Archimedes and the Measurement of the Circle: A New Interpretation

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In the third proposition of the *Dimensio Circuli*, ARCHIMEDES established $3\frac{1}{7}$ as an upper bound, $3\frac{10}{71}$ as a lower bound for the ratio of the perimeter to the diameter of the circle. EUTOCIUS remarks that APOLLONIUS derived bounds even more accurate.[1] I shall argue in the present study that ARCHIMEDES himself introduced such refinements, yielding bounds at least twenty times more accurate than those in the *Dimensio Circuli*. A close examination of these computations will affirm that the classical Greek mathematics, dominated by an interest in theoretical geometry, yet included expertise in practical arithmetic.

HERO reports as follows of ARCHIMEDES' measurement of the circle (*Metrica* I, 25):

> Archimedes proves in his work on plinthides and cylinders that of every circle the perimeter has to the diameter a greater ratio than 211875:67441, but a lesser ratio than 197888:62351. But since these numbers are not well-suited for practical measurements, they are brought down to very small numbers, such as 22:7.[2]

As is evident from Plate I, the manuscript forms of the numerals are clear. They reappear in the lower margin of the same folio. Yet the approximations to π derived from them are disappointing.[3] The fraction alleged to be a lower bound, 211875/67441 (=3.14163491...), is in fact an upper bound. Moreover, the upper

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[1] A critical Greek text of the *Dimensio Circuli* has been edited by J.L. HEIBERG in ARCHIMEDES, *Opera*, I, pp. 231–243. English translations and commentary appear in the editions by T.L. HEATH and E.J. DIJKSTERHUIS. For EUTOCIUS' commentary, see J.L. HEIBERG, *Archimedis Opera*, III, pp. 227–61; esp. p. 258 for his remark on APOLLONIUS.

[2] I have translated from the Greek text edited by H. SCHÖNE, pp. 64–66; *cf.* also E.M. BRUINS, *Codex*, II, p. 105.

[3] The Greeks never adopted a special symbol to denote the constant we represent as π. In ARCHIMEDES, HERO, PTOLEMY and others it is discussed always as "the ratio of the perimeter to the diameter of the circle." It is applied always as a *ratio*, never as a number (in the modern sense of "real number"). With these reservations explicit, there should be no objection to our use of the symbol π in the present discussion.

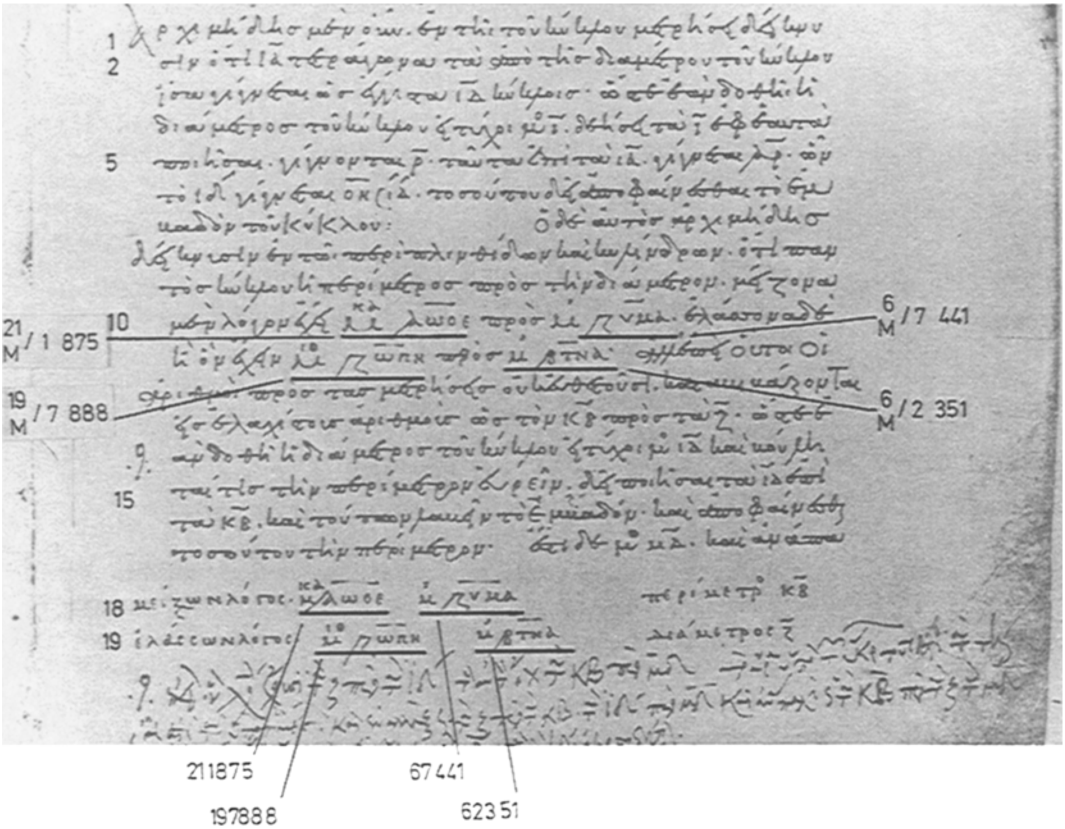

Plate I. From *Codex Constantinopolitanus*, f. 81r (reprinted by permission of E.M. BRUINS). In lines 1–7 HERO paraphrases ARCHIMEDES' *Dimensio Circuli*, prop. 2: that eleven times the square of the diameter equals fourteen times the area of the circle; he illustrates this rule in the case of the circle of diameter 10. In lines 7–13 HERO reports the refined ARCHIMEDEAN limits on the ratio of the circumference and the diameter; this ratio is said to be greater than 211875 ($^{\kappa\alpha}\mu,\alpha\omega o\epsilon$) to 67441 ($^{\varsigma}\mu,\zeta\nu\mu\alpha$) (line 10), less than 197888 ($^{\iota\theta}\mu,\zeta\omega\pi\eta$) to 62351 ($^{\varsigma}\mu,\beta\tau\nu\alpha$) (line 11). [One may note how the manuscript numerals differ in form from the modern typed convention.] HERO notes (lines 11–13) that these values are in practice reduced to small numbers, such as 22:7. In lines 14–17 he computes the perimeter of the circle of diameter 14 by means of the less accurate constant, 22/7. In lines 18 and 19 (at the left) the refined ratios are repeated as a marginal note; but the words "greater ratio" and "lesser ratio" prefacing them reverse the order of magnitude. At the far right of these lines are the words "perimeter 22" and "diameter 7", referring to the less accurate reduced value.

bound 197888/62351 (=3.17377...) is rather poorer an approximation than ARCHIMEDES' extant value $3\frac{1}{7}$ (=3.1428...).[4]

What right did we have to expect better? In the *Dimensio Circuli* ARCHIMEDES computes by means of the circumscribed polygon of 96 sides that $14688/4673\frac{1}{2} > \pi$;

[4] These may be compared with the approximation 3.141592654, correct to ten decimal places. The decimal equivalents are provided to facilitate comparisons. As ARCHIMEDES and HERO employed only rational (as opposed to radix) notations for fractional parts, we will later abandon the decimal notation and examine approximations strictly in terms of the ratios of integers. In Table 3 the decimal equivalents of the values discussed are listed for reference. (See p. 130.)

it follows *a fortiori* that $3\frac{1}{7}$ is an upper bound. The fractional error entailed by the former ratio is less than 1 part in 2546, or 1.84 in $4673\frac{1}{2}$. Similarly, ARCHIMEDES computes from the inscribed 96-gon that $6336/2017\frac{1}{4}$ is a lower bound; the fractional error is less than 1 part in 4599, or 0.44 in $2017\frac{1}{4}$. This indicates that the magnitude of the computed denominator is an index of the accuracy of the derived approximation. By analogy, we should expect that the ratios cited by HERO are not significantly more or less accurate than 1 part in 67441 and 62351, respectively, That the cited upper bound differs from π by more than 638 parts in 62351 thus points to a corruption in the text.

In the second place, the text-figures are inconsistent with HERO's remark that the large numbers "are brought down to very small numbers, such as 22:7". Under this procedure, a given ratio is replaced by one very nearly equal to it, but expressed in smaller numbers.[5] Such a step is employed by ARCHIMEDES in the *Dimensio Circuli*. The computed upper bound $14688/4673\frac{1}{2}$ is rounded off *upward* to $3\frac{1}{7}$; the computed lower bound $6336/2017\frac{1}{4}$ is rounded off *downward* to $3\frac{10}{71}$. By analogy with this procedure, the ratio 197888/62351, which lies between $3\frac{1}{5}$ and $3\frac{1}{6}$, ought to be rounded off to $3\frac{1}{5}$, an upper bound. Since the text treats 211875/67441 as a lower bound, this ought to be rounded off downward to $3\frac{1}{8}$. In neither case will the derived reduced value be $3\frac{1}{7}$, despite the fact that HERO explicitly cites this value.

Given the modest accuracy entailed by the text-ratios, their terms are disproportionately large. The ratios, as shown above, only justify the bounding inequalities $3\frac{1}{8} < \pi < 3\frac{1}{5}$. But in the *Dimensio Circuli* ARCHIMEDES obtains the equivalent accuracy by means of the inscribed and circumscribed 24-gons, leading to the inequalities $3\frac{1}{8} < \frac{24 \times 240}{1838\frac{9}{11}} < \pi < \frac{24 \times 153}{1162\frac{1}{8}} < 3\frac{1}{6}$. Thus, were the text-figures reported by HERO correct, the upper bound denominator of 62351 would be more than 53 times larger than necessary, while the lower bound denominator of 67441 would be more than 36 times larger than needed. This is a clear indication that any computation leading to the values reported by HERO must have differed radically from that of the *Dimensio Circuli*.

Third, the text is mistaken in claiming 211875/67441 to be a lower bound for π. Thus, if we accept the text as given, we must also accept a plain error in ARCHIMEDES' organization of the computation: an inadequate regard for the *direction* of rounding off at each stage in the process of computing. In the general way of deriving approximations, rounding off is effected to nearest whole-values, either upward or downward, and the final estimate may benefit from a favorable cancellation of errors. But in the computations given by ARCHIMEDES and those implied by HERO's report, the objective is to establish values as upper and lower bounds. This is a stronger result than mere approximation and necessitates control of the direction as well as the amount of all errors incurred in rounding off. That so able

[5] This procedure is to be effected by means of the EUCLIDEAN division algorithm, equivalent to the expansion and truncation of the continued fraction for the given ratio. EUCLID uses this algorithm for determining the greatest common measure of given integers (*Elements*, VII, 1–3). One may thus see the relation between the "reduction" cited by HERO and the more familiar "reduction to lowest terms."

a geometer and arithmetician as ARCHIMEDES should have been guilty of oversight on such a fundamental point is a charge that I, for one, care not to level.

These considerations lead me to conclude that the figures given by HERO are corruptions of those stemming from a more refined ARCHIMEDEAN computation. The project of discovering suitable emendations has already attracted scholarly comment. P. TANNERY has changed the upper bound into 195882/62351, the lower bound into 211872/67441. E. HOPPE has replaced the upper bound numerator by 195883, the lower bound numerator by 211871.[6] Certain features of their method are worthy of note.

