# *The Development of the Theory of Summable Divergent Series from 1880 to 1925*

JOHN TUCCIARONE

*Communicated by* M. KLINE

## 1. Introduction

Divergent series have appeared in mathematics for a long time. They were used widely throughout the $17^{\text{th}}$ and $18^{\text{th}}$ centuries, sometimes with great profit and sometimes without. Many mathematicians who used them were frequently in doubt about their validity while others seemed totally unconcerned. Throughout the period, controversy surrounded the use of divergent series mainly because contradictions were often derived through their application.

One of the earliest discussions regarding a divergent series occurred when the infinite series

$$1-1+1-1+1-+\cdots \tag{1}$$

was suggested by GUIDO GRANDI (1671–1742) toward the end of the $17^{\text{th}}$ century. GRANDI, in his book *Quadratura circuli et hyperbolae per infinitas hyperbolas geometrice exhibita*, published in 1703, had set $x=1$ in the expansion

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-+\cdots,$$

and had obtained

$$\tfrac{1}{2}=1-1+1-1+-\cdots.$$

GOTTFRIED LEIBNITZ (1646–1716) was asked what he thought about whether or not a "sum" might exist for series (1). In a letter to CHRISTIAN WOLF[1], he reasoned that the sum of $n$ terms of this series would either be 1 or 0 depending on whether $n$ is odd or even, so that the values 0 and 1 occur with equal frequency; therefore, according to the laws of probability, the most probable sum should be the arithmetic mean, $\frac{1}{2}$.

A precise definition for the concept of a convergent series was introduced at the beginning of the $19^{\text{th}}$ century by AUGUSTIN CAUCHY (1789–1857) and NIELS HENRIK ABEL (1802–1829). It was stated as follows: A series $\sum a_n$ of numbers is given. The numbers $S_i$, called partial sums, are associated with the series by the formula $S_i=\sum_{n=1}^{i} a_n$. If the sequence of partial sums has a limiting value, *i.e.*, if $\lim_{i\to\infty} S_i=S$, then the series $\sum a_n$ is said to be convergent and to have sum $S$. If the sequence of partial sums has no limit, then the series is said to be divergent.

[1] G. W. LEIBNITZ, *Acta Eruditorum Supplementum*, **5** (1713): 264–270.

CAUCHY and ABEL also used divergent series, as had other analysts, and they too were skeptical about their value. After working with them for a time, CAUCHY eventually urged others to abandon their use of divergent series. ABEL was more outspoken. A fine example of his attitude appears in a letter written by ABEL in 1826 to his former teacher HOLMBOË:

> Les séries divergentes sont, en général, quelque chose de bien fatal, et c'est une honte qu'on ose y fonder aucune démonstration... la partie la plus essentielle des Mathématiques est sans fondement. Pour la plus grande partie les résultats sont justes, il est vrai, mais c'est là une chose bien étrange. Je m'occupe à en chercher la raison, problème très intéressant.[1]

For many reasons, not the least of which were the pronouncements of CAUCHY and ABEL, research on divergent series was not forthcoming until the end of the 19th century. However, one very important contribution to the theory of divergent series was made during the early 1800's. It was a theorem first stated by KARL FRIEDRICH GAUSS (1777–1855) in 1812,[2] but first validly proved by ABEL in 1826.[3] The theorem is sometimes referred to as ABEL's Limit Theorem. When he proved it, ABEL probably did not have in mind an application to divergent series. However, SIMÉON DENIS POISSON (1781–1840) used it to define a summation method which can be applied to convergent series and many divergent series as well. It is a widely used method in modern analysis. Since the theorem of ABEL was one of the most significant results to come out of the early years in the history of divergent series, its proof is given here.

In 1826, ABEL stated the theorem: If the series $\sum_{n=0}^{\infty} V_n \alpha^n = f(\alpha)$ converges when $\alpha = \delta$, then it will also converge for $\alpha < \delta$ and $\lim_{\alpha \to \delta^-} f(\alpha) = f(\delta)$.[4] ABEL assumed that $\alpha > 0$. A more modern statement of the theorem would take into account the fact that an interval of convergence can be associated with every power series. ABEL sketched the following proof:

Let

$$\phi(\alpha) = V_0 + V_1\alpha + V_2\alpha^2 + \cdots + V_{m-1}\alpha^{m-1}, \tag{2}$$

$$\chi(\alpha) = V_m\alpha^m + V_{m+1}\alpha^{m+1} + \cdots. \tag{3}$$

ABEL rewrote (3) in the form

$$\chi(\alpha) = (\alpha/\delta)^m V_m \delta^m + (\alpha/\delta)^{m+1} V_{m+1} \delta^{m+2} + \cdots, \tag{4}$$

and then stated that if $\varrho$ is the largest term of

$$V_m\delta^m, \quad V_m\delta^m + V_{m+1}\delta^{m+1}, \quad V_m\delta^m + V_{m+1}\delta^{m+1} + V_{m+2}\delta^{m+2}, \ldots,$$

[1] N. H. ABEL, *Oeuvres d'Abel* (Paris, 1826), Vol. 2, p. 256. In translation: Divergent series are in general the work of the devil, and it is shameful to base any demonstration whatever on them ... The most essential part of mathematics is without basis. For the most part, the results are valid, it is true, but it is a curious thing. I am looking for the reason, and it is a very interesting problem.

[2] K. F. GAUSS, *Werke* (Göttingen: Königlichen Ges., 1876), Vol. 6, p. 143.

[3] N. H. ABEL, *Journal für die reine und angewandte Mathematik*, **1** (1826): 314.

[4] *Ibid.* The statement and subsequent proof given here is a free translation of the original German text.

then

$$\chi(\alpha) < (\alpha/\delta)^m \varrho. \tag{5}$$

This was a consequence of another theorem proved by ABEL on p. 314 of the same article and was based on the fact that the series $\sum V_n \delta^n$ is convergent and $\alpha \leqq \delta$.

From (5) ABEL concluded that for $\alpha \leqq \delta$, $m$ can be taken large enough to insure the existence of a number $\omega$ with the property that $\chi(\alpha) = \omega$. ABEL then noted that

$$f(\alpha) = \phi(\alpha) + \chi(\alpha)$$

and

$$f(\alpha) - \operatorname*{Lim}_{\alpha \to \delta^-} f(\alpha) = \phi(\alpha) - \operatorname*{Lim}_{\alpha \to \delta^-} \phi(\alpha) + \omega. \tag{6}$$

He stated that since $\phi(\alpha)$ was a function of $\alpha$, if $\alpha$ were chosen close enough to $\delta$,

$$\phi(\alpha) - \operatorname*{Lim}_{\alpha \to \delta^-} \phi(\alpha) = \omega; \qquad f(\alpha) - \operatorname*{Lim}_{\alpha \to \delta^-} f(\alpha) = \omega. \tag{7}$$

ABEL then stated that equations (6) and (7) together implied the theorem.

ABEL'S theorem suggested a method of summation. POISSON realized that for some series of the form $\sum a_n x^n$ whose radii of convergence equaled 1, and which were divergent at $x = 1$, the limit of the series as $x$ approaches 1 from below existed. That is, for this type of series, while the hypothesis of ABEL'S theorem did not apply because of the divergence at 1, nevertheless, the conclusion of the theorem might still hold. POISSON proposed the following definition: If the radius of convergence of the series $\sum a_n x^n$ is 1, and the series diverges when $x = 1$, then if the sum of the series in the interval of convergence is given by $f(x)$, and if $\operatorname*{Lim}_{x \to 1^-} f(x) = S$, then $S$ could be called the "sum" of the series $\sum a_n$. This definition, which is actually a refinement of one given earlier by EULER, is now referred to as ABEL Summability.

A more modern statement of ABEL'S theorem is:[1] If the series $\sum a_n x^n$ has a radius of convergence $r$, and converges for $x = r$, then $\operatorname*{Lim}_{x \to r^-} \sum a_n x^n = \sum a_n r^n$.

## 2. The Revival of Interest from 1880 to 1900

The last twenty years of the 19th century witnessed a rebirth of interest in divergent series. A number of important results were discovered during this period which proved beyond doubt that divergent series offered ground for fruitful research. The prominent contributors during this time were mathematicians such as GEORG FROBENIUS (1849–1917), OTTO HÖLDER (1859–1937), ERNESTO CESÀRO (1859–1906), and ÉMILE BOREL (1871–1956). All were striving for an extension of the traditional convergence concept, in order to "sum" divergent series. In their research, each of these men adhered to a general guideline; that is, they posed new definitions of summability which would sum ordinarily convergent series, and they then tried to frame the definitions to include in their scope as many divergent series as possible. It was clear to these men that to solve the problem by starting with a definition applicable only to divergent series could lead

[1] K. KNOPP, *Theory and Application of Infinite Series* (New York: Hafner, 1951), p. 177.

to difficulties. One might then be faced with a situation in which convergent series would be summable by the traditional methods and divergent series by others. In that case, the desirable uniformity of being able to treat both series by the same method would be lost.

In 1880, G. FROBENIUS launched the renaissance by proving a theorem which turned out to be fundamental in the theory of summability that was to follow. His theorem is similar to the limit theorem of ABEL proved in 1826 and is in fact an extension of a theorem proved by LAGRANGE in 1799.[1] In that year, LAGRANGE showed that if in a series $\sum a_n$, the ratio $(S_0+S_1+\cdots+S_{n-1})/n$ repeats itself at regular intervals, then $\operatorname*{Lim}_{x\to 1^-} \sum a_n x^n$ would always exist and have as its value,

$$\operatorname*{Lim}_{n\to\infty} (S_0+S_1+\cdots+S_{n-1})/n,$$

where the $S_i$ are the partial sums of the original series. In a paper written in 1836, J. L. RAABE arrived at the same conclusion, although his reasoning differed from LAGRANGE's.[2]

Instead of considering only those series treated by LAGRANGE, FROBENIUS generalized and considered all those series where $(S_0+S_1+\cdots+S_{n-1})/n$ tends to a limit as $n$ tends to infinity. For this kind of series, FROBENIUS proved the theorem:[3]

If $\sum a_n x^n$ is a series with a radius of convergence less than 1, then

$$\operatorname*{Lim}_{x\to 1^-} \sum a_n x^n = \operatorname*{Lim}_{n\to\infty} (S_0+S_1+\cdots+S_{n-1})/n, \tag{8}$$

where $S_n = \sum_{i=0}^{n} a_i$.

FROBENIUS offered the following proof: Given that

$$\operatorname*{Lim}_{n\to\infty} (S_0+S_1+\cdots+S_{n-1})/n = M, \tag{9}$$

the definition of limit implies that for $\varepsilon'>0$, there must exist a positive integer $N$ such that for all $n>N$,

$$\left|\frac{S_0+S_1+\cdots+S_{n+k-1}}{n+k} - M\right| < \varepsilon'. \tag{10}$$

Inequality (10) implies that the left hand side of the inequality is equal to some $\varepsilon \leq \varepsilon'$ for a certain value of $k$. In that case, one can write

$$\left|\frac{S_0+S_1+\cdots+S_{n+k}}{n+k+1} - M\right| = \varepsilon_{k+1}. \tag{11}$$

FROBENIUS rewrote equation (11) and obtained

$$\left|\frac{\dfrac{(S_0+S_1+\cdots+S_{n+k-1})(n+k)}{(n+k)} + S_{n+k}}{(n+k+1)} - M\right| = \varepsilon_{k+1}. \tag{12}$$

[1] J. L. LAGRANGE, *Mémoire d'Inst. Nat. Sci. et Arts*, **3** (1799).

[2] J. L. RAABE, *Journal für die reine und angewandte Mathematik*, **15** (1836): 355–364.

[3] G. FROBENIUS, *Journal für die reine und angewandte Mathematik*, **89** (1880): 262–264. *Gesammelte Abhandlungen* (Berlin: Springer, 1968), Vol. 2, pp. 8–10.

From (11) and (12) he obtained the following:

$$\left|\frac{(M+\varepsilon_k)(n+k)+S_{n+k}}{(n+k+1)}-M\right|=\varepsilon_{k+1}. \tag{13}$$

The quantity inside the absolute value sign may be either positive or negative. FROBENIUS did not distinguish these cases but assumed that the quantity was positive. There is no resulting loss of generality, because a similar argument can be applied to both cases. Accordingly, FROBENIUS stated that from (13) one could write

$$(M+\varepsilon_k)(n+k)+S_{n+k}-M(n+k+1)=\varepsilon_{k+1}(n+k+1). \tag{14}$$

He solved for $S_{n+k}$ and wrote

$$S_{n+k}=M+\varepsilon_{k+1}(n+k+1)-\varepsilon_k(n+k). \tag{15}$$

In a similar way, he found that

$$S_{n+k-1}=M+\varepsilon_k(n+k)-\varepsilon_{k-1}(n+k-1). \tag{16}$$

Subtraction of equation (16) from equation (15) yields

$$S_{n+k}-S_{n+k-1}=a_{n+k}=\varepsilon_{k+1}(n+k+1)-2\varepsilon_k(n+k)+\varepsilon_{k-1}(n+k-1). \tag{17}$$

Since each $\varepsilon_k<\varepsilon$, for each choice of $k$, equation (17) implies that

$$a_{n+k}<4\varepsilon(n+k). \tag{18}$$

FROBENIUS then defined the function

$$\begin{aligned} F(x)&=a_0+a_1x+a_2x^2+\cdots\\ &=\operatorname*{Lim}_{k\to\infty}(a_0+a_1x+a_2x^2+\cdots+a_{n+k}x^{n+k}). \end{aligned} \tag{19}$$

He used equation (15) with $k=0$ and the definition of $S_n$ to write

$$S_n=M+\varepsilon_1(n+1)-n\varepsilon_0=a_0+a_1+\cdots+a_n \tag{20}$$

or

$$M+\varepsilon_1(n+1)-n\varepsilon_0-a_0-a_1-\cdots-a_n=0. \tag{21}$$

Substitution of (17) and (21) in equation (20) gives an alternate way of expressing the function $F(x)$. That is,

$$F(x)=\operatorname*{Lim}_{k\to\infty}\left\{\begin{aligned} &M+(n+1)\varepsilon_1-n\varepsilon_0-a_0-a_1-\cdots-a_n+a_0+a_1x+\cdots\\ &\quad+a_nx^n+\{(n+2)\varepsilon_2-2(n+1)\varepsilon_1+n\varepsilon_0\}x^{n+1}\\ &\quad+\{(n+3)\varepsilon_3-2(n+2)\varepsilon_2+(n+1)\varepsilon_1\}x^{n+2}+\cdots\\ &\quad+\{(n+k+1)\varepsilon_{k+1}-2(n+k)\varepsilon_k+(n+k-1)\varepsilon_{k-1}\}x^{n+k}. \end{aligned}\right. \tag{22}$$

In order to simplify equation (22), FROBENIUS set

$$\begin{aligned} G(x)&=a_1x+a_2x^2+\cdots+a_nx^n-a_1-a_2-\cdots\\ &\qquad-a_n-n\varepsilon_0(1-x^{n+1})+(n+1)\varepsilon_1(1-x^n), \end{aligned} \tag{23}$$

and then wrote (22) in the form

$$F(x) = M + G(x) + \lim_{k\to\infty} \begin{cases} (n+1)\,\varepsilon_1 (x^n - 2x^{n+1} + x^{n+2}) \\ +(n+2)\,\varepsilon_2 (x^{n+1} - 2x^{n+2} + x^{n+3}) + \cdots \\ +(n+k+1)\,\varepsilon_{k+1} x^{n+k} - (n+k)\,\varepsilon_k x^{n+k+1}. \end{cases} \tag{24}$$

Now as $x$ approaches 1 from below, the terms $x^t$ also approach 1 so that the terms in the braces go to 0, giving the relation $F(x) = M + G(x)$. But as $x$ approaches 1 from below, $G(x)$ also geos to 0, and so, in the limit, $F(x) = M$, and FROBENIUS therefore wrote

$$\lim_{x\to 1} \sum_{n=0}^{\infty} a_n x^n = \lim_{n\to\infty} (S_0 + S_1 + \cdots + S_{n-1})/n, \tag{25}$$

which completed his proof.

