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Probability Theory and Related Fields

An Alternative Approach to Multiply Self-Decomposable Probability Measures on Banach Spaces

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Summary. For every $\alpha>0$ we define α-times self-decomposable probability measures on Banach spaces by random power series. We prove the Urbanik and integral representation for such measures and discuss some related limit problems.

1. Introduction, Notation and Preliminaries

The classical Lévy-Hinčin theory of *self-decomposable* (s.d.) *probability measures* (p.m.'s) (cf. [14], pp. 195 and also [15], pp. 319) has been developed by many researchers. The main and very significant contributions to the subject are due to Urbanik [30–33]. His ideas and methods have been advanced in recent papers by Berg and Forst [2], Kumar and Schreiber [11–13], Jajte [8], Jurek [9], Jurek and Vervaat [10], Sato [19, 20] and the author [21–27]. In this framework we introduce a concept of *α-times self-decomposable probability measures* (s.d.p.m'.s) ($\alpha>0$) on a Banach space. Our definition is based on random power series and stands for a natural generalization of that of n-times s.d.p.m.'s ($n=1, 2, \ldots$). Thus we are concerned with a continuous subclassification of infinitely divisible (i.d.) p.m'.s on a Banach space into decreasing classes L_α ($0\leqq\alpha\leqq\infty$) of α-times s.d.p.m'.s such that each L_α is closed under convolution operation, shifts, changes of scales and passages to weak limits. An alternative approach to classes L_α was given in [25] by interpolation of Levy measures corresponding to n-times s.d.p.m'.s on the space.

Throughout the paper we shall denote by X a real separable Banach space with the norm $\|\cdot\|$ and the topological dual space X^*. We shall consider only Borel σ-additive measures on X. Given $c>0$ and a measure μ let $T_c\mu$ denote a measure defined by $T_c\mu(E)=\mu(c^{-1}E)(E\subset X)$. It is clear that the class of all p.m'.s are invariant under T_c. In the sequel the convergence of p.m'.s will be understood in the weak sense.

Recall [22, 24] that *a* p.m. μ *on* X *is n-times* s.d. $(n=1,2,\ldots)$ *if for any* $c_1,\ldots,c_n$ *in the open interval* $(0,1)$ *there exist* p.m.'s $\mu_1,\ldots,\mu_n$ such that

$$\mu=T_{c_1}\mu*\mu_1, \qquad \mu_1=T_{c_2}\mu_1*\mu_2,\ldots,\mu_{n-1}=T_{c_n}\mu_{n-1}*\mu_n, \tag{1.1}$$

where the asterisk $*$ *denotes the convolution operation of measures.*

Let $L_n(X)$ denote the class of all n-times s.d.p.m.'s on X. The class of all *completely* s.d.p.m.'s on X is defined as the intersection of all $L_n(X)$'s and denoted by $L_\infty(X)$. In addition, we denote by $L_0(X)$ the class of all i.d.p.m.'s on X. It is known [22] that $L_\infty(X)\subset L_{n+1}(X)\subset L_n(X)\subset L_0(X)$ $(n=1,2,\ldots)$. Further, for every $\mu\in L_0(X)$ its characteristic functional (ch.f.) $\hat{\mu}$ is of the form

$$\hat{\mu}(y)=\exp(i\langle y,x_0\rangle-1/2\langle y,Ry\rangle+\int_{X\setminus\{0\}} K(x,y)\,M(dx)) \tag{1.2}$$

$(y\in X^*)$, where x_0 is a vector in X, R a Gaussian covariance operator, M a Levy measure (a generalized Poisson exponent). The kernel K is given by

$$K(x,y)=\exp i\langle y,x\rangle-1-i\langle y,x\rangle 1_{B_1}(x) \tag{1.3}$$

$(x\in X,\ y\in X^*)$, where 1_{B_1} denotes the indicator of the unit ball B_1 in X. Since the triple $[x_0,R,M]$ in (1.2) determines μ uniquely we shall write $\mu=[x_0,R,M]$. In particular, if $x_0=0$, $R=0$ we shall denote $[0,0,M]$ simply by $[M]$. Recall [6, 28, 29] that the class of all Levy measures on X is a cone. Moreover, if $M\geqq N$ and M is a Levy measure then so is N. This property of Levy measures will be repeatedly exploited later on.

Proceeding successively, let us consider a particular case of (1.1). It is easy to check that for $n=1$ (1.1) *holds if and only if*

$$\mu=\mathop{*}_{k=0}^{\infty} T_{c^k}\nu$$

where $c=c_1$ and $\nu=\mu_1$. More generally, we get the following proposition:

1.1. Proposition. *A* p.m. μ *on* X *is n-times* s.d. $(n=1,2,\ldots)$ *if and only if for every* $c\in(0,1)$ *there exists a* p.m. ν *such that*

$$\mu=\mathop{*}_{k=0}^{\infty}(T_{c^k}\nu)^{r_{k,n}} \tag{1.4}$$

where the power is taken in the convolution sense, $r_{k,n}$ *is the number of solutions of the equation* $x_1+\ldots+x_n=k$ $(n=1,2,\ldots,k=0,1,2,\ldots)$ *in non-negative integers and*

$$r_{k,n}=\binom{n+k-1}{k}. \tag{1.5}$$

We precede the proof of Proposition 1.1 by proving several lemmas.

1.2. Lemma. *Let* $Z, Z_{k_1,\ldots,k_n}$ $(k_1,\ldots,k_n=0,1,2,\ldots,\ n=1,2,\ldots)$ *be* i.i.d. *X-valued random variables such that for some* $c_1,\ldots,c_n\in(0,1)$ *the series*

$$\sum_{k_1,\ldots,k_n=0}^{\infty} c_1^{k_1}\ldots c_n^{k_n} Z_{k_1,\ldots,k_n} \tag{1.6}$$

is convergent with probability 1. *Then, it is absolutely convergent with probability* 1.

Consequently, if ν *is a* p.m. *on* X, $c\in(0,1)$, $n=1,2,\ldots$ *and the infinite convolution*

$$\mathop{*}_{k=0}^{\infty} (T_{c^k}\nu)^{r_{k,n}} \tag{1.7}$$

is convergent, then its limit does not depend on the order of convolution.

Proof. From results in [27] it follows that the series (1.6) is convergent with probability 1 if and only if

$$E\log^n(1+\|Z\|)<\infty. \tag{1.8}$$

Hence we get

$$\sum_{k_1,\ldots,k_n=0}^{\infty} c_1^{k_1}\ldots c_n^{k_n}\|Z_{k_1\ldots k_n}\|<\infty \qquad (P.1), \tag{1.9}$$

which proves the first statement of the Lemma. To prove the second statement it suffices to take ν as the distribution of Z and $c_1=\ldots=c_n=c$.

Given $c\in(0,1)$ let us denote by $K_{c,n}(X)$ $(n=1,2,\ldots)$ the class of all p.m.'s μ on X such that (1.4) holds for some p.m. ν. Further, we put

$$K_n(X)=\bigcap_{c\in(0,1)} K_{c,n}(X) \qquad (n=1,2,\ldots).$$

1.3. Lemma. *A* p.m. μ *on* X *belongs to* $K_{c,n}(X)$ $(n=1,2,\ldots)$ *if and only if there exist* p.m.'s $\mu_1,\ldots,\mu_n$ *such that*

$$\mu=T_c\mu*\mu_1, \qquad \mu_1=T_c\mu_1*\mu_2,\ldots,\mu_{n-1}=T_c\mu_{n-1}*\mu_n. \tag{1.10}$$

In other words, $\mu\in K_{c,n}(X)$ *if and only if it is n-times c-decomposable* (cf. [24], pp. 7).