It would appear that TANNERY obtained his values via the computations $67441\times\pi(=211872.15\ldots)$ and $62351\times\pi(=195881.44\ldots)$, using an approximation to π accurate to 9 or more places.[7] His method is arbitrary, in that he does not relate these particular figures to the computational procedure of the *Dimensio Circuli.* TANNERY notes that the emended ratios, when expanded as continued fractions and truncated, come into agreement with the value 355/113 used by ADRIAEN METIUS in 1625.[8] Except for their common implicit application of the EUCLIDEAN division, the relevance of METIUS for a computation by ARCHIMEDES is not clear. But other considerations make TANNERY's view untenable. As we have said, the expected fractional error in the ratios given by HERO is on the order of 1 part in 60,000. TANNERY's bounds are much closer than this: the upper bound differs from π by less than 1 part in 352,038 and the lower bound by less than 1 in 1,410,683. Thus, a consequence of his view is that HERO's values are not "rough" figures, as we have until now assumed them to be, but are the result of a

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[6] HEIBERG proposes the emendations 211875/67444 and 195888/62351 (*Archimedis Opera*, II, p. 542; *cf.* T. L. HEATH, *Greek Mathematics*, I, p. 233). These are defended by textual, but not mathematical, considerations.

[7] TANNERY, *Mémoires scientifiques*, III, pp. 149, 198–200. TANNERY does not actually specify the way he produced the emended numerators.

[8] METIUS ascribes to his father (ADRIAEN ANTHONISZOON) a pamphlet *(libellus)* in which he proved via "Archimedean demonstrations" that $\pi<3\frac{17}{120}$ and $\pi>3\frac{15}{106}$; as the value $3\frac{16}{113}$, or $\frac{355}{113}$, lies between these, he says this was thus adopted as the approximation for π. METIUS remarks further that this approximation errs in excess by less than 10^{-6}, as he knows from a comparison with the value computed by LUDOLPH VAN CEULEN [*Practica Geometriae*, 1625, pp. 88f, 178f]. This LUDOLPHINE value must have been the 20-place approximation published in 1596; his 32-place value was first published in 1615, after the first edition of METIUS' *Geometria* (1611). In 1584 SYMON VAN DER EYCKE published an alleged quadrature of the circle; at the instigation of ADRIAEN ANTHONISZOON, LUDOLPH refuted it, computing the bounds 3.141557587 and 3.141662746 via the ARCHIMEDEAN method of inscribed and circumscribed polygons ["Proefsteen", 1586; *cf.* B. DE HAAN, pp. 106–113, 121–124]. It would thus appear that ADRIAEN deduced from LUDOLPH's result the bounds $3\frac{15}{106}$ and $3\frac{17}{120}$ by means of the EUCLIDEAN division or an equivalent, and then proposed the intermediate value $3\frac{16}{113}$ as an appropriate approximation for practice. According to the division algorithm, this is the fraction $3+\frac{1}{7+}\frac{1}{16}$, intermediate between the proven bounds, $3+\frac{1}{7+}\frac{1}{15}$ and $3+\frac{1}{7+}\frac{1}{17}$, respectively. ADRIAEN's procedure does not establish how much closer a bound $3\frac{16}{113}$ is, or whether it errs by excess or by defect. That it is an upper bound, differing by less than 10^{-6}, may be deduced from LUDOLPH's later computations; for instance, the inequality $\pi<3.14159281$, established by the circumscribed 10240-gon, suffices [LUDOLPH, *De Circulo*, pp. 28f]. It is of interest to note that slightly earlier (1573) the same value $\frac{355}{113}$ was rigorously established as an upper bound for π by VALENTIN OTHO, by means of a computation accurate to nine decimal places [TROPFKE, *Geschichte*, 1923, IV, pp. 217f; CURTZE, 1895, p. 13]. The same value was employed even earlier by the Chinese (c. fifth century) [MIKAMI, 1909/10, pp. 195f]. The coincidence of these results indicates their common application both of the polygonal method and of the division algorithm as a technique for reduction.

reduction of rough values in still larger terms. If we follow TANNERY's suggestion and expand the continued fraction of 195882/62351, then truncate it, we find that 2862/911 is obtained as a convergent. This yields an approximation to π with fractional error less than 1 part in 314,788. But if the former ratio was itself the result of such a truncation process, it is a puzzle why the process was carried through to the equivalent of six quotients of the continued fraction, when virtually the same order of accuracy could be obtained by truncating after four. There is, finally, an irony in TANNERY's introduction of the METIUS-value. Although 195882/62351 is too accurate to be a "rough" figure in the context of the passage, it is not accurate enough to establish 355/113 as an upper bound for π. For the fractional error of this last ratio is less than 1 part in 11,810,498. These remarks show that TANNERY's casual comments on this passage entail complications he was but slightly aware of.[9]

HOPPE avoids such difficulties by developing his emendation from an estimate of the accuracy of the ratios. He argues that ARCHIMEDES employed the circumscribed and inscribed 384-gons in his computation, from which values accurate to four decimal places would be attainable.[10] On working out this computation, he found that ARCHIMEDES' initial values for $\sqrt{3}$, as given in the *Dimensio Circuli*, were not quite accurate enough for this process; but when he assumed somewhat better starting values, he found he could obtain suitably accurate bounds for π: namely, 3.141575 and 3.141617. He then supposed that HERO's report of the denominators was correct and thus deduced as numerators 211871 and 195883, respectively. We may raise a number of criticisms against HOPPE's argument. First, it suffers from a strong element of arbitrariness. Why, for instance, should we assume that the reported denominators are correct? If we adopt the numerators instead, or allow for corruptions in both numerators and denominators, we might discover many more values from which to choose. Second, the emendations HOPPE requires are not well attested in the extant mathematical manuscripts. Instances of the corruption of $\alpha(=1)$ into $\varepsilon(=5)$, necessary to change 21187$\underline{1}$ into 211875, are found. But I know of no instances of the corruption of ε into $\zeta(=7)$ or of $\gamma(=3)$ into $\eta(=8)$, required to change 19$\underline{5}$88$\underline{3}$ into 197888. HOPPE's judgment on this matter is guided by the numeral forms conventional in modern printed editions of the Greek texts. But as we shall discuss further below, the manner of writing numerals actually employed in the manuscripts could be quite different. Third, HOPPE's grounds for introducing the 384-gons are rather dubious. He

[9] $195882/62351 = 3 + \frac{1}{7+}\frac{1}{16-}\frac{1}{8+}\frac{1}{1+}\frac{1}{60}$; $2862/911 = 3 + \frac{1}{7+}\frac{1}{16+}\frac{1}{8}$. Accepting this consequence, that the fractions cited by HERO might themselves have been "reduced" from fractions in even larger terms, I inquired whether a fraction resembling one of the HERONIAN fractions might be found to lie between π and 355/113, thus to verify the latter as an upper bound for π. A computer program was devised to determine all such intermediate fractions, for integral denominator ranging from 60,000 to 75,000. This yielded 207 values, of which only about a dozen bore any near resemblance to HERO's figures, in most cases requiring three or more mistranscriptions plus changes in the ordering of digits. The value of greatest interest was 211891/67447; for this would demand only three alterations of the text-value 211875/67441, a number within the range of scribal error. But on closer inspection, the value proves to be too good, entailing a fractional error of less than 1 part in 785,000,000. We should then be establishing a bound over 70 times more accurate than necessary to verify 355/113 as an upper bound. I thus conclude that the HERONIAN figures do not point to a calculation which successfully established 355/113 as an upper bound. But we shall take up an alternative approach to this problem below.

[10] E. HOPPE, "Zweite Methode des Archimedes", pp. 104–7.

refers to the comment of a late Indian mathematician GANECA to the effect that ARYABHATA (b. 476 A.D.) obtained the approximations 3927/1250 and 754/240 for π via inscribed and circumscribed 384-gons.[11] As both values (equivalent respectively to 3.1416 and 3.14167) are upper bounds, the relevance of *inscribed* polygons is not clear. But more important, the suggestion of a direct influence of ARCHIMEDES or HERO on ARYABHATA would require very careful justification. As we shall argue, the existence of any such influence has no bearing on our understanding of the HERO-passage; for the key to the problem lies in ARCHIMEDES' *Dimensio Circuli*, not in the later Indian arithmetic.

Disdainful of such efforts at emendation, E.M. BRUINS has opted to accept HERO's text exactly as given.[12] As his approach ignores the incongruity of the text-ratios, our earlier arguments already undermine his attempt. For instance, BRUINS' premisses that "great numbers do not necessarily indicate high accuracy" and that "the context gives the strong impression that the big numbers were used only to derive a first rough approximation, as 22:7" are, as we have shown, untenable and inconsistent. But as BRUINS presumes to supply the details of a computation leading to the manuscript figures, a few additional remarks are justified.

BRUINS' ability to construct a complete computation might unduly impress the unwary.[13] But there are at least two major technical difficulties entailed by it. First, as the computation produces the text values from the inscribed and circumscribed 16-gons, he must begin with bounding values for $\sqrt{2}$. Those BRUINS adopts correspond to the inequalities $1093/773<\sqrt{2}<1104/780$. These may be obtained, as BRUINS shows, via a "pythmen" technique facilitated by adroit use of the fact that square numbers are formed by the successive summation of consecutive odds. Here, in effect, one begins with the estimate $11/8<\sqrt{2}<10/7$; multiplying the numerators by 10, one seeks the best denominators within that rank, so determining $110/78<\sqrt{2}<110/77$; then the numerators are corrected to yield $110/78<\sqrt{2}<109/77$; continuing to the next decimal order, similar adjustments at last yield $1103/780<\sqrt{2}<1104/780$ and $1093/773<\sqrt{2}<1094/773$. Now, I will admit that the Greeks were capable of developing such a method, in principle.[14] But would ARCHIMEDES have troubled himself to obtain bounds to the root in this way? Greek arithmeticians, at least from the late fifth century B.C., were in possession of the much more efficient method, called the "side and diameter" numbers.[15] Here, the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \ldots, \frac{a}{b}, \frac{a+2b}{a+b}, \ldots$$

[11] For details on the Indian mathematics, HOPPE follows CANTOR [*Geschichte*, I, pp. 646, 654].

[12] *Codex Constantinopolitanus*, III, pp. 245ff.

[13] BRUINS' computation appeared in somewhat more detail in his Dutch articles of 1943 and 1946. I have seen only excerpts from them, but most of the argument has been communicated to me in private correspondence by the author.

[14] BRUINS asserts that this method is to be found in the Babylonian texts and the Greek papyri. He gives no specific references. But I find in NEUGEBAUER's studies of Babylonian mathematics no evidence of such a procedure. Moreover, the Greek methods of root-extraction – namely, THEON of ALEXANDRIA's long-method, HERO's approximation rule, and the PYTHAGOREAN "side and diameter" numbers – have no relation to the method proposed by BRUINS. Explications of the Greek methods are given by HEATH, *op. cit.*, I, pp. 60–3, II, pp. 323–6.

[15] References to the texts on "side and diameter" numbers in THEON OF SMYRNA, IAMBLICHUS and PROCLUS are given by HEATH, *op. cit.*, I, pp. 91–93.

is generated, in which the values alternate as upper and lower bounds for $\sqrt{2}$.[16] The inequalities

$$\frac{1093}{773} < \frac{239}{169} < \sqrt{2} < \frac{99}{70} < \frac{1104}{780}$$

show that the "side and diameter" rule readily provides better estimates in lower terms than does the procedure elaborated by BRUINS. Furthermore, ARCHIMEDES' values for $\sqrt{3}$ in the *Dimensio Circuli* ($1351/780 > \sqrt{3} > 265/153$), like the fractions in the "side and diameter" sequence, coincide with the convergents of the respective continued fraction developments.[17] This further weakens the view that he should have employed the inefficient procedure suggested by BRUINS.

