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Probability Theory and Related Fields

Non-Zero-Sum Discrete Parameter Stochastic Games with Stopping Times

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§ 1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n,\ n=0,1,2,\ldots)$ an increasing sequence of sub-σ-fields of $\mathscr{F}$. Let X be the class of all $(\mathscr{F}_n)$-adapted sequences $(x(n))$ of random variables such that

$$E[\sup_n |x(n)|] < \infty, \qquad \lim_n x(n) = 0 \ \text{a.s.}$$

We notice that X is a complete lattice, that is to say, every non-empty bounded subset of X has a greatest lower bound and a least upper bound. For any $(\mathscr{F}_n)$-stopping time T, let Π_T denote the class of all $(\mathscr{F}_n)$-stopping times $S \geqq T$.

Now, we are given

$$g_i, h_i \in X, \qquad i=1,2. \tag{1}$$

Let us consider the non zero-sum stochastic game whose payoff functions are given by

$$\begin{aligned} J_1(T,S) &= E[1_{(T<S)} g_1(T) + 1_{(S \leqq T)} h_1(S)] \\ J_2(T,S) &= E[1_{(S \leqq T)} g_2(S) + 1_{(T<S)} h_2(T)], \qquad (T,S) \in \Pi_0 \times \Pi_0. \end{aligned} \tag{2}$$

We call $(T^*, S^*) \in \Pi_0 \times \Pi_0$ a Nash point for the game if

$$\begin{aligned} J_1(T^*, S^*) &\leqq J_1(T, S^*) \\ J_2(T^*, S^*) &\leqq J_2(T^*, S), \qquad (T,S) \in \Pi_0 \times \Pi_0. \end{aligned} \tag{3}$$

Our interest lies in finding a Nash point, which is an extension of a saddle point in the zero-sum game:

$$h_1 = -g_2, \qquad h_2 = -g_1. \tag{4}$$

This problem was first considered by A. Bensoussan and A. Friedman [2] in the diffusion case. However, it is not known whether the regularity of the solution of the quasi variational inequality holds, which is needed for the existence of a Nash point.

The purpose of this paper is to give sufficient conditions for the existence of a Nash point by using the martingale method and the fixed point theorem for monotone mappings.

§ 2. Main Results

The main results of this paper are the following.

Theorem 1. *Suppose that there exists* $z_i \in X$, $i=1,2$, *such that*

$$z_i \leqq g_i, \tag{5}$$

$$(z_i(n)) \text{ is a submartingale}, \tag{6}$$

$$z_i(n) = E[z_i((T^* \wedge S^*) \vee n) \mid \mathscr{F}_n], \qquad n=0,1,2,\ldots, \tag{7}$$

$$z_1(S^*) = h_1(S^*), \qquad z_2(T^*) = h_2(T^*), \tag{8}$$

where

$$T^* = \min\{n \mid z_1(n) = g_1(n)\}, \qquad S^* = \min\{n \mid z_2(n) = g_2(n)\}. \tag{9}$$

Then (T^*, S^*) *is a Nash point for the game.*

Theorem 2. *We assume that for each* $i=1,2$,

$$h_i \leqq g_i, \tag{10}$$

$$h_i \text{ is a submartingale}, \tag{11}$$

$$g_i \text{ is a supermartingale}. \tag{12}$$

Then there exists $z_i \in X$, $i=1,2$, *satisfying* (5), (6), (7) *and* (8).

§ 3. Proof of Theorem 1

By (7), (8) and (9) we have

$$\begin{aligned} z_1(0) &= E[z_1(T^* \wedge S^*) \mid \mathscr{F}_0] \\ &= E[1_{(T^* < S^*)} z_1(T^*) + 1_{(S^* \leqq T^*)} z_1(S^*) \mid \mathscr{F}_0] \\ &= E[1_{(T^* < S^*)} g_1(T^*) + 1_{(S^* \leqq T^*)} h_1(S^*) \mid \mathscr{F}_0]. \end{aligned}$$

For any $T \in \Pi_0$, it follows from (6), (7), (8) and the optional sampling theorem that

$$\begin{aligned} z_1(0) &\leqq E[z_1(T \wedge S^*) \mid \mathscr{F}_0] \\ &= E[1_{(T < S^*)} z_1(T) + 1_{(S^* \leqq T)} z_1(S^*) \mid \mathscr{F}_0] \\ &\leqq E[1_{(T < S^*)} g_1(T) + 1_{(S^* \leqq T)} h_1(S^*) \mid \mathscr{F}_0], \end{aligned}$$

and thus

$$J_1(T^*, S^*) = E[z_1(0)] \leqq J_1(T, S^*).$$

Similarly we get

$$J_2(T^*, S^*) = E[z_2(0)] \leqq J_2(T^*, S).$$

The proof is complete.