As an example of how this theorem could be applied, consider the LEIBNITZ Series (1),

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + - \cdots. \tag{26}$$

Associate with (26) the series

$$\sum_{n=0}^{\infty} a_n = 1 - 1 + 1 - 1 + - \cdots,$$

so that the partial sums are 1, 0, 1, 0, *etc.* Application of equation (25) gives the following:

$$\begin{aligned} \lim_{x\to 1^-} \sum (-1)^n x^n &= \lim_{x\to 1^-} (1 - x + x^2 - x^3 + - \cdots) \\ &= \lim_{n\to\infty} (S_0 + S_1 + \cdots + S_{n-1})/n. \end{aligned}$$

But $S_0/1 = 1$; $(S_0 + S_1)/2 = \frac{1}{2}$; $(S_0 + S_1 + S_2)/3 = 2/3$, *etc.*, so that the sequence $(S_0 + S_1 + \cdots + S_{n-1})/n$ is

$$1, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{1}{2}, \ldots, \tag{27}$$

and this sequence converges to $\frac{1}{2}$. Hence, by FROBENIUS' Theorem,

$$1 - 1 + 1 - 1 + - \cdots = \tfrac{1}{2}.$$

The theorem FROBENIUS proved in 1880 gave considerable support to those who were seeking summability methods based on averaging techniques. One such man was OTTO HÖLDER. In 1882 he utilized the concept of taking repeated arithmetic means to define a method of summation for divergent series.[1] He observed that if one were to apply FROBENIUS' theorem to the divergent series $\sum_{n=1}^{\infty} a_n$, the expression $\lim_{n\to\infty} (S_1 + \cdots + S_n)/n$ may not even exist. However, he noted that if one took the sequence of approximations

$$\frac{S_1}{1}, \quad \frac{\frac{S_1}{1} + \frac{S_1+S_2}{2}}{2}, \quad \frac{\frac{S_1}{1} + \frac{S_1+S_2}{2} + \frac{S_1+S_2+S_3}{3}}{3}, \ldots,$$

[1] O. HÖLDER, *Mathematische Annalen*, **20** (1882): 535–549.

and considered the limit of this sequence as $n$ goes to infinity, this limit may exist even when the former did not. He concluded that by taking limits of sequences of repeated means, the chances of eventually obtaining a convergent sequence may improve. Once this convergent sequence was reached, HÖLDER suggested that its limit be the value assigned as the sum of the original series. Of course, this meant that one would have to specify the number of times repeated means were used to obtain a "sum" so as to distinguish sums of different kinds of divergent series.

HÖLDER's definition was precisely the following: For a series $\sum_{n=1}^{\infty} a_n$, let

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n, \\ S_n^{(1)} &= (S_1 + S_2 + \cdots + S_n)/n, \\ S_n^{(2)} &= (S_1^{(1)} + S_2^{(1)} + \cdots + S_n^{(1)})/n, \\ &\vdots \\ S_n^{(k)} &= (S_1^{(k-1)} + S_2^{(k-1)} + \cdots + S_n^{(k-1)})/n. \end{aligned} \tag{28}$$

Then, if $r$ is the smallest value of $k$ for which

$$\operatorname*{Lim}_{n\to\infty} S_n^{(k)} = S$$

exists, then $S$ is defined to be the sum of the series $\sum a_n$. HÖLDER then proved an extension of FROBENIUS' theorem, couched in the same terms and influenced by ABEL's Limit Theorem. HÖLDER's theorem is the following:

If $\operatorname*{Lim}_{n\to\infty} S_n^{(k)} = S$ exists for some $k$, then

$$\operatorname*{Lim}_{x\to 1^-} \sum_{n=1}^{\infty} a_n x^{n-1} = S. \tag{29}$$

The proof supplied by HÖLDER in 1882 will now be discussed in detail.

The partial summation formula for infinite series, first demonstrated by ABEL in 1826, was employed by HÖLDER as a starting point in the proof.[1] This formula states that

$$\sum_{v=1}^{n} a_v b_v = \sum_{v=1}^{n} \left\{ (b_v - b_{v+1}) \sum_{r=1}^{v} a_r \right\} + b_{n+1} \sum_{r=1}^{n} a_r. \tag{30}$$

Using formula (30), HÖLDER wrote

$$\sum_{v=1}^{n} a_v x^{v-1} = \sum_{v=1}^{n} S_v (x^{v-1} - x^v) + x^n S_n. \tag{31}$$

Since we are assuming in the hypothesis of the theorem that $\operatorname*{Lim}_{n\to\infty} S_n^{(k)}$ exists, we may say that for each $n$,

$$|S_n^{(k)}| < A, \quad \text{where} \quad A > 0, \tag{32}$$

and, since

$$S_n^{(k-1)} = n S_n^{(k)} - (n-1) S_{n-1}^{(k)}, \tag{33}$$

[1] N. H. ABEL, *op. cit.*, p. 314.

(32) and (33) together imply that

$$|S_n^{(k-1)}| < nA + (n-1)A < 2nA. \tag{34}$$

Similarly,

$$|S_n^{(k-2)}| < 4n^2 A,$$

and, in general,

$$|S_n| < 2^k n^k A. \tag{35}$$

From inequality (35) we can deduce that

$$\lim_{n\to\infty} x^n S_n = 0 \quad \text{for} \quad |x| < 1.$$

In equation (31) HÖLDER took the limit as $n$ goes to infinity and obtained

$$\begin{aligned}\sum_{v=1}^{\infty} a_v x^{v-1} &= \sum_{v=1}^{\infty} S_v (x^{v-1} - x^v) \\ &= \sum_{v=1}^{\infty} (1-x) S_v x^{v-1} \\ &= (1-x) \sum_{v=1}^{\infty} S_v x^{v-1}.\end{aligned} \tag{36}$$

HÖLDER then considered the cases which exist for a bounded sequence of partial sums $S_n$.

*Case 1.* The $S_n$ converge.

Since the hypothesis of the theorem is that the Lim $S_n$ exists, HÖLDER argued that one should be able to choose a number $\{S_m\}$ lying between the upper and lower limits of the sequence

$$S_m, S_{m+1}, S_{m+2}, \ldots,$$

and such that the series can be expressed in the form

$$\sum_{v=1}^{\infty} a_v x^{v-1} = (1-x)\left(\sum_{v=1}^{m-1} S_v x^{v-1} + \{S_m\} \sum_{v=m}^{\infty} x^{v-1}\right). \tag{37}$$

By the geometric series test, HÖLDER concluded that

$$\sum_{v=m}^{\infty} x^{v-1} = x^{m-1}/(1-x), \quad \text{for} \quad |x| < 1. \tag{38}$$

Therefore, substituting (38) in (37) gives

$$\sum_{v=1}^{\infty} a_v x^{v-1} = (1-x) \sum_{v=1}^{m-1} S_v x^{v-1} + \{S_m\} x^{m-1}. \tag{39}$$

Since the $S_n$ converge, $m$ can be chosen large enough so that if $\lim_{n\to\infty} S_n = S$, and $\varepsilon > 0$,

$$S - \varepsilon < \{S_m\} < S + \varepsilon.$$

Since, as $x$ goes to 1 from below, $(1-x)$ goes to 0, taking limits in equation (37) gives

$$\lim_{x\to 1^-} \sum_{v=1}^{\infty} a_v x^{v-1} = S, \tag{40}$$

which is the theorem of FROBENIUS considered above.

*Case 2*. The $S_n$ oscillate, but the $S_n^{(1)}$ converge.

HÖLDER started this part of his proof by again using the partial summation formula of ABEL. HÖLDER wrote

$$(1-x)\sum_{v=1}^{\infty} S_v x^{v-1} = (1-x)\left[\sum_{v=1}^{n} v S_v^{(1)}(x^{v-1}-x^v) + n S_n^{(1)} x^n\right]. \tag{41}$$

Since

$$\lim_{n\to\infty} n S_n^{(1)} x^n = 0,$$

we have the following equation:

$$\begin{aligned}(1-x)\sum_{v=1}^{\infty} S_v x^{v-1} &= (1-x)^2 \sum_{v=1}^{\infty} v S_n^{(1)} x^{v-1} \\ &= (1-x)^2\left[\sum_{v=1}^{m-1} v S_v^{(1)} x^{v-1} + \{S_m^{(1)}\}\sum_{v=m}^{\infty} v x^{v-1}\right].\end{aligned} \tag{42}$$

Here, as in Case 1, $\{S_m^{(1)}\}$ designates a number lying between the upper and lower limits of the sequence

$$S_m^{(1)},\ S_{m+1}^{(1)},\ S_{m+2}^{(1)},\ \ldots.$$

HÖLDER then added and subtracted the expression

$$\{S_m^{(1)}\}\sum_{v=1}^{m-1} v x^{v-1}$$

in equation (42) and obtained

$$(1-x)^2\left[\sum_{v=1}^{m-1} v S_v^{(1)} x^{v-1} - \{S_m^{(1)}\}\sum_{v=1}^{m-1} v x^{v-1} + \{S_m^{(1)}\}\sum_{v=1}^{\infty} v x^{v-1}\right]. \tag{43}$$

Since

$$(1-x)^2\sum_{v=1}^{\infty} v x^{v-1} = 1,$$

expression (43) can be written in the form

$$(1-x)^2\left[\sum_{v=1}^{m-1} v S_v^{(1)} x^{v-1} - \{S_m^{(1)}\}\sum_{v=1}^{m-1} v x^{v-1}\right] + \{S_m^{(1)}\}. \tag{44}$$

Finally, HÖLDER wrote

$$\begin{aligned}\sum_{v=1}^{\infty} a_v x^{v-1} &= (1-x)\sum_{v=1}^{\infty} S_v x^{v-1} \\ &= (1-x)^2\left[\sum_{v=1}^{m-1} v S_v^{(1)} x^{v-1} - \{S_m^{(1)}\}\sum_{v=1}^{m-1} v x^{v-1}\right] + \{S_m^{(1)}\}.\end{aligned} \tag{45}$$

HÖLDER then treated equation (45) in the same manner in which he had treated equation (37) in Case 1. That is, on the assumption that the $S_n^{(1)}$ converge to $S$, if limits are taken in equation (45), the result will be

$$\lim_{x\to 1^-}\sum_{v=1}^{\infty} a_v x^{v-1} = S.$$

This process may be repeated indefinitely by finite induction. Thus, in the $r^{\text{th}}$ case, that is, where the $S_n^{(r-2)}$ do not converge, but the $S_n^{(r-1)}$ do, the same argument will apply.

In order to demonstrate how this technique worked, HÖLDER supplied an example in his paper. He considered the series

$$-\frac{1}{(1+x)^2} = -1+2x-3x^2+4x^3-+\cdots = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}. \qquad (46)$$

This series diverges when $x=1$ because the right member is an alternating series whose $n^{\text{th}}$ term does not go to 0. However, consider its partial sums,

$$S_1=-1; \quad S_2=1; \quad S_3=-2; \quad S_4=2; \quad \textit{etc.}$$

This sequence is

$$-1,\ 1,\ -2,\ 2,\ -3,\ 3,\ -4,\ 4,\ \ldots,$$

an unbounded, oscillating sequence. HÖLDER then went on to consider the terms $S_n^{(1)}$. He obtained the sequence

$$-1,\ 0,\ 0,\ -\tfrac{2}{3},\ -\tfrac{3}{5},\ 0,\ -\tfrac{4}{7},\ 0,\ \ldots.$$

Here again, the terms oscillate between 0 and $-\frac{1}{2}$, so HÖLDER went further to consider the terms $S_n^{(2)}$. Here he found the following sequence

$$-1,\ -\tfrac{1}{2},\ -\tfrac{5}{9},\ -\tfrac{5}{12},\ -\tfrac{34}{75},\ \ldots,$$

and HÖLDER showed that this sequence is in fact convergent, with limit $-\frac{1}{4}$. Hence, according to his theorem,

$$\begin{aligned}\operatorname*{Lim}_{x\to1^-}\sum_{n=1}^{\infty}(-1)^n n x^{n-1} &= \operatorname*{Lim}_{x\to1^-}(-1+2x-3x^2+-\cdots)\\ &= \operatorname*{Lim}_{n\to\infty} S_n^{(2)} = -\tfrac{1}{4}.\end{aligned}$$

On the basis of this outcome, it would seem quite natural to assign the value $-\frac{1}{4}$ as the "sum" of the original series, and characterize it as the $S^{(2)}$-Sum, or Second HÖLDER Sum.

In summary, HÖLDER's technique is the following: Given a series $\sum_{n=1}^{\infty} a_n$, form the sequence of partial sums $S_1, S_2, \ldots$. If this sequence converges, its limit is the sum of the series. If the sequence does not converge, form the sequence of arithmetic means $S_1^{(1)}, S_2^{(1)}, \ldots$. If this sequence converges, call the original series "summable (H, 1)," the sum being the limit of the sequence.[1] If this sequence does not converge, take the sequence of its arithmetic means, and so on indefinitely, until a convergent sequence is reached.

The question that immediately arises is whether it can be determined in advance if one of these sequences will converge. Since it is conceivable that one could follow this procedure forever without encountering a convergent sequence, it would be desirable to have such a test available. However, HÖLDER did not

[1] This notation is due to G. H. HARDY.

consider this precise question himself, and it was only to be answered later by other mathematicians who studied summability of divergent series. On of these men was ERNESTO CESÀRO.

