Some mathematicians dismissed the method completely; others had the attitude that it was a useful short cut for a demonstration and that it could be "saved". The means suggested for this were the two MONTUCLA had mixed up. Thus one group would argue as had JOHN HARRIS that "this *Method of Indivisibles* is only the Ancient *Method of Exhaustion* a little disguised and concentrated" and would suggest the introduction of a real exhaustion proof if necessary (HARRIS 1704–1710, vol. 1, article: indivisibles). Another group would rescue the method by replacing the indivisibles by infinitesimals, so that a method based on the assumption $F = \Sigma l \Delta x$, where Δx is an infinitesimal latitude, was obtained.

A very few 18^{th}-century mathematicians were aware that CAVALIERI had been rather scrupulous in trying to untie the Gordian knot for finding a new method of integration as exact as the Greek method of exhaustion. Thus, in his introduction to *A Treatise of Fluxions*, COLIN MACLAURIN wrote about CAVALIERI:

> In proposing it [the method of indivisibles] he strove to avoid supposing the magnitude to consist of indivisible parts and to abstract from the contemplation of infinity ... (MACLAURIN 1742, pp. 38–39]).

However, such a statement did not disturb the general opinion of indivisibles; we can therefore observe that the 18^{th}-century treatment of CAVALIERI's method contributed much to cement the idea that it was based on the assumption $F = \Sigma 1$.

X.5. All earlier confusion about CAVALIERI's method is reflected in historical expositions from this century. The method has been explained by setting $F = \Sigma 1$ or $F = \Sigma 1 \cdot \Delta x$ and by arithmetical calculations. In 1941 BOYER pointed out the great difference between CAVALIERI's method and methods, like the arithmetical ones, which discarded certain terms. Thus BOYER brought some clarity into the understanding of CAVALIERI's method; however he retained the interpretation of CAVALIERI's *omnes*-concepts as sums, and so did most writers on the subject.

In recent years this view has been criticized and modified; thus GIUSTI presents CAVALIERI's collections of lines as magnitudes (GIUSTI 1980, pp. 33–38) as I have done in this paper and as has been done also in an earlier publication (PEDERSEN 1980, pp. 32–36). GIUSTI has further explained how CAVALIERI's lack of explicitness made the interpretation of the *omnes*-concepts as sums possible (GIUSTI 1980, pp. 38–39), a point which has also been touched upon in Section VIII.6. In his comments accompanying the Italian translation of *Geometria* LOMBARDO-RADICE often stressed that "all the lines" was conceptionally far from a sum. He preferred to see the *omnes*-concept as a precursor of CANTOR's concept of a set (LOMBARDO-RADICE 1966, *e.g.* p. 194). It is true, as shown in Section III.5, that "all the lines" form a set, but that does not necessarily mean that CAVALIERI had any explicit understanding of the concept of set.

Altogether, CAVALIERI's concepts were so special and had so little direct influence on further development that it does not make much sense to relate them to later concepts. The comparisons between CAVALIERI's various concepts and concepts like the LEIBNIZian integral, the CAUCHY integral, and NEWTON's fluxions which have been made particularly by certain Italian historians of science, therefore do not have much historical value; at most they can have interest as examples of how the way of thought has changed in the course of time.

Concluding remarks

The preceding pages contain many details about CAVALIERI's theory and quite some calculations, although only a small part of those CAVALIERI carried out. My presentation of the topic has been circumstantial because I am of the opinion that only through familiarity with CAVALIERI's concepts and techniques is it possible to understand how elaborate and special his method was. To understand this is essential for an evaluation of the method, and particularly for seeing its role in the transition from the Greek method of exhaustion to the LEIBNIZian calculus in the proper perspective.

Rounding off this paper I shall focus on three points which are central for a proper historical perspective on CAVALIERI'S method of indivisibles. The first point is that in the transition process CAVALIERI had a very isolated position. He was the only one among the leading 17^{th}-century mathematicians who attempted to extend the Greek theory of magnitudes to quantities involving infinitely many elements, *viz.* the *omnes*-concepts, and thus the only one to create a new method of integration which was not a complete break with Greek tradition.

The second point is that CAVALIERI, despite his isolation, had an influence on the development of more heuristic methods of integration; an influence he gained by inspiring TORRICELLI to create his own version of the method of indivisibles.

The third point is that although the content of *Geometria* in general was very little known, the mere fact that the book existed stimulated several mathematicians to write down their own investigations concerning new methods of integration. In this way CAVALIERI had an indirect influence on the acceptance of infinitesimal methods, although he carefully avoided the use of infinitesimals.

In reviewing CAVALIERI'S method it is also important to observe that his principle had a long life and survived the creation of the calculus. It has now disappeared from mathematical textbooks, but is still used in less rigorous deductions.

My last remark will concern the name CAVALIERI chose for his theory: *the method of indivisibles.* If the reader wonders why CAVALIERI used that expression at all, I have gained one of the aims of this paper, namely to show that CAVALIERI'S method is independent of theories concerning the composition of the continuum; it just happened that he decided to use the term indivisibles as an alternative for "all the lines" and "all the planes".

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List of Symbols

$\mathcal{O}_F(l)$	"all the lines" introduced in Section III.1, transcribed as $\int_0^a l(t)\,dt$ in Section III.6.
$\mathcal{O}_S(p)$	"all the planes" introduced in Section IV.1.
$\mathcal{O}_F(A(l))$	"all similar plane figures" belonging to the figure F, introduced in Section IV.1.

$\mathcal{O}_{F_1,F_2}(l \times l)$ "all the rectangles" of F_1 and F_2, introduced in Section IV.2.
$\mathcal{O}_{ON}(\alpha)$ "all the abscissae" of the line segment ON, introduced in Section IV.3.
$\mathcal{O}_F(c)$ "all the circumferences", introduced in Section VIII.2.
$\mathcal{O}_F(l^n)$ "all the powers", introduced in Section VIII.5.
The *ut-unum* principle, presented in Section V.2.
The generalized *ut-unum* principle, presented in Section VI.3.

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