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Probability Theory and Related Fields

On the Derivation of Reaction-Diffusion Equations as Limit Dynamics of Systems of Moderately Interacting Stochastic Processes

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Summary. We consider systems of "moderately" interacting particles, which are divided into a finite number of different subpopulations, and show that in the limit as the population size tends to infinity the empirical processes of the subpopulations converge to the solution of a system of reaction-diffusion equations.

1. Introduction

Systems of reaction-diffusion equations are used very frequently to establish deterministic mathematical models in natural sciences, in particular, in biology [12, 18, 26], or chemistry [21]. They are intended to describe large systems of interacting individuals, molecules, or particles. It is the aim of this paper to present a method, which allows to derive a fairly general class of such systems as limit dynamics of interacting stochastic many-particle systems. Our considerations are based on [7, 17, 23, 24], where some simple examples had been investigated.

In particular, we study for any $N \in \mathbf{N}$ a population of about N individuals in $\mathbf{R}^d$, which is divided into K subpopulations. The individuals may move around in $\mathbf{R}^d$, may die, or give birth to new individuals, and may change their subpopulation. We rescale the interaction between the individuals in a suitable (moderate) way as the population size tends to infinity. Essentially, this means that for any fixed particle the drift coefficients, the birth-, death- and transition rates depend on the configuration of the remaining particles in a neighbourhood, which is macroscopically small, i.e. its volume tends to 0 as $N \to \infty$, and microscopically large, i.e. it contains an arbitrarily large number of individuals as $N \to \infty$. It is shown that for large N the empirical processes of the different subpopulations converge to the solution of a system of reaction-diffusion equations. For that we consider regularized versions of these empirical processes and study their asymptotic properties as $L^2(\mathbf{R}^d)$-valued stochastic processes.

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This paper is organized as follows: In the next section we describe our model for the N-particle system and give a heuristic study of its limit behaviour. Section 3 contains the formulation of our results, which finally are proved in Sect. 4.

2. Description of the Model

We consider for any $N \in \mathbf{N}$ a population in $\mathbf{R}^d$ consisting of $\approx N$ individuals, which is divided into K subpopulations. We enumerate the members of this population by $1, 2, \ldots$, and denote by $M(N, r, t)$, $r = 1, \ldots, K$, the set of all individuals belonging at time t to the rth subpopulation. Moreover, let $M(N, t) = \bigcup_{r=1}^{K} M(N, r, t)$ be the set of all individuals being alive at time t. For convenience we assign to any new-born a new number, which has not been used before.

Let $P_N^k(t) \in \mathbf{R}^d$, $k \in M(N, t)$, be the position of the kth individual at time $t \geqq 0$. For the description of the subpopulations and the total population no distinction between individuals of the same species is needed. Hence, it is convenient to consider the measure valued empirical processes

$$t \to S_{N,r}(t) = \frac{1}{N} \sum_{k \in M(N,r,t)} \delta_{P_N^k(t)}, \qquad r = 1, \ldots, K,$$

and

$$t \to S_N(t) = \sum_{r=1}^{K} S_{N,r}(t) = \frac{1}{N} \sum_{k \in M(N,t)} \delta_{P_N^k(t)},$$

where δ_a denotes Dirac measure at $a \in \mathbf{R}^d$.

2.A. *Densities for the N-particle System*

An essential ingredient of our model is the assumption that the dynamics of any individual depends on the configuration of the remaining population in a small neighbourhood. This interaction is formulated mathematically by using smooth versions of the empirical processes

$$\begin{aligned} s_{N,r}(x, t) &= (S_{N,r}(t) * V_N)(x), \\ \hat{s}_{N,r}(x, t) &= (S_{N,r}(t) * \hat{V}_N)(x), \end{aligned} \tag{2.1}$$

where "$*$" denotes the convolution. V_N and $\hat{V}_N$ are probability densities, which are obtained from a fixed symmetric, sufficiently smooth function V_1 by the scaling

$$V_N(x) = \alpha_N^d V_1(\alpha_N x), \qquad \hat{V}_N(x) = \hat{\alpha}_N^d V_1(\hat{\alpha}_N x), \tag{2.2}$$

where

$$\alpha_N = N^{\beta/d}, \qquad \hat{\alpha}_N = N^{\hat{\beta}/d}, \tag{2.3}$$

for fixed scaling exponents β and $\hat{\beta}$ satisfying

$$0 < \beta < \frac{d}{d+2}, \qquad 0 < \hat{\beta} < \frac{\beta}{d+1}. \tag{2.4}$$

The functions $s_{N,r}(x, t)$ and $\hat{s}_{N,r}(x, t)$ formally represent the density or concentration of the rth subpopulation near x at time t. This can seen most easily when V_1 is the indicator function of a ball with volume 1 and centre 0, although such a choice does not satisfy our assumptions below on V_1. In this case

$$\begin{aligned} s_{N,r}(x, t) &= \frac{1}{N}\,\alpha_N^d \text{ times the number of particles of the } r\text{th subpopulation} \\ &\quad \text{at time } t \text{ in the ball } B(\alpha_N^{-d}, x) \text{ with volume } \alpha_N^{-d} \text{ and centre } x \\ &= \frac{\text{number of particles of } M(N, r, t) \text{ in } B(\alpha_N^{-d}, x)}{\text{volume of } B(\alpha_N^{-d}, x)} \\ &\quad \times \frac{1}{\text{size of the total population}}. \end{aligned}$$

The space element $\Delta x = B(\alpha_N^{-d}, x)$ is macroscopically small, since its volume $\alpha_N^{-d} = N^{-\beta}$ tends to 0 as $N \to \infty$. On the other hand, if the particles are distributed sufficiently smooth, one expects that the number of individuals of any subpopulation in $B(\alpha_N^{-d}, x)$ is of order $N \times \text{volume}(B(\alpha_N^{-d}, x)) = N^{1-\beta}$, which tends to ∞ as $N \to \infty$, i.e. Δx is microscopically large. Hence the functions $s_{N,r}$, and similarly $\hat{s}_{N,r}$, meet the usual heuristic picture of a "population density" or "one-particle distribution function", as it is called in the context of statistical physics. For a discussion of this concept cf. [28], p. 75.

The introduction of two versions, $s_{N,r}$ and $\hat{s}_{N,r}$, of the concentrations has technical reasons, cf. (4.13).

2.B. *Dynamics of the N-particle System*

In our model we assume that any individual can change its state in three ways:

- change of the position in space,
- change of the internal state, i.e. transition from one subpopulation to another,
- birth and death.

The motion through space is described by a stochastic differential equation. Let $k \in M(N, r, t)$. Then

$$dP_N^k(t) = F_{N,r}(P_N^k(t), t)\,dt + \sigma_r\,dW^k(t), \tag{2.5}$$

where $W^k(.)$ are independent $\mathbf{R}^d$-valued standard Brownian motions, σ_r, $r = 1, \ldots, K$, fixed nonsingular matrices, and $F_{N,r}(x, t)$, $r = 1, \ldots, K$, vector-, i.e. $\mathbf{R}^d$-, valued functions depending on the position x and the densities $s_{N,q}(x, t)$, $\hat{s}_{N,q}(x, t)$, $q = 1, \ldots, K$, introduced in Sect. 2.A. More precisely, we assume

$$\begin{aligned} F_{N,r}(x, t) &= G_{N,r}(x, t) - \sum_{q=1}^{K} D_{N,qr}(x, t)\,\nabla s_{N,q}(x, t), \\ G_{N,r}(x, t) &= \breve{G}_r(x, \hat{s}_{N,1}(x, t), \ldots, \hat{s}_{N,K}(x, t)), \\ D_{N,qr}(x, t) &= \breve{D}_{qr}(x, \hat{s}_{N,1}(x, t), \ldots, \hat{s}_{N,K}(x, t)), \end{aligned} \tag{2.6}$$

where $\check{G}_r(\check{D}_{qr})$ are fixed $\mathbf{R}^d-(\mathbf{R}^d\otimes\mathbf{R}^d-)$ valued functions. The second contribution to $F_{N,r}$ can be utilized to model repulsion or attraction between different species q and r. This may be particularly useful in the description of predator-prey systems.

However, let us mention here that we have to assume certain relations between the matrix valued functions $\check{D}_{qr}$ and the diffusion matrices σ_r (cf. (3.7, 8)) to prevent the system from collapsing due to too strong attraction. Essentially this means that the main contribution to the second summand of $F_{N,r}$ should be repulsion between different individuals.

Furthermore, any individual $k\in M(N,r,t)$ at position $P_N^k(t)=y$ may induce discontinuous changes in the population structure. At time t it is supposed to

– leave $M(N,r,t)$ and enter $M(N,q,t)$, $q\neq r$, with intensity $t_{N,rq}(y,t)$, or to

– give birth to a new individual $k^*\in M(N,q,t)$, which at the moment of its birth is located at y too, with intensity $b_{N,rq}(y,t)$, or to

– die with intensity $d_{N,r}(y,t)$.