When CESÀRO began his researches into the area of summability of series, near the end of the 19th century, he used as an approach the study of multiplication of series.[1] In fact, it appears that CESÀRO's original intention was solely to enrich the theory of multiplication of series. However, in doing this, he devised a summation technique for use in his proofs, and it was this technique, in actuality an extension of HÖLDER's method, which has since become widely acclaimed.

In order to understand how the research of CESÀRO led to his new method, it is necessary to examine the theorems on the multiplication of series which furnished the background for CESÀRO's inquiry.

In 1821, CAUCHY gave a definition of a product of two infinite series, thereafter referred to as the "Cauchy Product."[2] The definition is:

Given two infinite series $\sum u_n$, $\sum v_n$, their CAUCHY Product is $\sum w_n$, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1.$$

At the same time, CAUCHY had proved that if $\sum u_n$ and $\sum v_n$ were two absolutely convergent series with sums $U$ and $V$, respectively, then the CAUCHY Product series $\sum w_n$ is also absolutely convergent, with sum $UV$.

ABEL continued the study by proving the following theorem in 1826.[3] If the series $\sum u_n$, $\sum v_n$, and $\sum w_n$ are all convergent, and $w_n$ is as described above, and if $U$, $V$, and $W$ are the respective sums of the series, then $W = UV$.

One further result which CESÀRO used as a basis for his paper was a theorem proved by FRANZ MERTENS (1840–1927) in 1875.[4] The theorem stated that if at least one of the two convergent series $\sum u_n$ and $\sum v_n$ converges absolutely, then the CAUCHY Product series $\sum w_n$ will converge, and again, its sum will be $UV$.

CESÀRO desired to extend these criteria and, as a preliminary to his main theorem, proved the following:

If the sequences $U_1, U_2, U_3, \ldots$, and $V_1, V_2, V_3, \ldots$ converge to $U$ and $V$, respectively, then

$$\lim_{n\to\infty} (U_1 V_n + U_2 V_{n-1} + \cdots + U_n V_1)/n = UV. \tag{47}$$

*Proof*. CESÀRO let $A$ be the greatest whole number less than or equal to $n/2$ for which

$$\lim_{n\to\infty} A/n = \lim_{n\to\infty} (n-A)/n = \tfrac{1}{2}. \tag{48}$$

Given an $\varepsilon > 0$, he chose $n$ so that

$$|V_r - V| < \varepsilon \quad \text{and} \quad r > n - A. \tag{49}$$

He then wrote

$$\begin{aligned} &|U_1(V_n - V) + U_2(V_{n-1} - V) + \cdots + U_v(V_{n-v+1} - V)| \\ &\qquad - (|U_1| + |U_2| + \cdots + |U_v|) < \varepsilon. \end{aligned} \tag{50}$$

1 E. CESÀRO, *Bulletin des sciences mathématiques*, (2) **14** (1890): 114–120.
2 A. CAUCHY, *Analyse Algébrique* (Paris, 1821), p. 147.
3 N. H. ABEL, *op. cit.*, p. 318.
4 F. MERTENS, *Journal für die reine und angewandte Mathematik*, **79** (1875): 182.

By applying two theorems of CAUCHY on arithmetic means,[1] he obtained

$$\lim_{n\to\infty} (U_1 + U_2 + \cdots + U_n)/n = U \tag{51}$$

and

$$\lim_{n\to\infty} (|U_1| + |U_2| + \cdots + |U_n|)/n = |U|. \tag{52}$$

Applying (51) and (52) to (50) yields

$$\lim_{n\to\infty} (U_1(V_n - V) + \cdots + U_v(V_{n-v+1} - V))/n = 0 \tag{53}$$

and

$$\lim_{n\to\infty} (U_1 V_n + U_2 V_{n-1} + \cdots + U_v V_{n-v+1})/n = \frac{UV}{2}. \tag{54}$$

Similarly, we have

$$\lim_{n\to\infty} (U_{v+1} V_{n-v} + U_{v+2} V_{n-v-1} + \cdots + U_n V_1)/n = \frac{UV}{2}. \tag{55}$$

The addition of equations (54) and (55) gives

$$\lim_{n\to\infty} (U_1 V_n + U_2 V_{n-1} + \cdots + U_n V_1)/n = UV, \tag{56}$$

which completes CESÀRO'S proof.

Using this theorem to give a generalization of ABEL'S theorem on the product of series, CESÀRO stated that if $\sum u_n = U$, and $\sum v_n = V$, and $\sum w_n$ converges, then $\sum w_n = UV$.

This follows directly from (56) if one considers the following:

$$W_n = w_1 + w_2 + \cdots + w_n = u_1 V_n + u_2 V_{n-1} + \cdots + u_n V_1 \tag{57}$$

and

$$W_1 + W_2 + \cdots + W_n = U_1 V_n + U_2 V_{n-1} + \cdots + U_n V_1. \tag{58}$$

The application of (56) to the above yields

$$\begin{aligned}\lim_{n\to\infty} (W_1 + W_2 + \cdots + W_n)/n &= \lim_{n\to\infty} (U_1 V_n + \cdots + U_n V_1)/n \\ &= UV.\end{aligned} \tag{59}$$

CESÀRO pointed out that if $\sum u_n$ and $\sum v_n$ converge, then $\sum w_n$ cannot diverge to infinity. If that were the case, then

$$\lim_{n\to\infty} (W_1 + W_2 + \cdots + W_n)/n$$

would not exist, thus contradicting the fact that this limit must equal $UV$. But he noted that even if the two series converge, the series $\sum w_n$ may not converge but may oscillate. He considered the following example:

$$\sum u_n = \sum_{n=1}^{\infty} (-1)^{n+1}/n; \qquad \sum v_n = \sum_{n=1}^{\infty} (-1)^{n+1}/\log(n+1). \tag{60}$$

---

[1] In his *Analyse Algébrique* of 1821, CAUCHY proved the theorems: If a sequence $\{x_n\}$ converges to $x$, then (a) the sequence $\{(x_1 + x_2 + \cdots + x_n)/n\}$ converges to $x$, and (b) the sequence $\{(|x_1| + |x_2| + \cdots + |x_n|)/n\}$ converges to $|x|$.

Then

$$\sum w_n = \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \frac{1}{\log(n+1)} + \frac{1}{2\log n} + \cdots + \frac{1}{n\log 2} \right]. \tag{61}$$

Series (61) converges because

$$|w_n| > \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{\log(n+1)}, \tag{62}$$

and the term on the right side of inequality (62) does not go to 0 as $n$ goes to infinity. However, the series does oscillate; and furthermore, the $w_n$ have a mean value of 0. If one were to apply the result of ABEL's theorem to this example, he would find that

$$\lim_{n\to\infty} (w_1 + w_2 + \cdots + w_n)/n = 0.$$

Hence, if $\sum w_n$ does oscillate, $UV$ must lie between the extreme limits of $\sum w_n$.[1] CESÀRO showed that in all cases the oscillation will be such that

$$\lim_{n\to\infty} (W_1 + W_2 + \cdots + W_n)/n = UV. \tag{63}$$

In order to present the proof of (63), CESÀRO first generalized his original theorem in the following way: If for the sequences $\{a_n\}$, $\{b_n\}$,

$$\lim_{n\to\infty} a_n/n^{r-1} = a; \qquad \lim_{n\to\infty} b_n/n^{s-1} = b, \tag{64}$$

for some integers $r$, $s$, each greater than 0, then

$$\lim_{n\to\infty} (a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1)/n^{r+s-1} = (r-1)!\,(s-1)!\,ab/(r+s-1)!. \tag{65}$$

CESÀRO then proved the main result of his paper of 1890, that is, an extension of ABEL's theorem for the case in which the series $\sum u_n$, $\sum v_n$, and $\sum w_n$ are not necessarily convergent but have sequences of partial sums whose arithmetic means converge to $U$, $V$, and $W$, respectively. He showed that, in this case, $UV = W$.

CESÀRO first considered the case where $r = s = 2$ in (64). Then equation (65) and properties (64) imply that

$$\lim_{n\to\infty} (a_1 b_n + \cdots + a_n b_1)/n^3 = \tfrac{1}{6} \lim_{n\to\infty} a_n/n \lim_{n\to\infty} b_n/n. \tag{66}$$

CESÀRO set

$$a_n = U_1 + U_2 + \cdots + U_n, \qquad b_n = V_1 + V_2 + \cdots + V_n,$$

so that

$$\lim_{n\to\infty} \frac{a_n}{n} = U; \qquad \lim_{n\to\infty} \frac{b_n}{n} = V. \tag{67}$$

Using the fact that

$$W_1 + W_2 + \cdots + W_n = U_1 V_n + U_2 V_{n-1} + \cdots + U_n V_1,$$

[1] Note that these limits can be $-\infty$ and $+\infty$.

CESÀRO obtained the equations

$$a_1 b_1 = U_1 V_1 = W_1 = \frac{2 \cdot 1}{2} W_1,$$

$$\begin{aligned} a_1 b_2 + a_2 b_1 &= U_1 (V_1 + V_2) + (U_1 + U_2) V_1 = 2 U_1 V_1 + (U_1 V_2 + U_2 V_1) \\ &= 2 W_1 + (W_1 + W_2) = \frac{3 \cdot 2}{2} W_1 + \frac{2 \cdot 1}{2} W_2, \end{aligned}$$

$$\begin{aligned} a_1 b_3 + a_2 b_2 + a_3 b_1 &= U_1 (V_1 + V_2 + V_3) + (U_1 + U_2)(V_1 + V_2) + (U_1 + U_2 + U_3) V_1 \\ &= 3 U_1 V_1 + 2 (U_1 V_2 + U_2 V_1) + (U_1 V_3 + U_2 V_2 + U_3 V_1) \\ &= 3 W_1 + 2 (W_1 + W_2) + (W_1 + W_2 + W_3) \\ &= \frac{4 \cdot 3}{2} W_1 + \frac{3 \cdot 2}{2} W_2 + \frac{2 \cdot 1}{2} W_3, \end{aligned}$$

and, in general,

$$a_1 b_n + \cdots + a_n b_1 = \frac{(n+1)(n)}{2} W_1 + \frac{(n)(n-1)}{2} W_2 + \cdots + \frac{2 \cdot 1}{2} W_n. \tag{68}$$

In equation (68), the right member can be rewritten as

$$\begin{aligned} &\tfrac{1}{2} (n+1)(n+2)(W_1 + W_2 + \cdots + W_n) \\ &\quad (n + \tfrac{3}{2})(W_1 + 2 W_2 + 3 W_3 + \cdots + n W_n) \\ &\quad + (W_1 + 4 W_2 + 9 W_3 + \cdots + n^2 W_n). \end{aligned} \tag{69}$$

CESÀRO now reasoned that since the hypothesis of the theorem states that the series $\sum w_n$ has partial sums whose arithmetic means converge to $W$, this implies that

$$\lim_{n \to \infty} (W_1 + W_2 + \cdots + W_n)/n = W. \tag{70}$$

CESÀRO then applied a theorem which CAUCHY proved in 1821, and which stated that if one is given statement (70), the following may be deduced:[1]

$$\lim_{n \to \infty} (W_1 + 2^{k-1} W_2 + 3^{k-1} W_3 + \cdots + n^{k-1} W_n)/n^k = \frac{W}{k}. \tag{71}$$

CESÀRO divided both sides of equation (68) by $n^3$, substituted (69) for the right member of (68), and took limits. He obtained the equation

$$\begin{aligned} &\lim_{n \to \infty} (a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1)/n^3 \\ &= \lim_{n \to \infty} \frac{(n+1)(n+2)}{2} (W_1 + W_2 + \cdots + W_n) - (n + \tfrac{3}{2})(W_1 + 2 W_2 + \cdots + n W_n) \\ &\quad + (W_1 + 4 W_2 + \cdots + n^2 W_n)/2. \end{aligned} \tag{72}$$

Separating the limits in (72), CESÀRO wrote the right member of (72) as

$$\begin{aligned} &\lim_{n \to \infty} \left( \frac{n^2 + 3n + 2}{2n} \right) \left( \frac{1}{n} \right) (W_1 + W_2 + \cdots + W_n) \\ &\quad - \lim_{n \to \infty} \left( \frac{n + \frac{3}{2}}{n} \right) \left( \frac{1}{n^2} \right) (W_1 + 2 W_2 + \cdots + n W_n) \\ &\quad + \lim_{n \to \infty} \left( \frac{1}{2} \right) \left( \frac{1}{n^3} \right) (W_1 + 4 W_2 + 9 W_3 + \cdots + n^2 W_n). \end{aligned} \tag{73}$$

[1] A. CAUCHY, *op. cit.*, p. 132.

By applying CAUCHY's theorem, (71), to each of the three limits, CESÀRO obtained for (73) the value

$$\frac{W}{2} - \frac{W}{2} + \left(\frac{1}{2} \cdot \frac{W}{3}\right) = \frac{W}{6}. \tag{74}$$

Hence, equation (72) implies that

$$\operatorname*{Lim}_{n \to \infty} (a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1)/n^3 = \frac{W}{6}, \tag{75}$$

but equation (65) implies that

$$\operatorname*{Lim}_{n \to \infty} (a_1 b_n + \cdots + a_n b_1)/n^3 = \frac{UV}{3!} = \frac{UV}{6}. \tag{76}$$

Therefore, equations (75) and (76) together imply that $W = UV$, which completes CESÀRO's proof.

The essence of this important theorem is that if two convergent series are multiplied, the CAUCHY Product may be either convergent or not, and if it is convergent, its sum will be the product of the sums of the original two series. If it is not convergent, but does have a sequence of partial sums whose arithmetic means converge to some number, that number will still be the product of the sums of the two original series.

CESÀRO called a series which does not converge, but whose partial sums form a sequence whose arithmetic means converge, a simply indeterminate series. However, CESÀRO did not stop there. If this mean value, or "first mean" did not exist, he prescribed a method for iterating the procedure until "a mean" was found. Herein lies the true significance of CESÀRO's research as far as the theory of summability is concerned. The method he set forth has since been adopted by many mathematicians as a convenient summation technique.