Second, BRUINS must obtain 211875/67441 (an upper bound for π) by a computation involving *inscribed* polygons. As we have seen, this is possible only if the direction of rounding off is handled improperly. By contrast, rounding off in the *Dimensio Circuli* is effected in the correct sense without exception. BRUINS evades this objection by persistently emphasizing the crudeness of the computation underlying the values given by HERO; he treats these as the work of a hypothetical "young" ARCHIMEDES, later to be supplanted by the mature author of the *Dimensio Circuli*. But this will hardly do. There is no indication that the *Dimensio Circuli* was written late in the sequence of ARCHIMEDEAN treatises.[18] Those works generally accepted as early (such as *De Sphaera et Cylindro* I and *Quadratura Parabolae*) already reveal on ARCHIMEDES' part a complete grasp of rigorous and formal geometrical argument.[19] Further, would not the mature ARCHIMEDES have striven to suppress such an embarrassingly flawed youthful effort? Yet if HERO was able to quote from the tract "On Plinthides and Cylinders", it must have had to remain in circulation under ARCHIMEDES' name for at least three centuries after his death. Perhaps, then, the "Plinthides" was a later forgery – and perhaps HERO, although a competent mathematician and editor, was easily duped by it. But speculation on such questions is completely otiose. They disappear as soon as we acknowledge that the figures given by HERO are corrupt and need to be emended.

Appropriate emendation is not difficult. First, the given fraction 211875/67441 is in fact quite a good approximation to π. It differs by just more than $\frac{9}{10}$ part in 67441, a fractional error precisely of the magnitude we should have expected. It is also the *best* approximation to π found in any extant Greek mathematical work. Unfortunately, HERO seems to treat it as a lower bound although it exceeds π. I deem it rash to discard such a value, to seek a different value, or to belittle the computation which produced it. The simplest resolution of this difficulty is to accept the figure as given, but as an *upper* bound, and to suggest that in the course

[16] The continued fraction for $\sqrt{2}$ is $1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\dots$, all subsequent quotients being 2 *ad infinitum*. The convergent fractions, *i.e.*, those obtained via successive truncation of the continued fraction, are precisely those formed via the "side and diameter" rule.

[17] For further details, see the Appendix.

[18] I intend to criticize in a separate note the claim of such editors as HEIBERG, HEATH and DIJKSTERHUIS, that the *Dimensio Circuli* was a relatively late ARCHIMEDEAN tract.

[19] B. L. VAN DER WAERDEN, for instance, describes the arguments of the *Quadratura Parabolae* as "elegant" [*Science Awakening*, p. 218]. HEATH judges that the "treatises are, *without exception*, monuments of mathematical exposition" [*op. cit.*, II, p. 20; italics are mine].

of HERO's writing of the *Metrica* or its subsequent transcriptions the two bounding ratios came to be interchanged.

This leaves the fraction 197888/62351 as a corruption of the original lower bound. As we have seen, most commentators have chosen to retain HERO's denominator and alter the numerator. But the numerator has an overriding claim to be the correct figure: it factors into $2^8 \times 773$ – suggestive of its origin as the value of the perimeter of a regular polygon. What of the denominator? The value 62989 yields an upper bound; 62990 yields a lower bound, as do all larger integral values. The fractional error entailed by the ratio 197888/62991 is just less than 1.3 parts in 62991; moreover, the replacement of 62991 by 62351 requires us to account for the corruption of only two digits.

Following HERO's suggestion, we may "bring down" the terms of these fractions – that is, by means of the EUCLIDEAN division. The successive quotients of 197888/62991 are 3, 7, 15, 4, 1, ..., which when truncated after three terms yield $333/106 (=3+\frac{1}{7+\frac{1}{15}})$ as *a fortiori* a lower bound for π. Similarly, the quotients derived for 211875/67441 are 3, 7, 16, 1, 1, 4, ... and these, when truncated after four terms yield 377/120 as an upper bound. To summarize, we take HERO's passage to refer to the following "rough" inequalities: $197888/62991 < \pi <$

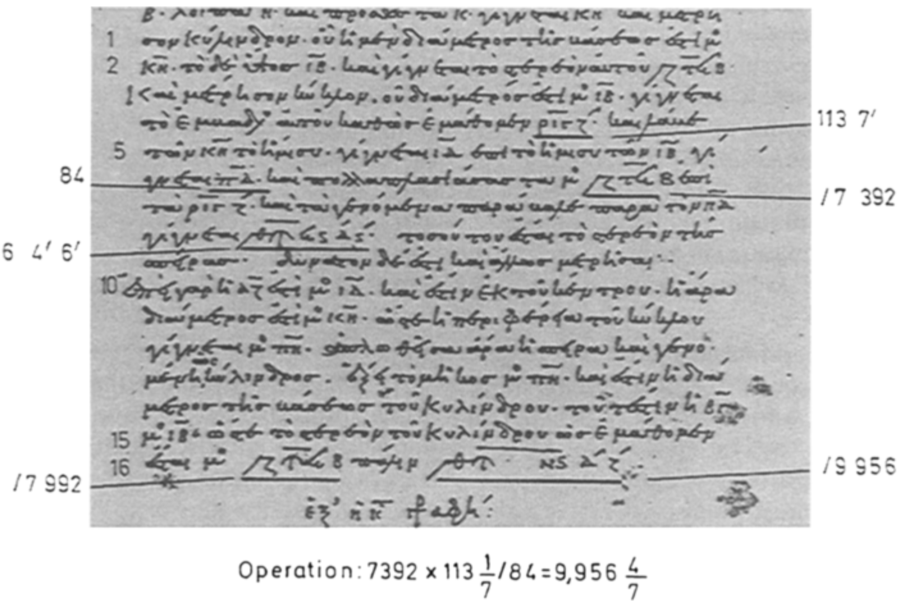

Plate II. From *Codex Constantinopolitanus*, f. 96r (reprinted by permission of E. M. BRUINS). This passage contains part of a computation of the volume of a torus. The cylinder of diameter 28 and height 12 has volume 7392 (,ζτqβ) (line 2); the circle of diameter 12 has area $113\frac{1}{7}$(ριγζ') (lines 3–4). HERO next forms $\frac{1}{2}28 \times \frac{1}{2}12 = 84$; he then finds the volume of the torus as $7392 \times 113\frac{1}{7}/84$ (lines 5–8). But the result is incorrectly reported as $9996\frac{4}{6}$ (,θϡqϛδ'ϛ') in line 8. The same computation is effected in a different order in lines 9–16. Here, the end-result is correctly given as $9956\frac{4}{7}$ (,θϡνϛδ'ζ') at the end of line 16; but the intermediate term is mistranscribed as 7992 (,ζϡqβ) earlier in the same line. This passage thus illustrates that in the orthography employed in the *Codex* or its prototypes, the characters for 900 and 300 (ϡ, τ in modern conventional type-face) can be confused, as can the characters for 90 and 50 (q, ν).

211875/67441. In ARCHIMEDES' treatment, these were rounded off without substantial loss of accuracy, so as to yield the inequalities $333/106<\pi<377/120$.[20]

We now defend these emendations by considering two issues: (1) Do they find support in corruptions which are actually detectable in the manuscript of the *Metrica?* (2) Can an ARCHIMEDEAN computation be framed which produces the emended values?

In answer to (1) we must decide whether 62991 could be miscopied as 62351; that is, whether the figure ϡ (=900) may have been corrupted to τ (=300) and the figure ϙ (=90) corrupted into ν (=50). In a later section of the *Metrica* (*Codex Constantinopolitanus*, fol. 96r) we encounter a computation arising from the measurement of the volume of the torus. A portion of the page is reproduced in Plate II. Here, the operation $7392\times 113\frac{1}{7}/84=9956\frac{4}{7}$ is set up (lines 1–9); the same operation is then set up again in a different order (lines 9–16). The numeral 7392 is written correctly in lines 2 and 6; but it is miscopied as 7$\underline{9}$92 in line 16. Thus, the numerals for 900 and 300 have been confused. The result of computation is given correctly as $9956\frac{4}{7}$ in line 16; but it is given incorrectly as $99\underline{9}6\frac{4}{6}$ in line 8. Here, the numerals for 90 and 50 (as well as those for 6 and 7) have been confused. The close resemblance of the forms ϡ and τ make readily understandable how they might be interchanged; in fact, their confusion is a common error in mathematical manuscripts.[21] The confusion of ϙ and ν seems less easy to account for. In slightly earlier orthographies they are figured respectively as ч and ᴎ—that is, one is the inverse image of the other, so that the motions of writing them are closely related.[22] At any rate, the manuscript of the *Metrica* confirms that the alterations which might change 62991 into 62351 are quite within the range of admissible scribal error.

What of the reversal of the order of the ratios? Anyone who has worked with inequalities in practice will admit how easily such reversals can occur. But we have in the *Codex* itself an indication of how the reversal might have been made in the present case. In the lower margin of the same page (fol. 81r) containing the HERO-passage, the following scholium is attached: "greater ratio 211875 67441"

[20] The continued fraction expansion of π is $3+\frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\frac{1}{1+}\frac{1}{1+}\cdots$. Successive truncation gives rise to the following sequence of fractional approximations, alternating lower and upper bounds: $\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \cdots$. Additional materials on the continued fraction for π are given by PERRON (1913, pp. 61–63) and JOLLIFFE (1910, p. 31).

[21] In his critical edition of EUTOCIUS' commentary on the *Dimensio Circuli* (*Archimedis Opera*, III, pp. 234–256), HEIBERG indicates 90 instances of the mistranscription of one numeral for another; six of these involve the confusion between *tau* (300) and *sampi* (900).

[22] On these alternative forms of *nu* (50) and *koppa* (90) in the papyri, see the reproduction from Michigan Papyrus No. 621 by L. C. KARPINSKI, 1923; all 27 numerals appear in this plate. This unusual mistranscription suggests to me a possible link between the ARCHIMEDES-passage (*Metrica* I, 25) and the torus-passage (*Metrica* II, 13). The latter is derived by HERO from DIONYSODORUS and is intended to facilitate the measurement of architectural columns. The former is taken by HERO from a work on "plinthides and cylinders"; here the word *plinthide* derives from *plinthos*, the term for the base of a column, while one might well associate the shaft of a column with the cylinder. Now, EUTOCIUS knew of DIONYSODORUS as a commentator on ARCHIMEDES; for he cites DIONYSODORUS' construction filling a gap in the received argument of *De Sphaera et Cylindro* II, 4 [*cf.* T. L. HEATH, *Archimedes*, pp. 66, 72–4]. If, then, HERO knew of ARCHIMEDES' work on "plinthides" through a work by DIONYSODORUS, the mistranscriptions under discussion may have been introduced into the *Metrica* by HERO himself following a corrupted DIONYSODORAN manuscript. On DIONYSODORUS, see HEATH, *Greek Mathematics*, II, pp. 218f.

and directly below this: "lesser ratio 197888 62351". These words appear to claim that 211875/67441 has a greater value, 197888/67441 a lesser value, the comparison in either case being made with the ratio of the circumference and diameter of the circle. But this reverses the order of the ratios as given in the text proper; for there 211875/67441 is claimed as the lower bound on π and 197888/62351 the upper bound. Of course, by assistance from the text, the scholium might be read as an ellipsis for "[the ratio of the circumference to the diameter is a] greater ratio [than that which] 211875 [has to] 67441" and similarly for the lesser ratio. But, taken in isolation, the words do not recommend this as the sense.