As in the case of the drift parameters, we assume that these intensities depend on the densities of the N-particle system, i.e.

$$\begin{aligned} t_{N,rq}(x,t)&=\check{t}_{rq}(x,\hat{s}_{N,1}(x,t),\dots,\hat{s}_{N,K}(x,t)),\\ b_{N,rq}(x,t)&=\check{b}_{rq}(x,\hat{s}_{N,1}(x,t),\dots,\hat{s}_{N,K}(x,t)),\\ d_{N,r}(x,t)&=\check{d}_{r}(x,\hat{s}_{N,1}(x,t),\dots,\hat{s}_{N,K}(x,t)).\end{aligned} \tag{2.7}$$

Of course we suppose

$$\check{t}_{rr}(\dots)\equiv 0, \qquad r=1,\dots,K. \tag{2.8}$$

To avoid technical complications, which would yield no further insight, we assume that

$$\text{the functions } \check{G}_r,\ \check{D}_{qr},\ \check{t}_{qr},\ \check{b}_{qr} \quad\text{and}\quad \check{d}_r \quad \text{are } C_b^\infty, \tag{2.9}$$

i.e. infinitely differentiable and uniformly bounded together with all their derivatives.

To obtain a condensed description of the discontinuous changes in the population structure we now introduce for any $k=1,2,\dots$, $r,q=1,\dots,K$ and $N\in\mathbf{N}$ the processes

$$t_{N,rq}^{*,k}(u)=\beta_{N,rq}^{c,k}\left(\int_0^u \mathbf{1}_{M(N,r,s)}(k)\,t_{N,rq}(P_N^k(s),s)\,ds\right),$$

$$b_{N,rq}^{*,k}(u)=\beta_{N,rq}^{b,k}\left(\int_0^u \mathbf{1}_{M(N,r,s)}(k)\,b_{N,rq}(P_N^k(s),s)\,ds\right),$$

$$d_{N,r}^{*,k}(u)=\beta_{N,r}^{d,k}\left(\int_0^u \mathbf{1}_{M(N,r,s)}(k)\,d_{N,r}(P_N^k(s),s)\,ds\right),$$

where $\beta_{N,rq}^{c,k}$, $\beta_{N,rq}^{b,k}$ and $\beta_{N,r}^{d,k}$ are independent standard Poisson processes. Hence the point processes $t_{N,rq}^{*,k}(u)$, $b_{N,rq}^{*,k}(u)$, $d_{N,r}^{*,k}(u)$, $0\leq u<\infty$, have intensities

$$\mathbf{1}_{M(N,r,u)}(k)\,t_{N,rq}(P_N^k(u),u), \qquad \mathbf{1}_{M(N,r,u)}(k)\,b_{N,rq}(P_N^k(u),u),$$
$$\mathbf{1}_{M(N,r,u)}(k)\,d_{N,r}(P_N^k(u),u)$$

for a jump of size $+1$ at time u. Therefore these processes mark the instants, where the kth individual leaves the rth and enters the qth subpopulation, gives birth to an individual of the qth subpopulation, or dies.

As final point in this subsection we now use Itô's formula to collect the different contributions to the time evolution of the individuals in stochastic differential equations describing the dynamics of the empirical processes $S_{N,r}$. Using the notation $\langle \mu, f\rangle = \int_{\mathbf{R}^d} f(x)\,\mu(dx)$ for any measure μ and realvalued function f on $\mathbf{R}^d$ we obtain for $f\in C_b^{2,1}(\mathbf{R}^d\times\mathbf{R}_+)$

$$
\begin{aligned}
&\langle S_{N,r}(t), f(.,t)\rangle\\
&\quad=\frac{1}{N}\sum_{k\in M(N,r,t)} f(P_N^k(t),t)\\
&\quad=\frac{1}{N}\sum_{k\in M(N,r,0)} f(P_N^k(0),0)\\
&\qquad+\frac{1}{N}\int_0^t \sum_{k\in M(N,r,s)}\Bigg(\Bigg(G_{N,r}(P_N^k(s),s)\\
&\qquad\qquad-\sum_{q=1}^{K} D_{N,qr}(P_N^k(s),s)\,\nabla s_{N,q}(P_N^k(s),s)\Bigg)\cdot\nabla f(P_N^k(s),s)\\
&\qquad\qquad+\frac{1}{2}\sum_{m,n=1}^{d} A_{r,mn}\nabla_m\nabla_n f(P_N^k(s),s)+\frac{\partial}{\partial s} f(P_N^k(s),s)\Bigg)\,ds\\
&\qquad+\frac{1}{N}\int_0^t \sum_{k\in M(N,r,s)} \nabla f(P_N^k(s),s)\cdot\sigma_r\,dW^k(s)\\
&\qquad-\frac{1}{N}\int_0^t \sum_{k\in M(N,r,s)} f(P_N^k(s),s)\Bigg(\sum_{q=1}^{K} t_{N,rq}^{*,k}(ds)+d_{N,r}^{*,k}(ds)\Bigg)\\
&\qquad+\frac{1}{N}\int_0^t \sum_{q=1}^{K}\sum_{k\in M(N,q,s)} f(P_N^k(s),s)\,(t_{N,qr}^{*,k}(ds)+b_{N,qr}^{*,k}(ds))\\
&\quad=\langle S_{N,r}(0), f(.,0)\rangle\\
&\qquad+\int_0^t\Bigg(\Bigg\langle S_{N,r}(s),\Bigg(G_{N,r}(.,s)-\sum_{q=1}^{K} D_{N,qr}(.,s)\,\nabla s_{N,q}(.,s)\Bigg)\cdot\nabla f(.,s)\\
&\qquad\qquad+\frac{1}{2}\sum_{m,n=1}^{d} A_{r,mn}\nabla_m\nabla_n f(.,s)+\frac{\partial}{\partial s} f(.,s)\\
&\qquad\qquad-\Bigg(\sum_{q=1}^{K} t_{N,rq}(.,s)+d_{N,r}(.,s)\Bigg) f(.,s)\Bigg\rangle\\
&\qquad+\sum_{q=1}^{K}\langle S_{N,q}(s),(t_{N,qr}(.,s)+b_{N,qr}(.,s))\,f(.,s)\rangle\Bigg)\,ds\\
&\qquad+M_{N,r}^1(f,t)+M_{N,r}^2(f,t),\qquad r=1,\dots,K,
\end{aligned}
\tag{2.10}
$$

where

$$
\begin{aligned}
&A_r = \sigma_r \sigma_r^T,\\
&M^1_{N,r}(f,t)\\
&\quad = \frac{1}{N} \int_0^t \sum_{k \in M(N,r,s)} \nabla f(P_N^k(s), s) \cdot \sigma_r \, dW^k(s),\\
&M^2_{N,r}(f,t)\\
&\quad = -\frac{1}{N} \int_0^t \sum_{k \in M(N,r,s)} f(P_N^k(s), s) \Big(\sum_{q=1}^K (t^{*,k}_{N,rq}(ds) - t_{N,rq}(P_N^k(s), s)\, ds)\\
&\qquad\qquad + (d^{*,k}_{N,r}(ds) - d_{N,r}(P_N^k(s), s)\, ds) \Big)\\
&\qquad + \frac{1}{N} \int_0^t \sum_{q=1}^K \sum_{k \in M(N,q,s)} f(P_N^k(s), s)((t^{*,k}_{N,qr}(ds) - t_{N,qr}(P_N^k(s), s)\, ds)\\
&\qquad\qquad + (b^{*,k}_{N,qr}(ds) - b_{N,qr}(P_N^k(s), s)\, ds)).
\end{aligned}
$$

Obviously these processes $M^1_{N,r}(f,.)$, $M^2_{N,r}(f,.)$ are martingales with respect to the natural filtration $\{\mathscr{F}_t\}_{t \geq 0}$ generated by the processes

$$t \to (P_N^k(t), \mathbf{1}_{M(N,q,t)}(k))\, \chi_{N,k}(t), \quad k = 1, 2, \ldots, \quad q = 1, \ldots, K, \ N \in \mathbf{N},$$

where $\chi_{N,k}$ is the indicator function of the lifetime of the kth individual.

(2.10) is a summarizing description of our model. The task in the remainder of this paper will be the investigation of the behavior of this equation in the limit $N \to \infty$. Of course, in the proofs of our results we later shall try to shorten the notation as far as possible. We shall consider mainly typical or most instructive parts of the interaction, e.g. the cross diffusion part determined by the functions $D_{N,qr}$.