Proof. Suppose that $\mu\in K_{c,n}(X)$ and the decomposition (1.4) holds for some p.m. ν. Then (1.10) holds with $\mu_n=\nu$, $\mu_{n-1}=\mathop{*}_{k=0}^{\infty} T_{c^k}\nu,\ldots,\ \mu_1=\mathop{*}_{k=0}^{\infty}(T_{c^k}\nu)^{r_{k,n-1}}$

(Remark, by Lemma 1.2, that the last infinite convolutions are convergent).

Next suppose that (1.10) holds. We shall prove that $\mu\in K_{c,n}(X)$.

Obviously, for $n=1$ (1.10) implies (1.4). Suppose that it is true for $n=k-1$ and assume that μ satisfies (1.10) for $n=k$. Then

$$\mu=\mathop{*}_{p=0}^{\infty}(T_{c^p}\mu_{k-1})^{r_{p,k-1}}, \qquad \mu_k=\mathop{*}_{m=0}^{\infty} T_{c^m}\mu_k$$

and hence

$$\mu=\mathop{*}_{p=0}^{\infty}\mathop{*}_{m=0}^{\infty}(T_{c^{p+m}}\mu_k)^{r_{p,k-1}}.$$

Since $r_{0,k-1}+r_{1,k-1}+\ldots+r_{m,k-1}=r_{m,k}$ we get

$$\mu=\mathop{*}_{m=0}^{\infty}(T_{c^m}\mu_k)^{r_{m,k}}\in K_{c,k}(X).$$

(Here we make rearrangement of infinite convolution of p.m.'s. Such a rearrangement is justified by Lemma 1.2.).

1.4. Corollary. *For any* $n=1,2,\ldots$ *and* $c\in(0,1)$

$$\begin{aligned} K_{c,n}(X) &\subset K_{c,n-1}(X),\\ K_n(X) &\subset K_{n-1}(X),\\ L_n(X) &\subset K_n(X)\subset L_0(X). \end{aligned} \tag{1.11}$$

Moreover, if $\mu\in K_{c,n}(X)$ *then for every* $p=1,2,\ldots$ $\mu\in K_{c^p,n}(X)$.

Proof. From (1.10) we get inclusions (1.11). Again by (1.10) and by Proposition 1.1 in [24] it follows that if μ is n-times c-decomposable i.e. $\mu\in K_{c,n}(X)$ then it is n-times c^p-decomposable for every $p=1,2,\ldots$.

Suppose that $\mu\in K_{c,1}(X)$ and $\hat{\mu}(y)\neq 0$ for every $y\in X^*$. Then there exists a unique p.m. ν such that

$$\mu=T_c\mu*\nu.$$

Let us denote ν by $R_c\mu$. Note, by (1.11), that $K_1(X)$ is contained in the domain of the transform R_c.

1.5. Lemma. *Let* μ *be a* p.m. *in* $L_{c,1}(X)$ *and* $\hat{\mu}(y)\neq 0$ *for every* $y\in X^*$. *Then, for every* $m=1,2,\ldots$

$$R_{c^m}\mu=\mathop{*}_{k=0}^{m-1} T_{c^k}(R_c\mu). \tag{1.12}$$

Proof. Since, for $m=1,2,\ldots,$

$$\begin{aligned} \mu &=\mathop{*}_{k=0}^{\infty} T_{c^k}(R_c\mu)\\ &=\mathop{*}_{p=0}^{\infty} T_{c^{pm}}\left(\mathop{*}_{k=0}^{m-1} T_{c^k}(R_c\mu)\right)\\ &=T_{c^m}\mu*\mathop{*}_{k=0}^{m-1} T_{c^k}(R_c\mu) \end{aligned}$$

and since $\hat{\mu}(y)\neq 0$ for every $y\in X^*$ we get (1.12).

1.6. Lemma. *For any* $c\in(0,1)$ *and* $n=1,2,\ldots$

$$R_c(K_{n+1}(X))\subset K_n(X). \tag{1.13}$$

Proof. Let μ belong to $K_{n+1}(X)$ and c be an arbitrary number in $(0,1)$. By (1.12) we get

$$R_c\mu=\mathop{*}_{k=0}^{m-1} T_{c^{k/m}}(R_{c^{1/m}}\mu)\qquad (m=1,2,\ldots). \tag{1.14}$$

Since, by Lemma 1.3, $R_{c^{1/m}}\mu$ belongs to $K_{c^{1/m},n}(X)$ we infer, by (1.14), that $R_c\mu$ belongs to $K_{c^{1/m},n}(X)$. Hence and by Corollary 1.4 it follows that $R_c\mu$ belongs to $K_{c^{p/m},n}$ for any $p,m=1,2,\ldots$. Consequently, $R_c\mu$ belongs to $K_{a,n}(X)$ for every $a\in(0,1)$. Thus $R_c\mu\in K_n(X)$, which completes the proof.

Proof of Proposition 1.1. By virtue of (1.11) it suffices to prove the inclusion

$$K_n(X) \subset L_n(X) \qquad (n=1,2,\dots). \tag{1.15}$$

We shall prove (1.15) by induction. For $n=1$ (1.17) is evident. Suppose that it is true for $n=k$ and assume that $\mu \in K_{k+1}(X)$. Given $c \in (0,1)$ we get, by Lemma 1.6, $R_c \mu \in K_k(X)$. Hence and by induction assumption it follows that $R_c \mu \in L_k(X)$. Consequently, $\mu \in L_{k+1}(X)$, which completes the proof.

Remark. By virtue of Theorem 2.6 [11] it follows that for every $\mu \in L_n(X)$ and $c \in (0,1)$ the measure ν in (1.4) is i.d.

Proposition 1.1 suggests us the following interpolation of classes $L_n(X)$ ($n = 1,2,\dots$):

For every number $\alpha > 0$ we put

$$\binom{\alpha}{k} = \begin{cases} 1 & k=0 \\ \alpha(\alpha-1)\dots(\alpha-k+1)/k! & k=1,2,\dots \end{cases} \tag{1.16}$$

and

$$\left|\begin{matrix}\alpha \\ k\end{matrix}\right| = \left|\binom{\alpha}{k}\right|. \tag{1.17}$$

A p.m. μ *on* X *is said to be* α*-times* s.d. *if for every* $c \in (0,1)$ *there exists an* i.d. p.m. $\mu_{c,\alpha}$ *on the space such that*

$$\mu = \mathop{*}_{k=0}^{\infty} (T_{c^k} \mu_{c,\alpha})^{r_{k,\alpha}} \tag{1.18}$$

where

$$r_{k,\alpha} = \binom{\alpha+k-1}{k} \qquad (k=0,1,2,\dots). \tag{1.19}$$

Let $L_\alpha(X)$ denote the class of all α-times s.d.p.m.'s on X.

It is easy to see that $\mu \in L_\alpha(X)$ if and only if for every $c \in (0,1)$ there exist a p.m. $\nu \in L_0(X)$ and a sequence $Z_0, Z_1, Z_2, \dots$ of independent X-valued random variables with distributions $\nu, \nu^{r_{1,\alpha}}, \nu^{r_{2,\alpha}}, \dots$ respectively such that μ is the distribution of the following random power series:

$$\sum_{k=0}^{\infty} c^k Z_k. \tag{1.20}$$

The paper is divided into the following sections: In §1 we give basic concepts, introduction, notations and prelimilaries. In §2 we present some characterizations of α-times s.d.p.m.'s on X. In §3 the Urbanik representation of p.m.'s in $L_\alpha(X)$ is given. Here we again emphasize the extreme point method stimulated by many results of Urbanik. In §4 we introduce the one - parameter semigroup J^α of operators on semi-finite measures. We prove that $J^\alpha G$ is a Levy measure if and only if G is a Levy measure. Further, following [26] we define α-integral on p.m.'s and obtain a complete characterization of α-integrable p.m.'s on X. As a consequence we show that every α-times s.d.p.m. on X can be represented as an α-integral. Section 5 is devoted to fundamental properties of classes $L_\alpha(X)$. Section 6 contains a concluding remark concerning an improved definition of fractional derivatives of p.m.'s. Finally, in Appendix we prove a representation of α-times monotone functions.