Now, the copyist has maintained an even spacing in setting the scholium in the lower margin; this merely indicates that he has dutifully copied it from his reference manuscript after he completed the page. But what was the significance of the scholium in the earlier manuscript? Its introduction in itself implies that a user of that manuscript (or, of course, some even earlier prototype) wished to call attention to these ratios. Why? Let it be supposed that the ratios as I have emended them appeared in the original manuscripts of the *Metrica* and that at some later stage of the manuscript tradition the figure 62991 was corrupted to 62351. Although the ratios were never intended for use in practical geometry, and although the ARCHIMEDEAN and HERONIAN texts by which the corrupted figure might be detected and corrected were soon inaccessible, one anomalous feature of the altered passage could be easily seen: the claimed (corrupt) lower bound 197888/62351 was actually greater than the claimed upper bound 211875/67441. In my view, the scholium was written at the time this anomaly first became apparent. It was thus appropriate to reverse the order of the ratios. The resultant corrupted text, identical now with that of the *Codex Constantinopolitanus*, would be deemed consistent with the other approximations to π then available. The ratio 197888/62351 is indeed an upper bound, as it exceeds the ARCHIMEDEAN upper bound $3\frac{1}{7}$. The ratio 211875/67441 could be compared with the PTOLEMAIC value 377/120; if the latter were misconstrued as "the" value of π, then the former would be recognized as a lower bound on it, as we have already seen. In this way, once these errors had crept into the manuscript tradition, the correct value would be irretrievable save by an arduous, and largely pointless, recomputation. To me the marvel is thus not that the text-figures, as extant, have been altered from their original form, but that in a tradition extending from ARCHIMEDES (c. 250 B.C.) through HERO (c. 60 A.D.) to the *Codex Constantinopolitanus* (XI. century) they survived with so few corruptions.

Let us now consider (2), the manner of computation which would lead to the emended text-numbers. A brief review of ARCHIMEDES' arithmetical technique in the *Dimensio Circuli* will make clear the role that the HERONIAN numbers would fill in a refined computation of the same type. The theory behind the computation is straightforward: the areas (and the perimeters) of the inscribed and circumscribed regular polygons tend toward the area (and perimeter) of the circle which they bound as the number of sides increases indefinitely. Both EUCLID (*Elements* XII, 2) and ARCHIMEDES (*Dimensio Circuli*, prop. I) make critical use of the fact that the difference between the areas of the circle and the inscribed polygon of $2n$-many sides is less than half the difference between those of the circle and its inscribed n-gon. ARCHIMEDES establishes the same result for the areas of the circumscribed polygons. Analogous results may be obtained for the perimeters, although these are not given explicitly. Thus, if we set out a sequence of inscribed or circumscribed polygons, each having twice as many sides as the one immediately preceding, then we know that convergence to the area and to the perimeter is faster than successive bisection of the difference at each stage. In fact, a better estimate of the convergence rate is obtainable. In the *Metrica* I, 32 HERO demonstrates that the area of a circular segment is greater than four-thirds the area of the largest inscribed triangle.[23] It follows at once that the rate of convergence of the areas of the polygons to the area of the circle in which they are inscribed is *slower* (but only slightly so) than that of successive quadrisection. A like conclusion holds for their perimeters. Applying a similar analysis to the areas and perimeters of the circumscribed polygons, one finds that they converge *faster* (but only slightly so) than successive quadrisection. This furnishes a rough guide to determine how many sides the inscribed or circumscribed polygons must have to approximate the circle they bound to within a preassigned degree. For instance, to increase the accuracy of the approximation by a factor of 1000 (the equivalent of obtaining three additional accurate decimal places for π), one will require 5 successive doublings of the number of the sides of the polygons (since $1000 \cong 4^5$).

To establish the upper bound, ARCHIMEDES starts with the circumscribed hexagon and an unexplained inequality, $265/153 < \sqrt{3}$. He successively doubles the number of sides four times to obtain that the ratio of the perimeter of the circumscribed 96-gon to the diameter of the circle is less than $14688/4673\frac{1}{2}$. Here, $14688 = 96 \times 153$. Since the circumference of the circle is less than the perimeter of the 96-gon and the fraction is less than $3\frac{1}{7}$, he concludes *a fortiori* that $\pi < 3\frac{1}{7}$. For the lower bound, he adduces the inequality $1351/780 > \sqrt{3}$. Starting from the inscribed hexagon and proceeding as above to the inscribed 96-gon, he obtains that the ratio of the perimeter of the 96-gon to the diameter of the circle exceeds $6336/2017\frac{1}{4}$. Here, $6336 = 96 \times 66$. As the circumference of the circle exceeds the perimeter of the inscribed 96-gon and the fraction is greater than $3\frac{10}{71}$, the latter follows as a lower bound for π.

Concealed within this computation are traces of a technique of approximation based on the EUCLIDEAN division algorithm, which yields results equivalent to continued-fraction expansions. The bounds ARCHIMEDES gives for $\sqrt{3}$, for instance, are obtainable by truncating the division algorithm after 9 and 12 quotients.[24] In rounding off $14688/4673\frac{1}{2}$ his application of such a procedure is explicit. He first observes that the fraction exceeds the value 3 by the amount $667\frac{1}{2}$; the fraction $667\frac{1}{2}/4673\frac{1}{2}$ is in turn exceeded by one-seventh; he so concludes that $3\frac{1}{7}$ is greater than the initial fraction. While ARCHIMEDES does not explain his method of approximating the lower bound in such detail, one finds that the division algorithm for $6336/2017\frac{1}{4}$, when truncated after three quotients, yields ARCHIMEDES' value $3\frac{10}{71} (= 3 + \frac{1}{7 + \frac{1}{10}})$. The use of the division algorithm is instanced in arithmetic work by others, such as the PYTHAGOREANS and ARISTARCHUS.[25] It is certainly beyond dispute that the division algorithm, as a theoretical device, was the foundation of the EUCLIDEAN theory of numbers (*Elements* VII) and the related theory of

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[23] HERO's proof is patterned after ARCHIMEDES' proof of the analogous result for parabolic segments in *Quadratura Parabolae*, 24.

[24] An alternative derivation is given in the Appendix.

[25] For references and discussion, see my *Evolution of the Euclidean Elements*, Chapter VIII, Section II.

irrationals (*Elements* X). But that the Greeks made use of this computational procedure in the practical manipulation of fractions and ratios of integers has not generally been recognized. The examples offered here ought to justify our introduction of the algorithm into parts of the discussion below.

We may presume that at ARCHIMEDES' time the approximation $3\frac{1}{7}$ was recognized in metrical practice.[26] We may suppose also that to frame a formal justification of this approximation ARCHIMEDES could be aware beforehand that $3\frac{1}{7}$ was in fact an upper bound and the amount of difference was on the order of 1/500.[27] This estimate influences his choice of the lower bound to $\sqrt{3}$, namely 265/153 (accurate to within 3×10^{-5}).[28] It was also in his power to predict the number of computations required: since the difference between the circumscribed and inscribed hexagons (assuming unit radius) is $2\sqrt{3}-3$, or slightly less then $\frac{1}{2}$, to reduce this difference to the order of 1/500 would require four successive quadrisections, or the construction of the circumscribed 96-gon through four doublings of the number of sides. In the event, by carefully controlling the degree of rounding-off ARCHIMEDES was just barely able to verify the expected upper bound

Table 1. ARCHIMEDES' Computation of Bounds for π

Upper Bounds				Lower Bounds				
n	A	B	C	n	A'	B'	C'	f
6	153	265	306	6	780	1351	1560	
12	153	571	$591\frac{1}{8}$	12	780	2911	$3013\frac{3}{4}$	
24	153	$1162\frac{1}{8}$	$1172\frac{1}{8}$	24	780	$5924\frac{3}{4}$		4/13
					240	1823	$1828\frac{9}{11}$	
48	153	$2334\frac{1}{4}$	$2339\frac{1}{4}$	48	240	$3661\frac{9}{11}$		11/40
					66	1007	$1009\frac{1}{6}$	
96	153	$4673\frac{1}{2}$		96	66	$2016\frac{1}{6}$	$2017\frac{1}{4}$	

These are the values reported by ARCHIMEDES in *Dimensio Circuli*, prop. 3. They are computed by the rule $B_{2n} = C_n + B_n$, $C_n^2 = A_n^2 + B_n^2$ from suitable initial values of A_6, B_6; the primed terms follow the same rule of formation. The numbers A_n, B_n express the ratio of the side of the circumscribed n-gon to the diameter of the circle (as do the numbers A'_n, B'_n); the numbers A_n, C_n express the ratio of the side of the inscribed n-gon to the diameter of the circle (as do A'_n, C'_n). In approximating the square roots, rounding-off of C is downward, of C' upward. Fractional errors incurred are on the order of 1 or 2 parts in 40,000 for the 12- and 24-gons, between 1 and 4 parts in 600,000 for the 48- and 96-gons.

The value nA_n is an upper bound for the perimeter of the n-gon circumscribed about the circle of diameter B_n; thus, $nA_n/B_n > \pi$ and one may deduce $\pi < 96 \times 153/4673\frac{1}{2} < 3\frac{1}{7}$. The value nA_n is also an upper bound of the perimeter of the n-gon inscribed in the circle of diameter C_n; this fact is useful for the computations reported in Table 4. Analogously, nA'_n is a lower bound for the perimeter of the n-gon circumscribed about the circle of diameter B'_n or inscribed in the circle of diameter C'_n. One deduces that $\pi > 96 \times 66/2017\frac{1}{4} > 3\frac{10}{71}$.

[26] Demotic (Egyptian) mathematical papyri from about the time of ARCHIMEDES employ the value 3. But better approximations were also known. The traditional Egyptian value was $4(\frac{8}{9})^2$, or just less than $3\frac{1}{6}$, while the ancient Babylonians used both 3 and $3\frac{1}{8}$. See R. A. PARKER, *Papyri*, p. 40 and O. NEUGEBAUER, *Exact Sciences*, pp. 46f, 51f, 78.

[27] In planning the *formal* computation, ARCHIMEDES would know beforehand (by what means we cannot say) the bounds to be established. Their difference ($\frac{1}{7} - \frac{10}{71} = \frac{1}{497}$) is thus an index of the degree of accuracy which must be maintained in the computation.

[28] The accuracy of this value may be estimated from the relation $265^2 + 2 = 3 \times 153^2$. Such identities are associated with the "side and diameter" numbers and have a bearing on the method by which ARCHIMEDES derived his estimates for $\sqrt{3}$. See also the Appendix.

by means of the 96-gon. For in rounding off his last result $14688/4673\frac{1}{2}$ upward to $3\frac{1}{7}$, he increases the difference by less than 1/32,713. Similar considerations might guide the planning out of the formal verification of the lower bound.

In Table 1 the figures computed by ARCHIMEDES are given. In each row, the numbers A, B, C specify the ratio of the three sides of a right triangle whose hypotenuse is proportional to C and whose vertex angle is one-half the angle subtended by the side of the circumscribed n-gon. By the PYTHAGOREAN relation, $A^2+B^2=C^2$. ARCHIMEDES demonstrates that if one bisects the vertex angle of a right triangle of legs A, B and hypotenuse C, the legs of the right triangle whose vertex equals the bisected angle have the ratio $(B+C):A$. Hence, in the table, we may keep A constant and derive B_{2n} as B_n+C_n. The same properties hold for the sides associated with the inscribed triangles of sides A', B', C'.

