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Tensorial Decomposition of Concept Lattices*

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Abstract. A tensor product for complete lattices is studied via concept lattices. A characterization as a universal solution and an ideal representation of the tensor products are given. In a large class of concept lattices which contains all finite ones, the subdirect decompositions of a tensor product can be determined by the subdirect decompositions of its factors. As a consequence, one obtains that the tensor product of completely subdirectly irreducible concept lattices of this class is again completely subdirectly irreducible. Finally, applications to conceptual measurement are discussed.

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1. Introduction

The fundamental interpretation of lattices as hierarchies of concepts has been elaborated in [14] based on the notion of the concept lattice $\mathfrak{B}(G, M, I)$ for a given context (G, M, I). In [15], it is shown how subdirect decompositions of $\mathfrak{B}(G, M, I)$ can be seen within (G, M, I). This gives rise to a construction method for concept lattices via subdirect products. In this paper we analyse tensorial decompositions of concept lattices which again provide us with construction methods. In a large class of concept lattices which contains all finite ones, the subdirect decompositions of a tensor product can be determined by the subdirect decompositions of its factors. In particular, it will be shown that the tensor product of completely subdirectly irreducible concept lattices of this class is again completely subdirectly irreducible.

2. A Tensor Product for Complete Lattices

A. G. Watermann [13], D. G. Mowat [7], and Z. Shmuely [11] have introduced a tensor product for complete lattices where the complete lattices are considered as $\vee$-semilattices with 0. Since it is the character of concept lattices to treat meets and joins equally, we propose for the analysis of concept lattices a tensor product which respects both operations. For complete lattices L_1 and L_2 we define the *tensor product* by

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$$L_1 \otimes L_2 := \mathfrak{B}(L_1 \times L_2, L_1 \times L_2, \nabla)$$

where

$$(x_1, x_2) \nabla (y_1, y_2) :\Leftrightarrow x_1 \leqslant y_1 \quad \text{or} \quad x_2 \leqslant y_2$$

for $(x_1, x_2), (y_1, y_2) \in L_1 \times L_2$. This construction was first considered by G. N. Raney [10] to study completely distributive complete lattices. It was named 'the topological product' by G. Kalmbach [6] and H.-J. Bandelt [2] because, if L_1 and L_2 are the lattices of all closed (open) subsets of topological spaces, $L_1 \otimes L_2$ is isomorphic to the lattice of all closed (open) subsets of their product space. This is an immediate consequence of Theorem 1 and the fact that the lattice of all closed subsets of a topological space S is isomorphic to $\mathfrak{B}(S, \mathfrak{A}, \notin)$ for every base $\mathfrak{A}$ of open sets. For the formulation of Theorem 1, we define the *direct product* of contexts $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K}_2 := (G_1, M_2, I_2)$ by

$$\mathbb{K}_1 \times \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, \nabla)$$

where

$$(g_1, g_2) \nabla (m_1, m_2) :\Leftrightarrow g_1 I_1 m_1 \quad \text{or} \quad g_2 I_2 m_2$$

for $(g_1, g_2) \in G_1 \times G_2$ and $(m_1, m_2) \in M_1 \times M_2$.

THEOREM 1. $\mathfrak{B}(\mathbb{K}_1) \otimes \mathfrak{B}(\mathbb{K}_2) \simeq \mathfrak{B}(\mathbb{K}_1 \times \mathbb{K}_2)$

Proof. To apply the basic theorem of concept lattices in [14; p. 449], we define mappings γ and μ from $\mathfrak{B}(\mathbb{K}_1) \times \mathfrak{B}(\mathbb{K}_2)$ into $\mathfrak{B}(\mathbb{K}_1 \times \mathbb{K}_2)$ by

$$\begin{aligned} &\gamma((A_1, B_1), (A_2, B_2)) := (A_1 \times A_2 \cup G_1 \times M_2' \cup M_1' \times G_2, B_1 \times M_2 \cup M_1 \times B_2), \\ &\mu((C_1, D_1), (C_2, D_2)) := (C_1 \times G_2 \cup G_1 \times C_2, D_1 \times D_2 \cup M_1 \times G_2' \cup G_1' \times M_2). \end{aligned} \qquad (*)$$