If, for the series $\sum u_n$ with partial suns $U_n$, the first mean did not exist, CESÀRO suggested writing the following:

$$\binom{r+n-1}{n-1} U_n^{(r)} = u_n + \binom{r+1}{1} u_{n-1} + \cdots + \binom{r+n-1}{n-1} u_1, \tag{77}$$

where $\binom{r+i}{i}$ is defined to be $(r+i)!/(r!)(i!)$. Then, if among the functions $U_n, U_n^{(1)}, U_n^{(2)}, \ldots, U_n^{(r)}$, the function $U_n^{(r)}$ is the first for which $\operatorname*{Lim}_{n\to\infty} U_n^{(r)}$ exists and has the value $U$, CESÀRO would call the series $\sum u_n$ "$r$-fold indeterminate with sum $U$."[1]

As a conclusion to his 1890 paper, CESÀRO extended ABEL's theorem on the multiplication of series to the case where the series $\sum u_n$ is $r$-fold indeterminate with sum $U$, and the series $\sum v_n$ is $s$-fold indeterminate with sum $V$. He showed that if $r$ and $s$ are positive integers, the CAUCHY Product series $\sum w_n$ has a "sum" which is the product of the "sums" of the original two series. That is, the series $\sum w_n$ will be $(r+s+1)$-fold indeterminate with sum $UV$. In other words, the relation which exists between "sums" is the same as if all three series were convergent.

---

[1] E. CESÀRO, *op. cit.*, p. 119. CESÀRO used the terminology "$r$-fois indéterminée."

CESÀRO's method has become widely used since its introduction. Many different methods, which were developed later, used his main approach. GODFREY HAROLD HARDY (1877–1947), one of the foremost mathematicians to work on divergent series in the first half of the 20th century, said of CESÀRO in 1957, "... his language now seems almost absurdly modest: 'il résulte de la une classification des séries indéterminées, qui est sans doute incomplète et pas assez naturelle ...' In fact, his classification is entirely natural."[1]

The first method of summation to gain recognition after CESÀRO introduced his was advanced by EMILE BOREL in 1895.[2] BOREL's approach was different in that instead of considering a weighted mean of a finite number of sums and then allowing the number of terms to become infinite, he considered a weighted mean of the infinite set of sums and allowed the weights to vary in a prescribed way. Thus, to define the sums of a series of functions $\sum u_n$, BOREL considered the expression

$$(C_0 S_0 + C_1 S_1 x + C_2 S_2 x^2 + \cdots)/(C_0 + C_1 x + C_2 x^2 + \cdots). \tag{78}$$

In (78), BOREL took $S_n$ to be the partial sums of the $\sum u_n$, $\sum C_n x^n = \phi(x)$ to be an entire function; and $\sum C_n S_n x^n$ to be convergent for all $x$. He then let $S = \operatorname*{Lim}_{x\to\infty} (1/\phi(x)) \sum C_n S_n x^n$.

As a first choice for the function $\phi(x)$, BOREL used $e^x$, and for the constants $C_n$, he used $1/n!$. Accordingly, BOREL defined a series of functions $\sum u_n$ to be "summable with sum $S$" if

$$\operatorname*{Lim}_{x\to\infty} e^{-x} T(x) = S, \tag{79}$$

where

$$T(x) = S_0 + (S_1/1!)\,x + (S_2/2!)\,x^2 + \cdots + S_n x^n/n!. \tag{80}$$

For example, consider the case where the $u_n$ are constants, *i.e.*, $\sum u_n = \sum (-1)^n$. As seen above, the $S_n$ are 1 or 0, depending on whether $n$ is even or odd. Hence,

$$\sum_{n=0}^{\infty} S_n x^n/n! = 1 + x^2/2! + \cdots = (e^x + e^{-x})/2$$

and

$$S = \operatorname*{Lim}_{x\to\infty} e^{-x}(e^x + e^{-x})/2 = \operatorname{Lim}(1 + e^{-x})/2 = \tfrac{1}{2}.$$

Therefore, the LEIBNITZ Series is "BOREL summable to $\frac{1}{2}$."

A comparison of the BOREL or B method and the CESÀRO method yields an interesting fact. For example, if $\sum u_n = \sum z^n$, where $z$ is a complex variable, and $z \neq 1$, then the partial sums $S_n$ can be written as

$$\begin{aligned} S_0 &= 1 = (1-z)/(1-z), \\ S_1 &= 1 + z = (1-z^2)/(1-z), \\ S_2 &= 1 + z + z^2 = (1-z^3)/(1-z), \end{aligned}$$

[1] G. H. HARDY, *Divergent Series* (Oxford: Clarendon, 1949), p. 8. The translation of HARDY's quotation: This results in a classification of indeterminate series which is without doubt incomplete and perhaps unnatural.

[2] E. BOREL, *Comptes rendus de l'Académie des sciences*, **121** (1895): 1125.

and, in general,

$$S_n = 1 + z + \cdots + z^n = (1 - z^{n+1})/(1 - z). \tag{81}$$

Then substitution of equation (81) in (79) gives

$$\begin{aligned} S &= \operatorname*{Lim}_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} S_n x^n/n! \\ &= \operatorname*{Lim}_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} (1 - z^{n+1})\, x^n/\big(n!\,(1 - z)\big). \end{aligned} \tag{82}$$

Equation (82) can be written in the form

$$\begin{aligned} S &= \operatorname*{Lim}_{x\to\infty} e^{-x}/(1 - z) \sum_{n=0}^{\infty} (1 - z^{n+1})\, x^n/n! \\ &= \operatorname*{Lim}_{x\to\infty} e^{-x}/(1 - z)\,[(1 - z) + (1 - z^2)\, x/1! + (1 - z^3)\, x^2/2! + \cdots]. \end{aligned} \tag{83}$$

Since the series in (83) is absolutely convergent for any $x$ and for $z$ such that $R(z) < 1$, it may be rearranged to give

$$\begin{aligned} S &= \operatorname*{Lim}_{x\to\infty} e^{-x}/(1 - z)\,[(1 + x/1! + x^2/2! + \cdots) - (z + z^2 x/1! + z^3 x^2/2! + \cdots)] \\ &= \operatorname*{Lim}_{x\to\infty} e^{-x}/(1 - z)\,(e^x - z e^{xz}) \\ &= \operatorname*{Lim}_{x\to\infty} \big[\big(1/(1 - z)\big) - \big(z/(1 - z)\big) e^{x(z-1)}\big]. \end{aligned} \tag{84}$$

If $R(z) < 1$, the limit in (84) becomes $1/(1 - z)$ as $x$ goes to infinity. Therefore, we might say that the geometric series $\sum z^n$ is B summable to $1/(1 - z)$ for $R(z) < 1$. However, the geometric series is not summable by CESÀRO's method outside the unit circle $|z| = 1$, but only for the boundary points, and even then, not at $z = 1$. BOREL's process sums the series in the half-plane $R(z) < 1$, and the sum is $1/(1 - z)$ there. Hence, the B method is more powerful than CESÀRO's method in the sense that it sums a series for values of the variable for which the CESÀRO process does not.

It is easy to see that the B method will assign to any convergent series its ordinary sum, for if $\sum u_n$ is a series having the sum $S$ in the usual sense, then

$$e^{-x} \sum_{n=0}^{\infty} S_n x^n/n! - S = e^{-x} \sum_{n=0}^{\infty} (S_n - S)\, x^n/n!. \tag{85}$$

Since for $n > m$, $S_n$ approaches $S$, one can choose $m$ so that for all $n > m$,

$$|S_n - S| < \frac{\varepsilon}{2}\,. \tag{86}$$

Then inequality (86) implies that

$$\begin{aligned} \left| e^{-x} \sum_{n=0}^{\infty} (S_n - S)\, x^n/n! \right| &\leqq e^{-x} \sum_{n=0}^{\infty} |S_n - S|\, x^n/n! \\ &\leqq e^{-x} \sum_{n=0}^{m} |S_n - S|\, x^n/n! + \varepsilon/2, \end{aligned} \tag{87}$$

where $\sum_{n=0}^{m} |S_n - S|\, x^n/n!$ is a polynomial of the $m^{\text{th}}$ degree. Now $e^{-x}$ times that polynomial goes to 0 as $x$ goes to infinity, and so, if $x_0$ is chosen large enough so that

$$e^{-x}\sum_{n=0}^{\infty}|S_n-S|x^n/n!<\varepsilon/2, \qquad x>x_0, \tag{88}$$

then (87) and (88) imply that for all $x>x_0$,

$$\left|e^{-x}\sum_{n=0}^{\infty}S_n x^n/n!-S\right|<\varepsilon.$$

BOREL also developed an integral form for his definition of summability in 1899.[1] He proceeded as follows: Given a series $\sum a_n$, define its sum to be $S=\int_0^{\infty}e^{-x}U(x)\,dx$, if the integral exists, where $U(x)=\sum_{n=0}^{\infty}a_n(x)\,x^n/n!$. There is a close relation between the first, or exponential method, and the second, the integral method. In fact, if

$$T(x)=S_0+S_1x/1!+S_2x^2/2!+\cdots,$$

we see that

$$T'(x)=S_1+S_2x/1!+\cdots=\sum_{n=0}^{\infty}S_{n+1}x^n/n!. \tag{89}$$

Therefore, if one integrates by parts,

$$\int_0^x e^{-t}U'(t)\,dt=e^{-x}U(x)-a_0(x)+\int_0^x e^{-t}U(t)\,dt, \tag{90}$$

where

$$U(x)=\sum_{n=0}^{\infty}a_n(x)\,x^n/n!.$$

We also have that

$$\begin{aligned} e^{-x}T(x)-a_0(x)&=\int_0^x\frac{d}{dt}\left(e^{-t}T(t)\right)dt=\int_0^x e^{-t}\left(T'(t)-T(t)\right)dt\\ &=\int_0^x e^{-t}\sum(S_{n+1}-S_n)\,t^n\,dt/n!\\ &=\int_0^x e^{-t}\sum a_{n+1}(x)\,t^n\,dt/n!\\ &=\int_0^x e^{-t}U'(t)\,dt. \end{aligned} \tag{91}$$

From equations (90) and (91) we obtain

$$e^{-x}U(x)-a_0(x)+\int_0^x e^{-t}U(t)\,dt=e^{-x}T(x)-a_0(x),$$

[1] E. BOREL, *Annales Scientifiques de l'Ecole Normale Supérieure*, (3) 16 (1899): 9–136.

or

$$e^{-x} T(x) = e^{-x} U(x) + \int_0^x e^{-t} U(t)\, dt. \tag{92}$$

Equation (92) illustrates the equivalence between the integral and exponential methods. If one were again to consider the geometric series $\sum z^n$, he would find that the integral method will sum that series to $1/(1-z)$ in the half-plane $R(z) < 1$.

### 3. The Emergence of the Theory

After CESÀRO and BOREL showed that divergent series could be systematically studied and used, research by others in this field began to expand. The doubts which ABEL and CAUCHY had engendered were forgotten. The immediate goal was to find the most powerful and general summation method, or to determine whether such existed. The years from 1890 to 1925 produced a considerable amount of research toward these goals, so that in this period the basic concepts of the theory of divergent series were solidified.

In the course of this solidification, a set of criteria came to be accepted by the researchers. That is to say, the men who studied divergent series adopted in an informal way a set of properties which they felt a new summation method should possess in order for it to be practical. Accordingly, as new techniques were developed, they were subjected to certain minimal requirements. Neither HÖLDER, CESÀRO nor BOREL presented a list of such criteria formally, but it was implicit in their work. Three main criteria were eventually singled out by K. KNOPP.[1] They are:

1) A new method should assign to a convergent series its ordinary sum. This condition is usually referred to as the condition of regularity.
2) At least one series, divergent in the former sense, should be summable by the new method.
3) For any two different summation methods, a series should have the same sum.

As an example of the application of these criteria to summation methods, consider the H (HÖLDER) and C (CESÀRO) methods. Regarding the condition of regularity, it should be clear that both methods are regular, since the first step in either process will supply a convergent sequence of partial sums.

The LEIBNITZ series provides a ready example of a divergent series which is assigned a sum by both methods.

As to the question of satisfying the third criterion, it should be observed that one must show that for any series to which the H and C methods apply, the H and C sums will be the same. This very theorem was proved in the early part of the 20th century. It is commonly referred to as the "Equivalence Theorem," and it states that if a series $\sum u_n$ is summable $(H, k)$ to $U$, then it is summable $(C, k)$ to $U$, and conversely.[2] K. KNOPP proved the direct implication in his

[1] K. KNOPP, *op. cit.*, p. 463.

[2] This notation was introduced by G. H. HARDY. If a series $\sum u_n$ is "summable $(C, k)$ to $U$," this implies that $k$ is the smallest positive integer such that $\lim_{n\to\infty} U_n^{(k)} = U$.

unpublished dissertation in 1907, and the converse was first proved by WALTER SCHNEE in 1909.[1]

It might be noted further that although the H and C methods are equivalent, a disadvantage exists in using the H method. It is not usually a simple matter to express $S_n^{(k)}$ as HÖLDER defined it, directly in terms of the original partial sums of the series. In CESÀRO's method, however, the successive sequences of means can be easily written in terms of the original series. CESÀRO himself provided the formula for doing so in his article of 1890. He stated that one could express the function $U_n^{(k)}$ in the form [2]

$$U_n^{(k)} = \left(S_n + \binom{k}{1} S_{n-1} + \cdots + \binom{k+n-2}{n-1} S_1\right) \Big/ \binom{k+n-1}{n-1}, \tag{93}$$

where the $S_n$ are the partial sums of the original series. Thus, if $n=3$ and $k=4$, formula (93) gives

$$U_3^{(4)} = \left(S_3 + \binom{4}{1} S_2 + \binom{5}{2} S_1\right) \Big/ \binom{6}{2} = (S_3 + 4S_2 + 10S_1)/15,$$

and from this, $U_3^{(4)}$ can be easily expressed in terms of the original series $\sum u_n$ as

$$\begin{aligned} U_3^{(4)} &= ((u_1+u_2+u_3) + 4(u_1+u_2) + 10(u_1))/15 \\ &= (15u_1 + 5u_2 + u_3)/15. \end{aligned}$$

The above consideration explains why the C method has become so popular over the years. So desirous have mathematicians been in trying to posit C summability, that a good deal of research has been aimed at determining which conditions must be required of a series to insure that it be summable $(C, k)$. Countless theorems have shown that necessary conditions usually involve the size of the terms in the series, rates of monotonicity, etc. For example, K. KNOPP proved in 1907 that if a series $\sum u_n$ is summable $(C, k)$, then both $u_n/n^k$ and $S_n/n^k$ go to 0 as $n$ goes to infinity.[3] Hence, if $u_n/n^k$ does not go to 0, one may conclude that the series is not summable $(C, k)$.

One of the first generalizations of the C method was made by NIELS ERIK NÖRLUND.[4] A similar definition had been given by G. F. VORONOI in 1902, but because his article appeared in an inaccessible journal, it went unnoticed.[5] In 1920, NÖRLUND gave the definition: If $p_0 > 0$, and $p_1, p_2, p_3, \ldots$ is a set of non-negative numbers, set

$$t_n = (p_n S_0 + p_{n-1} S_1 + \cdots + p_0 S_n)/(p_0 + p_1 + \cdots + p_n), \tag{94}$$

where the $S_n$ are the partial sums of the given series $\sum a_n$. Then, if $t_n$ approaches $S$ as $n$ goes to infinity, $S$ would be called the "sum" of the series $\sum a_n$. NÖRLUND

[1] W. SCHNEE, *Mathematische Annalen*, **67** (1909): 110–125.