In all computations of C a square root must be extracted and rounded off *downward* to ensure that the ratio nA/B remains an upper bound on π. Similarly, to compute C' square roots are extracted and rounded off *upward* so that nA'/C' remains a lower bound to π. The rounding-off technique which ARCHIMEDES employs is extremely subtle.[29] One indication of his adroitness is that twice in the calculation of the lower bound ARCHIMEDES adjusts the terms of the triangles, by factors of 4/13 and 11/40, respectively, in order to remove fractional remainders and thus facilitate further computation. The *choice* of the fractional remainders $\frac{3}{4}$ and $\frac{9}{11}$ was thus made after considerable forethought. As already explained, the values associated with the 96-gons are finally expanded to produce the bounds $3\frac{1}{7}$ and $3\frac{10}{71}$.

To obtain the refined estimates cited by HERO, we might first attempt merely to extend ARCHIMEDES' computation and examine the bounding approximations so derived. But ARCHIMEDES has introduced such a large fractional error at the 96-gon stage that this effort, however indefinitely extended, could never attain suitable accuracy. If, beginning with the same initial values, we reduce the successive losses incurred in rounding-off, we may attain a degree of accuracy adequate for establishing our emended upper bound by way of the circumscribed 384-gon; the inscribed 768-gon will suffice for the emended lower bound (here, the 384-gon is not sufficient). Alternatively, we may begin with somewhat better approximations to $\sqrt{3}$, as HOPPE does in his treatment. The associated 384-gons will then permit the derivation of both bounds.

But these approaches ignore an important fact: the numerator of the proposed upper bound is $197888=2^8\times 773$ and that of the proposed lower bound is $211875=5^4\times 3\times 113$; the presence of powers of 2 and 5 as factors appropriately reflects the formation of these numbers as the perimeters of regular polygons, where the number of sides n arises from a process of successive doubling.[30] But how is one to

[29] I am including a discussion of ARCHIMEDES' technique of rounding-off in an article in preparation on HERO's rule for square roots.

[30] The appearance of a power of 5 is unexpected, perhaps, but need not be problematic. It may be explained, for instance, via the multiplication by a "myriad" (10,000) and the subsequent removal of the factors of 2 by successive bisection. In the *Dimensio Circuli* ARCHIMEDES modifies the terms in his computation by multiplying by fractions in small terms. Such an adjustment, by repeated multiplication by $\frac{5}{4}$, for instance, could also produce a final numerator divisible by a power of 5. In the computation proposed in Table 2 we discover the convenience of multiplying the final terms by 125/128, thus obtaining a fraction whose terms are integers (rather than integers with fractional remainders) without change in the value of the fraction itself.

account for the presence of the large prime factors 773 and 113, if the initial sides are 780 and 153, respectively? The artificial introduction of such large and unwieldy factors in the course of the computation does not well conform to the structure of ARCHIMEDES' argument in the *Dimensio Circuli*. It is thus clear that the factors 773 and 113 must be divisors of the sides A' and A of the initial triangles. Unfortunately, if we examine the sequence of convergent fractions for $\sqrt{3}$, we find none in the appropriate range of accuracy (*i.e.*, of difference no less than 10^{-10}) which possesses a denominator factorable by either prime. The same holds for the sequence of "side and diameter" numbers, the convergents for $\sqrt{2}$, so that an alternative computation beginning from the square is also ruled out.

We meet success, however, when we choose the decagon as our starting-point. The geometry of the pentagon and decagon is dominated by a ratio, $(\sqrt{5}+1):2$, much studied by the Greeks; they styled it the "extreme and mean ratio", but it is most familiar now as the "golden section".[31] Just as in the case of the "side and diameter" numbers, which approximate $\sqrt{2}$, there is here a very easily computed alternating sequence which converges to this ratio: namely,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots, \frac{a}{b}, \frac{a+b}{a}, \ldots.$$ [32]

We may assume that, just as ARCHIMEDES had at his disposal the "side and diameter" numbers and some analogous sequence for $\sqrt{3}$, so also he knew how to form the related sequence for the extreme and mean ratio, or some equivalent, when occasion demanded. (A more detailed discussion of the method he may have employed is presented in the Appendix.) Within this sequence we find a value 6765/4181, a lower bound to the extreme and mean ratio, whose denominator 4181 is the product 37×113. We recognize this at once as suited to the side of the circumscribed polygons which produce our desired upper bound for π. Corresponding to the initial value $A = 4181$, $C = 2 \times 6765 = 13530$, while $B = 12867\frac{7}{9}$ via the PYTHAGOREAN relation and a small downward rounding-off.

Five terms further on in the sequence there appears the upper bound 75025/46368. While the denominator of this is not exactly divisible by 773, it very nearly is so, and we easily deduce that $75025/46368 < 75045/46380 = 15 \times 5003/60 \times 773$. This suggests adopting $A' = 4 \times 773$ as initial side for the inscribed decagon, and $C' = 2 \times 5003$; via the PYTHAGOREAN relation and a small upward rounding-off, we obtain $B' = 9516\frac{4}{7}$. We thus have suitable initial triangles by which to effect a computation precisely in the manner of ARCHIMEDES in the *Dimensio Circuli*. The result is given in Table 2.

At the 640-gon stage we may examine the bounds derivable for π. The lower bound will be $nA'/C' = 640 \times 3092/629909$. If we relax this bound by raising the denominator to 629910 and then remove the common factor of 10, we obtain 197888/62991 as a lower bound for π. As already explained, a further relaxation

[31] The "extreme and mean ratio" is introduced by EUCLID in several contexts in Books II, IV, VI, X and XIII. For a review of this material and an interpretation of its significance for the studies on irrationals by THEAETETUS and EUDOXUS, see my book *Evolution*, Chapters II/II, VI/IV and VIII/IV.

[32] One sees in the rule of formation the so-called "FIBONACCI" sequence. References to the massive literature on this sequence may be found in L. E. DICKSON, *Theory of Numbers*, I, ch. XVII, pp. 393–407. On possible Greek precedents for studies of this sequence, see my discussion of S. HELLER's account of THEODORUS, *Evolution*, Chapter II/II.

Table 2. Scheme for a Refined Computation

Upper Bounds				Lower Bounds			
n	A	B	C	n	A'	B'	C'
10	4181	$12867\frac{7}{9}$	13530	10	3092	$9516\frac{2}{7}$	10006
20	4181	$26397\frac{7}{9}$		20	3092	$19522\frac{2}{7}$	$19765\frac{3}{4}$
	339	$2140\frac{45}{125}$	$2167\frac{4}{125}$				
40	339	$4307\frac{48}{125}$	$4320\frac{87}{125}$	40	3092	39288	$39409\frac{1}{2}$
80	339	$8628\frac{10}{125}$	$8634\frac{92}{125}$	80	3092	$78697\frac{1}{2}$	$78758\frac{1}{4}$
160	339	$17262\frac{102}{125}$	$17266\frac{18}{125}$	160	3092	157456	$157486\frac{1}{2}$
320	339	$34528\frac{120}{125}$	$34530\frac{78}{125}$	320	3092	$314942\frac{1}{2}$	314958
640	339	$69059\frac{73}{125}$		640	3092	629901	629909

The numbers in this table are formed according to the same method as that in Table 1. Fractional errors here in the square root approximations are much smaller, ranging from 1 to 3 parts in 2,500,000 for the lower bounds, somewhat smaller generally for the upper bounds. The terms of the 20-gon (upper bound) have been adjusted by a factor of 3/37; this follows the precedent to be seen with the 24- and 48-gons (lower bounds) in Table 1.

From the 640-gons, one deduces (1) $\pi < 640 \times 339/69059\frac{73}{125} = 211875/67441 < 377/120$ and (2) $\pi > 640 \times 3092/629909 > 197888/62991 > 333/106$.

by means of the EUCLIDEAN division yields 333/106 as a lower bound. The upper bound is formed as $nA/B = 640 \times 339/69059\frac{73}{125}$. Here, the fraction may be simplified without any loss of accuracy by adjusting each term by the factor $125/128 (= 5^3/2^7)$, yielding 211875/67441. This upper bound may be relaxed to establish 377/120 as an upper bound. In this way, we have detailed an ARCHIMEDEAN procedure leading exactly to the bounds cited by HERO as we have emended them.

It is possible to present a sketch of the preliminary planning of this formal computation, as we did for the *Dimensio Circuli*. ARCHIMEDES required first an awareness of the degree of accuracy to be established. Later computers were to prearrange this by the selection of a denominator of form 10^n, the choice of n being arbitrary. But ARCHIMEDES was concerned with finding *optimal* bounding fractions, not decimal approximations. The EUCLIDEAN division would of necessity guide his thinking. In the *Dimensio Circuli* he has already established that the first two quotients in the algorithm for π are 3 and 7; he has also found that the third quotient is at least 10. A closer approximation for π will be needed to determine that quotient exactly. If the quotient is in fact 10, then 223/71 will remain as the best lower bound attainable at this stage of the division procedure. If the quotient is 11, then $(223+22)/(71+7)$ will be the optimal lower bound; if the quotient is $10+n$, then $(223+22n)/(71+7n)$ will be the optimum. For argument's sake, suppose the quotient is in fact 20; then $3\frac{20}{141}$ will be a lower bound and $3\frac{21}{148}$ will be an upper bound. Their difference is $1/141 \times 148$, or a bit less than 1/20,000; this entails a fractional error of no more than 1 part in 60,000 associable with either bound. As we have seen, this is the order of accuracy verifiable by the text-fractions. Now, the limits established by means of the 96-gons each entail a fractional error of about 1 part in 2500. Thus, the hypothesized refined computation requires that the accuracy be increased by at least a factor of 24. As $4^2 < 24 < 4^3$, it will be expected that two steps beyond the 96-gons (*i.e.*, the 384-gons) will not suffice, but that three steps (the 768-gons) will more than do. This estimate also suggests the idea of directing attention to one of the intermediate constructible polygons, such as the

640-gon. Such a reformulation of the problem starting from the decagon introduces a new theoretical feature, the manipulation of the extreme and mean ratio.

If the computation implied by the HERONIAN figures did indeed have the object of determining the third quotient of π under the EUCLIDEAN division, it could not be fully successful, however. The fractions associated with the quotients 15, 16 and 17 are, respectively, 333/106, 355/113 and 377/120. The computation verifies that the first of these is a lower bound for π and the third an upper bound. But to determine the quotient, we must know also whether the second fraction is an upper or a lower bound. It is in fact an upper bound, but entails a fractional error of less than 1 part in 11,800,000. This is almost 200 times closer than we have planned for in the above sketch. I think it unlikely that ARCHIMEDES would have expected this outcome – at least, the text-figures give no evidence that he did so. It thus again becomes clear what difficulties are implicit in TANNERY's hypothesis regarding this passage.