§ 2. A Characterization of Measures in $L_\alpha(X)$

In this Section we modify the classical definition of s.d.p.m'.s to the case of α-times s.d.p.m'.s. Namely we replace the role of the operator T_c by $T_{c,\alpha}$ which is defined on the whole of $L_0(X)$ as follows:

$$T_{c,\alpha}\mu = \mathop{*}_{k=1}^{\infty} (T_{c^k}\mu)^{\left|\binom{\alpha}{k}\right|} \tag{2.1}$$

where $0<c<1$, $0<\alpha<1$ and $\mu \in L_0(X)$. Such an operator was introduced and studied in [26]. It should be noted that if $0<\alpha<1$ then $\sum_{k=1}^{\infty}\left|\binom{\alpha}{k}\right| = 1$ and hence the right-hand side of (2.1) is convergent for every $\mu \in L_0(X)$. Further, for $\alpha = 1$ $T_{c,\alpha}$ is reduced to T_c.

Recall [26] that if μ is s.d. then it is $T_{c,\alpha}$-decomposable for any $0<c<1$ and $0<\alpha<1$. The same is true for the more general case. Namely, we get the following theorem:

2.1. Theorem. *An* i.d.p.m. μ on X is α-times s.d., *where* $0<\alpha<1$, *if and only if for every* $0<c<1$ *there exists a* p.m. $\mu_{c,\alpha}$ *in* $L_0(X)$ *such that*

$$\mu = T_{c,\alpha}\mu * \mu_{c,\alpha}. \tag{2.2}$$

Proof. Suppose that $\mu \in L_\alpha(X)$ with $0<\alpha<1$. Then for every $0<c<1$ there exists an i.d.p.m. $\mu_{c,\alpha}$ such that the equation (1.18) holds. By (1.18) and (2.1) we get

$$\begin{aligned} T_{c,\alpha}\mu &= \mathop{*}_{n=1}^{\infty} \mathop{*}_{k=0}^{\infty} (T_{c^{k+n}}\mu_{c,\alpha})^{\left|\binom{\alpha}{n}\right| r_{k,\alpha}} \\ &= \mathop{*}_{m=1}^{\infty} (T_{c^m}\mu_{c,\alpha})^{\sum_{n=1}^{m}\left|\binom{\alpha}{n}\right| r_{m-n,\alpha}} \end{aligned}$$

which together with the fact that

$$\sum_{n=1}^{m}\left|\binom{\alpha}{n}\right| r_{m-n,\alpha} = r_{m,\alpha} \qquad (m=1,2,\ldots) \tag{2.3}$$

implies the formula

$$T_{c,\alpha}\mu = \mathop{*}_{m=1}^{\infty} (T_{c^m}\mu_{c,\alpha})^{r_{m,\alpha}}. \tag{2.4}$$

Hence and by (1.18) we get (2.2).

Conversely, suppose that for every c in $(0,1)$ the decomposition (2.2) holds for some p.m. $\mu_{c,\alpha} \in L_0(X)$. Let $\mu = [x_0, R, M]$. Then we get the formula

$$T_{c,\alpha}\mu=\left[\sum_{k=1}^{\infty}\left|\begin{matrix}\alpha\\k\end{matrix}\right|c^k x_0,\ \sum_{k=1}^{\infty}\left|\begin{matrix}\alpha\\k\end{matrix}\right|c^{2k}R,\ \sum_{k=1}^{\infty}\left|\begin{matrix}\alpha\\k\end{matrix}\right|T_{c^k}M\right] \tag{2.5}$$

which, by (2.2), implies that $\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}T_{c^k}M$ is a Levy measure and

$$\mu_{c,\alpha}=\left[(1-c)^\alpha x_0,\ (1-c^2)^\alpha R,\ \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}T_{c^k}M\right]. \tag{2.6}$$

Further, since

$$\sum_{k=0}^{\infty} r_{k,\alpha}x^k=(1-x)^{-\alpha}\qquad(0<x<1) \tag{2.7}$$

and

$$\sum_{n=0}^{m}(-1)^{m-n}r_{n,\alpha}\binom{\alpha}{m-n}=\begin{cases}1 & m=0\\ 0 & m=1,2,\dots\end{cases} \tag{2.8}$$

it follows, by (2.6), that

$$\mu=\mathop{*}_{n=0}^{\infty}(T_{c^n}\mu_{c,\alpha})^{r_{n,\alpha}} \tag{2.9}$$

which shows that $\mu\in L_\alpha(X)$. Thus the Theorem is completely proved.

2.2. Corollary. *Suppose that $\mu\in L_0(X)$ and $\alpha_1,\alpha_2,\dots$ is a sequence of numbers from the interval* $(0,1]$. *Then $\mu\in L_\alpha(X)$, where $\alpha=\alpha_1+\alpha_2+\dots$, if and only if for every $c\in(0,1)$ there exists a sequence $\mu_1,\mu_2,\dots$ of* p.m.'s *in $L_0(X)$ such that*

$$\mu=T_{c,\alpha_1}\mu*\mu_1,\qquad \mu_1=T_{c,\alpha_2}\mu_1*\mu_2,\dots. \tag{2.10}$$

Proof. Suppose first that $\alpha<\infty$ and $\mu\in L_\alpha(X)$. Then, for every $c\in(0,1)$ there exists a p.m. $\mu_{c,\alpha}$ in $L_0(X)$ such that (1.18) holds. Define $\mu_0=\mu_{c,\alpha}$ and

$$\mu_n=\mathop{*}_{k=0}^{\infty}(T_{c^k}\mu_{n-1})^{r_{k,\alpha_n}}\qquad(n=1,2,\dots). \tag{2.11}$$

By virtue of Theorem 2.1 it follows that p.m.'s $\mu_1,\mu_2,\dots$ satisfy (2.10).

To prove "if" part of the Corollary one may assume, without loss of generality, that $0<\alpha<1$. Given $0<c<1$ let $\mu_1,\mu_2,\dots$ be a sequence of p.m.'s in $L_0(X)$ satisfying (2.10). Then it is easy to check that

$$\mu=T_{c,s_n}\mu*\mu_n, \tag{2.12}$$

where $s_n=\alpha_1+\dots+\alpha_n$, $n=1,2,\dots$. Letting $n\to\infty$ we infer, by a simple reasoning, that $T_{c,s_n}\mu$ converges to $T_{c,\alpha}\mu$ and μ_n converges to some p.m. $\mu_{c,\alpha}$ and hence (2.2) holds. Thus the Corrollary is proved for the case $\alpha<\infty$.

To prove the Corollary for the case $\alpha=\infty$ it suffices to observe, by virtue of the case $\alpha<\infty$, that for any $0<\gamma<\beta<\infty$ we get the inclusion

$$L_\beta(X)\subset L_\gamma(X) \tag{2.13}$$

which will be proved in Theorem 5.1. Thus the Corollary is proved.

2.3. Corollary. *An* i.d.p.m. $\mu=[x_0, R, M]$ *belongs to* $L_\alpha(X)$, *where* $0<\alpha<\infty$, *if and only if*

$$\Delta_c^\alpha M(E) := \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} T_{c^k} M(E) \geqq 0 \tag{2.14}$$

for any $0<c<1$ *and Borel subset* E *of* X *separated from* 0.

Proof. By virtue of Theorem 2.1 in [22] and Corollary 2.2 it suffices to prove the Corollary for the case $0<\alpha<1$. This, however, is an easy consequence of (2.2) and (2.5). Thus the Corollary is fully proved.