First we have to show that $(A_1 \times A_2 \cup G_1 \times M_2' \cup M_1' \times G_2, B_1 \times M_2 \cup M_1 \times B_2)$ is a concept of $\mathbb{K}_1 \times \mathbb{K}_2$. $(m_1, m_2) \in B_1 \times M_2 \cup M_1 \times B_2$ means $m_1 \in B_1$ or $m_2 \in B_2$. This is equivalent to $g_1 I_1 m_1$ or $g_2 I_2 m_2$ for all $(g_1, g_2) \in A_1 \times A_2 \cup G_1 \times M_2' \cup M_1' \times G_2$ because $m_1 \notin B_1$ and $m_2 \notin B_2$ would imply $\neg g_1 I_1 m_1$ and $\neg g_2 I_2 m_2$ for some $(g_1, g_2) \in A_1 \times A_2$. Hence,

$$B_1 \times M_2 \cup M_1 \times B_2 = (A_1 \times A_2 \cup G_1 \times M_2' \cup M_1' \times G_2)'.$$

Now, let $(g_1, g_2) \in A_1 \times A_2 \cup G_1 \times M_2' \cup M_1' \times G_2$. Since $G_1 \times M_2' \cup M_1' \times G_2 \subseteq X'$ for every $X \subseteq M_1 \times M_2$, we may assume that $(g_1, g_2) \in A_1 \times A_2$. This is equivalent to $g_1 I_1 m_1$ for all $m_1 \in B_1$ and $g_2 I_2 m_2$ for all $m_2 \in B_2$. Hence, $(g_1, g_2) \in (B_1 \times M_2 \cup M_1 \times B_2)'$. Let $(h_1, h_2) \in (B_1 \times M_2 \cup M_1 \times B_2)'$. If $(h_1, h_2) \notin A_1 \times A_2$ then w.l.o.g. $\neg h_1 I_1 m_1$ for some $m_1 \in B_1$. It follows that $h_2 I_2 m_2$ for all $m_2 \in M_2$, whence $h_2 \in M_2'$ and $(h_1, h_2) \in G_1 \times M_2'$. We conclude

$$A_1 \times A_2 \cup G_1 \times M_2' \cup M_1' \times G_2 = (B_1 \times M_2 \cup M_1 \times B_2)'.$$

Therefore, we have that γ (and dually μ) is a mapping from $\mathfrak{B}(\mathbb{K}_1)\times\mathfrak{B}(\mathbb{K}_2)$ into $\mathfrak{B}(\mathbb{K}_1\times\mathbb{K}_2)$. Since

$$\{(g_1,g_2)\}'=\{g_1\}'\times M_2\cup M_1\times\{g_2\}'\quad\text{for } (g_1,g_2)\in G_1\times G_2,$$

it follows that

$$\gamma((\{g_1\}'',\{g_1\}'),(\{g_2\}'',\{g_2\}'))=(\{(g_1,g_2)\}'',\{(g_1,g_2)\}').$$

Thus, the image of γ is supremum-dense in $\mathfrak{B}(\mathbb{K}_1\times\mathbb{K}_2)$; dually, the image of μ is infimum-dense in $\mathfrak{B}(\mathbb{K}_1\times\mathbb{K}_2)$. Finally, $((A_1,B_1),(A_2,B_2))\nabla((C_1,D_1),(C_2,D_2))$ is equivalent to $A_1\subseteq C_1$ or $A_2\subseteq C_2$ and so, because of (*), equivalent to $\gamma((A_1,B_1),(A_2,B_2))\leqslant\mu((C_1,D_1),(C_2,D_2))$. Now, the asserted isomorphism is a consequence of the theorem cited above. □

Theorem 1 shows the independence of the tensor product (up to isomorphism) with respect to underlying contexts. If we comprehend the supremum-dense subset given by the objects and the infimum-dense subset given by the attributes as a 'double base' of the concept lattice, we may understand the tensor product to be constructed by the 'direct product of double bases'. This is the reason why we prefer the name 'tensor product' for $L_1\otimes L_2$. It is natural to introduce two 'tensorial operations' between L_1 and L_2 (cf. (*)):

$$x_1\,\triangle\!\!\!\!\!\bigcirc\,x_2:=([0,x_1]\times[0,x_2]\cup L_1\times\{0\}\cup\{0\}\times L_2,[x_1,1]\times L_2\cup L_1\times[x_2,1]),$$
$$x_1\,\triangledown\!\!\!\!\!\bigcirc\,x_2:=([0,x_1]\times L_2\cup L_1\times[0,x_2],[x_1,1]\times[x_2,1]\cup L_1\times\{1\}\cup\{1\}\times L_2).$$