Table 3. Approximations to π

1. Source	2. Value	3. Decimal equivalent	4. Difference ($\times 10^{-5}$)	5. Fractional error
HERO	197888/62351	3.17377427	3218	1/97 (638.71/62351)
ARCHIMEDES	$3\frac{1}{7}$	3.14285714	126.45	1/2484
ARCHIMEDES	$14688/4673\frac{1}{2}$	3.14282657	123.40	$1/2546\ (1.836/4673\frac{1}{2})$
PTOLEMY	377/120	3.14166666	7.402	1/42446
HERO	211875/67441	3.14163491	4.226	1/74344 (0.91/67441)
HOPPE	195883/62351	3.14161761	2.50	1/125,855
TANNERY	195882/62351	3.14160157	0.893	1/352,038
	355/113	3.14159292	0.027	1/11,810,498
	211891/67447	3.141592658	0.0004	1/785,398,163
	π	3.141592654	0	
TANNERY	211872/67441	3.14159042	−0.223	−1/1,410,683
HOPPE	211871/67441	3.14157559	−1.706	−1/184,203
HERO-emended	197888/62991	3.14152815	−6.450	−1/48707 (−1.3/62991)
	333/106	3.14150943	−8.323	−1/37750
ARCHIMEDES	$6336/2017\frac{1}{4}$	3.14090965	−68.30	$-1/4599\ (-0.44/2017\frac{1}{4})$
ARCHIMEDES	$3\frac{10}{71}$	3.14084507	−74.76	−1/4202
	197888/62988	3.14167778	8.513	1/36905
	197888/62989	3.14162790	3.525	1/89125
	π	3.14159265		
	197888/62990	3.14157802	−1.463	−1/214,795
	197888/62991	3.14152815	−6.450	−1/48707
	333/106	3.14150943	−8.323	−1/37750
	197888/62992	3.14147828	−11.44	−1/27468

In this table, the decimal equivalents (3) have been simply truncated. Designating the value in (2) or (3) as p, one obtains the number in (4) as $p-\pi$; the number is understood to be multiplied by 10^{-5}. The fractional error in (5) is formed as $(p-\pi)/\pi$; this has been put into unit-reciprocal form and the denominator simply truncated. The values in (4) and (5) are upper bounds of the respective differences and fractional errors.

The reconstructed computation conforms closely to the pattern given by ARCHIMEDES in the *Dimensio Circuli*. To be sure, the extraction of so many square roots to the requisite degree of accuracy amounts to a laborious computing effort, even if facilitated by special arithmetic techniques. But this project would not overreach the abilities of a competent Greek arithmetician.[33] No such reconstruction can presume to be perfect, digit for digit. But the scheme provided above serves to show that an ARCHIMEDEAN computation can be devised which results in the text-ratios as I have emended them; it also answers an implicit challenge posed by such other interpreters of the HERO-passage, as BRUINS, who have included the details of computation in their argument. In view of other components in my restoration – the retention of one of the text-ratios unaltered, the small modification of the other strictly in accordance with possible transcriptional error, and accounting for the significance of the factors 773 and 113 – I feel confident that my argument, certainly in its broader outline, is substantially correct.

Is there any documentation, other than the passage from HERO's *Metrica*, which might confirm that ARCHIMEDES carried through an investigation of this type? Let me point to three possible items of supporting evidence.

First, we have the report of EUTOCIUS that in a work called the *Ocytocium* ("Easy Delivery") APOLLONIUS derived bounds for π better than the ARCHIMEDEAN limits and that the computation involved "myriads" (multiples of 10,000). The fragments of that work which have survived have nothing to do with the geometry of the circle *per se*, but merely illustrate techniques for the ready manipulation of large numbers.[34] Now, the ARCHIMEDEAN limits cited by HERO are also of the order of myriads. It would thus appear that ARCHIMEDES' younger contemporary APOLLONIUS achieved at best but a small improvement in accuracy over the refined ARCHIMEDEAN values. Without denying this view, I may suggest that an alternative explanation is possible. Given the largely arithmetic content of APOLLONIUS' book, we may surmise that APOLLONIUS used the ARCHIMEDEAN computation as a vehicle for explaining and facilitating the arithmetical operations, in particular the computation of squares and square roots of large numbers. This

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[33] I did computations, like those given in Table 2, with the assistance of a miniature electronic calculator. With practice, I came to require between five and ten minutes to complete the computation of a bound associated with the 640-gon. As an experiment, I worked out long-hand the step in the 160-gon which verifies that $(3092^2+157456^2)^{\frac{1}{2}} < 157486\frac{1}{2}$; it took 20 minutes. Now, as far as the long-hand methods are concerned, the Greek numeral system is no more or less efficient than our own. I thus estimate that a Greek computist using these methods would require about the same time as I did. As each bound requires six root-extractions, at least two hours of continuous computing would be necessary. Assuming that the computation would be done twice, perhaps even three times, as a check against error, we may estimate that a Greek computer would need at least 12 hours of uninterrupted and tedious effort to effect the computation of Table 2. Of course, the effort would be spaced over a longer interval, or distributed among a number of computers. But granted the use of such techniques as HERO's rule for roots (see note 51), ARCHIMEDES or a skilled Greek computer could reduce this time considerably.

[34] EUTOCIUS in *Archimedis Opera*, ed. J. L. HEIBERG, III, p. 258. HEIBERG takes parts of PAPPUS' *Collectio* to be fragments of the *Ocytocium* (*Apollonii Opera*, II, pp. 124–132). This material develops techniques for the ready manipulation of large numbers. While explicitly citing APOLLONIUS, PAPPUS does not name the *Ocytocium*. As G. HUXLEY argues ("*Okytokion*", pp. 203f), the connection can be drawn via the aphorism cited by PAPPUS, the letters of which are interpreted as numerals and then multiplied together; for the aphorism cites ARTEMIS the goddess specifically renowned as guardian over childbirth (*ōkytokos*). But I consider HUXLEY pushes the argument too far in using ARTEMIS' association with the moon as justifying the ascription of APOLLONIUS' lunar theory to the *Ocytocium*. An interesting speculation by HULTSCH, linking this work to ARCHIMEDES' *Sand-Reckoner* and *Cattle-Problem* is mentioned by HEATH, *Archimedes*, p. xxxv.

being so, by the time of EUTOCIUS (VI. century), long after ARCHIMEDES' original tract had been lost, the appearance of these computations in the APOLLONIAN work might have led him to infer that APOLLONIUS – and not ARCHIMEDES – was their originator. In this way, the testimonia of EUTOCIUS and of HERO become compatible.

Second, the upper bound 377/120 which we derived by rounding off the upper ARCHIMEDEAN limit coincides with the value used by PTOLEMY in his *Syntaxis* (VI, 7). PTOLEMY remarks that his own value 3; 8, 30 (expressed in sexagesimal notation) lies between ARCHIMEDES' values $3\frac{1}{7}$ and $3\frac{10}{71}$. This comment does not of course indicate ARCHIMEDES as the source of PTOLEMY's value, and it is generally accepted that the value was in fact derived from the table of chords in Book I.[35] In brief, PTOLEMY computes the chords of 60° and 36°, and from these the chord of their difference 24°. By successive bisections of the angle, he at last obtains chd $1\frac{1}{2}° = 1; 34,15$ and chd $\frac{3}{4}° = 0; 47,8$. He next introduces a critical inequality: chd a/chd $b < a/b$. Using this, he argues that chd $1° = 1; 2,50$ very nearly.[36] Now, as PTOLEMY assumes that the diameter (D) of the circle is 120 units, and since chd $a = D \sin \frac{a}{2}$ so that chd 1° is the side of the inscribed 360-gon, we may deduce that π is very nearly $360 \times$ chd $1°/120 = 3; 8,30$. Such is the argument linking the table of chords with PTOLEMY's explicit value for π in Book VI. As a derivation, this accords with the ARCHIMEDEAN in utilizing the successive bisection of an initial vertex angle. But unlike ARCHIMEDES, who had to round off always in the same sense, PTOLEMY could round off in either direction, as convenience recommended. He could thus hope to benefit from a favorable cancellation of errors. Only in this way could he have obtained so accurate a value as 3; 8,30 via polygons of so few sides, or have ended with an upper bound via *inscribed* polygons. By ARCHIMEDEAN standards, such a method as this lacks rigor. In particular, one could not have an adequate estimate of the degree of accuracy of the value computed. It seems to me that the central importance of the values of π and chd 1° for PTOLEMY's astronomy would advise him not to leave such a matter to chance. If, however, PTOLEMY and his predecessors in trigonometry, knew of the ARCHIMEDEAN study proving 377/120 to be a good upper bound for π, the obvious convenience of the value within the sexagesimal notation would encourage its adoption. In constructing the table of chords, this value could *guide* the computations for small angles, rather than be derived from them. Indeed, there is evidence of just such manipulation in PTOLEMY's table. Although PTOLEMY usually rounds off to the *nearest* three-place sexagesimal value, he departs from this practice in taking 0; 47,8 for chd $\frac{3}{4}°$, as the four-place value is 0; 47,7,25. By this he brings it into consistency with chd $1\frac{1}{2}°$.[37] But he thereby also ensures that the resultant value of chd 1° is compatible with his value for π.

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[35] *Cf.* T. L. HEATH, *Greek Mathematics*, I, p. 233.

[36] PTOLEMY, *Syntaxis*, I, 10; *cf.* the editions by J. L. HEIBERG, I, pp. 45f and by R. C. TALIAFERRO, p. 20. The argument is summarized by B. L. VAN DER WAERDEN, *op. cit.*, pp. 206f.

[37] PTOLEMY's value for chd $\frac{3}{4}°$ is thus subordinate to that for chd $1\frac{1}{2}°$. This means in effect that his ultimate value for chd 1°, the basis for his value for π, is founded on the inscribed 240-gon. PTOLEMY's rather flexible manipulation of these inequalities troubled THEON, who worked out the chords to four sexagesimal places so as to indicate their consistency (*Commentaires*, ed. A. ROME, I, pp. 492–495).

Third, it is generally recognized that the extant *Dimensio Circuli* is a post-ARCHIMEDEAN revision of a portion of ARCHIMEDES' original and far more comprehensive treatment of the circle.[38] Notably, in *Metrica* I, 37 HERO draws from the *Dimensio Circuli* a theorem on the area of circular sectors: the area equals half the product of the radius and the arc of the sector. The same theorem is discussed at considerable length by PAPPUS, also relying explicitly on an ARCHIMEDEAN work.[39] This theorem is not contained in the extant *Dimensio Circuli*, although, as an extension of the first proposition of that tract, its relevance to it is clear. Hence, an ARCHIMEDEAN tract more extensive than the extant *Dimensio Circuli* circulated in antiquity and was still accessible to PAPPUS (late III. century), although not to EUTOCIUS (VI. century). Is it unreasonable to suppose that ARCHIMEDES' refined computation was contained in the lost portion of this work? The fact that HERO draws the ratios of the large numbers from another work "On Plinthides and Cylinders" is not incompatible with this view.[40] Of the latter tract we know nothing more, but the title suggests it might have been akin to *Metrica* II, devoted to the measurement of solids. In the preface to such a work, ARCHIMEDES might well have mentioned his bounds to π, for these have an obvious pertinence to a study of cylindriform solids. By HERO's time, the portions of the *Dimensio Circuli* containing the refined computation might already have been lost; or HERO might have found it as convenient to refer to the "Plinthides" as to consult the full text of the *Dimensio*.

It is becoming more widely appreciated that Greek mathematics was founded on a strong tradition of practical arithmetical competence, and the present study of the technique implicit in ARCHIMEDES' work on the circle ought to contribute to this understanding.[41] An older conventional view, that the Greeks devoted themselves to theoretical geometry to the detriment of practical mathematics, is thus to be seen as a distortion occasioned by the selective survival of documents. But it is true that from the theoretical standpoint, little of what we have attributed to ARCHIMEDES here is of real mathematical profundity. There may be a certain fascination with computing square roots with ease to great orders of accuracy or with other feats of calculation, but no notable insight is required or gained by it.[42] Is it possible that ARCHIMEDES' computation was produced toward a more theoretically interesting end? We have mentioned that HERO presents an inequality relating to the area of circular segments: that the segment exceeds four-thirds of the greatest inscribable triangle (*Metrica* I, 32). While HERO names no source here, ARCHIMEDES is cited in the same chapter for the analogous result on parabolic segments, and HERO's proof for circular segments follows the same pattern used by ARCHIMEDES for the parabola in *Quadratura Parabolae*, prop. 24. I am led to conclude that ARCHIMEDES was likewise the source for HERO's in-

[38] E. J. DIJKSTERHUIS, *Archimedes*, p. 222; T. L. HEATH, *Greek Mathematics*, II, p. 50.