§ 3. The Urbanik Representation

In this Section we prove the *Urbanik representation* for α-times s.d.p.m'.s on X. The method of the proof is based on the fundamental work [33] by Urbanik and analogous to that in [22].

For every subset E of X we put $\tau(E)=\{tx: x\in E, t>0\}$. It is clear that the set $\tau(E)$ is invariant i.e. $\tau(\tau(E))=\tau(E)$.

The following lemma is an analogue of Lemma 3.4 [22] and its proof will be omitted.

3.1. Lemma. *Let* M *be a Levy measure corresponding to a* p.m. $\mu\in L_\alpha(X)$ $(\alpha>0)$. *Then there exists a decomposition* $M=\sum_{k=1}^{\infty} M_k$, *where every* M_k $(k=1,2,\ldots)$ *is a Levy measure corresponding to a* p.m. *in* $L_\alpha(X)$, M_k *are concentrated on disjoint sets* $\tau(E_k)$, $0\in E_k$ *and* E_k *are compact.*

This Lemma reduces our problem of examining Levy measures corresponding to p.m'.s in $L_\alpha(X)$ to the case of measures concentrated on $\tau(E)$, where E is compact and $0\notin E$.

Let Φ be a weight function on X in the Urbanik sense [33]. Let $\bar{\tau}(E)$ denote a compactification of $\tau(E)$ where E is a compact subset of X and $0\notin E$. In what follows the set $\bar{\tau}(E)$ will be defined as $(\tau(E)\cap S)$ $x[-\infty,\infty]$ with the product topology, where S denotes the unit sphere in X. Then the mapping $e^{-t}x\mapsto(x,t)$ is an embedding of $\tau(E)$ into a dense subset of $\bar{\tau}(E)$ and the function Φ as well as the norm $\|\cdot\|$ can be extended from $\tau(E)$ onto $\bar{\tau}(E)$ by continuity (cf. [22]).

Given a finite Borel measure N on $\bar{\tau}(E)$ we put

$$M(U)=\int_U \frac{N(du)}{\Phi(u)} \qquad (U\subset\bar{\tau}(E)). \tag{3.1}$$

For $0<c<1$ and $\alpha>0$ define $\Delta_c^\alpha M$ in the same manner as in (2.14). Further, let $H_\alpha(E)$ denote the class of all finite measures N on $\bar{\tau}(E)$ for which the corresponding measures M defined by (3.1) fulfil the condition that for every $0<c<1$ the measures $\Delta_c^\alpha M$ are non-negative. Let $P_\alpha(E)$ be the subset of $H_\alpha(E)$ consisting of p.m'.s. The set $P_\alpha(E)$ is convex and compact. Moreover, its extreme points must be concentrated on orbits of elements of $\bar{\tau}(E)$. Obviously, all measures $\delta_z, z\in\bar{\tau}(E)\setminus\tau(E)$, are extreme points of $P_\alpha(E)$ and its other extreme points are concentrated on sets $\tau(\{x\})$, where $x\in\tau(E)$.

Let us fix $x\in\tau(E)$ with $\|x\|=1$ and let N be an extreme point of $P_\alpha(E)$ concentrated on $\tau(\{x\})$. Define, for $b\in R^1$,

$$g_N(b)=M(\{e^{-t}x: t<b\}) \tag{3.2}$$

where M is defined by (3.1). Then, by Corollary 2.3 it follows that for every $t>0$, $c=e^{-t}$ and for every set U of the form $U=\{e^{-t}x:\ a\leqq t<b\}$ we get the formula

$$\Delta_t^\alpha g_N(b)-\Delta_t^\alpha g_N(a)=\Delta_c^\alpha M(U)\geqq 0, \tag{3.3}$$

where the fractional difference operator Δ_t^α is defined by (A.4) in Appendix. Hence it follows that g_N is α-times monotone (see Appendix for definition of α-times monotone functions). By Theorem A.2 it follows that there exists a unique non-negative left-continuous monotone non-decreasing function P_N on R^1 such that

$$g_N(t)=I^\alpha P_N(t) \tag{3.4}$$

where I^α is the integral of fractional order $\alpha>0$ defined by (A.5) which together with (3.1), (3.2) and (3.3) implies that for any $a<b$

$$N(\{e^{-t}x: a\leqq t<b\})=\int_a^b \Phi(e^{-t}x)\, I^{\alpha-1}P_N(t)\,dt. \tag{3.5}$$

Consequently, we have

$$\int_{-\infty}^{\infty} \Phi(e^{-t}x)\, I^{\alpha-1}P_N(t)\,dt=1. \tag{3.6}$$

Conversely, every non-negative left-continuous monotone non-decreasing function P_N normalized by the condition (3.6) determines, by (3.5), a p.m. N concentrated on $\tau(\{x\})$. Moreover, by Theorem A.2 the corresponding function g_N defined by (3.4) is α-times monotone which shows that $N\in P_\alpha(E)$. Hence we conclude that a measure N in $P_\alpha(E)$ is an extreme point if and only if the corresponding function P_N cannot be decomposed into a non-trivial convex combination of two functions P_{N_1} and P_{N_2} with the stated properties. But this is the case if and only if $P_N(t)=0$ for $t\leqq t_0$ and $P_N(t)=v$ for $t>t_0$, where t_0 and v are some constants. By (3.5) we get the formula

$$N(\{e^{-t}x: a\leqq t<b\})=v\int_a^b \Phi(e^{-t}x)\, 1_{(t_0,\infty)}(t)(t-t_0)^{\alpha-1}\,dt. \tag{3.7}$$

The constant v is determined by (3.6). Namely,

$$v^{-1}=\int_{t_0}^{\infty} \Phi(e^{-t}x)(t-t_0)^{\alpha-1}\,dt. \tag{3.8}$$

Conversely, one can prove that for every $t_0 \in R^1$ the measure N defined by (3.7) is an extreme point of $P_\alpha(E)$ concentrated on $\tau(\{x\})$.

Let z be an arbitrary element of $\tau(E)$. Substituting $x = z/\|z\|$ and $t_0 = -\log\|z\|$ into formula (3.7) and (3.8), we get all extreme points N_z^α of $P_\alpha(E)$ concentrated on $\tau(E)$ as follows:

$$N_z^\alpha(U) = v_\alpha(z) \int_0^\infty 1_U(e^{-t} z)\, \Phi(e^{-t} z)\, t^{\alpha-1}\, dt, \tag{3.9}$$

where U is a Borel subset of $\tau(E)$ and

$$v_\alpha^{-1}(z) = \int_0^\infty \Phi(e^{-t} z)\, t^{\alpha-1}\, dt. \tag{3.10}$$

It should be noted, by definition of a weight function (cf. [33]), that the right-hand side of (3.10) is always finite.