2.C. *Heuristic Derivation of the Limit Dynamics*

First we observe that (2.2.4) imply

$$\lim_{N \to \infty} V_N = \lim_{N \to \infty} \hat{V}_N = \delta_0 \quad \text{(in the sense of distributions).} \tag{2.11}$$

Now we assume that in some yet unspecified sense

$$\lim_{N \to \infty} S_{N,r}(t) = S_r(t), \qquad r = 1, \ldots, K, \ t \geq 0,$$

where the measures $S_r(t)$ have smooth densities $s_r(.,t)$ with respect to Lebesgue measure. Hence by (2.1, 11) for $r = 1, \ldots, K$, $t \geq 0$

$$\lim_{N \to \infty} s_{N,r}(.,t) = \lim_{N \to \infty} \hat{s}_{N,r}(.,t) = s_r(.,t),$$

and

$$\lim_{N \to \infty} \nabla s_{N,r}(.,t) = \nabla s_r(.,t).$$

Furthermore one may show that the quadratic variations of the martingales $M^1_{N,r}(f,.)$ and $M^2_{N,r}(f,.)$ tend to 0 as $N\to\infty$. Hence they can be neglected in this limit and we formally obtain from (2.6, 7, 10)

$$\begin{aligned}
&\langle s_r(.,t), f(.,t)\rangle \\
&\quad=\langle s_r^*(.), f(.,0)\rangle \\
&\quad+\int_0^t \Bigg(\Bigg\langle s_r(.,s), \Bigg(G_{\infty,r}(.,s)-\sum_{q=1}^K D_{\infty,qr}(.,s)\,\nabla s_q(.,s)\Bigg)\cdot \nabla f(.,s) \\
&\qquad\qquad+\frac{1}{2}\sum_{m,n=1}^d A_{r,mn}\nabla_m\nabla_n f(.,s)+\frac{\partial}{\partial s}f(.,s) \\
&\qquad\qquad-\Bigg(\sum_{q=1}^K t_{\infty,rq}(.,s)+d_{\infty,r}(.,s)\Bigg) f(.,s)\Bigg\rangle \\
&\qquad+\sum_{q=1}^K \langle s_q(.,s),(t_{\infty,qr}(.,s)+b_{\infty,qr}(.,s))\,f(.,s)\rangle\Bigg)\,ds, \\
&\quad f\in C_b^{2,1}(\mathbf{R}^d\times\mathbf{R}_+),\qquad r=1,\ldots,K,\ t\geqq 0,
\end{aligned}\tag{2.12}$$

where $s_r^*(.)$ is the density of $S_r(0)$, and

$$\begin{aligned}
G_{\infty,r}(x,t)&=\check{G}_r(x,s_1(x,t),\ldots,s_K(x,t)),\\
D_{\infty,rq}(x,t)&=\check{D}_{rq}(x,s_1(x,t),\ldots,s_K(x,t)),\\
t_{\infty,rq}(x,t)&=\check{t}_{rq}(x,s_1(x,t),\ldots,s_K(x,t)),\\
b_{\infty,rq}(x,t)&=\check{b}_{rq}(x,s_1(x,t),\ldots,s_K(x,t)),\\
d_{\infty,r}(x,t)&=\check{d}_r(x,s_1(x,t),\ldots,s_K(x,t)),
\end{aligned}\tag{2.13}$$

Using integration by parts we notice that (2.12) is a weak form of the system of reaction-diffusion equations

$$\begin{aligned}
\frac{\partial}{\partial t}s_r(x,t)&=\nabla\cdot\Bigg(s_r(x,t)\Bigg(-G_{\infty,r}(x,t)+\sum_{q=1}^K D_{\infty,qr}(x,t)\,\nabla s_q(x,t)\Bigg)\Bigg)\\
&\quad+\frac{1}{2}\sum_{m,n=1}^d A_{r,mn}\nabla_m\nabla_n s_r(x,t)\\
&\quad-\Bigg(\sum_{q=1}^K t_{\infty,rq}(x,t)+d_{\infty,r}(x,t)\Bigg)s_r(x,t)\\
&\quad+\sum_{q=1}^K (t_{\infty,qr}(x,t)+b_{\infty,qr}(x,t))\,s_q(x,t),\qquad t\geqq 0,
\end{aligned}\tag{2.14}$$

$$s_r(x,0)=s_r^*(x),\qquad r=1,\ldots,K.\tag{2.15}$$

For the exact derivation of (2.14, 15) we shall show that for $T\in(0,\infty)$ the expression $\sup_{t\leqq T}\sum_{r=1}^K \|s_{N,r}(.,t)-s_r(.,t)\|_2^2$ vanishes as $N\to\infty$.

3. The Results

After some technical assumptions we present our main results and then conclude this section with a discussion, in particular, of the relations to other work.

3.A. Assumptions

We assume that the *interaction potential* V_1 can be written as a convolution product

$$V_1(x)=(W_1 * W_1)(x)=\int_{\mathbf{R}^d} W_1(x-y)\,W_1(y)\,dy \tag{3.1}$$

for some symmetric probability density W_1 satisfying

$$\tilde{W}_1 \in C_b^2(\mathbf{R}^d), \tag{3.2}$$

$$|\tilde{W}_1(\tau)| \leqq C\exp(-C'|\tau|), \tag{3.3}$$

$$|\Delta \tilde{W}_1(\tau)| \leqq C(1+|\tau|^2)\,|\tilde{W}_1(\tau)|, \tag{3.4}$$

and

$$v \to |\tilde{W}_1(v\tau)|, \quad v \geqq 0, \text{ is decreasing for any fixed } \tau \in \mathbf{R}^d. \tag{3.5}$$

Here and in the rest of this paper $\tilde{f}(\tau)=(2\pi)^{-d/2}\int_{\mathbf{R}^d} f(x)\exp(-i\tau x)\,dx$ is the Fourier transform of $f \in L^2(\mathbf{R}^d)$. Moreover we denote by $C, C', C'', \ldots$ finite positive constants, which may vary from place to place. Sometimes we write $C(a, b, \ldots)$ to express dependence of a constant C on parameters $a, b, \ldots$. In analogy to (2.1, 2) we define

$$W_N(x)=\alpha_N^d\, W_1(\alpha_N x), \qquad \hat{W}_N(x)=\hat{\alpha}_N^d\, W_1(\hat{\alpha}_N x),$$

$$h_{N,r}(x,t)=(S_{N,r}(t) * W_N)(x), \qquad r=1, \ldots, K.$$

Like $s_{N,r}$ or $\hat{s}_{N,r}$ the functions $h_{N,r}$ represent densities for the N-particle system.

Remarks:

(i) (3.1–5) are satisfied, e.g. if V_1 is a Gaussian probability density.

(ii) (2.2), (3.1, 3) imply W_N, V_N, $\hat{V}_N \in C_b^\infty$ for any $N \in \mathbf{N}$. Therefore the functions $s_{N,r}(\cdot, t)$, $\hat{s}_{N,r}(\cdot, t)$ and $h_{N,r}(\cdot, t)$ are smooth too with e.g.

$$\left\| \frac{\partial^{k_1+\ldots+k_d}}{(\partial x_1)^{k_1}\ldots(\partial x_d)^{k_d}}\,\hat{s}_{N,r}(\cdot, t)\right\|_\infty \leqq C(k_1, \ldots, k_d)\langle S_N(t), 1\rangle\, \hat{\alpha}_N^{k_1+\ldots+k_d+d},$$

$$k_1, \ldots, k_d \in \mathbf{N}, \; r=1, \ldots, K, \; t \geqq 0. \tag{3.6}$$

We still need some ellipticity of the principal contributions to the right side of (2.14), i.e., the highest order differential expressions. More precisely, we assume the existence of a symmetric positive definite $E=(E_{ij})_{i,j=1,\ldots,K}$, such that

$$\sum_{j,q=1}^{K} E_{ij}\,(\check{D}_{qi}(\ldots)x_q)\cdot x_j \geqq 0, \qquad i=1,\ldots,K,\ x_1,\ldots,\ x_K \in \mathbf{R}^d, \tag{3.7}$$

and

$$\sum_{j,q=1}^{K} E_{qj}\,x_q\cdot A_j x_j \geqq C \sum_{q=1}^{K} |x_q|^2,\ x_1,\ldots,x_K \in \mathbf{R}^d. \tag{3.8}$$

These conditions are satisfied for example in the following situations:

(i) For $d=1$, if the functions $\check{D}_{rq}(\ldots)\equiv \check{d}_{rq}$ are constant for some symmetric positive definite matrix $(\check{d}_{rq})_{r,q=1,\ldots,K}$ and if A_r is independent of r. In this case we can choose $E_{ij}=\check{d}_{ij}$.

(ii) For arbitrary $d\geqq 1$ and positive definite A_r, if $\check{D}_{rq}(\ldots)\equiv 0$ for $r\neq q$, and if the $d\times d$-matrices $\check{D}_{qq}(\ldots)$ are nonnegative definite. Here we can use $E_{ij}=\delta_{ij}$.

Unfortunately, there seems to exist so far no general result assuring the unique existence of a sufficiently regular solution $(s_1,\ldots,s_K)$ of parabolic systems like (2.14, 15). For this reason we base our further considerations on the following hypothesis:

For some $T>0$ the system (2.14, 15) has a unique C_b^∞-solution $(s_1,\ldots,s_K)$ in $[0,T]$. The functions $s_r(.,t)$ are positive and together with their partial derivatives of any order integrable uniformly in $t\leqq T$. $(\mathscr{H})$

A discussion of this hypothesis can be found in subsection 3.C.

3.B. The Limit Theorem

First we formulate a result concerning the asymptotics of the densities $h_{N,r}$ as $N\to\infty$.