[39] PAPPUS, *Commentaires*, ed. A. ROME, pp. 253–260.

[40] J. L. HEIBERG admits the possibility that the "Plinthides" and the tract "On the Measurement of the Circle" are one and the same (*Archimedis Opera*, II, p. 542). For an alternative view on the relation of these works, see note 22 above.

[41] The depth of the arithmetical technique implicit in PTOLEMY's work has been emphasized by O. PEDERSEN, "Logistics and the theory of functions", pp. 29–50.

[42] C. BOYER calls attention to this important consideration in his remarks on the computing feats of early circle-squarers; *cf. History of Mathematics*, pp. 224f.

equality on the circular segments. Now, this inequality may readily be modified to provide an estimate of the rate of convergence of the inscribed polygons to the bounding circle, as we have already mentioned. But it can also provide the means of improving the approximation to π derived by the polygonal method. Specifically, if p_n is the perimeter of the inscribed n-gon, c the circumference of the circle, then $\frac{1}{3}(4p_{2n} - p_n) < c$. Many such inequalities associated with the sequences of inscribed and circumscribed polygons were discovered by HUYGENS and GREGORY in the seventeenth century for the purpose of increasing the rate of convergence.[43] In doing such a study, SNELL made much of the advantage of his own method in comparison with the standard ARCHIMEDEAN technique. But ARCHIMEDES himself had results suited for the same type of inquiry.

Denoting as π_{2n} the ratio of the perimeter of the inscribed $2n$-gon to the diameter of the circle, we obtain from the above inequality $\frac{4}{3}\pi_{2n} - \frac{1}{3}\pi_n < \pi$.[44] By means of this formula, ARCHIMEDES' values for the dodecagons suffice for establishing the bounds which were verified via 96-gons by the standard method (Table 4a – 1). The stronger result we established via 640-gons can be obtained via 24-gons, if the formula is used (Table 4a – 2). From the geometry of the cir-

Table 4a. Improved Convergence – Lower Bounds

(1)			(2)		
π_6	$6 \times 153/306$	3.0	π_{12}	$12 \times 153/591\frac{1}{7}$	3.10584823
π_{12}	$12 \times 780/3013\frac{3}{4}$	3.10576524	π_{24}	$24 \times 240/1838\frac{5}{7}$	3.13262372
p_{12}	$\frac{4}{3}\pi_{12} - \frac{1}{3}\pi_6$	3.14102032	p_{24}		3.14154889
π_{96}	$6336/2017\frac{1}{4}$	3.14090965	π_{640}	197888/62991	3.14152815
f_1	$(\pi_{12} - \pi)/\pi$	−1/87	f_1		−1/350
f_2	$(p_{12} - \pi)/\pi$	−1/5489	f_2		−1/71786
	f_1/f_2	62		f_1/f_2	204
	$5(12/\pi)^2$	72		$5(24/\pi)^2$	291
	$3\frac{10}{71} < \pi_{96} < p_{12} < \pi$			$\frac{333}{106} < \pi_{640} < p_{24} < \pi$	

Remarks: In (1), π_6 and π_{12} are calculated from values of ARCHIMEDES, found in Table 1; as π_6 must here be an *upper* bound for the ratio of the perimeter of the inscribed hexagon to the diameter of the circle, values of A and C from the left-hand table are used. All values have been simply truncated. The fractional errors f_1 and f_2 are thus upper bounds of the accuracy, while the numbers f_1/f_2 and $5(12/\pi)^2$ are lower bounds of, respectively, the actual and the theoretical factors of improvement of the rate of convergence. Discrepancies between these two numbers indicate the significance of errors due to rounding off. In (2), π_{12} and π_{24} stem from a recomputation of the ARCHIMEDEAN values (*cf.* Table 1), where an accuracy equivalent to three correct places beyond the decimal-point has been maintained.

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43 On the convergence-improvement formulae of GREGORY and HUYGENS, see J. E. HOFMANN, "Gregorys Näherungen", pp. 24–37. SNELL's lower-bound formula, differing from the HERONIAN, was published in the *Cyclometricus* (1621). On this and related studies by NICOLAUS of CUSA and VIETA, see HOFMANN, *Geschichte*, I, pp. 127, 152, 163, and TROPFKE, *Geschichte*, II, pp. 213, 217–220.

44 Using the power-series expression for sin x, one may verify the inequality $x > \frac{1}{6}(8 \sin x - \sin 2x) > \sin x$. The rightmost member correlates with the procedure of approximation via inscribed polygons; the central member is equivalent to the HERONIAN improvement-formula. The angle x is half the central angle subtended by a side of the inscribed n-gon, *i.e.*, π/n. The inequality also shows that use of the formula improves the accuracy of an approximation associable with an n-gon by a factor of $\frac{5}{x^2} - \frac{1}{4}$, where $x = \pi/n$.

Table 4b. Improved Convergence – Upper Bounds

(3)			(4)		
π_{12}	$12\times 153/591\frac{1}{8}$	3.10594206	π_{24}	$24\times 153/1172\frac{17}{100}$	3.13265140
π'_{12}	$12\times 153/571$	3.21541155	π'_{24}	$24\times 153/1162\frac{1}{7}$	3.15968039
p'_{12}	$\frac{1}{3}\pi'_{12}+\frac{2}{3}\pi_{12}$	3.14243189	p'_{24}		3.14166106
π'_{96}	$14688/4673\frac{1}{2}$	3.14282657	π'_{640}	211875/67441	3.14163491
				377/120	3.14166666
	f_1	1/42		f_1	1/173
	f_2	1/3743		f_2	1/45919
	f_1/f_2	87		f_1/f_2	264
	$\frac{20}{3}(12/\pi)^2$	97		$\frac{20}{3}(24/\pi)^2$	389
	$3\frac{1}{7}>\pi'_{96}>p'_{12}>\pi$			$\frac{377}{120}>p'_{24}>\pi'_{640}>\pi$	

Table 4b. (Continued)

(5)			(6)		
π_{80}	$80\times 339/8634.7827$	3.14078546	π_{160}	$160\times 339/17266.2366$	3.14139098
π'_{80}	$80\times 339/8628.1256$	3.14320876	π'_{160}	$160\times 339/17262.9084$	3.14199662
p'_{80}		3.14159323	p'_{160}		3.14159286
	355/113	3.14159292		355/113	3.14159292
	f_1	1/1943		f_1	1/7776
	f_2	1/5435281		f_2	1/14,749,261
	f_1/f_2	2796		f_1/f_2	1896
	$\frac{20}{3}(80/\pi)^2$	4323		$\frac{20}{3}(160/\pi)^2$	17292
	$p'_{80}>\frac{355}{113}$			$\frac{355}{113}>p'_{160}>\pi$	

Remarks: In (3), π'_{12} and π_{12} are computed from ARCHIMEDEAN values (Table 1); as both are to be upper bounds, they employ data from the left-hand table. Computations in (4) are based on ARCHIMEDEAN values, recomputed to an accuracy of three places beyond the decimal-point. Values in (5) and (6) stem from numbers as in Table 2, but recomputed to an accuracy of four places beyond the decimal-point.

The comparisons in (1) and (3) show that results slightly stronger than those obtained via 96-gons under the standard polygonal approximation can be established via dodecagons and the convergence-improving formulas. Similarly, (2) and (4) show that the formulas educe from 24-gons bounds of equivalent accuracy to those obtained from 640-gons by the standard method. The computations in (5) and (6) show that the upper-bound formula can establish 355/113 as an upper bound for π via 160-gons, the 80-gons being just barely insufficient; the equivalent result by the standard polygonal method would require polygons of no fewer than $6144(=6\times 2^{10})$ sides.

cumscribed polygons, a similar formula for upper bounds may be obtained: $\frac{1}{3}\pi'_n+\frac{2}{3}\pi_n>\pi$. Here, π'_n denotes the ratio of the perimeter of the circumscribed n-gon to the diameter of the circle.[45] As before, the bounds verified by the 96-gons and 640-gons under the standard method require, respectively, only 12-gons and 24-gons if the formula is applied. Moreover, this formula permits one to verify 355/113 as an upper bound via 160-gons if the values in our refined computation

[45] This formula may be verified by means of the power-series for

$$\tan x\left(=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{272x^7}{7!}+\cdots\right).$$

One uses the inequality $\tan x>\frac{1}{3}(\tan x+2\sin x)>x$, where as above $x=\pi/n$. The approximation resulting from the formula is closer by a factor of $20/3\,x^2+2\frac{2}{3}$ than that derived from the circumscribed n-gons by the standard method.

(Table 2) are determined to a higher degree of accuracy (Table 4b–6).[46] In this way, a result requiring under the standard method no fewer than ten square-root computations, each to an accuracy of 1 part in 10^9 or better, can be obtained via the extraction of four square roots to an accuracy of 1 part in 10^8. We do not know to what extent ARCHIMEDES investigated the refinement of the bounds for π. It seems to me probable that if he succeeded in verifying bounds as accurate as 355/113, he did so with assistance from convergence-improving inequalities derived from geometric theory, rather than by laborious extension of the standard polygonal method. Unfortunately, such theoretical aspects, if part of ARCHIMEDES' study of the circle, proved too subtle for the uses of later commentators like HERO and EUTOCIUS.

We have argued that the numbers cited by HERO require but a few emendations, each readily defensible by manuscript considerations, to yield relatively good bounds for π: namely, $197888/62991 < \pi < 211875/67441$. We have also seen how the standard ARCHIMEDEAN method of inscribed and circumscribed polygons, initiated by suitable approximations to the extreme and mean ratio in the context of the decagons and carried through to the 640-gons, can yield these ratios. But the *Dimensio Circuli* and HERO's comment indicate that these ratios were intended as "rough" results. In singling them out for preservation, HERO betrays his naive awe at the large numbers, as well as an appreciation of the computational effort required for deriving them. By contrast, ARCHIMEDES' reduction of the ratios by means of the EUCLIDEAN division results in ostensibly less impressive inequalities: $333/106 < \pi < 377/120$; yet these in fact entail virtually the same degree of accuracy as the former, despite the much smaller terms. Explaining such reduction as effected for mere computational convenience, HERO and EUTOCIUS so miss ARCHIMEDES' subtler recognition of the power of small numbers.

Appendix: Archimedes' Method of Approximating Roots

In the *Dimensio Circuli* ARCHIMEDES introduces without explanation the approximation $265/153 < \sqrt{3}$ and later the second approximation $1351/780 > \sqrt{3}$. These values have stimulated a massive scholarly commentary.[47] Since both values are convergents within the continued-fraction expansion of the root, it has been presumed that some equivalent procedure, based on the EUCLIDEAN division, was used in their derivation. From the mathematical point of view, this is certainly correct; for it is hardly by mere accident that ARCHIMEDES chose

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46 In theory, the formula can establish 355/113 as an upper bound via the 80-gons. Using the standard method, one requires 10240-gons; *cf.* note 8.