Now remark that every element of $H_\alpha(E)$ is of the form tN_1 with $N_1 \in P_\alpha(E)$ and $t > 0$. Further, if a measure M concentrated on $\tau(E)$ is a Levy measure corresponding to a p.m. in $L_\alpha(X)$ then the measure N associated with M in (3.1) belongs to $H_\alpha(E)$. In this case, by Krein-Milman-Choquet theorem (see [5] and [18], Chap. 3), there exists a finite measure m on $\tau(E)$ such that

$$\int_{\tau(E)} f(x)\, M(dx) = \int_{\tau(E)} v_\alpha(z) \int_0^\infty f(e^{-t} z)\, t^{\alpha-1}\, dt\, m(dz) \tag{3.11}$$

for every M-integrable function f on $\tau(E)$. Hence and by Lemma 3.1 and by the same arguments as in [22] it follows that for every Levy measure M corresponding to a p.m. in $L_\alpha(X)$ there exists a finite measure m on X such that $m(\{0\}) = 0$ and

$$\int_X f(x)\, M(dx) = \int_X v_\alpha(z) \int_0^\infty f(e^{-t} z)\, t^{\alpha-1}\, dt\, m(dz) \tag{3.12}$$

for every M-integrable function f on X, which together with (1.2) yields the following theorem:

3.2. Theorem (The Urbanik representation). *Let Φ be a weight function on X (in Urbanik's sense [33]). A* p.m. *μ on X is α-times* s.d. ($\alpha > 0$) *if and only if there exist a finite measure m on X vanishing at* 0, *a covariance operator R and an element $x_0 \in X$ such that*

$$\hat{\mu}(y) = \exp\Big(i\langle y, x_0\rangle - 1/2\langle y, Ry\rangle + \int_X v_\alpha(x) \int_0^\infty K(e^{-t} x, y)\, t^{\alpha-1}\, dt\, m(dx)\Big) \tag{3.13}$$

($y \in X^*$), *where $v_\alpha(x)$ is defined by* (3.10) *and the kernel K is given by* (1.3).

§ 4. The Integral Representation

Consider a Levy measure M corresponding to a p.m. in $L_\alpha(X)$. By virtue of (3.12) we get the formula

$$M(E)=\int_X \int_0^\infty 1_E(e^{-t}x)\,t^{\alpha-1}\,dt\, v_\alpha(x)\, m(dx) \tag{4.1}$$

($E\subset X\setminus\{0\}$), where m is a finite measure on X vanishing at 0 and $v_\alpha(x)$ is given by (3.10).

Putting $G(dx)=\Gamma(\alpha)\, v_\alpha(x)\, m(dx)$ and taking into account (4.1) we get a measure G which is finite outside every neigh-bourhood of 0 and

$$M(E)=1/\Gamma(\alpha)\int_X \int_0^\infty 1_E(e^{-t}x)\,t^{\alpha-1}\,dt\, G(dx) \tag{4.2}$$

($E\subset X$Ä$\{0\}$). Moreover, $M(\{0\})=0$ and since $M(B_1')<\infty$ it follows that

$$\int_{B_1'} \log^\alpha \|x\|\; G(dx)<\infty. \tag{4.3}$$

Let $G_\alpha(X)$ ($\alpha\geqq 0$) denote the class of all measures G on X such that G are finite outside every neighbourhood of 0, $G(\{0\})=0$ and the condition (4.3) is satisfied.

Following [25] we introduce an operator $J^\alpha(\alpha>0)$ from $G_\alpha(X)$ into $G_0(X)$ by

$$J^\alpha G(E)=1/\Gamma(\alpha)\int_X \int_0^\infty 1_E(e^{-t}x)\,t^{\alpha-1}\,dt\, G(dx), \tag{4.4}$$

where $G\in G_\alpha(X)$ and $E\subset X\setminus\{0\}$. Recall ([25], Theorems 2.2 and 2.9) that J^α is one-to-one and

$$J^\alpha J^\beta G=J^{\alpha+\beta} G \tag{4.5}$$

for any $\alpha,\beta>0$ and $G\in G_{\alpha+\beta}(X)$.

It is natural to ask whether J^α and its converse transform Levy measures into themselves? In what follows we shall give an affirmative answer to this question.

The following theorem was proved by Jurek and Vervaat in [10]:

4.1. Theorem. *A* p.m. μ *on* X *is* s.d. *if and only if there exists an X-valued process $Y(t)$ with independent and stationary increments such that* $Y(0)=0$ $(P.1)$ *and μ is the distribution of the pathwise Laplace-Stieltjes integral* $\int_0^\infty e^{-t}\,Y(dt)$.

Suppose that μ and $Y(t)$ are the same as above with Levy measures M and G_t, respectively. By Theorem 4.1 and by a simple computation it follows that $M=J^1G_1$. Hence and by the fact that J^1 is one-to-one it follows that if $G\in G_1(X)$ and if one of G and J^1G is a Levy measure then the other is also a Levy measure. Moreover, by iterating the operator J^1 we get

4.2. Corollary. *For any $n=1,2,\ldots$ and $G\in G_n(X)\,J^nG$ is a Levy measure if and only if G is a Levy measure.*

The more general case is also true. Namely, we get

4.3. Theorem. *For any $\alpha>0$ and $G\in G_\alpha(X)\,J^\alpha G$ is a Levy measure if and only if G is a Levy measure.*

Proof. Define two auxiliary operators J_1^α and J_2^α as follows:

$$J_1^\alpha G(E) = 1/\Gamma(\alpha) \int_X \int_0^1 1_E(e^{-t}x)\, t^{\alpha-1}\, dt\, G(dx) \tag{4.6}$$

and

$$J_2^\alpha G = J^\alpha G - J_1^\alpha G \tag{4.7}$$

$(G \in G_\alpha(X),\ E \subset X \setminus \{0\})$.
Let G be a measure in $G_\alpha(X)$. Since $J^\alpha G$ is a Levy measure for any finite G one may assume, without loss of generality, that $G(B_1') = 0$. Consider the following cases:

Case 1: $\alpha > 1$ and G is a Levy measure. In this case we get the inequalities:

$$J_1^\alpha G \leqq 1/\Gamma(\alpha) J_1^1 G \tag{4.8}$$

and

$$J_2^\alpha G \leqq [\alpha]!/\Gamma(\alpha) J_2^{[\alpha]+1} G \tag{4.9}$$

where $[\alpha]$ is the integer part of α. Further, by Corollary 4.2 it follows that $J^1 G$ and $J^{[\alpha]+1} G$ are Levy measures, which by (4.6) and (4.7) implies that $J_1^1 G$ and $J_2^{[\alpha]+1} G$ are Levy measures. Hence and by (4.8) and (4.9) it follows that $J^\alpha G$ is a Levy measure.

Case 2: $\alpha > 1$ and $J^\alpha G$ is a Levy measure. By virtue of (4.6) and (4.7) we get the inequalities

$$[\alpha]!/\Gamma(\alpha) J_1^{[\alpha]+1} G \leqq J_1^\alpha G, \tag{4.10}$$

$$1/\Gamma(\alpha) J_2^1 G \leqq J_2^\alpha G \tag{4.11}$$

and for $n = 1, 2, \ldots$

$$J_2^{n+1} G \leqq J^n(J_2 G). \tag{4.12}$$

Further, since $J^\alpha G$ is a Levy measure it follows that J_1^α and $J_2^\alpha G$ are Levy measures, which by (4.10) and (4.11) implies that $J_1^{[\alpha]+1} G$ and $J_2^1 G$ are Levy measures. On the other hand, by assumption that $G(B_1') = 0$ it follows that $J_2^1 G(B_1') = 0$. Hence and by Case 1 we infer that for every $n = 1, 2, \ldots$ $J^n(J_2^1 G)$ is a Levy measure, which by (4.12) implies that $J_2^{n+1} G$ is a Levy measure. Thus $J_1^{[\alpha]+1} G$ and $J_2^{[\alpha]+1} G$ are Levy measures. Hence $J^{[\alpha]+1} G$ is a Levy measure and by Corollary 4.2 we conclude that G is a Levy measure.

Case 3: $0 < \alpha \leqq 1$.
Since for every $n = 1, 2, \ldots$ $J^{\alpha+n} G = J^n(J^\alpha G)$ it follows, by Corollary 4.2 and Cases 1 and 2, that if one of G and $J^\alpha G$ is a Levy measure then the other is also a Levy measure.

Finally, combining Cases 1, 2 and 3, the proof of the Theorem is completed.

Consider a p.m. $\mu = (x_0, R, M]$ in $L_\alpha(X)$. By (4.2), (4.4) and Theorem 4.3 we get the following theorem:

4.4. Theorem. *Every α-times* s.d.p.m. *μ on X is of the unique form*

$$\mu=[x_0, R, J^\alpha G] \tag{4.13}$$

where $x_0 \in X$, R is a Gaussian covariance operator and G a Levy measure in $G_\alpha(X)$.