Theorem 1. *Assume* (2.1–9), (3.1–5, 7, 8), $(\mathscr{H})$,

$$\lim_{N\to\infty} P\left[\sum_{r=1}^{K} \|h_{N,r}(.,0)-s_r^*(.)\|_2^2 \geqq \delta\right]=0, \qquad \delta>0, \tag{3.9}$$

and

$$\lim_{n\to\infty}\ \sup_{N\in\mathbf{N}} P[\langle S_N(0),1\rangle \geqq n]=0. \tag{3.10}$$

Then for any $\delta>0$

$$\lim_{N\to\infty} P\left[\sum_{r=1}^{K}\left(\sup_{t\leqq T}\|h_{N,r}(.,t)-s_r(.,t)\|_2^2 + \int_0^T \|\nabla h_{N,r}(.,t)-\nabla s_r(.,t)\|_2^2\,dt\right)\geqq \delta\right]=0, \tag{3.11}$$

where the system $s_1,\ldots,s_K$ *is the unique solution of* (2.14, 15).

(3.9) is satisfied, e.g. if (3.10) holds, and if conditioned on $M(N, q, 0)$, $q=1, \ldots, K$, for any $r=1, \ldots, K$ the positions $P_N^k(0)$, $k \in M(N, r, 0)$, are i.i.d. with density $\langle s_r^*, 1\rangle^{-1} s_r^*$, cf. [24].

A convenient metric on the space of positive finite measure $\mathscr{M}(\mathbf{R}^d)$ on $\mathbf{R}^d$ is defined by

$$d(\mu, v) = \sup\{\langle \mu - v, f\rangle : f \in C_b^1(\mathbf{R}^d), \|f\|_\infty + \|\nabla f\|_\infty \leqq 1\},$$

which generates the weak-$*$-topology, cf. [11]. As the convolution kernel W_N, which transforms $S_{N,r}(t)$ into $h_{N,r}(.,t)$, tends to δ_0 as $N \to \infty$, we obtain from (3.11) the following corollary.

Theorem 2. *Assume the conditions of Theorem 1, and*

$$\lim_{n\to\infty} \sup_{N\in\mathbf{N}} P\left[\sum_{r=1}^{K} \langle S_{N,r}(0), \psi^2\rangle \geqq n\right] = 0, \tag{3.12}$$

where $\psi(x) = \log(2+x^2)$. *Then*

$$\lim_{N\to\infty} P\left[\sum_{r=1}^{K} \left(\sup_{t\leqq T} d(S_{N,r}(t), s_r(.,t))\right) \geqq \delta\right] = 0, \qquad \delta > 0. \tag{3.13}$$

3.C. Discussion

1. The hypothesis ($\mathscr{H}$). The main problem in the verification of ($\mathscr{H}$) is the proof of the Hölder continuity of the functions s_r constituting a weak solution of (2.14, 15). Together with (2.9) and classical results on linear parabolic systems (cf. [15]) this can be used to obtain step by step the existence and boundedness of higher partial derivatives of s_r, $r=1, \ldots, K$. This Hölder continuity has been derived e.g. in cases, where the principal part on the right side of (2.14) is uniformly elliptic and diagonal, i.e. $\check{D}_{qr}(\ldots) \equiv 0$ for $q \neq r$, $E_{ij} = \delta_{ij}$ (cf. [30, 31]). For systems with $E_{ij} \not\equiv \delta_{ij}$ we define $\varphi_r(x,t) = \sum_{i=1}^{K} E_{ri} s_i(x,t)$, $r=1, \ldots, K$. Then we can write (2.14) as

$$\frac{\partial}{\partial t} b_r(\varphi_1(x,t), \ldots, \varphi_K(x,t)) = \nabla \cdot \left(\sum_{p=1}^{K} F_{rp}(x, b(\varphi_1(x,t), \ldots, \varphi_K(x,t))) \nabla \varphi_p(x,t)\right)$$
$$+ \text{minor terms}, \qquad r=1, \ldots, K, \tag{3.14}$$

where

$$b(y_1, \ldots, y_K) = (b_1(y_1, \ldots, y_K), \ldots, b_K(y_1, \ldots, y_K)),$$
$$b_r(y_1, \ldots, y_K) = \sum_{q=1}^{K} (E^{-1})_{rq} y_q = (E^{-1} y)_r,$$

and

$$F_{rp}(x, b) = b_r \sum_{q=1}^{K} \check{D}_{qr}(x, b_1, \ldots, b_K)(E^{-1})_{qp} + A_r (E^{-1})_{rp}.$$

(3.7, 8) and the positivity of the functions s_r imply that $(F_{pr}(\ldots))_{p,r=1,\ldots,K}$ is uniformly elliptic. Such systems are studied e.g. in [1]. To our knowledge however, regularity results like the Hölder continuity of the solutions of (3.14), which are applicable in the present situation, are not available so far.

Notice that for our calculations $(\mathscr{H})$ could be weakened by demanding less differentiability properties (e.g. C_b^3) of the functions s_r. However, as the analytical difficulties to verify these less restrictive properties remain the same, and in order to save some notation, we use our C_b^∞-assumptions.

2. Possible Extensions. Sometimes it is desireable to introduce long-range interaction between different particles. The description of stochastic many-particle systems, which model such situations, leads in the limit of large population sizes to integro-differential equations. This limit is sometimes called McKean-Vlasov limit and has been studied extensively in the literature, e.g. in [6, 16, 22], and in a situation involving different species in [19, 20]. It should cause no serious problems to derive results quite analogous to Theorems 1 or 2 for a model supplemented by such long-range interactions.

Similarly, it is fairly easy to include certain jump components into the spatial motion of the particles, such that the intensities for the jumps depend on $\hat{s}_{N,1}, \ldots, \hat{s}_{N,K}$.

In cases, where some diffusion matrices A_r vanish, one has to expect difficulties. It seems that one may proceed along similar lines as here, as long as $(\mathscr{H})$ is satisfied. However, in general $(\mathscr{H})$ can get false for some $T>0$, even if the initial data are smooth. This shows the fairly simple example of the porous medium equation, where the first spatial derivative may become discontinuous, cf. [3]. Then one has to do additional work to find useful estimates for the behaviour of the empirical processes in those regions of space, where the solution of the limit equation gets degenerate, cf. [25].

The use of two scaling parameters β and $\hat{\beta}$, and hence of two versions $s_{N,r}$ and $\hat{s}_{N,r}$ for the particle densities, is a consequence of the desire to handle density dependent cross diffusion terms. A careful check of the proof of Theorem 1 reveals that one can use $0<\beta=\hat{\beta}<\dfrac{d}{d+2}$ instead of (2.4), if the matrices $\check{D}_{qr}$ only depend on x. If the $\check{D}_{qr}$ vanish, we even can allow $0<\beta<1$, similarly as in [17] or [23].

3. Relations to Other Work. As far as our model for the N-particle system is regarded, there exist relations to [7, 17, 23, 24], where for quite simple models the interaction between the particles, i.e. its range and strength, is rescaled in the same way as here as $N\to\infty$. In particular, the proofs of our results are based on methods, which have been applied in [24] to the model with $K=1$, where $\check{G}_1$, $\check{b}_{11}$, and $\check{d}_1$ vanish and $\check{D}_{11}$ is the identity matrix.

Different approaches to derive reaction-diffusion equations as limit dynamics of many-particle systems are investigated e.g. in [2, 4, 8–10, 13, 14]. The model in [9, 10] corresponds to a situation, which in our notation may be described as follows: $K=1$, $\check{D}_{11}$, $\check{b}_{11}$, and $\check{G}_1$ vanish, $\check{d}_1(x,s)=s$, $\beta=\hat{\beta}=1$. This scaling is essentially the reason that our methods are not applicable in that situation. An extension of [9, 10] (i.e. $\check{b}_{11}\neq 0$) is studied in [4], and a further modification in [8]. These papers start with models in discrete space, like [2, 13, 14], which can be regarded as high density versions of [9, 10]. One has reactions within small cells containing many particles and diffusion between these cells. The idea to let the number of particles in each cell tend to infinity corresponds to our use of (2.1–4) to rescale the N-particle densities $s_{N,r}$ and $\hat{s}_{N,r}$.

The system (2.14, 15) is a standard equation of population dynamics, cf. [26], p. 169ff. Very often such equations exhibit a strange behaviour like the formation of travelling waves or spatial and temporal patterns. Our results indicate that in stochastic descriptions of large populations such phenomena may occur too. Of course, it is a long way to exact results on e.g. temporal patterns of stochastic models. An interesting result in this direction is [29], where time periodicity of the distribution of a model for the so-called Brusselator is proved. Travelling waves in many-particle systems are discussed e.g. in [5, 27].

Furthermore, [27] and the references therein give a survey on hydrodynamic behaviour of many-particle systems. This means that a limiting reaction-diffusion equation is obtained by leaving the dynamics fixed and rescaling space and time. This is in contrast to the situation here, where the change of the dynamics as $N \to \infty$ can not be obtained by a pure space-time scaling.

4. Proofs

First we shall give the derivation of Theorem 1 without the proof of a technical lemma, which is deferred to the end of this section, together with the proof of Theorem 2.

4.A. Proof of Theorem 1

To simplify our calculations and shorten the formulas we first assume that $\check{G}_r$, $\check{t}_{rq}$, $\check{b}_{rq}$ and $\check{d}_r$ vanish for any $q, r = 1, \ldots, K$. Then the size of the different subpopulations remains constant in time, and interaction between the individuals comes in only through the cross diffusion matrices $D_{N,qr}$. At the end of this subsection we shall indicate how to handle the general case.