As direct consequence of these definitions we obtain the following identities:

$$x_1\,\triangle\!\!\!\!\!\bigcirc\,1\vee 1\,\triangle\!\!\!\!\!\bigcirc\,x_2=x_1\,\triangledown\!\!\!\!\!\bigcirc\,x_2,\qquad x_1\,\triangledown\!\!\!\!\!\bigcirc\,0\wedge 0\,\triangledown\!\!\!\!\!\bigcirc\,x_2=x_1\,\triangle\!\!\!\!\!\bigcirc\,x_2,$$
$$\bigwedge_{t\in T}x_1^t\,\triangle\!\!\!\!\!\bigcirc\,x_2^t=\Big(\bigwedge_{t\in T}x_1^t\Big)\,\triangle\!\!\!\!\!\bigcirc\,\Big(\bigwedge_{t\in T}x_2^t\Big),\qquad \bigvee_{t\in T}x_1^t\,\triangledown\!\!\!\!\!\bigcirc\,x_2^t=\Big(\bigvee_{t\in T}x_1^t\Big)\,\triangledown\!\!\!\!\!\bigcirc\,\Big(\bigvee_{t\in T}x_2^t\Big),$$
$$\bigvee_{t\in T}x_1\,\triangle\!\!\!\!\!\bigcirc\,x_2^t=x_1\,\triangle\!\!\!\!\!\bigcirc\,\Big(\bigvee_{t\in T}x_2^t\Big),\qquad \bigwedge_{t\in T}x_1\,\triangledown\!\!\!\!\!\bigcirc\,x_2^t=x_1\,\triangledown\!\!\!\!\!\bigcirc\,\Big(\bigwedge_{t\in T}x_2^t\Big),$$
$$\bigvee_{t\in T}x_1^t\,\triangle\!\!\!\!\!\bigcirc\,x_2=\Big(\bigvee_{t\in T}x_1^t\Big)\,\triangle\!\!\!\!\!\bigcirc\,x_2,\qquad \bigwedge_{t\in T}x_1^t\,\triangledown\!\!\!\!\!\bigcirc\,x_2=\Big(\bigwedge_{t\in T}x_1^t\Big)\,\triangledown\!\!\!\!\!\bigcirc\,x_2,$$
$$\bigwedge_{t\in T}(x_1^t\,\triangle\!\!\!\!\!\bigcirc\,1\vee 1\,\triangle\!\!\!\!\!\bigcirc\,x_2^t)=\bigvee_{S\subseteq T}\left(\Big(\bigwedge_{s\in S}x_1^s\,\triangle\!\!\!\!\!\bigcirc\,1\Big)\wedge\Big(\bigwedge_{t\in T\setminus S}1\,\triangle\!\!\!\!\!\bigcirc\,x_2^t\Big)\right),$$
$$\bigvee_{t\in T}(x_1^t\,\triangledown\!\!\!\!\!\bigcirc\,0\wedge 0\,\triangledown\!\!\!\!\!\bigcirc\,x_2^t)=\bigwedge_{S\subseteq T}\left(\Big(\bigvee_{s\in S}x_1^s\,\triangledown\!\!\!\!\!\bigcirc\,0\Big)\vee\Big(\bigvee_{t\in T\setminus S}0\,\triangledown\!\!\!\!\!\bigcirc\,x_2^t\Big)\right).$$

These identities form a basis for calculating in the tensor product $L_1\otimes L_2$. This follows from Theorem 2 which we prefer to formulate with the complete embeddings $\epsilon_1\colon L_1\to L_1\otimes L_2$ and $\epsilon_2\colon L_2\to L_1\otimes L_2$ defined by

$$\epsilon_1(x_1):=x_1\,\triangle\!\!\!\!\!\bigcirc\,1=x_1\,\triangledown\!\!\!\!\!\bigcirc\,0\quad\text{and}\quad\epsilon_2(x_2):=1\,\triangle\!\!\!\!\!\bigcirc\,x_2=0\,\triangledown\!\!\!\!\!\bigcirc\,x_2$$

(notice that $x_1\,\triangle\!\!\!\!\!\bigcirc\,x_2=\epsilon_1(x_1)\wedge\epsilon_2(x_2)$ and $x_1\,\triangledown\!\!\!\!\!\bigcirc\,x_2=\epsilon_1(x_1)\vee\epsilon_2(x_2)$). ϵ_1 and ϵ_2 preserve 0 and 1, which, in general, we assume for complete homomorphisms in this paper.