Conversely, for any x_0, R and G as above the formula (4.13) *defines an α-times* s.d.p.m. *on X.*

We now proceed to give an *integral representation* for measures in $L_\alpha(X)$. Using the same terminology as in [26] we say that *μ is a simple Poisson* p.m. *if $\mu=[G]$, where G is supported by a finite subset of $X\setminus\{0\}$.* Obviously such a measure G is a Levy measure and moreover for every $\alpha>0$ $J^\alpha G$ is a Levy measure.

For every simple Poisson p.m. $\mu=[G]$ we put

$$I^\alpha \mu=[J^\alpha G]. \tag{4.14}$$

Further, *a p.m. v on X is said to be α-integrable if there exists a sequence $\{v_n\}$ of simple Poisson* p.m.'s *such that for some point $x \in X$ $v_n * \delta_x$ converges to v and $I^\alpha v_n * \delta_x$ converges to some* p.m., *say $I^\alpha v$.* The limit measure $I^\alpha v$ depends on v and α only and will be called *α-integral* of v. Such a concept was introduced in [26]. Further, we obtained in [26] a characterization of α-integrable p.m.'s on X under some geometric conditions. Our further aim is to prove a complete characterization of α-integrable p.m.'s on X.

The following theorem was proved in [26]:

4.5. Theorem. *A* p.m. *$v=[x_0, R, G]$ on X is α-integrable if and only if $J^\alpha G$ is a Levy measure.*

Now, by Theorem 4.3 it follows that for $v=[x_0, R, G]$ $J^\alpha G$ is a Levy measure if and only if $G \in G_\alpha(X)$. Furthermore, by Lemma 2.5 in [26] it follows that $G \in G_\alpha(X)$ if and only if

$$\int_X \log^\alpha(1+\|x\|)\, v(dx)<\infty. \tag{4.15}$$

Hence and by Theorem 4.5 we get the following

4.6. Theorem. *An* i.d.p.m. *v on X is α-integrable* ($\alpha>0$) *if and only if the condition* (4.15) *is satisfied.*

Proceeding successively, if $v=[x_0, R, G]$ is α-integrable then

$$I^\alpha v=[x_0, 2^{-\alpha} R, J^\alpha G] \tag{4.16}$$

(cf. [26], Lemma 2.1) which together with Theorem 4.4 implies the following *integral representation* of measures in $L_\alpha(X)$:

4.7. Theorem. *An* i.d.p.m. *μ on X is α-times* s.d. *if and only if there exists a unique* i.d.p.m. *v such that*

$$\mu=I^\alpha v. \tag{4.17}$$

As a consequence of the above Theorem we get

4.8. Corollary. *The set* $\{\mu=I^{\alpha}[G]$: *G is concentrated on a finite subset of* $X \setminus\{0\}\}$ *is dense in* $L_\alpha(X)$.

§5. The Continuity and Monotonicity of $L_\alpha(X)$

The aim of this Section is to prove some fundamental properties of the classes $L_\alpha(X)$. Namely, we get the following theorem:

5.1. Theorem. (i) *Every* $L_\alpha(X)$ $(0\leqq\alpha\leqq\infty)$ *is closed under convolution operation, shifts, changes of scales and passages to weak limit.*

(ii) *For any* $0\leqq\gamma<\beta\leqq\infty$

$$L_\beta(X) \subsetneqq L_\gamma(X), \tag{4.18}$$

$$L_\beta(X)=\bigcap_{\gamma<\beta} L_\gamma(X) \tag{4.19}$$

and

$$L_\alpha(X)=\overline{\bigcup_{\gamma>\alpha} L_\gamma(X)} \tag{4.20}$$

where the bar denotes the closure in the weak topology.

Proof. (i) It follows directly from definition of $L_\alpha(X)$.

(ii) Let $\mu=[x_0, R, M]$ belong to $L_\beta(X)$. Without loss of generality one may assume that $0<\gamma<\beta<\infty$, $x_0=0$ and $R\equiv 0$. Then, for every $c\in(0,1)$ there exists an i.d.p.m. $\mu_{c,\beta}=[M_{c,\beta}]$ such that

$$M=\sum_{k=0}^{\infty} r_{k,\beta} T_{c^k} M_{c,\beta}.$$

Since, $r_{k,\beta-\gamma}\leqq r_{k,\gamma}$ $(k=0,1,2,\ldots)$ we infer that $\sum_{k=0}^{\infty} r_{k,\beta-\gamma} T_{c^k} M_{c,\beta}$ is a Levy measure. Putting

$$\mu_{c,\gamma}=\left[\sum_{k=0}^{\infty} r_{k,\beta-\gamma} T_{c^k} M_{c,\beta}\right]$$

and taking into account the fact that $r_{m,\beta}=\sum_{k+p=m} r_{k,\beta-\gamma} r_{p,\gamma}$ $(m=0,1,2,\ldots)$ we get

$$\mu=\mathop{*}_{k=0}^{\infty} (T_{c^k}\mu_{c,\gamma})^{r_{k,\gamma}} \in L_\gamma(X),$$

which implies the inclusion (2.13).

Next we shall prove that the inclusion (2.13) is strict. Indeed, let $\nu=[\delta_1]$ be the Poisson measure on R^1 and $\mu=I^{\gamma}\nu$. Then $\mu\in L_\gamma(R^1)$ but $\mu\notin L_\beta(R^1)$. Contrary to this let us assume that $\mu\in L_\beta(R^1)$. By Theorem 4.7 we should get a p.m. τ such that $\mu=I^{\beta}\tau$. Hence $\nu=I^{\beta-\gamma}\tau\in L_{\beta-\alpha}(R^1)$. By Corollary 2.3 it follows that

$N:=\sum_{k=0}^{\infty}(-1)^k\binom{\beta-\gamma}{k}T_{c^k}\delta_1$ should be a non-negative measure and hence $N(\{c\})=-(\beta-\gamma)\geqq 0$. This contradiction shows that $\mu\notin L_\beta(R^1)$. Now consider R^1 as a subspace of X we conclude, by the above arguments, that (4.18) holds.

Next suppose that $\mu=[x_0,R,M]$ belongs to $L_\gamma(X)$ for every $0<\gamma<\beta$. By Corrollary 2.3 it follows that $\Delta_c^\gamma M(E)\geqq 0$ for any $0<c<1$ and Borel subset E of X separated from 0, where $\Delta_c^\gamma M(E)$ is defined by (2.14). Letting $\gamma\nearrow\beta$ we get the relation $\Delta_c^\beta M(E)\geqq 0$ which again by Corollary 2.3 implies that $\mu\in L_\beta(X)$. Thus $\bigcap_{\gamma<\beta}L_\gamma(X)\subset L_\beta(X)$. Since, by (4.18) the converse inclusion also holds we get the equality (4.19).

Finally, let $\mu=I^\alpha[G]=[J^\alpha G]$ where G is concentrated on a finite subset of $X\setminus\{0\}$. Then the relation $\mu=\lim_{n\to\infty}[J^{\alpha+1/n}G]$ holds. Hence and by Corollary 4.8 it follows that $L_\alpha(X)\subset\overline{\bigcup_{\gamma>\alpha}L_\gamma(X)}$, which by (4.18) implies (4.20). Thus the Theorem is fully proved.