§ 4. Proof of Theorem 2

In order to prove the theorem, we shall prepare for two lemmas. For any $(x_1, x_2) \in X \times X$, we define the subclasses $C_1(x_2)$ and $C_2(x_1)$ of X by

$$\begin{aligned} C_1(x_2) = \{x \in X \mid x \leqq g_1,\ & x \text{ is a submartingale}, \\ & x(S) = h_1(S) \text{ for any } S \in \Pi_{S_2} \text{ where} \\ & S_2 = \min\{n \mid x_2(n) \geqq g_2(n)\}\}, \\ C_2(x_1) = \{x \in X \mid x \leqq g_2,\ & x \text{ is a submartingale}, \\ & x(S) = h_2(S) \text{ for any } S \in \Pi_{S_1} \text{ where} \\ & S_1 = \min\{n \mid x_1(n) \geqq g_1(n)\}\}. \end{aligned} \tag{13}$$

Lemma 3. *Under* (10) *and* (11), *the class* $C_1(x_2) \times C_2(x_1)$ *has a maximal element* (x_1^*, x_2^*), i.e.

$$x_1^* \geqq y_1, \quad x_2^* \geqq y_2, \quad (y_1, y_2) \in C_1(x_2) \times C_2(x_1). \tag{14}$$

Proof. We shall show the existence of x_1^* in $C_1(x_2)$. By (10) and (11) we note that

$$h_1 \in C_1(x_2), \quad h_2 \in C_2(x_1), \quad (x_1, x_2) \in X \times X. \tag{15}$$

Let $Y = \{y_\lambda\}$ be any totally ordered subset of $C_1(x_2)$. Since $y_\lambda \leqq g_1$, we can define $y = (y(n)) \in X$ by

$$y(n) = \operatorname*{ess\,sup}_\lambda y_\lambda(n).$$

Clearly $y \leqq g_1$, and also

$$E[y(n+1) \mid \mathscr{F}_n] \geqq E[y_\lambda(n+1) \mid \mathscr{F}_n] \geqq y_\lambda(n),$$

which implies that y is a submartingale. Let $S \in \Pi_{S_2}$ be arbitrary. Since Y is directed upwards, there exists, for any fixed n, a sequence (λ_k) such that

$$y(n) = \lim_k y_{\lambda_k}(n).$$

Hence we have on the set $\{S = n\}$,

$$y(S) = y(n) = \lim_k y_{\lambda_k}(n) = \lim_k y_{\lambda_k}(S) = h_1(S).$$

This implies $y(S) = h_1(S)$. Consequently, we have $y \in C_1(x_2)$ and y is an upper bound of Y.

By Zorn's lemma, $C_1(x_2)$ has a maximal element x_1^*. Similarly $C_2(x_1)$ has a maximal element x_2^*. It is easy to see that

$$y_1, y_2 \in C_1(x_2) \text{ (resp. } C_2(x_1)) \Rightarrow y_1 \vee y_2 \in C_1(x_2) \text{ (resp. } C_2(x_1)). \tag{16}$$

Therefore (x_1^*, x_2^*) satisfies (14). The proof is complete.

Lemma 4. *Let* $(A, \leqq)$ *be a complete lattice and let* $f\colon A \to A$ *be a monotone non-decreasing mapping. Suppose that the set* $B = \{x \in A \mid x \leqq f(x)\}$ *is bounded above and not empty. Then* f *has at least one fixed point.*

Proof. The proof follows from a slight modification of the Knaster-Birkhoff theorem (see [1, Th. 9.25]). By assumption, B has a least upper bound $a \in A$. For any $x \in B$, we have

$$x \leqq f(x) \leqq f(a),$$

and then $a \leqq f(a)$. Furthermore,

$$f(a) \leqq f(f(a)),$$

which implies $f(a) \in B$ and $f(a) \leqq a$. Thus the lemma is proved.