THEOREM 2. *Let $\tilde{\epsilon}_1: L_1 \to L$ and $\tilde{\epsilon}_2: L_2 \to L$ be complete homomorphisms between complete lattices satisfying the distributive laws*

$$\bigvee_{t\in T} (\tilde{\epsilon}_1(x_1^t) \wedge \tilde{\epsilon}_2(x_2^t)) = \bigwedge_{S\subseteq T} \Big(\bigvee_{s\in S} \tilde{\epsilon}_1(x_1^s) \vee \bigvee_{t\in T\setminus S} \tilde{\epsilon}_2(x_2^t)\Big),$$

$$\bigwedge_{t\in T} (\tilde{\epsilon}_1(x_1^t) \vee \tilde{\epsilon}_2(x_2^t)) = \bigvee_{S\subseteq T} \Big(\bigwedge_{s\in S} \tilde{\epsilon}_1(x_1^s) \wedge \bigwedge_{t\in T\setminus S} \tilde{\epsilon}_2(x_2^t)\Big).$$

Then there exists a unique complete homomorphism $\alpha: L_1 \otimes L_2 \to L$ with $\tilde{\epsilon}_1 = \alpha \circ \epsilon_1$ and $\tilde{\epsilon}_2 = \alpha \circ \epsilon_2$.

Proof. First we show that the following is valid for all $X \subseteq L_1 \times L_2$:

$$\bigvee_{(x_1, x_2)\in X} (\tilde{\epsilon}_1(x_1) \wedge \tilde{\epsilon}_2(x_2)) = \bigwedge_{(y_1, y_2)\in X'} (\tilde{\epsilon}_1(y_1) \vee \tilde{\epsilon}_2(y_2)) \qquad (**)$$

By assumption,

$$\begin{aligned}
&\bigvee_{(x_1, x_2)\in X} (\tilde{\epsilon}_1(x_1) \wedge \tilde{\epsilon}_2(x_2)) \\
&\quad = \bigwedge_{S\subseteq X} \Big(\bigvee_{(x_1, x_2)\in S} \tilde{\epsilon}_1(x_1) \vee \bigvee_{(x_1, x_2)\in X\setminus S} \tilde{\epsilon}_2(x_2)\Big) \\
&\quad = \bigwedge_{S\subseteq X} \Big(\tilde{\epsilon}_1\Big(\bigvee_{(x_1, x_2)\in S} x_1\Big) \vee \tilde{\epsilon}_2\Big(\bigvee_{(x_1, x_2)\in X\setminus S} x_2\Big)\Big).
\end{aligned}$$

For

$$S \subseteq X, y_1^S := \bigvee_{(x_1, x_2)\in S} x_1 \quad \text{and} \quad y_1^S := \bigvee_{(x_1, x_2)\in X\setminus S} x_2,$$

we have $(x_1, x_2) \nabla (y_1^S, y_2^S)$ for all $(x_1, x_2) \in X$ and so $(y_1^S, y_2^S) \in X'$. If

$$(y_1, y_2) \in X' \quad \text{and} \quad S := \{(x_1, x_2) \in X \mid x_1 \leqslant y_1\},$$

then $\tilde{\epsilon}_1(y_1^S) \vee \tilde{\epsilon}_2(y_2^S) \leqslant \tilde{\epsilon}_1(y_1) \vee \tilde{\epsilon}_2(y_2)$. All together yield (**). Now we define α: $L_1 \otimes L_2 \to L$ by

$$\alpha(A, B) := \bigvee_{(x_1, x_2)\in A} (\tilde{\epsilon}_1(x_1) \wedge \tilde{\epsilon}_2(x_2)) = \bigwedge_{(y_1, y_2)\in B} (\tilde{\epsilon}_1(y_1) \wedge \tilde{\epsilon}_2(y_2)).$$

Using (**), we obtain

$$\begin{aligned}
&\bigvee_{t\in T} \alpha(A_t, B_t) \\
&\quad = \bigvee_{t\in T} \bigvee_{(x_1, x_2)\in A} (\tilde{\epsilon}_1(x_1) \wedge \tilde{\epsilon}_2(x_2)) \\
&\quad = \bigwedge_{(y_1, y_2)\in (\cup A_t)'} (\tilde{\epsilon}_1(y_1) \vee \tilde{\epsilon}_2(y_2)) = \alpha \bigvee_{t\in T} (A_t, B_t)
\end{aligned}$$