[2] E. CESÀRO, *op. cit.*, p. 119.

[3] K. KNOPP, *op. cit.*, p. 484. The complete proof is given there.

[4] N. E. NÖRLUND, *Lunds Universitets Arsskrift*, (2) **16** (1920).

[5] G. F. VORONOI, *Proc. of 11th Congr. of Russian Nat. & Sci.*, published in St. Petersburg in 1902. A translation appears in *Annals of Math.*, (2) **33** (1932): 422–428, by A. TAMARKIN.

stated as a condition that $p_n/(p_0+p_1+\cdots+p_n)$ goes to 0 as $n$ goes to infinity. Although VORONOI had not stated this explicitly, it seems clear from his definition that he also implied it.

As an example of how this generalization would be applied, consider the case in which $p_n=1$ for all $n$. In that case, one has

$$t_n=(S_0+S_1+\cdots+S_n)/(n+1),$$

and this is precisely the (C, 1) mean which CESÀRO defined. In fact, in general it is the case that if

$$p_n=\binom{n+k-1}{k-1}, \quad \text{where } k>0,$$

then

$$\begin{aligned} t_n &= \frac{\left[\binom{n+k-1}{k-1}S_0+\binom{n+k-2}{k-1}S_1+\cdots+\binom{k-1}{k-1}S_n\right]}{\left[\binom{k-1}{k-1}+\cdots+\binom{n+k-1}{k-1}\right]} \\ &= \left[\binom{n+k-1}{k-1}S_0+\cdots+\binom{k-1}{k-1}S_n\right]\Big/\binom{n+k}{k}, \end{aligned} \tag{95}$$

and the expression on the right side of (95) is CESÀRO'S $(C, k)$ mean.

Another extension of CESÀRO'S methods which was introduced during the early 1900's, and which was typical of much of the work then being done, was formulated by SYDNEY CHAPMAN. He, along with HARDY, were two British mathematicians who devoted most of their lives to the study of divergent series. The number of articles published by the two, either jointly or independently, staggers the imagination. HARDY in particular was most prolific in his writings. He alone was the author of over 300 scholarly publications.

Although KNOPP had considered non-integral orders of C summability, CHAPMAN was the first to give a systematic treatment of them. He showed that the "weights" in the C method could be real or complex valued functions as well as positive integers. In 1911, CHAPMAN published his results, beginning with the definitions:[1]

$$\begin{aligned} S_n^{(r)} &= S_n+\binom{r}{1}S_{n-1}+\cdots+\binom{r+n-1}{n}S_0, \\ A_n^{(r)} &= \binom{r+n}{n}=(r+1)(r+2)\ldots(r+n)/n!. \end{aligned} \tag{96}$$

In these definitions $S_n$ represents the $n^{\text{th}}$ partial sum of the given series, and $r$ may take on any real or complex values except negative integral ones. CHAPMAN stated that if

$$\lim_{n\to\infty} S_n^{(r)}/A_n^{(r)}=S$$

exists, the given series is to be called summable $(C, r)$ to $S$.

[1] S. CHAPMAN, *Proc. of the London Math. Society*, (2) **9** (1911): 369–409.

CHAPMAN noted that it is usually easier to find the sum of a non-convergent series by using an integral order than by using the above method for non-integral values of $r$. The importance of CHAPMAN's method, however, was that it gave better information regarding the degree of non-convergence.

Dissatisfied with this generalization and those advanced by other mathematicians, CHAPMAN continued to search for the ultimate generalization, and in 1912 he published a lengthy article in which he set forth the most abstract bases for a theory of summability that had been conceived up to that time.[1] He started out by defining a classification for infinite forms. By an infinite form, CHAPMAN meant a sequence of functions of one or more variables expressible in closed form. He called an infinite form "pure" and of the "first type" if the generating form is a function only of integral valued variables $m, n, p, \ldots$; it is "pure" and of the "second type" if none of the variables are integral. It is a "mixed form" when some variables are of the first type and some of the second.

Examples of the first type are infinite series, infinite products, infinite continued fractions; of the second type, infinite integrals; of mixed forms, an infinite series integrated over an infinite range.

CHAPMAN formulated a general principle of summability for all infinite forms in the following way:

> When the sequence of finite forms, which define or generate an infinite form, does not tend to a limit as the variables tend to infinity in the assigned order through the sequence of values constituting the domains of these variables, then we may agree that the number represented by the given infinite form is to be the limit of a sequence of associated finite forms, different from the members of the original sequence; the second sequence must of course be judiciously chosen, so that the limit to which it tends is usefully related to the original sequence. The number of its variables may be the same or greater than the original number; and the additional variables, if any, may or may not be required to tend to infinity.[2]

Each particular method of correlating another sequence to a given one will thus correspond to a method of "summation" and the resulting value will be called the "sum."

For example, CHAPMAN would consider $\sum_{n=1}^{\infty}(-1)^{n-1}x^n$ an infinite form. The sequence of finite forms generating it would be the partial sums $x$, $x-x^2$, $x-x^2+x^3$, *etc.* Since when $x=1$ the sequence of finite forms does not tend to a limit as the variables tend to infinity, CHAPMAN would obtain a sequence of "associated finite forms" and take the limit of that associated sequence as the pseudo-limit of the original sequence.

Hence, CHAPMAN might judiciously choose as the associated sequence the following:

$$S_1/1, \quad (S_1+S_2)/2, \quad (S_1+S_2+S_3)/3, \quad (S_1+S_2+S_3+S_4)/4, \quad \text{etc.}$$

---

[1] S. CHAPMAN, *Quarterly Journal of Math.*, **43** (1912): 1–52.
[2] *Ibid.*, p. 2.

In that case, the limit of the associated sequence when $x=1$ turns out to be $\frac{1}{2}$. CHAPMAN would then assign this value as the pseudo-limit of the original sequence of partial sums for the series $\sum(-1)^{n-1}x^n$.

Various restrictions may be imposed upon a definition of summability for convenience in use, and CHAPMAN noted that the most important is the condition of consistency. He formulated a so-called "generalized condition of consistency."[1] This states that a parametric method of summation will satisfy the generalized condition of consistency over any part of the domain of the parameter if, in that domain, the values of the parameter can be so ordered that any form, which is summable with parameter $r_1$, shall also be summable with any other parameter $r_2$, which succeeds $r_1$ in the ordered set of $r$. For example, if $r=(r_1, r_2, r_3, \ldots)$ corresponds to the degrees of summability in the CESÀRO definition and if a series is summable $r_k$, it will be automatically summable $r_n$ for $n>k$ in the sequence $r$.

In infinite form which is summable by such a general definition is said to be summable $(K, h, \ldots, r)$, where $K$ indicates the particular method and $h, \ldots, r$ indicates the additional variables or parameters introduced, if any. Thus, an infinite form is summable $(K, h, \ldots, r)$ if the subsidiary sequence converges to $S$, which is then called the "sum" of the infinite series.

CHAPMAN'S approach, as general as it may have been, still originated from the averaging concepts of CESÀRO. During the years from 1890 to 1925, extensions in the theory of divergent series stemmed from other sources as well. After CESÀRO had defined his technique, mathematicians realized that although it had a great number of advantages, other methods such as ABEL'S and BOREL'S were even more powerful. And so they turned their attention to those directions.

For example, around the turn of the century, some mathematicians began to realize that many theorems which hypothesize the convergence of a series may be generalized by weakening the hypothesis to that of summability. They found that this was frequently the case when they substituted ABEL-Summability or A summability as a criterion in place of convergence, especially when dealing with a series of complex numbers. They discovered that under the hypothesis of summability, the ABEL Limit Theorem could be extended to apply to the complex plane, and thereby prescribe a circle of convergence for a convergent series.

The ideas used to extend the ABEL Limit Theorem were derived from the theorems of CAUCHY referred to in Section 2. One generalization of those theorems was made by J. JENSEN in 1884 and was subsequently applied by CESÀRO in 1890.[2] Another generalization was made by OTTO STOLZ in 1889,[3] but before this was published, STOLZ used the idea to reformulate ABEL'S theorem in 1875, in the context of the complex plane.[4] The statement of ABEL'S theorem for the complex plane is: If $z$ is complex and the series $\sum a_n z^n$ has a radius of convergence 1, then

$$\lim_{z\to 1^-} \sum a_n z^n = \sum a_n$$

if $z$ approaches 1 along the positive real axis from the origin. STOLZ showed that this theorem remains true under the given hypothesis, no matter how $z$

[1] *Ibid.*, p. 3.

[2] JENSEN, *Tidskrift for Mathematik*, (5) **2** (1884): 81–84.

[3] O. STOLZ, *Mathematische Annalen*, **33** (1889): 237.

[4] O. STOLZ, *Zeitschrift für Math. u. Phys.*, **20** (1875): 369.

approaches 1, as long as the path of approach is within the unit circle $|z|<1$ and the path lies between two rays from $z=1$.

Once the extension of ABEL's theorem was made for complex modes of approach, the question as to whether the conclusion would remain valid if convergence was replaced by summability in the hypothesis was then investigated. In two theorems similar to ABEL'S, EMANUEL LASKER and ALFRED PRINGSHEIM proved that the conclusion does remain valid if one substitutes $(H, k)$ summability for convergence, for any positive integer $k$.[1] KNOPP later replaced the $(H, k)$ summability with $(C, k)$. He showed that if $\sum a_n z^n$ has a radius of summability 1 and is summable $(C, k)$ to $S$ at $z=1$, for some positive integer $k$, then

$$\operatorname*{Lim}_{z\to 1^-} \sum a_n z^n = S,$$

for any method of approach described above.

If, in KNOPP's theorem, $k=0$, the theorem reduces to the one STOLZ proved in 1889. When $k=1$, we obtain the theorem FROBENIUS proved in 1880, and when $k=2, 3, \ldots$, we obtain the theorem of HÖLDER proved in 1882. The main consequence of this chain of theorems is that the $(C, k)$ summability of a series to $S$ in the complex plane involves its $A$ summability to $S$. That is, the range of the A method includes that of the C method, and in that regard the A method is superior to the C.

Another direction in which the theory of divergent series advanced was in the exponential and integral methods of BOREL. Once his basic definitions had been made, BOREL continued to strive for a theory of summability analogous to the theory of convergence. For example, at the turn of this century, BOREL introduced the notion of Absolute Summability.[2] This criterion for a summation method provided conclusions which closely resembled those obtained for absolute convergence of a series. BOREL stated his definition as follows: A series $\sum u_n$ is absolutely summable if not only the integral $\int_0^\infty e^{-x} U(x)\, dx$ is convergent, where $U(x)=\sum S_n x^n/n!$, but also the integrals $\int_0^\infty e^{-x}|U(x)|\,dx$, $\int_0^\infty e^{-x}|U'(x)|\,dx, \ldots, \int_0^\infty e^{-x}|U^{(n)}|(x)\,dx, \ldots$, where $n$ is an index of differentiation.

BOREL stated that all convergent series were automatically absolutely summable, but HARDY later found this to be in error.[3] HARDY provided the example: a series whose general term is $(-1)^{\sqrt{n}}/\sqrt{n}$ where $n$ is integral, and 0 for those values of $n$ which are not perfect squares. In this case HARDY pointed out that the integral $\int e^{-x}|U(x)|\,dx$ has no meaning and yet the series converges. HARDY further showed that absolute convergence and not mere convergence is necessary to imply that a series is absolutely summable.

---

[1] E. LASKER, *Phil. Trans. of the Royal Soc.*, (A) **196** (1901): 431; A. PRINGSHEIM, *Acta Mathematica*, **28** (1904): 1.

[2] E. BOREL, *Leçons sur les Séries Divergentes* (Paris: Gauthier-Villars, 1901), p. 128.

[3] G. H. HARDY, *Quarterly Journal of Math.*, **35** (1904): 22–66.

One noteworthy property of absolute summability is that if $\sum u_n$ and $\sum v_n$ are absolutely summable with sums $U$ and $V$, respectively, then the series $\sum (a u_n + b v_n)$ is also absolutely summable with sum $aU + bV$. A more important consequence is: given the above conditions, the CAUCHY Product series $\sum w_n$ is also absolutely summable with sum $W = UV$. HARDY showed that if one were to suppose that only one of the factor series is absolutely summable by BOREL's method, then the product series is still summable although not necessarily absolutely. This may be regarded as an extension of MERTEN's theorem on the multiplication of convergent series.

A similar generalization was made during this period in the attempt to find a concept analogous to uniform convergence. In 1903, HARDY suggested the following definition of "uniform summability."[1] If instead of a series of constants, we have a series of real-valued functions $\sum u_n(a)$, that series will be called uniformly summable for $b < a < c$ if for these values of $a$, $\int_0^\infty e^{-x} u(x, a)\, dx$ converges uniformly, where $u(x, a) = \sum u_n(a)\, x^n/n!$.

Such a definition shows once again that convergence and summability possess analogous properties.

Further extensions of BOREL's methods were advanced in the early 1900's. Because power series are so important as representations in the theory of analytic functions, an attempt was made by BOREL at this time to determine the domain of absolute summability for power series. Former analysts had determined the domain of convergence in the complex plane, and BOREL wanted to determine in what region of the plane certain fundamental operations could be validly performed on an absolutely summable series. The import of BOREL's research on this subject was felt in the theory of analytic continuation.

The analytic continuation of a function $f(z)$ from a region $R_1$ of the complex plane in which it is analytic, to a region $R_2$ overlapping $R_1$, refers to a function $g(z)$, analytic in $R_2$ and such that $g(z) = f(z)$ in the intersection of $R_1$ and $R_2$.[2] Here $f(z)$ and $g(z)$ can be regarded as local representations of one and the same analytic function, which would be completely described when all of its local representations are found.

Many mathematicians sought the most effective way of continuing a function. Most techniques proceeded by an attempt to extend the domain beyond the circle of convergence for a function $f(z)$ defined by a power series $\sum a_n z^n$. A different technique utilized the notion of summability. In this case, a summation method is used to evaluate the power series of an analytic function in a star-like domain extending beyond the circle of convergence.

The star-like domain usually refers to the largest star-shaped region, including the interior of the circle of convergence, of the series $\sum a_n z^n$ such that there exists a function which is holomorphic throughout this region and coincides with the sum of that series inside the circle of convergence. A region of this kind is characterized by the property: If $P$ is a point of the region, then all points of the segment $0P$ belong to the region, where $0$ is the origin.

---

[1] G. H. HARDY, *Trans. of the Cambr. Phil. Soc.*, **19** (1904): 297–321.