Proof of Theorem 2. Let us define the mapping $m_i\colon X \to X$, $i = 1, 2$, by

$$\begin{aligned} m_1(x_2) &= x_1^* \\ m_2(x_1) &= x_2^*, \qquad (x_1, x_2) \in X \times X, \end{aligned} \tag{17}$$

where (x_1^*, x_2^*) is a maximal element of $C_1(x_2) \times C_2(x_1)$. We define the mapping $m\colon X \to X$ and the subset K of X by

$$m = m_1 \cdot m_2, \qquad K = \{x \in X \mid x \leqq m(x)\}. \tag{18}$$

We first show that m is monotone non-decreasing. For any $x_1, x_1' \in X$, $x_1 \leqq x_1'$, we have by (13)

$$C_2(x_1') \subset C_2(x_1).$$

Hence $m_2(x_1), m_2(x_1') \in C_2(x_1)$, and also by (16)

$$m_2(x_1) \vee m_2(x_1') \in C_2(x_1).$$

Thus

$$m_2(x_1') \leqq m_2(x_1) \vee m_2(x_1') = m_2(x_1).$$

Furthermore, by (13)

$$C_1(m_2(x_1)) \subset C_1(m_2(x_1')).$$

Hence $m(x_1), m(x_1') \in C_1(m_2(x_1'))$, and then

$$m(x_1) \vee m(x_1') \in C_1(m_2(x_1')).$$

Consequently

$$m(x_1) \leqq m(x_1) \vee m(x_1') = m(x_1').$$

Next, by (15) and (17)

$$m_1(x_2) \geqq h_1, \qquad x_2 \in X.$$

Therefore we get

$$m(h_1)=m_1(m_2(h_1))\geqq h_1.$$

This implies that K is not empty. For any $x\in K$, we have

$$m(x)\in C_1(m_2(x)),$$

and by (13)

$$x\leqq m(x)\leqq g_1.$$

This implies that K is bounded above. Consequently m has a fixed point x^* by Lemma 4.

Finally, we shall show that the pair (z_1, z_2) defined by

$$z_1=m(x^*)\in C_1(z_2), \qquad z_2=m_2(x^*)\in C_2(z_1) \tag{19}$$

satisfies (5), (6), (7) and (8). It is easy to see that (5), (6) and (8) are verified. For simplicity we set

$$u(n)=E[z_1(R\vee n)\,|\,\mathscr{F}_n], \qquad R=T^*\wedge S^*.$$

The optional sampling theorem yields $u\geqq z_1$. By (5) and (12) we have

$$u(n)\leqq E[g_1(R\vee n)\,|\,\mathscr{F}_n]\leqq g_1(n).$$

For any $S\in\Pi_{S^*}$

$$u(S)=E[z_1(R\vee S)\,|\,\mathscr{F}_S]=z_1(S)=h_1(S).$$

Moreover,

$$\begin{aligned} E[u(n+1)\,|\,\mathscr{F}_n]&=E[z_1(R\vee(n+1))\,|\,\mathscr{F}_n]\\ &=E[1_{(R>n)}z_1(R)+1_{(R\leqq n)}z_1(n+1)\,|\,\mathscr{F}_n]\\ &\geqq E[1_{(R>n)}z_1(R)+1_{(R\leqq n)}z_1(n)\,|\,\mathscr{F}_n]=u(n). \end{aligned}$$

Consequently u belongs to $C_1(z_2)$. By maximality $u=z_1$. Similarly (7) is verified for $i=2$. Thus the theorem is established.

§ 5. Remarks

Let us consider the case of (4). In [4], J.M. Bismut showed the existence of a saddle point under Mokobodzki's assumption:

There exist two submartingales $x, y\in X^{(-)}$ such that

$$h_1\leqq x-y\leqq g_1, \tag{20}$$

where

$$X^{(-)}=\{x\in X\,|\,x\leqq 0\}.$$

In this case, (11) and (12) are stronger than (20). Also, we can see that (20) is equivalent to the following condition:

There exist $x, y\in X^{(-)}$ such that

$$x\leqq Q(y+g_1), \qquad y\leqq Q(x-h_1), \tag{21}$$

where Qx denote the Snell envelope of x, i.e., $Qx(n)=\operatorname*{ess\,inf}_{T\geqq n}E[x(T)\,|\,\mathscr{F}_n]$.

Indeed, it is clear that (20) implies (21). We define the mapping w: $X^{(-)} \to X^{(-)}$ by

$$w(x) = Q(Q(x - h_1) + g_1).$$

Then the assumption of Lemma 4 is fulfilled by (21) and w has a fixed point in $X^{(-)}$. Thus the assertion follows.

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