The idea of the proof is

– to write down Itô's formula for the random function $t \to \sum_{q,r=1}^{K} E_{qr} \langle h_{N,q}(\cdot, t) - s_q(\cdot, t), h_{N,r}(\cdot, t) - s_r(\cdot, t)\rangle$,

– to show that the martingale part vanishes in the limit $N \to \infty$,

– to derive a suitable estimate for the bounded variation part, such that an application of Gronwall's inequality gets possible.

We obtain from (2.1, 2, 10), (3.1), the symmetry of the matrix E, the symmetry of V_N, which implies $\nabla V_N(0) = 0$, and Itô's formula

$$\sum_{q,r=1}^{K} E_{qr} \langle h_{N,q}(\cdot, t), h_{N,r}(\cdot, t)\rangle$$

$$= \frac{1}{N^2} \sum_{q,r=1}^{K} E_{qr} \sum_{\substack{k \in M(N,q,t), \\ l \in M(N,r,t)}} V_N(P_N^k(t) - P_N^l(t))$$

$$
\begin{aligned}
&= \frac{1}{N^2} \sum_{q,r=1}^{K} E_{qr} \sum_{\substack{k \in M(N,q,0),\\ l \in M(N,r,0)}} V_N(P_N^k(0) - P_N^l(0)) \\
&\quad + \frac{2}{N^2} \sum_{q,r=1}^{K} E_{qr} \int_0^t \sum_{\substack{k \in M(N,q,s),\\ l \in M(N,r,s),\\ k \neq l}} \Bigg(\Bigg(- \sum_{p=1}^{K} D_{N,pq}(P_N^k(s), s) \nabla s_{N,p}(P_N^k(s), s) \Bigg) \\
&\qquad \cdot \nabla V_N(P_N^k(s) - P_N^l(s)) \\
&\qquad + \frac{1}{2} \sum_{m,n=1}^{d} A_{q,mn} \nabla_m \nabla_n V_N(P_N^k(s) - P_N^l(s)) \Bigg) ds \\
&\quad + \frac{2}{N^2} \sum_{q,r=1}^{K} E_{qr} \int_0^t \sum_{\substack{k \in M(N,q,s),\\ l \in M(N,r,s)\\ k \neq l}} \nabla V_N(P_N^k(s) - P_N^l(s)) \cdot \sigma_q \, dW^k(s) \\
&= \sum_{q,r=1}^{K} E_{qr} \langle h_{N,q}(\cdot, 0), h_{N,r}(\cdot, 0) \rangle \\
&\quad - 2 \sum_{q,r=1}^{K} E_{qr} \int_0^t \Bigg(\sum_{p=1}^{K} \langle S_{N,q}(s), (D_{N,pq}(\cdot, s) \nabla s_{N,p}(\cdot, s)) \cdot \nabla s_{N,r}(\cdot, s) \rangle \\
&\qquad + \tfrac{1}{2} \langle A_q \nabla h_{N,q}(\cdot, s), \nabla h_{N,r}(\cdot, s) \rangle \Bigg) ds \\
&\quad - \frac{1}{N} \sum_{q=1}^{K} E_{qq} \sum_{m,n=1}^{d} A_{q,mn} \nabla_m \nabla_n V_N(0) \int_0^t \langle S_{N,q}(s), 1 \rangle \, ds \\
&\quad + \frac{2}{N} \sum_{q,r=1}^{K} E_{qr} \int_0^t \sum_{k \in M(N,q,s)} \nabla s_{N,r}(P_N^k(s), s) \cdot \sigma_q \, dW^k(s).
\end{aligned}
\tag{4.1}
$$

By additionally using (2.14) we obtain in the same way as in (4.1)

$$
\begin{aligned}
&\sum_{q,r=1}^{K} E_{qr} \langle h_{N,q}(\cdot, t), s_r(\cdot, t) \rangle \\
&\quad = \sum_{q,r=1}^{K} E_{qr} \langle h_{N,q}(\cdot, 0), s_r(\cdot, 0) \rangle \\
&\qquad - \sum_{q,r=1}^{K} E_{qr} \int_0^t \Bigg(\sum_{p=1}^{K} (\langle S_{N,q}(s), (D_{N,pq}(\cdot, s) \nabla s_{N,p}(\cdot, s)) \cdot \nabla s_r(\cdot, s) * W_N \rangle \\
&\qquad\quad + \langle s_q(\cdot, s), (D_{\infty,pq}(\cdot, s) \nabla s_p(\cdot, s)) \cdot \nabla h_{N,r}(\cdot, s) \rangle) \\
&\qquad\quad + \tfrac{1}{2} \langle A_q \nabla h_{N,q}(\cdot, s), \nabla s_r(\cdot, s) \rangle + \tfrac{1}{2} \langle \nabla h_{N,q}(\cdot, s), A_r \nabla s_r(\cdot, s) \rangle \Bigg) ds \\
&\qquad + \frac{1}{N} \sum_{q,r=1}^{K} E_{qr} \int_0^t \sum_{k \in M(N,q,s)} \nabla (s_r(\cdot, s) * W_N)(P_N^k(s)) \cdot \sigma_q \, dW^k(s),
\end{aligned}
\tag{4.2}
$$

and

$$
\begin{aligned}
&= \sum_{q,r=1}^{K} E_{qr} \langle s_q(.,0), s_r(.,0)\rangle \\
&\quad -2 \sum_{q,r=1}^{K} E_{qr} \int_0^t \bigg(\sum_{p=1}^{K} \langle s_q(.,s), (D_{\infty,pq}(.,s) \nabla s_p(.,s)) \cdot \nabla s_r(.,s)\rangle \\
&\qquad\qquad + \tfrac{1}{2} \langle A_q \nabla s_q(.,s), \nabla s_r(.,s)\rangle \bigg) ds. \qquad (4.3)
\end{aligned}
$$

After adding (4.1–3) and combining the different terms in a suitable way we obtain

$$
\begin{aligned}
&\sum_{q,r=1}^{K} E_{qr} \langle h_{N,q}(.,t) - s_q(.,t), h_{N,r}(.,t) - s_r(.,t)\rangle \\
&\quad = \sum_{q,r=1}^{K} E_{qr} (\langle h_{N,q}(.,t), h_{N,r}(.,t)\rangle \\
&\qquad\qquad -2\langle h_{N,q}(.,t), s_r(.,t)\rangle + \langle s_q(.,t), s_r(.,t)\rangle) \\
&\quad = \sum_{q,r=1}^{K} E_{qr} \langle h_{N,q}(.,0) - s_q(.,0), h_{N,r}(.,0) - s_r(.,0)\rangle \\
&\qquad -2 \sum_{p,q,r=1}^{K} E_{qr} \int_0^t (\langle S_{N,q}(s), (D_{N,pq}(.,s) \nabla s_{N,p}(.,s)) \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N)\rangle \\
&\qquad\qquad + \langle s_q(.,s), (D_{\infty,pq}(.,s) \nabla s_p(.,s)) \cdot \nabla(s_r(.,s) - h_{N,r}(.,s))\rangle) ds \\
&\qquad - \sum_{q,r=1}^{K} E_{qr} \int_0^t \langle A_q \nabla(h_{N,q}(.,s) - s_q(.,s)), \nabla(h_{N,r}(.,s) - s_r(.,s))\rangle ds \\
&\qquad - \frac{1}{N} \sum_{q=1}^{K} E_{qq} \sum_{m,n=1}^{d} A_{q,mn} \nabla_m \nabla_n V_N(0) \langle S_{N,q}(0), 1\rangle t \\
&\qquad + \frac{2}{N} \sum_{q,r=1}^{K} E_{qr} \int_0^t \sum_{k \in M(N,q,s)} (\nabla s_{N,r}(P_N^k(s),s) - \nabla(s_r(.,s) * W_N)(P_N^k(s))) \cdot \sigma_q \, dW^k(s).
\end{aligned}
\qquad (4.4)
$$

For the integrand of the first integral on the right side of (4.4) we obtain

$$
\begin{aligned}
&\langle S_{N,q}(s), (D_{N,pq}(.,s) \nabla s_{N,p}(.,s)) \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N)\rangle \\
&\quad + \langle s_q(.,s), (D_{\infty,pq}(.,s) \nabla s_p(.,s)) \cdot \nabla(s_r(.,s) - h_{N,r}(.,s))\rangle \\
&= \langle S_{N,q}(s), (D_{N,pq}(.,s) \nabla(s_{N,p}(.,s) - s_p(.,s) * W_N)) \\
&\quad \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N)\rangle \\
&\quad + \langle S_{N,q}(s) - s_q(.,s), (D_{N,pq}(.,s) \nabla s_p(.,s) * W_N) \\
&\quad \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N)\rangle \\
&\quad + \langle s_q(.,s), ((D_{N,pq}(.,s) - D_{\infty,pq}(.,s)) \nabla s_p(.,s) * W_N) \\
&\quad \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N)\rangle \\
&\quad + \langle s_q(.,s), (D_{\infty,pq}(.,s) \nabla(s_p(.,s) * W_N - s_p(.,s))) \\
&\quad \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N)\rangle \\
&\quad + \langle s_q(.,s), (D_{\infty,pq}(.,s) \nabla s_p(.,s)) \\
&\quad \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N - h_{N,r}(.,s) + s_r(.,s))\rangle. \qquad (4.5)
\end{aligned}
$$

By the ellipticity conditions (3.7, 8) we conclude

$$\sum_{p,q,r=1}^{K} E_{qr}\langle S_{N,q}(s), (D_{N,pq}(.,s)\nabla(s_{N,p}(.,s)-s_p(.,s)*W_N)) \cdot \nabla(s_{N,r}(.,s)-s_r(.,s)*W_N)\rangle \geqq 0, \qquad 0\leqq s<\infty, \tag{4.6}$$

and

$$\sum_{q,r=1}^{K} E_{qr}\langle A_q \nabla(h_{N,q}(.,s)-s_q(.,s)), \nabla(h_{N,r}(.,s)-s_r(.,s))\rangle \geqq C\sum_{k=1}^{K}\|\nabla h_{N,k}(.,s)-\nabla s_k(.,s)\|_2^2, \qquad 0\leqq s<\infty. \tag{4.7}$$

To obtain useful upper bounds for the remaining terms on the right sides of (4.4, 5) we need the following facts, which are proved in subsection 4.B.