[2] E. HILLE, *Analytic Function Theory* (New York: Ginn & Co., 1962), Vol. 2, p. 69.

The star, or "MITTAG-LEFFLER Star" as it is frequently called because of the work done in this field by that mathematician,[1] is the domain obtained for a function $f(z)$ by drawing rays through the origin to every singular point of $f(z)$ and removing from the plane the parts of the rays beyond the singular points.[2]

As an example, consider the geometric series $\sum z^n$ which converges for $|z|<1$. Its "star" is the entire complex plane cut along the half-ray on the real axis from 1 to infinity. The reason for this is that when $z$ approaches 1-along a line from the origin, the sum of the series tends to infinity. Every analytic continuation tends to infinity as $z$ goes to 1 because $f(z)=1/(1-z)$ coincides with the sum of the series inside the circle of convergence.

BOREL suggested another important method of analytic continuation when he defined the so-called "Polygon of Summability."[3] He showed that if all the singular points of the function $f(z)=\sum a_n z^n$ were joined to the origin by straight lines, and perpendicular lines were drawn through the singular points to these lines, a convex region, which BOREL called the polygon of summability, would be formed. If the singular points are finite in number, the region so defined will generally be a closed rectilinear polygon. If the singular points are not finite in number, the polygon may be curvilinear, or at least in part.

BOREL showed that if $f(z)=\sum a_n z^n$ is absolutely summable for $z=z_0$, it must also be absolutely summable on the segment $0z_0$, that is, from the origin to the point $z_0$.[4] Moreover, the sum of the series on $0z_0$ will be an analytic function which will have no singular point in the circle described on $0z_0$ as diameter. Using this fact, BOREL showed that the power series representing a function, which is regular at the origin, is absolutely summable by his method (either the exponential or integral) inside the BOREL polygon of summability and is not summable at any point outside the polygon.

Thus, BOREL determined a region of absolute summability for a series representation of a function, and, hence, the Borel method of summability provides a means of analytic continuation. The polygon will extend beyond the circle of convergence at all non-singular points. Therefore, the series $\sum z^n$ is B summable to $1/(1-z)$ whenever $R(z)<1$. That is, the polygon of summability for this series is the half-plane, $R(z)<1$.

## 4. Further Developments from 1900 to 1925

After the investigations of FROBENIUS, HÖLDER, and CESÀRO reawakened interest in divergent series, and after the researches of HARDY, CHAPMAN, BOREL and others had begun, the theory of divergent series and summability became a legitimate branch of theoretical mathematics. From that point on, the research directed itself to related areas of mathematics and theoretical physics. In this last section some of the more important concepts and applications discovered between 1900 and 1925 will be treated.

---

[1] G. MITTAG-LEFFLER (1846–1927).

[2] G. H. HARDY, *Divergent Series*, *op. cit.*, p. 197.

[3] E. BOREL, *Leçons*, *op. cit.*, p. 158.

[4] *Ibid.*, p. 152.

In the early 1800's, the German mathematician P. G. DIRICHLET (1805–1859) made a special study of a certain type of infinite series. Its importance lay in the field of series of analytic functions. The study of these series, commonly called DIRICHLET Series, was continued in the early 20th century, and it was only then that their actual power was realized. These series provide an excellent example of the advantage of using the concept of summability.

A DIRICHLET series is defined to be any series of the form $\sum a_n/M_n^z$. One of the first important results concerning such series was derived partly by J. JENSEN in 1884 and by EUGENE CAHEN in 1894.[1] It stated: To every DIRICHLET series, for which $M_n = n$ and for which $z$ is complex, there corresponds a real number $A$ with the property that the series converges when $R(z) > A$ and diverges when $R(z) < A$. Also, if there is a number $A' > A$, and $A' \neq \infty$, then $\sum a_n/n^z$ will be uniformly convergent for $R(z) \geqq A'$, and so the series would represent an analytic function for $R(z) > A$. Correspondingly, if a DIRICHLET series diverges at $z = z_0$, it will diverge for every $z$ whose real part $R(z) < R(z_0)$.

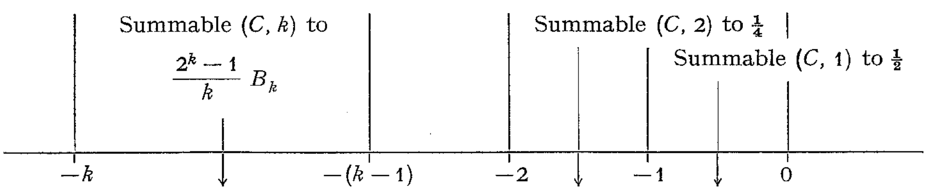
The number $A$ is sometimes called the "abscissa of convergence" since it marks the point on the real axis where the convergence begins. For example, the series $\sum 1/2^n n^z$ converges for all $z$, and hence $A = -\infty$. The series $\sum 2^n/n^z$ diverges for all $z$, and therefore $A = \infty$. The series $\sum 1/n^z$ is analytic for $R(z) > 1$, and for this series, $A = 1$. This series is usually called the "RIEMANN Zeta Function."

To understand what the concept of summability does for a DIRICHLET series, let us consider another example, namely, the DIRICHLET series

$$\sum_{n=1}^{\infty} (-1)^{n-1}/n^z = 1/1^z - 1/2^z + 1/3^z - + \cdots. \tag{97}$$

This series converges for $R(z) > 0$ and diverges for all other values of $z$.[2] At $z = 0$, it reduces to $\sum(-1)^{n-1}$, and this series is summable $(C, 1)$ to $\frac{1}{2}$. At $z = -1$, the series reduces to $\sum(-1)^{n-1} n$, and this is summable $(C, 2)$ to $\frac{1}{4}$. In fact, if $z = -(k-1)$, the series (97) will be summable $(C, k)$ to $(2^k - 1) B_k/k$, where $B_k$ is the $k^{\text{th}}$ BERNOULLI Number.

More generally, this DIRICHLET series is summable by one of the methods outside its convergence region, namely, $R(z) > 0$. In fact, it happens that the series is summable $(C, k)$ for all $z$ such that $R(z) > -k$, and the order of summability is exactly $k$ throughout the strip $-k < R(z) \leqq -(k-1)$. Therefore this series has the following range of summability:




Hence, whereas formerly, a "sum" of the series could be associated only with each point of the right half-plane $R(z) > 0$, we can now associate a sum with

[1] J. JENSEN, *Tidskrift for Math.*, (5) **2** (1884): 81; E. CAHEN, *Annales Scientifiques de l'Ecole Normale Supérieure*, (3) **11** (1894): 75.

[2] K. KNOPP, *op. cit.*, p. 345.

every point of the entire plane, thus defining a function of $z$ in the whole plane, using "sum" for its value.

Analogous properties belong to every DIRICHLET series. Therefore, call $A$, $\lambda_0$, since $(C, 0)$ summability coincides with convergence, and let $\lambda_i$ be the boundary points. That is, $R(z) < \lambda_k$ would imply that the series is $(C, k)$ summable, and the series is not summable $(C, k)$ for $R(z) > \lambda_k$. Now we have that $\lambda_0 \geqq \lambda_1 \geqq \lambda_2 \geqq \cdots$, and so the $\lambda_i$ either have a limit or go to infinity. Call the limit $D$ if it exists, and one can say: a DIRICHLET series is summable $(C, k)$ for all $z$ such that $R(z) > D$; if $D$ is finite, then the line $R(z) = D$ is called the "boundary of summability" of the series.

In 1909, MARCEL RIESZ worked on this very problem of extending the domain of definition of a DIRICHLET series by substituting summability in place of convergence as the criterion. In that year, RIESZ published an article on this subject in which he observed that the process of taking arithmetic means could be greatly improved by also multiplying by suitable weighting factors.[1] He proceeded as follows: Given a DIRICHLET series of the form $\sum a_n/M_n^z$, where $M_n^z = (e^{\lambda_n})^z$, where the $\lambda_n$ are a positive increasing sequence going to infinity, let

$$a_n e^{-\lambda_n z} = c_n; \qquad \sum_{i=1}^{n} c_i = S_n, \tag{98}$$

and let

$$\sigma(\lambda) = S_n, \qquad \lambda_{n-1} < \lambda < \lambda_n. \tag{99}$$

RIESZ then defined

$$\begin{aligned} S_n^{(1)} &= c_1 + c_2 + \cdots + c_n - \frac{\lambda_1 c_1 + \lambda_2 c_2 + \cdots + \lambda_n c_n}{\lambda_n} \\ &= \big(S_1(\lambda_2 - \lambda_1) + S_2(\lambda_3 - \lambda_2) + \cdots + S_{n-1}(\lambda_n - \lambda_{n-1})\big)/\lambda_n \\ &= 1/\lambda_n \int_0^{\lambda_n} \sigma(\lambda)\, d\lambda \end{aligned} \tag{100}$$

and, more generally,

$$S_n^{(k)} = c_1\left(1 - \frac{\lambda_1}{\lambda_n}\right)^k + c_2\left(1 - \frac{\lambda_2}{\lambda_n}\right)^k + \cdots + c_n\left(1 - \frac{\lambda_n}{\lambda_n}\right)^k. \tag{101}$$

The sum of the series $\sum c_n$ was then defined by RIESZ to be $\operatorname{Lim}_{n\to\infty} S_n^{(k)}$. He called summability by these means, summability of order $k$ with exponents $\lambda_1, \lambda_2, \ldots, \lambda_n$.

For $\lambda_n = n$, this definition is equivalent to $(C, k)$ summability but differs from it in form except for $k = 0, 1$. For $\lambda_n = \log n$, RIESZ called the method "logarithmic summability," and in that case, the limit is equivalent to

$$\operatorname*{Lim}_{n\to\infty} \frac{S_1/1 + S_2/2 + \cdots + S_n/n}{\log n}.$$

By applying the RIESZ method of summation to DIRICHLET series we obtain some interesting results. If the function $f(z)$, represented in a part of the plane by the convergent series $\sum a_n e^{-\lambda_n z}$, is regular in the domain $R(z) > C$, and on the line $R(z) = C$; if it does not admit any other singular points than poles and a finite number of algebraic critical points of an order $\leqq m$; and if there exists a number

[1] M. RIESZ, *Comptes rendus de l'Académie des sciences*, **149** (1909): 18–21.

$m'$ such that the function $f(z)$ in the half-plane $R(z) \geqq C$ satisfies the condition $|f(z)| < C|z|^{m'}$, for $z$ large enough, then the series is summable by RIESZ means of order $k$ at every regular point of the line $R(z) = C$, where $k$ is a positive number such that $k > m'$ and $k > m - 1$, and the series will represent $f(z)$ at these points. RIESZ showed that the summability will be uniform on every finite portion of the line which contains only regular points.

RIESZ later modified his definition by introducing a continuous parameter $\omega$.[1] His final expression for the sum was

$$R^{(k)}(\omega) = \sum_{\lambda_1}^{\lambda_n < \omega} c_n \left(1 - \frac{\lambda_n}{\omega}\right)^k = k/\omega^k \int_0^\omega \sigma(\lambda)\,(\omega - \lambda)^{k-1}\, d\lambda. \tag{102}$$

Therefore, if $\lim\limits_{\omega \to \infty} R^{(k)}(\omega) = A$, RIESZ would say that $\sum c_n$ is summable $(R, n, k)$ to $A$.[2]

Through the use of RIESZ's methods, a way was opened up for defining a function which is analytic in the entire complex plane, even when the domain of CAUCHY convergence for the series defining that function is much smaller.

A second important application of summability to mathematics, which was made during the early years of the 20th century, concerned the theory of FOURIER Series. In 1822, J. FOURIER (1768–1830) published his famous treatise on the conduction of heat in which he developed a theory of infinite trigonometric series.[3] This theory has since become known as FOURIER Series. FOURIER stated therein that almost any real-valued function can be represented by a trigonometric series of the form

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n x + b_n \sin n x). \tag{103}$$

As a result of FOURIER's theory, mathematicians were able, for the first time, to express as functions what would not have been considered functions in the time of EULER. The position of these series in higher analysis is respected because they are more effective than power series for many purposes. Resulting from efforts to analyze empirical data in physics, FOURIER series have had profound effects on the development of such fields as integral and differential equations and analytic functions. Almost every study in periodic motion, including acoustics, electrodynamics, and heat, involves these series in some way.

The FOURIER series associated with a function $f(x)$, defined and RIEMANN integrable over the interval $(0, 2\pi)$, may be formally expressed in the form (103), where the FOURIER coefficients are defined as:

$$a_n = 1/\pi \int_0^{2\pi} f(x) \cos n x\, dx; \qquad b_n = 1/\pi \int_0^{2\pi} f(x) \sin n x\, dx. \tag{104}$$

The question of convergence, and whether the series really represented the function, was first considered by DIRICHLET.[4] He showed that if $f(x)$ satisfies

[1] *Ibid.*, pp. 909–912.

[2] This symbolism was introduced by G. H. HARDY in 1910.

[3] J. FOURIER, *Théorie Analytique de la Chaleur* (Paris, 1822).

[4] G. DIRICHLET, *Journal für die reine und angewandte Math.*, **4** (1829): 157.

certain fairly general conditions, the FOURIER series will converge to the value of $f(x)$ at all points of continuity of the function.

It was assumed for a long time that every function $f(x)$ which is continuous at $x_0$ possesses a FOURIER series which converges at that point and has sum $f(x_0)$ there. However, DU BOIS-REYMOND was the first to discredit this supposition when, in 1873, he noted that additional conditions are required for the $a_n$ and $b_n$ to exist.[1] The fact that FOURIER series of continuous functions need not converge everywhere endangered the whole theory of the representation of functions by their FOURIER series.

This situation was salvaged by LEOPOLD FEJÉR at the turn of the century.[2] He proved that the FOURIER series of a continuous function $f(x)$ will always be summable on $(0, 2\pi)$ to $f(x)$ by the method of the arithmetic mean. That is, if $S_n(x)$, the partial sums of the FOURIER series at $x_0$, are defined as follows:

$$S_n(x_0) = \tfrac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos k x_0 + b_k \sin k x_0),$$

or its equivalent integral form due to DIRICHLET,

$$S_n = \frac{1}{\pi} \int_0^{2\pi} f(x+t) \frac{\sin(n+t/2)}{2 \sin t/2} \, dt, \tag{105}$$

then the expression

$$\sigma_n(x) = \big(S_0(x) + S_1(x) + \cdots + S_n(x)\big)/(n+1) \tag{106}$$

tends to $f(x)$ as $n$ goes to infinity.

This major breakthrough, together with the work of LEBESGUE, who put FOURIER series on a new basis with his theory of the integral, showed that summability, rather than convergence, should be the main criterion for the theory of FOURIER series.