and dually

$$\bigwedge_{t\in T} \alpha(A_t, B_t) = \alpha \bigwedge_{t\in T} (A_t, B_t).$$

Thus α is a complete homomorphism. By definition,

$$\begin{aligned}&\alpha(\epsilon_1(x_1))\\ &\quad = \alpha([0,x_1] \times L_2 \cup L_1 \times \{0\}, [x_1, 1] \times L_2 \cup L_1 \times \{1\})\\ &\quad = \tilde{\epsilon}_1(x_1) \wedge \tilde{\epsilon}_2(1) = \tilde{\epsilon}_1(x_1);\end{aligned}$$

hence $\tilde{\epsilon}_1 = \alpha \circ \epsilon_1$ and analogously $\tilde{\epsilon}_2 = \alpha \circ \epsilon_2$. Since $\epsilon_1(L_1) \cup \epsilon_2(L_2)$ is a generating subset of the complete lattice $L_1 \otimes L_2$, α is unique. □

We conclude this section by listing some laws for the tensor product which can be easily deduced from Theorem 1; we use the notation $\mathbb{1}$ for a one-element lattice, $\mathbb{2}$ for a two-element lattice, and L^d for the dual of the lattice L:

$$\begin{aligned}&L_1 \otimes L_2 \simeq L_2 \otimes L_1; \quad L_1 \otimes (L_2 \otimes L_3) \simeq (L_1 \otimes L_2) \otimes L_3,\\ &\quad \mathbb{1} \otimes L \simeq \mathbb{1}, \quad \mathbb{2} \otimes L \simeq L,\\ &L_1 \otimes (L_2 \times L_3) \simeq (L_1 \otimes L_2) \times (L_1 \otimes L_3), \quad (L_1 \otimes L_2)^d \simeq L_1^d \otimes L_2^d.\end{aligned}$$

3. An Ideal Representation of Tensor Products

Since $L_1 \circledast L_2$ is supremum-dense in $L_1 \otimes L_2$, we obtain a set representation of $L_1 \otimes L_2$ by assigning to each of its elements w the subset

$$\sigma(w) := \{(x_1, x_2) \in L_1 \times L_2 \mid x_1 \circledast x_2 \leqslant w\}$$

of $L_1 \times L_2$. To use this representation as a construction method for tensor products, we have to characterize $\sigma(L_1 \otimes L_2)$. For that purpose we introduce the following definitions: If $\mathfrak{T}$ is an order filter of the power set $\mathfrak{P}(S)$ of the set S, then $\mathfrak{T}^{\#}$ is the order filter $\{S \backslash T \mid T \subseteq S \text{ and } T \notin \mathfrak{T}\}$. A subset J of $L_1 \times L_2$ is called a *G_κ-ideal* (cf. [11]) where κ is any cardinal number, if J satisfies the following conditions:

(i) $(0, 1) \in J$ and $(1, 0) \in J$.
(ii) $(x_1, x_2) \in J$ and $(y_1, y_2) \leqslant (x_1, x_2)$ always imply $(y_1, y_2) \in J$.
(iii) For each index set S with $|S| \leqslant \kappa$ and every order filter $\mathfrak{T}$ of $\mathfrak{P}(S)$, $(x_1^s, x_2^s) \in J$ $(s \in S)$ implies

$$\left(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_1^t, \bigwedge_{T\in\mathfrak{T}^{\#}} \bigvee_{t\in T} x_2^t\right) \in J.$$

A complete lattice L has *$\bigvee$-breadth* κ if κ is the smallest cardinal number such that for every family $(x_t)_{t\in T}$ of elements of L there exists a subset U of T with $|U| \leqslant \kappa$ and $\bigvee_{t\in T} x_t = \bigvee_{u\in U} x_u$.

THEOREM 3. *σ is an isomorphism from $L_1 \otimes L_2$ onto the lattice of all G_κ-ideals of $L_1 \times L_2$ where κ is the $\bigvee$-breadth of $L_1 \otimes L_2$.*

Proof. Clearly, $\sigma(w)$ satisfies condition (i) and (ii). Let $(x_1^s, x_2^s) \in \sigma(w)$ for all $s \in S$ and let be an order filter of (S). Then

$$\begin{aligned}
&\Big(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_1^t\Big) \circledast \Big(\bigwedge_{T\in\mathfrak{T}^\#} \bigvee_{t\in T} x_2^t\Big) \\
&= \Big(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_1^t \circledast 0\Big) \wedge \Big(\bigwedge_{T\in\mathfrak{T}^\#} \bigvee_{t\in T} 0 \circledast x_2^t\Big) \\
&\leqslant \bigwedge_{T\subseteq S} \Big(\Big(\bigvee_{t\in T} x_1^t \circledast 0\Big) \vee \Big(\bigvee_{s\in S\setminus T} 0 \circledast x_2^s\Big)\Big) \\
&= \bigvee_{s\in S} (x_1^s \circledast 0 \wedge 0 \circledast x_2^s) = \bigvee_{s\in S} x_1^s \circledast x_2^s \leqslant w;
\end{aligned}$$

hence

$$\Big(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_1^t, \bigwedge_{T\in\mathfrak{T}^\#} \bigvee_{t\in T} x_2^t\Big) \in \sigma(w).$$