Lemma 1. *Assume f, $\nabla f\in L^2(\mathbf{R}^d)$. Then*

$$\|f-f*W_N\|_2^2\leqq C\alpha_N^{-2}\|\nabla f\|_2^2. \tag{4.8}$$

An analogous result holds, if we replace W_N, α_N by $\hat{W}_N$, $\hat{\alpha}_N$.

Next let $U_N(x)=|x|\,W_N(x)$. Then for any finite positive measure μ on $\mathbf{R}^d$ and any $\varepsilon>0$

$$\|\mu*U_N\|_2^2\leqq C\alpha_N^{2\varepsilon-2}\|\mu*W_N\|_2^2+\langle\mu,1\rangle^2\exp(-C'\alpha_N^{\varepsilon}). \tag{4.9}$$

Moreover, for any finite signed measure μ on $\mathbf{R}^d$

$$\|\mu*\hat{V}_N\|_2^2\leqq\|\mu*V_N\|_2^2\leqq\|\mu*W_N\|_2^2. \tag{4.10}$$

The constants in (4.8, 9) are independent of f and μ.

By (2.1), (3.1) we can write the second term on the right side of (4.5) as

$$\begin{aligned}
&\langle S_{N,q}(s)-s_q(.,s), (D_{N,pq}(.,s)\nabla s_p(.,s)*W_N) \cdot \nabla(s_{N,r}(.,s)-s_r(.,s)*W_N)\rangle\\
&=\langle S_{N,q}(s)-s_q(.,s), \int_{\mathbf{R}^d} W_N(u)(D_{N,pq}(.-u,s)\nabla s_p(.-u,s)*W_N) \cdot \nabla(h_{N,r}(.-u,s)-s_r(.-u,s))\,du\rangle\\
&\quad+\langle S_{N,q}(s)-s_q(.,s), \int_{\mathbf{R}^d} W_N(u)(D_{N,pq}(.,s)\nabla s_p(.,s)*W_N - D_{N,pq}(.-u,s)\nabla s_p(.-u,s)*W_N)\\
&\qquad \cdot \nabla(h_{N,r}(.-u,s)-s_r(.-u,s))\,du\rangle\\
&=J_{N,pqr}^1(s)+J_{N,pqr}^2(s).
\end{aligned} \tag{4.11}$$

By (2.6, 9, 13), (3.6), (4.8–10), ($\mathscr{H}$) we obtain for the terms on the right side of (4.11) and the remaining expressions in (4.5) the following estimates, which hold uniformly in $0 \leq s \leq T$ and $N \in \mathbf{N}$. The constant $\tilde{C}$ will be choosen later in a suitable way.

$$
\begin{aligned}
&|J_{N,pqr}^1(s)| \\
&\quad = |\langle S_{N,q}(s) - s_q(.,s)) * W_N, (D_{N,pq}(.,s) \nabla s_p(.,s) * W_N) \\
&\qquad \cdot \nabla(h_{N,r}(.,s) - s_r(.,s))\rangle| \\
&\quad \leq C \|h_{N,q}(.,s) - s_q(.,s) * W_N\|_2 \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2 \\
&\quad \leq C(\|h_{N,q}(.,s) - s_q(.,s)\|_2 + \|s_q(.,s) - s_q(.,s) * W_N\|_2) \\
&\qquad \cdot \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2 \\
&\quad \leq C\tilde{C}(\|h_{N,q}(.,s) - s_q(.,s)\|_2^2 + \alpha_N^{-2}) + \frac{1}{\tilde{C}} \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2^2, \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
&|J_{N,pqr}^2(s)| \\
&\quad \leq \langle S_{N,q}(s) + s_q(.,s), \int_{\mathbf{R}^d} W_N(u)|u| \, \|\nabla(D_{N,pq}(.,s) \nabla s_p(.,s) * W_N)\|_\infty \\
&\qquad \cdot |\nabla h_{N,r}(.-u,s) - \nabla s_r(.-u,s)| \, du\rangle \\
&\quad \leq C \hat{\alpha}_N^{d+1}(\langle S_N(s), 1\rangle + 1) \\
&\qquad \cdot \langle (S_{N,q}(s) + s_q(.,s)) * U_N, |\nabla h_{N,r}(.,s) - \nabla s_r(.,s)|\rangle \\
&\quad \leq C \hat{\alpha}_N^{d+1}(\langle S_N(s), 1\rangle + 1) \\
&\qquad \cdot (\alpha_N^{\varepsilon-1}(\|h_{N,q}(.,s)\|_2 + 1) + (\langle S_N(s), 1\rangle + 1) \exp(-C' \alpha_N^{\varepsilon})) \\
&\qquad \cdot \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2 \\
&\quad \leq C\tilde{C} \hat{\alpha}_N^{2d+2} \alpha_N^{2\varepsilon-2}(\langle S_N(s), 1\rangle^4 + 1)(\|h_{N,q}(.,s) - s_q(.,s)\|_2^2 + 1) \\
&\qquad + \frac{1}{\tilde{C}} \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2^2 \\
&\qquad (\text{with } 0 < \varepsilon < 1 - \hat{\beta}(d+1)/\beta), \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
&|\langle s_q(.,s), ((D_{N,pq}(.,s) - D_{\infty,pq}(.,s)) \nabla s_p(.,s) * W_N) \\
&\qquad \cdot \nabla(s_{N,r}(.,s) - s_r(.,s) * W_N)\rangle| \\
&\quad \leq C \sum_{l=1}^{K} \|\hat{s}_{N,l}(.,s) - s_l(.,s)\|_2 \|\nabla s_{N,r}(.,s) - \nabla s_r(.,s) * W_N\|_2 \\
&\quad \leq C \sum_{l=1}^{K} (\|\hat{s}_{N,l}(.,s) - s_l(.,s) * \hat{V}_N\|_2 + \|s_l(.,s) * \hat{V}_N - s_l(.,s)\|_2) \\
&\qquad \cdot \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2 \\
&\quad \leq C\left(\sum_{l=1}^{K} \|h_{N,l}(.,s) - s_l(.,s)\|_2 + \hat{\alpha}_N^{-1}\right) \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2 \\
&\quad \leq C\tilde{C}\left(\sum_{l=1}^{K} \|h_{N,l}(.,s) - s_l(.,s)\|_2^2 + \hat{\alpha}_N^{-2}\right) + \frac{1}{\tilde{C}} \|\nabla h_{N,r}(.,s) - \nabla s_r(.,s)\|_2^2, \qquad (4.14)
\end{aligned}
$$

$$\begin{aligned}&|\langle s_q(.,s),(D_{\infty,pq}(.,s)\,\nabla(s_p(.,s)*W_N-s_p(.,s)))\\&\quad\cdot\nabla(s_{N,r}(.,s)-s_r(.,s)*W_N)\rangle|\\&\leqq C\hat{C}\alpha_N^{-2}+\frac{1}{\hat{C}}\|\nabla h_{N,r}(.,s)-\nabla s_r(.,s)\|_2^2,\end{aligned}\tag{4.15}$$

$$\begin{aligned}&|\langle s_q(.,s),(D_{\infty,pq}(.,s)\,\nabla s_p(.,s))\\&\quad\cdot\nabla(s_{N,r}(.,s)-s_r(.,s)*W_N-h_{N,r}(.,s)+s_r(.,s))\rangle|\\&=|\langle s_q(.,s)\,D_{\infty,pq}(.,s)\,\nabla s_p(.,s)-(s_q(.,s)\,D_{\infty,pq}(.,s)\,\nabla s_p(.,s))*W_N,\\&\quad\cdot\nabla h_{N,r}(.,s)-\nabla s_r(.,s)\rangle|\\&\leqq C\tilde{C}\alpha_N^{-2}+\frac{1}{\tilde{C}}\|\nabla h_{N,r}(.,s)-\nabla s_r(.,s)\|_2^2.\end{aligned}\tag{4.16}$$