§ 6. Concluding Remark

The definition of *fractional differentiation* on p.m'.s given in [26] can be improved as follows: *A* p.m. μ *in* $L_\alpha(X)$ $(\alpha>0)$ *is said to be* α*-differentiable if the following limit exists*

$$D^\alpha\mu:=\lim_{t\searrow 0}\mu_{c,\alpha}^{t^{-\alpha}} \tag{6.1}$$

where $t=-\log c$ *and* $\mu_{c,\alpha}$ *is given by* (1.12). The limit measure $D^\alpha\mu$ is called α-*derivative* of μ. *The problem of characterization of* α*-differentiable* p.m'.s *on* X *is open.* A partial solution of this problem is given in [26]. *We conjecture that every* α*-times* s.d.p.m. *on* X *is* α*-differentiable.* Further, in the same way as in [26] one can prove the following theorem:

6.1. Theorem. *A* p.m. μ *on* X *is stable if and only if it is the solution of the following fractional differential equation:*

$$\mu^\beta=D^\alpha\mu*\delta_x \tag{6.2}$$

for some $\alpha,\beta>0$ *and* $x\in X$. *In particular,* μ *is Gaussian if and only if it satisfies* (6.2) *for* $\beta=2^\alpha$.

Appendix

For every $\alpha>0$ we define α*-times monotone functions* on R^1 by their fractional differences. Such a concept seems to be first introduced and studied by Williamson [35] but his approach was the fractional differentiation. Our method is based on papers [4, 16, 34] by Marchaud, Butzer and Wetsphal. The case $\alpha=1,2,\dots$ was treated in [22] with applications to multiply s.d.p.m'.s.

Given a function f on R^1 and non-negative numbers $t, t_1, \dots, t_n$ we put

$$\Delta_t f(x) = \Delta_t^1 f(x) = f(x) - f(x-t) \tag{A.1}$$

and

$$\Delta_{t_1,\dots,t_n} f(x) = \Delta_{t_1} \dots \Delta_{t_n} f(x) \tag{A.2}$$

($x \in R^1$). In particular, for $t_1 = \dots = t_n = t$ we get

$$\Delta_{t_1,\dots,t_n} f(x) = \Delta_t^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kt). \tag{A.3}$$

Similarly, for every $\alpha > 0$ we put

$$\Delta_t^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kt) \quad (x \in R^1). \tag{A.4}$$

Further, following [4, 16, 34] we define *integral and derivative of fractional order* $\alpha > 0$ respectively by

$$I^\alpha f(x) = 1/\Gamma(\alpha) \int_{-\infty}^{x} (x-u)^{\alpha-1} f(u)\, du \tag{A.5}$$

and

$$D^\alpha f(x) = s - \lim_{t \searrow 0} t^{-\alpha} \Delta_t^\alpha f(x) \tag{A.6}$$

($x \in R^1$).

Recall [22] that f is *n-times monotone* ($n = 1, 2, \dots$) *if* $f(-\infty) = 0$ *and for any* $x > y$ *and* $t_1, \dots, t_n > 0$

$$\Delta_{t_1,\dots,t_n} f(x) \geqq \Delta_{t_1,\dots,t_n} f(y). \tag{A.7}$$

Further, for every n-times monotone function f on R^1 there exists a unique non-negative left-continuous monotone non-decreasing function q such that

$$f(x) = 1/(n-1)! \int_{-\infty}^{x} (x-u)^{n-1} q(u)\, du = I^n q(x) \tag{A.8}$$

(see [22], Proposition 4.1).

The following theorem gives an equivalent definition of n-times monotone functions:

A.1. Theorem. *A function f is n-times monotone ($n = 1, 2, \dots$) if and only if $f(-\infty) = 0$ and for every $t > 0$ $\Delta_t^n f(x)$ is a monotone non-decreasing function in x.*

Proof. Let K_n denote the class of all n-times monotone functions on R^1 and H_n the class of all functions f such that for every $t > 0$ $\Delta_t^n f(x)$ is monotone non-decreasing in x. It is evident that $K_n \subset H_n$. Hence to prove the Theorem it suffices to show that

$$H_n \subset K_n. \tag{A.9}$$

We shall prove (A.9) by induction. The case $n = 1$ is clear. Suppose that (A.9) holds for some $n = k$. We will show it for $n = k + 1$.

Accordingly, let f belong to H_{k+1} and $t>0$. It is easy to check that for any $x\in R^1$ and $n, m=1,2,\ldots$

$$\Delta^n_{mt} f(x)=\sum_{j=0}^{n(m-1)} \binom{n+j-1}{j} \Delta^n_t f(x-jt) \tag{A.10}$$

which implies that

$$\Delta_{t/m}\Delta^k_t f(x) \geqq \Delta_{t/m}\Delta^k_t f(y) \tag{A.11}$$

for any $x>y$ and $m=1,2,\ldots$. Consequently, we get

$$\Delta_{tp/q}\Delta^k_t f(x) \geqq \Delta_{tp/q}\Delta^k_t f(y) \tag{A.12}$$

for any $p, q=1,2,\ldots$, $x>y$ and $t>0$. Hence for any t, $t_{k+1}>0$ $\Delta_{tt_{k+1}}\Delta^k_t f(x)$ is monotone non-decreasing in x, which means that $\Delta_{t_{k+1}} f(x)$ belongs to H_k. By induction assumption it follows that for any $t_1, \ldots, t_{k+1}>0$ $\Delta_{t_1,\ldots,t_k}\Delta_{t_{k+1}} f(x)$ is monotone non-decreasing in x. Thus f belongs to K_{k+1} which proves (A.9) and completes the proof of the Theorem.

The above Theorem enables us to generalize the concept of n-times monotone function to the fractional case. Namely, *a function f is said to be α-times monotone* ($\alpha>0$) *if* $f(-\infty)=0$ *and for every* $t>0$ $\Delta^\alpha_t f(x)$ *is monotone non-decreasing in* x.

It is the same as in the integer case we get the following theorem:

A.2. Theorem. *For every α-times* ($\alpha>0$) *monotone function f there exists a unique non-negative left-continuous monotone non-decreasing function p on R^1 such that*

$$f(x)=I^\alpha p(x) \qquad (x\in R^1). \tag{A.13}$$

Conversely, for every above-mentioned function p such that $I^\alpha p(x)<\infty$ ($x\in R^1$) *the formula* (A.13) *defines an α-times monotone function.*

Proof. The proof of the second part of the Theorem is easy and will be omitted. We shall prove the first party by considering the following steps:

Step 1: $0<\alpha<1$. It is clear that if f is α-times monotone then for any $x>y$ and $t>0$

$$f(x)-f(y) \geqq \alpha(f(x-t)-f(y-t)). \tag{A.14}$$

Therefore f is continuous. Moreover, it is not hard to check that (A.14) implies that f is absolutely continuous on every half-line $(-\infty, a]$ ($a\in R^1$). Consequently, there exists a non-negative function q such that

$$f(x)=\int_{-\infty}^{x} q(t)\,dt \qquad (x\in R^1) \tag{A.15}$$

and for any $x>y$

$$q(x) \geqq \alpha q(y). \tag{A.16}$$

Consequently, for every $x \in R^1$ $I^{1-\alpha} q(x) < \infty$. Moreover, since for $t > 0$

$$t^{-\alpha} \Delta_t^\alpha f(x) = \int_{-\infty}^{x} t^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} 1_{(-\infty, x-kt)}(u)\, q(u)\, du$$
$$= \int_{-\infty}^{x} t^{-\alpha} \sum_{0 \leqq k \leqq \frac{x-u}{t}} (-1)^k \binom{\alpha}{k} q(u)\, du$$

and

$$\lim_{t \to 0+} t^{-\alpha} \sum_{0 \leqq k \leqq \frac{x-u}{t}} (-1)^k \binom{\alpha}{k} = \lim_{n \to \infty} (x-u)^{-\alpha} n^\alpha \sum_{k=1}^{n} (-1)^k \binom{\alpha}{k} = (x-u)^{-\alpha}/\Gamma(1-\alpha)$$