This shows that $\sigma(w)$ is a G_λ-ideal of $L_1 \times L_2$ for each cardinal number λ, in particular for $\lambda = \kappa$. Conversely, let J be a G_κ-ideal of $L_1 \times L_2$ and let

$$w := \bigvee_{(x_1, x_2)\in J} x_1 \circledast x_2 .$$

Then there exists a family of elements (x_1^s, x_2^s) $(s \in S)$ in J with $|S| \leqslant \kappa$ and $w = \bigvee_{s\in S} x_1^s \circledast x_2^s$. Now, let $(y_1, y_2) \in \sigma(w)$. It follows:

$$\begin{aligned}
y_1 \circledast y_2 \leqslant w = \bigvee_{s\in S} x_1^s \circledast x_2^s &= \bigwedge_{T\subseteq S} \Big(\Big(\bigvee_{t\in T} x_1^t \circledast 0\Big) \vee \Big(\bigvee_{s\in S\setminus T} 0 \circledast x_2^s\Big)\Big) \\
&= \bigwedge_{T\subseteq S} \Big(\Big(\bigvee_{t\in T} x_1^t\Big) \circledast 1 \vee 1 \circledast \Big(\bigvee_{s\in S\setminus T} x_2^s\Big)\Big) \\
&= \bigwedge_{T\subseteq S} \Big(\bigvee_{t\in T} x_1^t\Big) \circledast \Big(\bigvee_{s\in S\setminus T} x_2^s\Big) .
\end{aligned}$$

Therefore, for $T \subseteq S$, $[y_1, 1] \times L_2 \cup L_1 \times [y_2, 1]$ contains $[\bigvee_{t\in T} x_1^t, 1] \times [\bigvee_{s\in S\setminus T} x_2^s, 1]$ and so

$$y_1 \leqslant \bigvee_{t\in T} x_1^t \quad \text{or} \quad y_2 \leqslant \bigvee_{s\in S\setminus T} x_2^s .$$

For the order filter $\mathfrak{T} := \{T \subseteq S \mid y_1 \leqslant \bigvee_{t\in T} x_1^t\}$ of $\mathfrak{P}(S)$, we obtain

$$(y_1, y_2) \leqslant \Big(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_1^t, \bigwedge_{T\in\mathfrak{T}^\#} \bigvee_{t\in T} x_2^t\Big);$$

hence $(y_1, y_2) \in J$. This proves $J = \sigma(w)$. Because

$$v = \bigvee_{(x_1, x_2)\in\sigma(v)} x_1 \circledast x_2$$

for all $v \in L_1 \otimes L_2$, we conclude that σ is a bijection from $L_1 \otimes L_2$ onto the lattice of all G_K-ideals of $L_1 \times L_2$. As σ and σ^{-1} are order-preserving, σ is an isomorphism. □

To each G_K-ideal J of $L_1 \times L_2$ we may assign a mapping $\tau_J: L_1 \to L_2$ defined by $\tau_J x := \bigvee\{y \in L_2 \mid (x, y) \in J\}$ $(x \in L_1)$. Since J is already a *G-ideal* of $L_1 \times L_2$ in the sense of Z. Shmuely [11], i.e., J satisfies, besides (i) and (ii), the following condition:

(iv) $(x_1^s, x_2^s) \in J$ $(s \in S)$ implies

$$\left(\bigvee_{s\in S} x_1^s, \bigwedge_{s\in S} x_2^s\right) \in J \quad\text{and}\quad \left(\bigwedge_{s\in S} x_1^s, \bigvee_{s\in S} x_2^s\right) \in J$$