We note that it is essentially the estimate (4.13) for $J_{N,pqr}^2$, where we use the fact that $s_{N,r}$ and $\hat{s}_{N,r}$ are obtained by employing different scalings of V_1. Next we obtain from (2.2, 3)

$$|\nabla_m\nabla_n \mathrm{V}_N(0)|\leqq CN^{\beta(d+2)/d}.\tag{4.17}$$

For the martingale

$$M_N^*(t)=\frac{2}{N}\sum_{q,r=1}^{K}E_{qr}\int_0^t\sum_{k\in M(N,q,s)}(\nabla s_{N,r}(P_N^k(s),s)-(\nabla s_r(.,s)*W_N)(P_N^k(s)))\cdot\sigma_q\,dW^k(s)$$

on the right side of (4.4) we derive from (2.2, 3), (3.1) and Doob's inequality

$$\begin{aligned}&E[\sup_{t\leqq T}|M_N^*(t)|\mathscr{F}_0]^2\\&\leqq 4E[|M_N^*(T)|^2|\mathscr{F}_0]\\&=\frac{16}{N}\sum_{q=1}^{K}E\left[\int_0^T\left\langle S_{N,q}(s),\left|\sum_{r=1}^{K}E_{qr}\sigma_q^T(\nabla s_{N,r}(.,s)-\nabla s_r(.,s)*W_N)\right|^2\right\rangle ds\middle|\mathscr{F}_0\right]\\&\leqq\frac{C}{N}\sum_{q,r=1}^{K}E\left[\int_0^T\langle h_{N,q}(.,s),|\nabla h_{N,r}(.,s)-\nabla s_r(.,s)|^2\rangle\,ds|\mathscr{F}_0\right]\\&\leqq\frac{C}{N}\alpha_N^d\langle S_N(0),1\rangle\sum_{r=1}^{K}E\left[\int_0^T\|\nabla h_{N,r}(.,s)-\nabla s_r(.,s)\|_2^2\,ds\middle|\mathscr{F}_0\right].\end{aligned}\tag{4.18}$$

Since the matrix E is positive definite, we have

$$\frac{1}{\lambda}\sum_{i=1}^{K}|y_i|^2\leqq\sum_{i,j=1}^{K}E_{ij}y_iy_j\leqq\lambda\sum_{i=1}^{K}|y_i|^2,\qquad y_1,\ldots,y_K\in\mathbf{R},\tag{4.19}$$

for some $\lambda>0$. Now we may collect the different contributions (2.3, 4), (4.4–7, 11–19) to arrive at

$$\sum_{r=1}^{K} E\left[\sup_{t \leq T} \|h_{N,r}(.,t)-s_r(.,t)\|_2^2+\int_0^T \|\nabla h_{N,r}(.,s)-\nabla s_r(.,s)\|_2^2\,ds \,\middle|\, \mathscr{F}_0\right]$$
$$\leq C(\langle S_N(0),1\rangle^4+1)$$
$$\cdot \sum_{r=1}^{K}\Big(\|h_{N,r}(.,0)-s_r^*(.)\|_2^2+\tilde{C}TE[\sup_{t\leq T}\|h_{N,r}(.,t)-s_r(.,t)\|_2^2\,|\,\mathscr{F}_0]$$
$$+\tilde{C}T(\hat{\alpha}_N^{-2}+\hat{\alpha}_N^{2d+2}\,\alpha_N^{2\varepsilon-2}+N^{\beta(d+2)/2-1})$$
$$+\frac{1}{\tilde{C}}E\left[\int_0^T \|\nabla h_{N,r}(.,s)-\nabla s_r(.,s)\|_2^2\,ds\,\middle|\,\mathscr{F}_0\right]\Big). \tag{4.20}$$

Suppose now that additionally

$$P[\langle S_N(0),1\rangle \geq n_0]=0, \qquad N\in\mathbf{N}, \tag{4.21}$$

holds for some n_0. Then by (2.4), (3.9), and since we may choose $\tilde{C}$ arbitrarily large, we immediately derive (3.11), at least if $T\leq(2C\tilde{C}(n_0^4+1))^{-1}=T_{n_0}$. By iteration of our arguments in the time intervals $[T_{n_0}, 2T_{n_0}]$, $[2T_{n_0}, 3T_{n_0}]$, ... we obtain its validity for arbitrary T. Now we obviously can replace (4.21) by (3.10) to finish the proof of Theorem 1 in this particular case, where $\check{G}_r$, $\check{t}_{qr}$, $\check{b}_{qr}$ and $\check{d}_r$ vanish.

In the case, where $\check{b}_{rq}\not\equiv 0$, we have to add to the right side of (4.4) the terms

$$2\sum_{p,q,r=1}^{K} E_{qr}\int_0^t (\langle S_{N,p}(s), b_{N,pq}(.,s)(s_{N,r}(.,s)-s_r(.,s)*W_N)\rangle$$
$$+\langle s_p(.,s), b_{\infty,pq}(.,s)(s_r(.,s)-h_{N,r}(.,s))\rangle)\,ds$$
$$+\frac{2}{N}\sum_{p,q,r=1}^{K} E_{qr}\int_0^t \sum_{k\in M(N,p,s)} (s_{N,r}(P_N^k(s),s)-(s_r(.,s)*W_N)(P_N^k(s)))$$
$$(b_{N,pq}^{*,k}(ds)-b_{N,pq}(P_N^k(s),s)\,ds). \tag{4.22}$$

These terms can be estimated in exactly the same way as the corresponding expressions in (4.4), cf. (4.5, 18). We only have to notice that now the size $\langle S_{N,r}(t),1\rangle$, $r=1,\dots,K$, and $\langle S_N(t),1\rangle$ of the subpopulations and the total population depend on time. In particular, we obtain instead of (4.20) an inequality involving $\sup_{t\leq T}\langle S_N(t),1\rangle^4$ in place of $\langle S_N(0),1\rangle^4$. However, the arguments following (4.20) may easily be generalized to the present situation, since the uniform boundedness of the functions $\check{b}_{rq}$ (cf. (2.9)) implies that the process $t\to\langle S_N(t),1\rangle$ is stochastically dominated by $\frac{1}{N}Y_{N\langle S_N(0),1\rangle}(t)$, where $Y_k(.)$ is a Yule process with birth rate $\sum_{q,r=1}^{K}\|\check{b}_{qr}\|_\infty$ and $Y_k(0)=k$. Especially, (3.10) implies

$$\lim_{n\to\infty}\sup_{N\in\mathbf{N}} P[\sup_{t\leq T}\langle S_N(t),1\rangle\geq n]=0. \tag{4.23}$$

In the slightly more general case, where $\check{G}_r$, $\check{t}_{rq}$ and $\check{d}_r$ don't vanish, one has to find estimates for more expressions like (4.22). However, since (4.23) remains valid, one observes no further difficulties.

4.B. Proof of Lemma 1

$$\begin{aligned}\|f-f*W_N\|_2^2 &= \int_{\mathbf{R}^d} |\tilde f(\tau)|^2 \,|1-(2\pi)^{d/2}\,\tilde W_N(\tau)|^2\, d\tau\\ &= \int_{\mathbf{R}^d} |\tilde f(\tau)|^2 \,|1-(2\pi)^{d/2}\,\tilde W_1(\tau/\alpha_N)|^2\, d\tau\\ &= \int_{\mathbf{R}^d} |\tilde f(\tau)|^2 \frac{|1-(2\pi)^{d/2}\,\tilde W_1(\tau/\alpha_N)|^2}{|\tau/\alpha_N|^2}\,|\tau/\alpha_N|^2\, d\tau\\ &\leqq \alpha_N^{-2}(2\pi)^d\,\|\nabla\tilde W_1\|_\infty^2 \int_{\mathbf{R}^d} |\tilde f(\tau)|^2\,|\tau|^2\, d\tau\\ &\quad (\text{since } \tilde W_1(0)=(2\pi)^{-d/2}).\end{aligned}$$

By (3.2) we obtain (4.8). Next we observe

$$\begin{aligned}&\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} \mu(dy)\, W_N(x-y)\,|x-y|\Big)^2 dx\\ &\quad\leqq \int_{\mathbf{R}^d}\left(\int_{\mathbf{R}^d} \mu(dy)\, W_N(x-y)\left(A\,|x-y|^2+\frac{1}{A}\right)\right)^2 dx, \qquad A>0,\end{aligned}$$

and

$$(2\pi)^{-d/2}\int_{\mathbf{R}^d} e^{-i\tau x}\, W_N(x)\,x^2\,dx = -\Delta\tilde W_N(\tau) = -\alpha_N^{-2}(\Delta\tilde W_1)(\tau/\alpha_N). \tag{4.24}$$