The result spurred a number of mathematicians to apply summability techniques to the major classes of functions. POISSON studied the effect of using ABEL summability on FOURIER series; FEJÉR made analogous studies of LAPLACE functions; C. N. MOORE studied BESSEL functions; HAAR studied STURM-LIOUVILLE functions. In each case, a given method of summation generated analogous results to those already obtained for FOURIER series.

A third direction in which the theory of summability grew during the first quarter of the 20th century was in the introduction of summation methods involving matrices. The first encounter the theory of summability had with the concepts of matrix algebra occurred in the theorem of OTTO TOEPLITZ in 1911.[3] His theorem was an extension of CAUCHY's theorem on the convergence of means of a convergent sequence. TOEPLITZ's theorem provided a basis upon which summability methods could be stutied as a class of linear transformations.

---

[1] DU BOIS-REYMOND, *Göttingen Nachrichten*, (1873): 571.

[2] L. FEJÉR, *Mathematische Annalen*, **58** (1903): 51–69.

[3] O. TOEPLITZ, *Prace matematyczno-fizyczne*, **22** (1911): 113–119.

The theorem of TOEPLITZ stated that if $x_0, x_1, x_2, \ldots$ is a sequence converging to 0, and the system

$$\begin{array}{lllll} a_{00} & & & & \\ a_{10} & a_{11} & & & \\ a_{20} & a_{21} & a_{22} & & \\ \vdots & & & & \\ a_{n0} & a_{n1} & a_{n2} & \ldots & a_{nn} \end{array} \tag{107}$$

is such that the numbers $a_{ij}$ satisfy the conditions:

a) for each $p \geqq 0$, $a_{np}$ goes to 0 as $n$ goes to infinity,

b) there exists a constant $C$ such that for each $n$,

$$|a_{n0}| + |a_{n1}| + \cdots + |a_{nn}| < C,$$

then the sequence $x'_0, x'_1, x'_2, \ldots$, converges to 0, where

$$x'_n = a_{n0} x_0 + a_{n1} x_1 + \cdots + a_{nn} x_n. \tag{108}$$

TOEPLITZ supplied the following proof: If for $\varepsilon > 0$, $m$ is such that $|x_n| < \varepsilon/2C$, for $n > m$, then

$$|x_n| < |a_{n0} x_0 + \cdots + a_{nm} x_m| + \frac{\varepsilon}{2}. \tag{109}$$

$n_0$ was then chosen greater than $m$ so that for each $n > n_0$,

$$|a_{n0} x_0 + \cdots + a_{nm} x_m| < \frac{\varepsilon}{2}, \tag{110}$$

according to condition (a) of the hypothesis. Hence, for $n > n_0$, the substitution of (110) in (109) gives

$$|x'_n| < |a_{n0} x_0 + \cdots + a_{nm} x_m| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{111}$$

and the proof was complete.

It should be noted that if each row of the array has an infinite number of terms, and if all the conditions are met, the conclusion of the theorem is still valid. In fact, it is usually the infinite case which is encountered in applications of the theorem.

The extension of the above theorem to an infinite "TOEPLITZ matrix" is important because of its role in the application of summability in the complex plane. Consider matrix A:

$$\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0j} & \cdots \\ a_{10} & a_{11} & \cdots & a_{1j} & \cdots \\ \vdots & & & & \\ a_{i0} & a_{i1} & \cdots & a_{ij} & \cdots \\ \cdot & \cdot & \cdots & & \cdots \end{pmatrix}, \tag{112}$$

with the conditions:

a) for each $j \geqq 0$, $a_{ij}$ goes to 0 as $i$ goes to infinity;

b) there exists a constant $C$ such that for each $n$,

$$|a_{i0}|+|a_{i1}|+\cdots+|a_{in}|<C.$$

Then if $z_0, z_1, z_2, \ldots$ is a null sequence, so is $z'_0, z'_1, z'_2, \ldots$, where $z'_i=\sum_{n=0}^{\infty} a_{in} z_n$. If a third condition is added, namely,

c) $a_{ij}$ is such that $\sum a_{in}$ goes to 1 as $i$ goes to infinity, then if $z_0, z_1, \ldots$ goes to $p$, so does $z'_0, z'_1, \ldots$ go to $p$, and $\operatorname{Lim}_{i\to\infty} \sum_{n=0}^{\infty} a_{in} z_n = p$.

An infinite matrix $A$ satisfying conditions (a), (b), and (c) above, has come to be referred to in the literature as a "regular transformation" or a "T-matrix." It has been found that a T-matrix will transform a convergent series into one which still converges and some divergent series into convergent ones by using methods similar to the one to be considered next. This is the value of using a transformation—it may serve as a device for assigning a "sum" to an otherwise divergent series.

A generalization of the concept of T-matrix introduced by TOEPLITZ was the following: Given an infinite matrix $A$ as above, with the following conditions:

a) for each $j \geqq 0$, $a_{ij}$ goes to $\alpha_j$ as $i$ goes to infinity;

b) there exists a $C$ such that for each $n$,

$$|a_{i0}|+|a_{i1}|+\cdots+|a_{in}|<C;$$

c) $a_{ij}$ is such that $\sum a_{in}$ goes to $\alpha$ as $i$ goes to infinity; the matrix will then be called a "K-matrix" from the researches of T. KOJIMA and I. SCHUR.[1] It is easy to see that a T-matrix is a special case of the more general K-matrix.

If one defines the sum and product of two infinite matrices $A$ and $B$, in a way analogous to that for finite matrices, namely, $A+B=(a_{ij}+b_{ij})$ and $AB=\left(\sum_{k=0}^{\infty} a_{ik} b_{kj}\right)$, the usual operations may be performed with infinite matrices. However, as might have been anticipated, there are many distinctions. For one, the product matrix $AB$ in the infinite case may not even exist because $\sum a_{ik} b_{kj}$ may diverge for some or all values of $i, j$.

Although the sum of two T-matrices is not a T-matrix, it has been shown that the product of two T-matrices always exists and is itself a T-matrix.[2] Since the matrices considered in this section are T-matrices, multiplication of two of these matrices is a valid operation.

Following the original investigation by TOEPLITZ, in 1917 W. A. HURWITZ & L. L. SILVERMAN studied certain kinds of linear transformations in connection with analytic functions and summability.[3] However, it was FELIX HAUSDORFF

[1] T. KOJIMA, *Tohoku Math. Journal,* **12** (1917): 291–326; I. SCHUR, *Journal für die reine und angewandte Math.,* **151** (1920): 79–111.

[2] R. G. COOKE, *Infinite Matrices and Sequence Spaces* (London: Macmillan, 1950), p. 83.

[3] W. A. HURWITZ & L. L. SILVERMAN, *Trans. of the Amer. Math. Soc.,* **18** (1917): 1–20.

who rediscovered the entire class of transformations and developed a theory based on regular transformations.[1] His initial interest was sponsored by the generalization of summability definitions, but the consequences of his research are not restricted to that subject only.

The starting point in HAUSDORFF'S theory depended on the transformation $t_m = \Delta^m S_0 = \sum_{n=0}^{m} (-1)^n \binom{m}{n} S_n$, where the $S_n$ are the partial sums of a given series. According to this definition, the following values of $t_m$ are obtained:

$$\begin{aligned} t_0 &= S_0 \\ t_1 &= \Delta^1 S_0 = S_0 - S_1 \\ t_2 &= \Delta^2 S_0 = S_0 - 2S_1 + S_2 \\ t_3 &= \Delta^3 S_0 = S_0 - 3S_1 + 3S_2 - S_3 \\ &\vdots \end{aligned}$$

and the matrix associated with the transformation is of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 1 & -1 & 0 & 0 & 0 & . & . & . \\ 1 & -2 & 1 & 0 & 0 & 0 & . & . \\ 1 & -3 & 3 & -1 & 0 & 0 & 0 & . \\ \vdots & . & . & . & . & . & . & . \end{pmatrix}. \tag{113}$$

Here, if we write the transformation $t_m = \Delta^m S_0$ as a product of matrices $t = dS$, the matrix $d$ will be (113), and $S$ will be a matrix of zeroes except for the terms $S_n$ along the main diagonal. It can be easily demonstrated that $d$ is its own reciprocal.

Now if we let $t = dv = \mu u$ and $u = dS$, we obtain the transformation in the form $t = (d\mu d) S = \lambda S$. Here $\mu$ is a matrix understood to be diagonal. The transformation so defined, namely, $t = \lambda S$, is called a "HAUSDORFF transformation" and its associated matrix is called a "HAUSDORFF matrix."[2]

In his paper of 1921, HAUSDORFF showed that summability $(H, k)$ and summability $(C, k)$ are special kinds of HAUSDORFF transformations. That is, consider the diagonal matrix

$$\begin{pmatrix} \mu_0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \mu_1 & 0 & 0 & 0 & \\ 0 & 0 & \mu_2 & 0 & 0 & \dots \\ \vdots & & & \ddots & & \end{pmatrix}.$$

HAUSDORFF proved that if $\mu_n = 1/(n+1)^k$, then HÖLDER'S method is given by the transformation $t = \lambda S$.

[1] F. HAUSDORFF, *Mathematische Zeitschrift*, 9 (1921): 74–109.

[2] G. H. HARDY, *Divergent Series*, *op. cit.*, p. 249.

As an example of how this is applied, consider the case where $k=1$. In that case $t=(d\mu d)S=$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1/2 & 0 & 0 & \dots \\ 0 & 0 & 1/3 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix} \begin{pmatrix} S_0 & 0 & 0 & 0 & \dots \\ 0 & S_1 & 0 & 0 & \dots \\ 0 & 0 & S_2 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix}.$$

If these matrices are multiplied, the resulting matrix is

$$\begin{pmatrix} S_0 & 0 & 0 & 0 & \dots \\ 1/2 S_0 & 1/2 S_1 & 0 & 0 & \dots \\ 1/3 S_0 & 1/3 S_1 & 1/3 S_2 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix}.$$

The sums of the row elements of this matrix are

$$S_0/1, \quad (S_0+S_1)/2, \quad (S_0+S_1+S_2)/3, \quad \text{etc.};$$

one will immediately recognize the HÖLDER means for $k=1$.

HAUSDORFF also proved that if $\mu_n = \binom{n+k}{k}^{-1}$, then CESÀRO'S method is given by the transformation $t=\lambda S=(d\mu d)S$. Again, this is easy to see if, for example, we consider the case $k=2$. Then we have $t=\lambda S=(d\mu d)S=$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1/3 & 0 & 0 & \dots \\ 0 & 0 & 1/6 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix} \begin{pmatrix} S_0 & 0 & 0 & 0 & \dots \\ 0 & S_1 & 0 & 0 & \dots \\ 0 & 0 & S_2 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix}.$$

Again, when these matrices are multiplied, we obtain the matrix

$$\begin{pmatrix} S_0 & 0 & 0 & 0 & \dots \\ 2/3 S_0 & 1/3 S_1 & 0 & 0 & \dots \\ 3/6 S_0 & 2/6 S_1 & 1/6 S_2 & 0 & \dots \\ \vdots & & & \ddots & \end{pmatrix}.$$

If one compares the rows of this matrix with the following: $S_0$, $(2S_0+S_1)/3$, $(3S_0+2S_1+S_2)/6$, *etc.*, one will recognize the CESÀRO means for $k=2$.

HAUSDORFF proved a great number of theorems concerning relationships among the factors $\mu_n$. One especially noteworthy result involved the concept of a regular linear transformation, that is, one which satisfies condition (c) above. HAUSDORFF showed that in order for a transformation of the type $t=\lambda S$ to be regular, for real $\mu_n$ values, one must be able to express the sequence $(\mu_n)$ as the difference of two totally monotone sequences. This theorem helped to point out the similarities with the theory of functions and sequences of bounded variation and the STIELTJES integral.

A fourth direction which the study of divergent series and summabilitity took during the early 1900's was into an area of inquiry that originated with a theorem proved in 1897 by ALFRED TAUBER.[1] His theorem inspired a host of others, all of which assumed as hypotheses, a summability criterion, and which obtained as conclusions, ordinary convergence. Theorems of this kind which arose in the ensuing years were characterized by HARDY as "TAUBERIAN theorems." The term has come to signify any theorem where ordinary convergence is deduced from a hypothesis of summability, and additional conditions.

TAUBER'S theorem stated the following: If a series $\sum a_n$ is $A$ summable to $S$, that is, if $\lim_{x\to 1^-} \sum a_n x^n = S$, and if $\lim_{n\to\infty} n a_n = 0$, then $\sum a_n$ is convergent in the usual sense.[2]

The real power behind TAUBERIAN theorems lies in the fact that they provide strong tests for the convergence of large classes of series. Some of the more important of these theorems, which were proved before 1925, include HARDY'S proof of a theorem on CESÀRO summability.[3] His theorem stated: If a series $\sum a_n$ is summable $(C, k)$ to $S$ and $|a_n| < M/n$, where $M$ is constant, then $\sum a_n$ is convergent. The condition $|a_n| < M/n$ is frequently written as $a_n = o(n^{-1})$.

EDMUND M. LANDAU also proved a theorem of this type in 1910.[4] He stated that if a series of real numbers $\sum a_n$ is summable $(C, k)$ to $S$ and $n a_n > -H$, for some constant $H$, then the series $\sum a_n$ is convergent.

One of the most famous TAUBERIAN theorems introduced during this period is the so-called LITTLEWOOD theorem.[5] It is a deeper and more difficult theorem and is an attempt at the generalization of TAUBER'S first theorem. The theorem states the following: A series $\sum a_n$ which is $A$ summable to $S$, such that $(n a_n)$ are bounded, that is, $a_n = O(n^{-1})$, converges to $S$.

The reader can see that this is precisely the original theorem of TAUBER with the condition for small $o$ being replaced by the new condition for large $O$. LITTLEWOOD'S proof of this theorem, and several proofs which came subsequently, were not elegant. Most of the techniques involved repeated differentiation. However, in 1930, JOVAN KARAMATA found a fairly simple proof of the theorem, and his proof has become the standard.[6] A still more powerful way of proving this theorem came later, with the TAUBERIAN theorems of NORBERT WIENER.[7] But, in spite of the power of WIENER'S method, it makes use of the theory of FOURIER transforms and is not elementary.

In 1912, HARDY & LITTLEWOOD proved a TAUBERIAN theorem for BOREL summability.[8] They stated that if the series $\sum a_n$ is $B$ summable to $S$, and $a_n = o(n^{-\frac{1}{2}})$, then the series $\sum a_n$ converges to $S$.

After 1925, the history of TAUBERIAN theorems really became the history of one man; for many years, NORBERT WIENER dominated this field of research.