(choose $\mathfrak{T}$ or $\mathfrak{T}^\#$ equal to $\{S\}$), we obtain $(x, \tau_J x) \in J$. From this, it easily follows that $\tau_J \bigvee_{t\in T} x_t = \bigwedge_{t\in T} \tau_J x_t$ (cf. [11, Lemma 3.3]). For $a \in L_1$, let $\mathfrak{T} := \{X \subseteq L_1 \mid a \leqslant \bigvee X\}$. Then $\mathfrak{T}^\# = \{L_1 \backslash X \mid a \nleqslant \bigvee X\}$. As $a = \bigwedge_{X\in\mathfrak{T}} \bigvee X$, we have $(a, \bigwedge_{Y\in\mathfrak{T}^\#} \bigvee_{y\in Y} \tau_J y) \in J$ and so

$$\tau_J a \geqslant \bigwedge_{Y\in\mathfrak{T}^\#} \bigvee_{y\in Y} \tau_J y = \bigwedge_{x \ngeqslant a} \bigvee_{y \nleqslant x} \tau_J y \geqslant \tau_J a$$

hence $\tau_J a = \bigwedge_{x\ngeqslant a} \bigvee_{y\nleqslant x} \tau_J y$. Now let $\tau: L_1 \to L_2$ be a map satisfying

$$\tau \bigvee_{t\in T} x_t = \bigwedge_{t\in T} \tau x_t \quad\text{and}\quad \tau a = \bigwedge_{x\ngeqslant a} \bigvee_{y\nleqslant x} \tau y$$

for all $x_t \in L_1$ $(t \in T)$ and $a \in L_1$; such mappings are called *tight Galois maps.* We define $J_\tau := \{(x, y) \in L_1 \times L_2 \mid y \leqslant \tau x\}$. Clearly, $(1, 0) \in J_\tau$ and, because of $\tau 0 = \tau \bigvee \emptyset = \bigwedge \tau\emptyset = \bigwedge \emptyset = 1$, also $(0, 1) \in J_\tau$, i.e., J_τ satisfies (i). If $(x, y) \in J_\tau$ and $(u, v) \leqslant (x, y)$, then $v \leqslant y \leqslant \tau x \leqslant \tau u$ and so $(u, v) \in J_\tau$; hence J_τ satisfies (ii). Let $(x_s, y_s) \in J_\tau$ $(s \in S)$ and let $\mathfrak{T}$ be an order filter on $\mathfrak{P}(S)$; we set $a := \bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_t$. For $x \ngeqslant a$ in L_1, it follows that $S_x := \{s \in S \mid x_s \leqslant x\} \notin \mathfrak{T}$ and so $S \backslash S_x \in \mathfrak{T}^\#$. Therefore,

$$\bigwedge_{T\in\mathfrak{T}^\#} \bigvee_{t\in T} y_t \leqslant \bigwedge_{x\ngeqslant a} \bigvee_{z\nleqslant x} \tau z = \tau a.$$

This yields

$$\left(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_t, \bigwedge_{T\in\mathfrak{T}^\#} \bigvee_{t\in T} y_t\right) \in J_\tau,$$

i.e., J_τ satisfies (iii). Thus, J_τ is a G_K-ideal. Obviously, $\tau = \tau_{J_\tau}$ and $J = J_{\tau_J}$; furthermore, $\tau_1 \leqslant \tau_2$ for tight Galois maps is equivalent to $J_{\tau_1} \subseteq J_{\tau_2}$. In this way we obtain from Theorem 3 the following theorem of G. N. Raney [10].

THEOREM 4. *The mapping* $w \mapsto \tau_{\sigma(w)}$ *is an isomorphism from* $L_1 \otimes L_2$ *onto the lattice of all tight Galois maps from* L_1 *into* L_2.

In [11], Z. Shmuely has shown that the lattice of all G-ideals of $L_1 \times L_2$ is isomorphic to the lattice of all Galois connections between L_1 and L_2. This lattice can be considered as the tensor product of L_1 and L_2 in the category of complete lattices and complete join morphisms (cf. [1, 7, 8]). We like to denote this tensor product by $L_1 \underline{\otimes} L_2$. In connection with Theorem 4, Raney deduced the following corollary in [10].