Hence (3.3, 4) imply

$$\begin{aligned}&\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} \mu(dy)\, W_N(x-y)\,|x-y|\Big)^2 dx\\ &\leqq (2\pi)^d \int_{\mathbf{R}^d} |\tilde\mu(\tau)|^2\,\Big|-A\,\alpha_N^{-2}(\Delta\tilde W_1)(\tau/\alpha_N)+\frac{1}{A}\,\tilde W_N(\tau)\Big|^2 d\tau\\ &\leqq C\int_{\mathbf{R}^d} |\tilde\mu(\tau)|^2\left(\frac{A^2}{\alpha_N^4}\left(1+\left|\frac{\tau}{\alpha_N}\right|^4\right)+\frac{1}{A^2}\right)|\tilde W_N(\tau)|^2\, d\tau\\ &\leqq C\left(\left(A^2\,\alpha_N^{4\varepsilon-4}+\frac{1}{A^2}\right)\int_{\mathbf{R}^d} |\tilde\mu(\tau)|^2\,|\tilde W_N(\tau)|^2\, d\tau\right.\\ &\qquad\left.+\frac{A^2}{\alpha_N^4}\int_{\{|\tau|\geqq\alpha_N^{1+\varepsilon}\}} |\tilde\mu(\tau)|^2\left|\frac{\tau}{\alpha_N}\right|^4 |\tilde W_N(\tau)|^2\, d\tau\right)\\ &\leqq C\left(\left(A^2\,\alpha_N^{4\varepsilon-4}+\frac{1}{A^2}\right)\|\mu*W_N\|_2^2+\frac{A^2}{\alpha_N^4}\int_{\{|\sigma|\geqq\alpha_N^{\varepsilon}\}} \|\tilde\mu\|_\infty^2\,\sigma^4\exp(-C'\,|\sigma|)\,\alpha_N^d\, d\sigma\right)\\ &\leqq C\big(\alpha_N^{2\varepsilon-2}\,\|\mu*W_N\|_2^2+\langle\mu,1\rangle^2\exp(-C'\alpha_N^{\varepsilon})\big)\\ &\qquad(\text{if we choose } A=\alpha_N^{1-\varepsilon}).\end{aligned}$$

We still have to prove (4.10).

$$\begin{aligned}\|\mu*\hat V_N\|_2^2 &= (2\pi)^d\int_{\mathbf{R}^d} |\tilde\mu(\tau)|^2\,|\tilde{\hat V}_N(\tau)|^2\, d\tau\\ &= (2\pi)^{2d}\int_{\mathbf{R}^d} |\tilde\mu(\tau)|^2\,|\tilde W_1(\tau/\hat\alpha_N)|^4\, d\tau\\ &\leqq (2\pi)^{2d}\int_{\mathbf{R}^d} |\tilde\mu(\tau)|^2\,|\tilde W_1(\tau/\alpha_N)|^4\, d\tau \qquad (by\ (3.5)) \qquad = \|\mu*V_N\|_2^2\end{aligned}$$

$$\leqq (2\pi)^d \int_{\mathbf{R}^d} |\tilde{\mu}(\tau)|^2 \, |\tilde{W}_1(\tau/\alpha_N)|^2 \, d\tau$$

(since $(2\pi)^{d/2} \, |\tilde{W}_1(\tau)| \leqq 1$ for the probability density W_1)

$$= \|\mu * W_N\|_2^2.$$

4.C. Proof of Theorem 2

Without restricting generality we can assume that $\check{G}_r$, $\check{t}_{rq}$ and $\check{d}_r$ vanish. Let us fix some $\delta \in (0, 1)$ and define the stopping time

$$T_\delta = \inf\left\{t \geqq 0: \sum_{r=1}^{K} \left(\|h_{N,r}(.,t) - s_r(.,t)\|_2^2 + \int_0^t \|\nabla h_{N,r}(.,s) - \nabla S_r(.,s)\|_2^2 \, ds\right) \geqq \delta\right\}.$$

Then we insert the function $(x,t) \to \psi(x)^2 = (\log(2+x^2))^2$ into (2.10). Calculations as in (4.11–16) yield

$$\sum_{r=1}^{K} \langle S_{N,r}(t \wedge T_\delta), \psi^2 \rangle$$
$$\leqq \sum_{r=1}^{K} \langle S_{N,r}(0), \psi^2 \rangle + C \int_0^{t \wedge T_\delta} \sum_{r=1}^{K} \langle S_{N,r}(s), \psi^2 \rangle \, ds + C' t + M_N^3(t \wedge T_\delta)$$

for some martingale M_N^3. By taking expectations on both sides and applying Gronwall's inequality we obtain

$$\sup_{t \leqq T} E\left[\sum_{r=1}^{K} \langle S_{N,r}(t \wedge T_\delta), \psi^2 \rangle \,\middle|\, \mathscr{F}_0\right] \leqq C \sum_{r=1}^{K} \langle S_{N,r}(0), \psi^2 \rangle + C'. \tag{4.25}$$

Applying (2.10) to the function $(x,t) \to \psi(x,t)$ shows

$$\sum_{r=1}^{K} \langle S_{N,r}(t \wedge T_\delta), \psi \rangle$$
$$\leqq \sum_{r=1}^{K} \langle S_{N,r}(0), \psi \rangle + C \int_0^{t \wedge T_\delta} \sum_{r=1}^{K} \langle S_{N,r}(s), \psi \rangle \, ds + C' t + M_N^4(t \wedge T_\delta) \tag{4.26}$$

for the martingale

$$M_N^4(t) = \frac{1}{N} \sum_{r=1}^{K} \int_0^t \sum_{k \in M(N,r,s)} \nabla \psi(P_N^k(s)) \cdot \sigma_r \, dW^k(s)$$
$$+ \frac{1}{N} \sum_{q,r=1}^{K} \int_0^t \sum_{k \in M(N,q,s)} \psi(P_N^k(s)) (b_{N,qr}^{*,k}(ds) - b_{N,qr}(P_N^k(s), s) \, ds).$$

Doob's inequality yields

$$E[\sup_{t \leqq T \wedge T_\delta} |M_N^4(t)| \, | \mathscr{F}_0]^2 \leqq 4 E[|M_N^4(T \wedge T_\delta)|^2 \, | \mathscr{F}_0]$$
$$\leqq \frac{C}{N^2} \sum_{r=1}^{K} E\left[\int_0^{T \wedge T_\delta} \sum_{k \in M(N,r,s)} \psi^2(P_N^k(s)) \, ds \,|\, \mathscr{F}_0\right]$$
$$= \frac{C}{N} \sum_{r=1}^{K} E\left[\int_0^{T \wedge T_\delta} \langle S_{N,r}(s), \psi^2 \rangle \, ds \,|\, \mathscr{F}_0\right]. \tag{4.27}$$

(4.25–27) imply

$$E\left[\sup_{t \leqq T \wedge T_\delta} \sum_{r=1}^{K} \langle S_{N,r}(t), \psi\rangle \middle| \mathscr{F}_0\right] \leqq C \sum_{r=1}^{K} \langle S_{N,r}(0), \psi^2\rangle + C'. \tag{4.28}$$

Similarly, we obtain

$$\sup_{t \leqq T} \sum_{r=1}^{K} \langle s_r(.,t), \psi\rangle \leqq C \sum_{r=1}^{K} \langle s_r^*(.), \psi\rangle, \tag{4.29}$$

where the right side is finite by (3.9, 12). Since W_N is a probability density, and by (3.2), (4.24), we obtain a slight modification of (4.8):

$$\begin{aligned} |f(x)-(f*W_N)(x)| &= \Big|\int_{\mathbf{R}^d} W_N(y)(f(x)-f(x-y))\,dy\Big| \\ &\leqq \|\nabla f\|_\infty \int_{\mathbf{R}^d} |y|\, W_N(y)\,dy \\ &\leqq C\alpha_N^{-1} \|\nabla f\|_\infty . \end{aligned}$$

Obviously, we can represent any $f \in \mathscr{B}_1 = \{g \in C_b^1(\mathbf{R}^d)\colon \|g\|_\infty + \|\nabla g\|_\infty \leqq 1\}$ as $f = f_R + \hat{f}_R$, where f_R, $\hat{f}_R \in \mathscr{B}_1$, and $\operatorname{supp}(f_R) \subseteq B_R = \{x \in \mathbf{R}^d\colon |x| \leqq R\}$, $\operatorname{supp}(\hat{f}_R) \subseteq \mathbf{R}^d \setminus B_{R-2}$, respectively. Hence for $r=1, \ldots, K$, $t \geqq 0$ uniformly in $f \in \mathscr{B}_1$

$$\begin{aligned} &|\langle S_{N,r}(t) - s_r(.,t), f\rangle| \\ &\quad \leqq |\langle S_{N,r}(t) - s_r(.,t), f_R\rangle| + \langle S_{N,r}(t) + s_r(.,t), |\hat{f}_R|\rangle \\ &\quad \leqq |\langle h_{N,r}(.,t) - s_r(.,t), f_R\rangle| + \langle S_{N,r}(t), |f_R - f_R * W_N|\rangle \\ &\qquad + \frac{C}{\psi(R)} \langle S_{N,r}(t) + s_r(.,t), \psi\rangle \\ &\quad \leqq C\Big(\|h_{N,r}(.,t) - s_r(.,t)\|_2 R^{d/2} + \alpha_N^{-1} \langle S_{N,r}(t), 1\rangle \\ &\qquad + \frac{C}{\psi(R)} \langle S_{N,r}(t) + s_r(.,t), \psi\rangle \Big). \end{aligned}$$

Consequently (3.13) follows by (3.11, 12), (4.23, 28, 29).

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