47 For surveys, see T. L. HEATH, *Archimedes*, pp. lxxx–iv, xc–xcix and E. J. DIJKSTERHUIS, *Archimedes*, pp. 234–8. HOFMANN explains the values via a formula for $\sqrt{x^2-1}$ ("Quadratwurzel", pp. 204f); but I show in my study of HERO's rule that this formula is not needed to account for the cases which troubled HOFMANN, and thus his overall argument is weakened. Methods based on recursive sequences have been proposed by TANNERY, DE LAGNY and HEILERMANN (*cf.* HEATH, *loc. cit.*). VOGEL, HULTSCH and HUNRATH introduce equivalent principles, related to continued-fraction manipulations; but their methods have an *ad hoc* character and they do not specify either the amount or the direction of error entailed by the derived approximations. These latter features were certainly critical considerations for ARCHIMEDES.

values falling within the convergent-sequence. But the manner of derivation I shall now propose develops from the principle of the "side and diameter" numbers.

The "side and diameter" numbers, attributable to the fifth- and fourth-century PYTHAGOREANS, are a means of approximating $\sqrt{2}$.[48] One sets $d_1 = 1$ and $s_1 = 1$ and defines $d_{n+1} = d_n + 2s_n$, $s_{n+1} = d_n + s_n$. The fractions d_n/s_n form an alternating sequence converging to $\sqrt{2}$. In fact, these same fractions are precisely the convergents derived from the continued-fraction expansion of $\sqrt{2}$, and the formula of the algorithm can readily be derived by applying the EUCLIDEAN division to the side and diameter of the square.[49] PROCLUS remarks that the difference $d_n^2 - 2s_n^2$ equals $+1$ or -1 in alternation and that this property is verifiable by means of the identity proved in *Elements* II, 10: namely, $(a+2b)^2 + a^2 = 2b^2 + 2(a+b)^2$.[50] Although no discussion of the convergence has survived from antiquity, the rate of convergence is easily obtained from PROCLUS' theorem. The difference of consecutive fractions in the sequence is

$$d_{n+1}/s_{n+1} - d_n/s_n = (2s_n^2 - d_n^2)/s_n(s_n + d_n) = \pm 1/s_n(s_n + d_n).$$

As each side and diameter is greater than double the preceding side and diameter, respectively, the magnitude of the difference of the fractions $1/s(s+d)$ will be diminished to less than one-fourth by each passage to a new side and diameter. We may presume that ARCHIMEDES had knowledge of such sequences and their properties.

An analogous sequence, converging to $\sqrt{3}$, may be generated according to the formula $d_{n+1} = d_n + 3s_n$, $s_{n+1} = d_n + s_n$, and the initial conditions $d_1 = 2$ and $s_1 = 1$. The first terms are these: $\frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \frac{1351}{780}, \frac{3691}{2131}$, *etc.* ARCHIMEDES' values appear as the eighth and eleventh terms of the sequence. But I doubt that ARCHIMEDES used this particular sequence to obtain them. In choosing 265/153 as the lower bound, he has judged that the associated fractional error (*i.e.*, less than 1 part in 70225) will suffice for establishing $3\frac{1}{7}$ as an upper bound for π (fractional error: 1 in 2484). Now, either of the adjacent fractions 97/56 or 362/209 is of sufficiently small fractional error (respectively, 1 in 18816 and 1 in 262114) to verify $3\frac{10}{71}$ as a lower bound for π (fractional error: 1 in 4202). Moreover, each has a factorable denominator (*e.g.*, $209 = 11 \times 19$) and ARCHIMEDES sometimes manipulates such factors to facilitate computation (see Table 1). Why, then, did he choose the unnecessarily accurate bound 1351/780 (fractional error: 1 in 3,654,115)? The answer, I believe, lies not in any intrinsic advantage to this value, but rather in the computing rule he used.

By means of a different formula, one may obtain the sequence containing every third convergent in the expansion of $\sqrt{3}$. If we start from the inequality $\sqrt{3} > \frac{5}{3}$, the HERONIAN rule for roots leads to the formation of $\frac{9}{5}$ as the associated upper bound.[51] Given any integers d, s, we may form an intermediate fraction

[48] HEATH cites and discusses the sources on the "side and diameter" numbers in *Greek Mathematics*, I, pp. 91–93.

[49] A derivation of the formula via the EUCLIDEAN division is given by HEATH, *Mathematics in Aristotle*, pp. 30–33.

[50] PROCLUS, *In Rem Publicam*, II, pp. 27–29.

[51] Under the HERONIAN rule (*Metrica* I, 8), to approximate $\sqrt{N}$, one forms from an initial estimate b the value N/b and then computes a closer estimate $b' = \frac{1}{2}(b + N/b)$. The procedure is recursive.

via $(9s+5d)/(5s+3d)$. This averaging principle is readily verified by theorems in the EUCLIDEAN proportion theory and arises often in the course of the use of the EUCLIDEAN division.[52] If, now, we set $d=5$, $s=3$, the resultant ratio is $(27+25)/(15+15)=\frac{26}{15}$. Assuming next $d=26$, $s=15$, we obtain $(135+130)/(75+78)=\frac{265}{153}$. From $d=265$, $s=153$, we have next $(1377+1325)/(765+795)=2702/1560=\frac{1351}{780}$. One sees that the values employed by ARCHIMEDES are obtained in immediate succession. Moreover, the first approximation $\frac{26}{15}$ is a value frequently used by HERO, but attributable also to ARCHIMEDES.[53] Hence, we have a derivation which, in contrast with others proposed, is direct in application and yields only those values actually used by ARCHIMEDES.

As for the "side and diameter" numbers, the convergence of this sequence may easily be estimated. By means of the geometric techniques of *Elements* II, one verifies $(9s+5d)^2-3(5s+3d)^2=2(3s^2-d^2)$. Since the initial value of $\frac{d}{s}$ is $\frac{5}{3}$, a lower bound, the left-hand side is initially 4. Hence, $52^2-3\times 30^2=4$, or $26^2-3\times 15^2=1$; hence, the difference d^2-3s^2 takes on the values 1 or -2 in alternation. Applied to the difference $\frac{d}{s}-\frac{9s+5d}{5s+3d}$ and the fact that each side and diameter is at least ten times larger than the preceding side and diameter, respectively, this establishes that each fraction in the sequence is at least $50(=\frac{1}{2}\times 10^2)$ times more accurate than its predecessor. In the knowledge of such properties, ARCHIMEDES could choose values from the sequence suitably accurate for the purposes of his computation.

We have argued that ARCHIMEDES computed more accurate bounds for π by means of the initial inequalities $\frac{6765}{4181}<\frac{1+\sqrt{5}}{2}<\frac{75025}{46368}$. Both fractions have been taken from the sequence of convergents to the "extreme and mean ratio", defined by the familiar "FIBONACCI" rule: $d_{n+1}=d_n+s_n$, $s_{n+1}=d_n$, where $d_1=1$ and $s_1=1$. But as before, the choice of an unnecessarily accurate upper bound has to be explained. For after $\frac{6765}{4181}$ follow the terms $\frac{10946}{6765}$, $\frac{17711}{10946}$, $\frac{28657}{17711}$, $\frac{46368}{28657}$, $\frac{75025}{46368}$, *etc.*, of which the first, third and fifth are upper bounds, each entailing a smaller fractional error than does $\frac{6765}{4181}$. Moreover, the factorizations $6765=15\times 451$ and $17711=89\times 199$ serve to discount reasons based on computational convenience. We shall thus seek an alternative sequence.

In the above sequence, the fourth term is the upper bound $\frac{5}{3}$ and the fifth the lower bound $\frac{8}{5}$. For any integers d, s, the fraction $(8d+5s)/(5d+3s)$ will lie between those fractions. From $d=5$, $s=3$, we obtain $(40+15)/(25+9)=\frac{55}{34}$. From $d=55$,

[52] From EUCLID, *Elements* VII, 12 one has that if $a:b=c:d$, then $(a+c):(b+d)$ is in the same ratio. It follows easily that if $\frac{a}{b}<\frac{c}{d}$, then $\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}$, and generally, $\frac{a}{b}<\frac{na+mc}{nb+md}<\frac{c}{d}$, for arbitrary integers m, n.

[53] A scholiast to DIOPHANTUS asserts that "Archimedes has proved that 30 equilateral triangles equal 13 squares" (*Archimedis Opera*, ed. HEIBERG, II, p. 542). Another scholiast, commenting on HERO, cites ARCHIMEDES for the proof that 13 times the square of the side of the hexagon equals five times the hexagon (*Heronis Opera*, ed. J. L. HEIBERG, IV, p. xxiv). Both claims are equivalent to the approximation of $\sqrt{3}$ by $\frac{26}{15}$. This value is used by HERO, as in his measurement of the equilateral triangle, *Metrica* I, 17.

$s=34$, we obtain $(440+170)/(275+102)=\frac{610}{377}$. From these terms we derive $\frac{6765}{4181}$, and from these in turn we derive $\frac{75025}{46368}$. Thus, the two bounds we seek are produced in succession. The decision to choose $\frac{6765}{4181}$ must be made on the issue of accuracy; for the prior lower bound $\frac{55}{34}$ entails too large a fractional error (*i.e.*, 1 part in 4181) to suit a computation planned for an accuracy of 1 part in 60,000. ARCHIMEDES could then choose the next term in the sequence as upper bound, namely, $\frac{75025}{46368}$ (fractional error: less than 1 part in $7\frac{1}{2}\times 10^9$), and then for convenience reduce this to $\frac{75045}{46380}=\frac{5003}{3092}$ (fractional error: 1 in 128,588), a value still accurate enough for the computation.[54] Such estimates of accuracy are easily made. By means of the geometry of *Elements* II one can establish that $d^2-s(d+s)$ equals $+1$ or -1 in alternation. From the difference $\frac{d}{s}-\frac{9d+5s}{5d+3s}=\frac{\pm 5}{s(5d+3s)}$ and the fact that each diameter and side is at least 11 times larger than the preceding, one deduces that any fraction in the sequence is at least 121 times more accurate than its predecessor. The choice of a value suitably close for the purposes of a given computation is thus straightforward.

By such refinements of the PYTHAGOREAN studies of the "side and diameter" numbers, ARCHIMEDES could select the approximations for $\sqrt{3}$ and for the extreme and mean ratio required for his computations of upper and lower limits for π. One cannot expect to specify in precise detail the *form* of ARCHIMEDES' investigation of such recursive sequences. But the view that he employed them for the present purposes is supported not only by the precedent of the PYTHAGOREAN studies, but also by the appearance of related techniques in DIOPHANTUS' *Arithmetica*. There, we may see both a familiarity with the problem of finding integral solutions for relations of form $A^2-nB^2=m$ and also a technique of examination based on linear substitutions of the unknowns.[55] Inasmuch as ARCHIMEDES' famed "cattle-problem" requires the solution of a relation of this same form, we may be assured that the knowledge of these parts of arithmetic theory had already been sufficiently advanced by his time.

Thus, our explanation of how ARCHIMEDES could obtain appropriate bounds for the extreme and mean ratio for computing refined estimates for π supports a new resolution of the much-discussed question of how he derived the bounds for $\sqrt{3}$ employed in the *Dimensio Circuli*.

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[54] Why should ARCHIMEDES have passed over the value $\frac{610}{377}$ (fractional error: 1 in 514,314)? I suppose that he wished initially to retain the possibility of establishing a very close bound for π, but then found the adopted value $\frac{75025}{46368}$ was unnecessarily close for the bound he could obtain and cumbersome besides, and so reduced it to the more manageable value $\frac{5003}{3092}$.

[55] In *Arithmetica* V, 9 and 11 and VI, 15 (lemma), DIOPHANTUS obtains integral solutions to $x^2=26y^2+1$, $x^2=30y^2+1$, and $x^2=3y^2-11$ by a method employing linear substitutions for x.

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