we get the formula

$$D^\alpha f(x) = \lim_{t \to 0+} t^{-\alpha} \Delta_t^\alpha f(x) = I^{1-\alpha} q(x) \tag{A.17}$$

which implies that $D^\alpha f(x)$ is a non-negative left-continuous monotone non-decreasing function. Now putting $q(x) = D^\alpha f(x)$ we get

$$f(x) - I^\alpha D^\alpha f(x) = I^\alpha q(x) \qquad (x \in R^1). \tag{A.18}$$

Step 2. $\alpha = n + \beta$ where $0 < \beta < 1$ and $n = 1, 2, \ldots$. Let f be α-times monotone. It is obvious that for $0 < \gamma < \alpha$ f is γ-times monotone. Moreover, by (A.10) it follows that for any $t > 0$, $x \in R^1$ and $m = 1, 2, \ldots$

$$\Delta_{mt}^n \Delta_t^\beta f(x) = \sum_{j=0}^{n(m-1)} \binom{n+j-1}{j} \Delta_t^{n+\beta} f(x - jt)$$

which implies that for $x > y$

$$\Delta_{t/m}^\beta \Delta_t^n f(x) \geqq \Delta_{t/m}^\beta \Delta_t^n f(y). \tag{A.19}$$

Hence and by Step 1 we get the relation

$$\lim_{m \to \infty} (t/m)^{-\beta} \Delta_{t/m}^\beta \Delta_t^n f(x) = D^\beta \Delta_t^n f(x)$$
$$= \Delta_t^n D^\beta f(x) \geqq \Delta_t^n D^\beta f(y). \tag{A.20}$$

Consequently, by Theorem A.1, $D^\beta f(x)$ is n-times monotone and by (A.8) it follows that

$$D^\beta f(x) = I^n p(x) \qquad (x \in R^1) \tag{A.21}$$

for some left-continuous monotone non-decreasing function p, which by a simple reasoning implies that (A.13) holds for $\alpha = n - \beta$.

Finally combining Steps 1, 2 and (A.8) we infer that for every $\alpha > 0$ the formula (A.13) holds. It is clear that the function p in (A.13) with the stated properties is unique which completes the proof of the theorem.

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References

1. Araujo, A., Giné, E.: Type, cotype and Levy's measures in Banach spaces. Ann. Probab. **6**, 637–643 (1978)
2. Berg, C., Forst, G.: Multiply self-decomposable probability measures on R_+ and Z_+. Z. Wahrscheinlichkeitstheor. Verw. Geb. **62**, 147–163 (1983)
3. Billingsley, P.: Convergence of probability measures. New York: Wiley 1968
4. Butzer, P.L., Westphal, U.: An access to fractional differentiation via fractional difference quotients. In: Proceedings of the Conference on fractional calculus and its applications to the mathematical sciences. New Haven, June 15–16, 1974. Lecture Notes in Mathematics **457**, 116–145, Berlin-Heidelberg-New York: Springer 1976
5. Choquet, G.: Le théorème de representation intégrale dans les esembles convexes compact. Ann. Inst. Fourier **10**, 333–344 (1960)
6. Dettweiler, E.: Grenzwertsätze für Wahrscheinlichkeitsmasse auf Badrikianschen Räumen. Thesis, Eberhard-Karls Universität zu Tübingen (1974)
7. Ito, K., Nisio, M.: On the convergence of sums of independent Banach space valued random variables. Osaka J. Math. **5**, 35–48 (1968)
8. Jajte, R.: V-decomposable measures on Hilbert space. Lecture Notes in Mathematics **828**, 108–127. Berlin-Heidelberg-New York: Springer 1980
9. Jurek, Z.J.: Limit distributions and one-parameter groups of linear operators on Banach spaces. J. Multivariate Anal. **13**, 578–604 (1983)
10. Jurek, Z.J., Vervaat, W.: An integral representation for self-decomposable Banach space valued radom variables. Z. Wahrscheinlichkeitstheor. Verw. Geb. **62**, 247–262 (1983)
11. Kumar, A., Schreiber, B.M.: Self-decomposable probability measures on Banach spaces. Studia Math. **53**, 55–71 (1975)
12. Kumar, A., Schreiber, B.M.: Characterization of subclasses of class L probability distributions. Ann. Probab. **6**, 279–293 (1978)
13. Kumar, A., Schreiber, B.M.: Representation of certain infinitely divisible probability measures on Banach spaces. J. Multivariate Anal. **9**, 288–303 (1979)
14. Lévy, P.: Théorie de l'addition des variables aléatoires. Paris: Gauthier-Villars 1937
15. Loéve, M.: Probability Theory (3d Edition). New York: Van Nostrand 1963
16. Marchaud, A.: Sur les derivées et sur les differences des fonctions de variables réelles J. de Math. **6**, 337–425 (1927)
17. Parthasarathy, K.R.: Probability measures on metric spaces. New York-London: Academic Press 1967
18. Phelps, R.R.: Lectures on Choquet's theorem. Toronto-New York-London: Van Nostrand 1966
19. Sato, K.I.: Urbanik's class L_m of probability measures. Ann. Sci. Coll. Lib. Arts Kanazawa Univ. **15**, 1–10 (1978)
20. Sato, K.I.: Class L of multivariate distributions and its subclasses. J. Multivariate Anal. **10**, 207–232 (1980)
21. Thu, N.V.: Multiply self-decomposable measure in generalized convolution algebras. Bull. Acad. Pol. Sci. Sér. Sci. Math., Astronom. et Phys. **26**, No. 9–10, 855–861 (1978)
22. Thu, N.V.: Multiply self-decomposable probability measures on Banach spaces. Studia Math. **66**, 161–175 (1979)
23. Thu, N.V.: A characterization of mixed-stable laws. Bull. Acad. Polon. Sci. Sér. Sci. Math., Astronom. et Phys. **27**, 629–630 (1979)
24. Thu, N.V.: Limit theorems for random fields. Diss. Math. **186** (1981)
25. Thu, N.V.: Universal multiply self-decomposable probability measures on Banach spaces. Probab. Math. Statist. **3**, No. 1, 71–84 (1982)
26. Thu, N.V.: Fractional calculus in Probability. Probab. Math. Statist. **3**, No. 2, 173–189 (1984)

27. Thu, N.V.: A characterization of some probability distributions. Lecture Notes in Mathematics **828**, 302–308. Berlin-Heidelberg-New York: Springer 1980
28. Tortrat, A.: Structure des lois indéfiniment divisibles dans un espace vectorial topologique (separe) X. Symposium on Probability Methods in Analysis. Lecture Notes in Mathematics **31**, 299–328. Berlin-Heidelberg-New York: Springer 1967
29. Tortrat, A.: Sur la structure des lois indéfiniment divisibles dans les espaces vectoriels. Z. Wahrscheinlichkeitstheor. Verw. Geb. **11**, 311–326 (1969)
30. Urbanik, K.: A representation of self-decomposable distributions. Bull. Acad. Polon. Sci. Sér. Sci. Math., Astronom. et Phys. **16**, 209–214 (1968)
31. Urbanik, K.: Limit laws for sequences of normed sums satisfying some stability conditions. In: Multivariate Analysis III (ed. P.R. Krishnaiah) pp. 225–237. New York: Academic Press 1973
32. Urbanik, K.: Levy's probability measures on Euclidean spaces. Studia Math. **44**, 119–148 (1972)
33. Urbanik, K.: Levy's probability measures on Banach spaces. Studia Math. **63**, 283–308 (1978)
34. Westphal, U.: An approach to fractional powers of operators via fractional differences. Proc. London Math. Soc. **29**, 557–576 (1974)
35. Williamson, R.E.: Multiply monotone functions and their Laplace transforms. Duke Math. J. **23**, 189–207 (1956)

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