The last two topics to be treated in this article are chosen because they represent two of the many attempts which were made by mathematicians during

---

[1] A. TAUBER, *Monatshefte für Math.*, **8** (1897): 273.

[2] A complete proof may be found in K. KNOPP, *op. cit.*, p. 500.

[3] G. H. HARDY, *Proc. of the London Math. Soc.*, (2) **8** (1910): 301–320.

[4] E. LANDAU, *Prace matematyczno-fizyczne*, **21** (1910): 103–113.

[5] J. E. LITTLEWOOD, *Proc. of the London Math. Soc.*, (2) **9** (1911): 434–448.

[6] J. KARAMATA, *Mathematische Zeitschrift*, **32** (1930): 319–320.

[7] N. WIENER, *Annals of Math.*, (2) **33** (1932): 1–100.

[8] G. H. HARDY & J. E. LITTLEWOOD, *Proc. of the London Math. Soc.*, (2) **11** (1912): 1–16.

the first quarter of this century to abstract, from all the various summation methods, a truly general approach to the study of divergent series. Consider a point of view enunciated by LLOYD L. SMAIL in 1918.[1] He proceeded by treating a function $f_i(n, x)$ defined for all positive integral values of $i$, and all positive values of $n$ and $x$. He called a series $\sum a_n$ summable $A_f$ to $S$ by the summation function $f_i(n, x)$ if

$$S = \lim_{x\to\infty} \lim_{n\to\infty} \sum_{i=0}^{n} a_i f_i(n, x).$$

In order to make summability $A_f$ an actual generalization of convergence, SMAIL subjected the function $f_i$ to certain restrictive conditions, namely: when $n$ and $x$ are fixed the sequence $(f_i)$ is positive and decreasing and $\operatorname{Lim}\operatorname{Lim} f_i(n, x) = 1$, for fixed $i$.

SMAIL then showed that this definition satisfies all the desirable properties of a summation method. He presented a table of special cases for the function $f_i$, and from the table, the reader can see the great generality of the method. Below are listed some of the many methods listed in SMAIL's table:

1) CESÀRO's method:

$$f_i(n, x) = \frac{n(n-1)\ldots(n-i+1)}{(k+n)(k+n-1)\ldots(k+n-i+1)}.$$

2) HÖLDER's method:

$$f_i(n, x) = \left(1 - \frac{i}{n+1}\right)^k.$$

3) RIESZ's method:

$$f_i(n, x) = (1 - h(i)/h|n|)^k.$$

4) BOREL's integral method:

$$f_i(n, x) = \int_0^x e^{-t} t^i / i!\, dt.$$

5) BOREL's exponential method:

$$f_i(n, x) = e^{-x}(E_n(x) - E_{i-1}(x)), \quad \text{where} \quad E_n(x) = \sum_{i=0}^{n} x^i/i!.$$

The other generalized approach to the study was begun by CHARLES N. MOORE in 1907.[2] He suggested the following: Suppose one has a divergent series. If one could find a set of functions which, when introduced as factors of the successive terms of the series, cause the series to converge, then MOORE would call that set of functions "convergence factors" for the given series. He sought the most general convergence factors which, when applied, would make a divergent series converge, and an already convergent series converge more rapidly.

Most of the theorems on convergence factors proved by MOORE and others at this time, began as extensions of the central theme behind the so-called DIRICHLET

[1] L. L. SMAIL, *Annals of Math.*, **20** (1918): 149.

[2] C. N. MOORE, *Trans. of the Amer. Math. Soc.* **8** (1907): 299–330.

theorem. This theorem states: Given a series $\sum a_n$, and a set of functions of some parameter $f_n$, if $\sum a_n$ converges, the series $\sum a_n f_n$ will also converge if and only if

$$\sum|\Delta f_n| = \sum|f_n - f_{n+1}| < \infty. \quad (135)$$ [1]

Also, if $\sum a_n$ is a series, not necessarily convergent, but one for which the partial sums are bounded, then the series $\sum a_n f_n$ will be convergent if and only if the $f_n$ go to 0 as $n$ goes to infinity.

In this theorem, the set of functions $f_n$ can be considered as a set of convergence factors. Many generalizations of this theorem were later proved for summable series. These generalizations were of two types: in the first, the summability of the series $\sum a_n f_n$ is inferred from the summability of the series $\sum a_n$; in the second, stronger conditions are imposed on the $f_n$, and then, for example, the CESÀRO summability of $\sum a_n f_n$ may be inferred.

An example of a principal theorem of the first type is the following: If $\sum a_n$ is summable $(C, k)$, $k$ an integer, and the $f_n$ go to 0 as $n$ goes to infinity, and

$$\sum (n+1)^k |\Delta^{k+1} f_n| < \infty,$$

where $\Delta^{k+1}$ is the $(k+1)^{\text{st}}$ finite difference, then $\sum a_n f_n$ is summable $(C, k)$. This theorem was first proved by NIELS BOHR in 1909.[2]

An example of a principal theorem of the second type is the following: If the series $\sum a_n$ is summable $(C, k)$ and $0 < s < k+1$, then the series $\sum \binom{n+s}{s}^{-1} a_n$ is summable $(C, k-s)$. This theorem was stated for integral $k$ by M. RIESZ in 1909 and was proved for all $k$ and integral $s$ by S. CHAPMAN in 1911.[3]

MOORE devoted much of his life's work toward building a complete theory of convergence factors. To date, his theory still ranks as one of the most widely used, uniform treatments of the study of summability. His work culminated with the publication by the American Mathematical Society, in 1938, of his monograph entitled *Summable Series and Convergence Factors*.

## 5. Conclusions

One might say that the history of infinite series began when the early Greek philosophers and physicists wondered as to whether an infinite sequence of numbers could be summed. Their curiosity was aroused by a study of natural phenomena. Accordingly, since they were driven by a desire to explain the world around them, answers to their questions were acceptable to them if they seemed intuitive and agreed with nature. It is fair to say that, with the exception of certain mathematicians such as CAUCHY and ABEL, this attitude permeated the development of the theory of infinite series to modern times. Throughout the history, intuition, coupled with pragmatism, were the main guides in the construction of the theory.

Most of the original methods used to sum divergent series can be considered primitive. Early researchers asked the question: If one can add an infinite number

[1] *Ibid.*, p. 304.
[2] N. BOHR, *Comptes rendus de l'Acad\'emie des sciences*, **148** (1909): 75–80.
[3] S. CHAPMAN, *Proc. of the London Math. Soc.*, (2) **9** (1911): 388.

of terms in some cases and obtain meaningful results, why not use the same technique in all cases? These men did not care particularly under which conditions their methods applied. In fact, logical considerations seem to have rarely entered the picture at all. These practitioners needed solutions to their physical problems, and so they used their intuition to devise representative answers. If the solution they obtained conformed to what their common sense and physical knowledge regarded as a meaningful result, they were satisfied.

Of course, after the methods were used for a time with some success, subsequent practitioners felt completely justified in using them; precedent had been established upon which they could rely. Their reliance was not based on a clear analysis of the concepts, but rather on utility or intuition.

In addition to these considerations, the development of the theory of divergent series and summability was hampered by many external influences. The greatest negative influence was the attitude of CAUCHY and ABEL toward divergent series. It would be senseless to accuse these men of jeopardizing the development of the theory, for no one can infer what might have transpired if they had not made their convictions known, but, nevertheless, the historian should try to gauge what the effects of such attitudes were. The plain historical fact is that from the year 1835 to the year 1880, almost no research was carried on in the subject of divergent series.

In truth, one can understand why denouncements were uttered by CAUCHY and ABEL. They recognized the danger of leaving too much to intuition, for, as history has verified many times, intuition by itself is normally insufficient. Often enough, the intuitions of the most renowned thinkers have later been contradicted. CAUCHY was striving for a mathematics based on logic which could stand the test of time. He saw no place in it for the loose generalities one seemingly had to accept in order to deal with divergent series.

It is entirely conceivable that the decrees of CAUCHY and ABEL could have terminated the research on the theory of divergent series and summability. The situation was dangerous, and divergent series facted the possibility of beind eliminated from theoretical mathematics forever. But in the final analysis, it was utility again which saved the day for divergent series. The mathematicians of the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ centuries realized how useful these series can be, especially in applications to astronomy, and this realization outweighed the pronouncements of CAUCHY and ABEL. And so, with the start of the 20$^{\text{th}}$ century, divergent series had finally achieved recognition. Today, divergent series and summability stand as a legitimate branch of theoretical mathematics.

The author wishes to thank Dr. MORRIS KLINE of New York University for his patience, encouragement and time, all of which were generously given in the preparation of this paper.

## Additional Bibliography

ANDERSEN, A. F., Bemerkung zum Beweis des Herrn Knopp für die Äquivalenz der Cesàro- und Hölder-Summabilität. *Math. Zeit.*, **28**, 356–359 (1928).

BOREL, E., Applications de la théorie des séries divergentes sommables. *Comptes Rendus*, **122**, 805–807 (1896).

BOREL, E., Fondements de la théorie des séries divergentes sommables. *Journal de Math.*, (5) **2**, 103–133 (1896).
BOREL, E., Le prolongement analytique et les séries sommables. *Math. Annalen*, **55**, 74–80 (1901).
BOREL, E., Mémoire sur les séries divergentes. *Ann. Sci. de l'Ecole Norm. Sup.*, (3) **16**, 9–136 (1899).
BROMWICH, T. J., On the limits of certain infinite series and integrals. *Math. Annalen*, **65**, 313–349 (1908).
BUHL, A., Application du procédé de sommation de M. E. Borel aux séries trigonométriques généralisees. *Comptes Rendus*, **143**, 445–446 (1906).
BUHL, A., Sur de nouvelles formules de sommabilité. *Bull. des Sciences*, **42**, 340–346 (1907).
BUHL, A., Sur la sommabilite des séries d'une variable réélle ou complexe. *Journal de Math.*, (6) **4**, 367–393 (1908).
CAJORI, F., *A History of Mathematics*. NewYork: Macmillan 1919.
CANTOR, M., *Vorlesungen über Geschichte der Mathematik*. 4 Vols. Leipzig: Teubner, 1900–1908.
CARMICHAEL, R. D., General aspects of the theory of summable series. *Bull. of the Amer. Math. Soc.*, **25**, 97–131 (1919).
CAUCHY, A., Sur un emploi légitime des séries divergentes. *Comptes Rendus*, **17**, 370–376 (1843).
CESÀRO, E., Contributions à la théorie des limites. *Bull. des Sciences*, (2) **13**, 51–54 (1899).
CESÀRO, E., Nuova contribuzione ai principii fondamentali dell' aritmetics assintotica. *Rendiconti Napoli*, (2) **7**, 208–209 (1893).
CHAPMAN, S., A note on the theory of summable integrals. *Bull. of the Amer. Math. Soc.*, **18**, 111–117 (1911).
CHAPMAN, S., On the summability of series of Legendre's functions. *Math. Annalen*, **72**, 211–227 (1912).
CHESSIN, A. S., On divergent series. *Bull. of the Amer. Math. Soc.*, (2) **2**, 72–75 (1895).
DIENES, P., La série de Taylor sur le cercle de convergence. *Comptes Rendus*, **140**, 489–491 (1905).
FORD, W. B., A set of criteria for the summability of divergent series. *Bull. of the Amer. Math. Soc.*, (2) **15**, 439–444 (1909).
FORD, W. B., On the relation between the sum formulas of Hölder and Cesàro. *Amer. Journal of Math.*, **32**, 315–326 (1910).
FORD, W. B., A conspectus of the modern theory of divergent series. *Bull. of the Amer. Math. Soc.*, **25**, 1–15 (1918).
FORD, W. B., *Studies on Divergent Series and Summability*. NewYork: Macmillan 1916.
HARDY, G. H., An extension of a theorem on oscillating series. *Proc. of the London Math. Soc.*, **12**, 156–173 (1913).
HARDY, G. H., Further researches in the theory of divergent series and integrals. *Trans. of the Cambr. Phil. Soc.*, **21**, 1–48 (1908).
HARDY, G. H., Generalization of a theorem in the theory of divergent series. *Proc. of the London Math. Soc.*, (2) **6**, 225–264 (1908).
HARDY, G. H., A note on divergent Fourier series. *Mess. of Math.*, (2) **33**, 137–144 (1904).
HARDY, G. H., & S. Chapman, A general view of the theory of summable series. *Quar. Journal of Math.*, **42**, 181–215 (1911).
HELLINGER, E., & O. TOEPLITZ, Grundlagen für eine Theorie der unendlichen Matrizen. *Göttingen Nach.*, 351–355 (1906).
HURWITZ, W. A., Report on topics in the theory of divergent series. *Bull. of the Amer. Math. Soc.*, **28**, 17–36 (1922).
JAMES, G., On the theory of summability. *Annals of Math.*, (2) **21**, 120–127 (1919).
KLEIN, F., *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*. Berlin, 1926.

KNOPP, K., Über das Eulersche Summierungsverfahren. *Math. Zeit.*, **15**, 226–253 (1922).
LE ROY, E., Sur les séries divergentes et les fonctions définies par un développement de Taylor. *Annales de Toulouse*, (2) **2**, 317–430 (1900).
LINDELÖF, E., Quelques applications d'une formule sommatoire générale. *Acta Soc. Fennicae*, **31**, 1–46 (1902).
PERRON, O., Beitrag zur Theorie der divergenten Reihen. *Math. Zeit.*, **6**, 286–310 (1920).
POISSON, S. D., Mémoire sur le calcul numérique des intégrales définies. *Mémoires Acad. Sci. Inst. France* **6**, 571–602 (1823).
PRINGSHEIM, A., Divergente Reihen. *Ency. d. math. Wissenschaften*, IA3, Secs. 38–40: 1198.
RICOTTI, E., Sulle serie divergenti sommabili. *Giornale di Mat. di Battaglini*, **48**, 79–111 (1910).
RIEMANN, B., *Gesammelte Mathematische Werke*. New York: Dover reprint, 1953.
RIESZ, M., Sur les séries de Dirichlet et les séries entières. *Comptes Rendus*, **149**, 909–912 (1909).
SCHUR, I., Über die Äquivalenz der Cesàroschen und Hölderschen Mittelwerte. *Math. Annalen*, **74**, 447–458 (1913).
SILVERMAN, L. L., On the definition of the sum of a divergent series. *Univ. of Missouri Studies*, 1–96, 1913.
SMAIL, L. L., *History and Synopsis of the Theory of summable Infinite Processes*. Eugene, Oregon: Univ. Press 1925.
TAKENAKA, S., A general view of the theory of summability. *Tohoku Math. Journal*, **21**, 193–221 (1922).
WHITTAKER, E. T., *A Course of Modern Analysis*. Cambridge: Univ. Press 1902.

St. John's University
Jamaica, New York

*(Received October 2, 1972)*