COROLLARY 5. *If L_1 or L_2 is completely distributive then $L_1 \otimes L_2 \simeq L_1 \underline{\otimes} L_2$.*

This corollary is an immediate consequence of Theorem 3 and the fact that every G-ideal J of $L_1 \times L_2$ is already a G_K-ideal if L_1 is completely distributive. To show this fact, let $(x_1^s, x_2^s) \in J$ $(s \in S)$ and let $\mathfrak{T}$ be an order filter of $\mathfrak{P}(S)$. Then $(\bigwedge_{T\in\mathfrak{T}} x_1^{\delta T}, \bigvee_{T\in\mathfrak{T}} x_2^{\delta T}) \in J$ for each element δ of the direct product $\Pi\mathfrak{T}$ and so

$$\left(\bigvee_{\delta\in\Pi\mathfrak{T}} \bigwedge_{T\in\mathfrak{T}} x_1^{\delta T}, \bigwedge_{\delta\in\Pi\mathfrak{T}} \bigvee_{T\in\mathfrak{T}} x_2^{\delta T}\right) \in J.$$

As $\{\delta T \mid T\in\mathfrak{T}\} \in \mathfrak{T}^{\#}$, it follows

$$\left(\bigvee_{\delta\in\Pi\mathfrak{T}} \bigwedge_{T\in\mathfrak{T}} x_1^{\delta T}, \bigwedge_{\delta\in\Pi\mathfrak{T}} \bigvee_{T\in\mathfrak{T}} x_2^{\delta T}\right) \geqslant \left(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_1^{t}, \bigwedge_{T\in\mathfrak{T}^{\#}} \bigvee_{t\in T} x_2^{t}\right),$$

hence

$$\left(\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} x_1^{t}, \bigwedge_{T\in\mathfrak{T}^{\#}} \bigvee_{t\in T} x_2^{t}\right) \in J.$$

Thus, J is a G_K-ideal of $L_1 \times L_2$. For applying Corollary 5, the following characterization of completely distributive concept lattices might be helpful (cf. [10], Theorem 5).

PROPOSITION 6. *The concept lattice $\mathfrak{B}(G, M, I)$ is completely distributive if and only if for $g \in G$ and $m \in M$ with $(g, m) \notin I$ there exist $h \in G$ and $n \in M$ such that $(g, n) \notin I$, $(h, m) \notin I$, and $h \in \{k\}''$ for all $k \in G \backslash \{n\}'$.*

Proof. In each concept lattice we have

$$\bigwedge_{T\in\mathfrak{T}} \bigvee_{t\in T} (A_t, B_t) \geqslant \bigvee_{\delta\in\Pi\mathfrak{T}} \bigwedge_{T\in\mathfrak{T}} (A_{\delta T}, B_{\delta T}).$$

Suppose that the two sides are not equal. Then there exist $g \in \bigcap_{T\in\mathfrak{T}} (\bigcap_{t\in T} B_t)'$ and $m \in \bigcap_{\delta\in\Pi\mathfrak{T}} (\bigcap_{T\in\mathfrak{T}} A_{\delta T})'$ with $(g, m) \notin I$. Let us assume the existence of $h \in G$ and $n \in M$ such that $(g, n) \notin I$, $(h, m) \notin I$, and $h \in \{k\}''$ for all $k \in G\backslash\{n\}'$. Then $(g, n) \notin I$ implies $n \notin \bigcap_{t\in T} B_t$ for all $T\in\mathfrak{T}$ and so $n \notin B_{\delta T}$ $(T\in\mathfrak{T})$ for some $\delta \in \Pi\mathfrak{T}$; hence $h \in A_{\delta T}$ for all $T\in\mathfrak{T}$. This together with $m \in (\bigcap_{T\in\mathfrak{T}} A_{\delta T})'$ contradicts $(h, m) \notin I$. Thus, the 'if'-part of the assertion is proved. Now, let us assume that $\mathfrak{T}(G, M, I)$ is completely distributive. We choose $g \in G$ and $m \in M$ with $(g, m) \notin I$. By distributivity,

$$(\{g\}'', \{g\}') = \bigvee_{(g, n)\notin I} \bigwedge_{(k, n)\notin I} (\{k\}'', \{k\}')$$

whence there is an $n \in M$ with $(g, n) \notin I$ and $m \notin (\bigcap_{(k, h)\notin I} \{k\}'')'$; furthermore, there is an $h \in \bigcap_{(k, n)\notin I} \{k\}''$ with $(h, m) \notin I$. This shows the 'only if'-part of the assertion. □

Now, let L_1 and L_2 be completely distributive and let $(x_1, x_2), (y_1, y_2) \in L_1 \times L_2$ such that $(x_1, x_2) \nabla (y_1, y_2)$ is not valid which is equivalent to $x_1 \not\leqslant y_1$ and $x_2 \not\leqslant y_2$. By Proposition 6, there are $u_i, v_i \in L_i$ with $x_i \not\leqslant v_i, y_i \not\geqslant u_i$, and $L_i = [0, v_i] \cup [u_i, 1]$ $(i = 1, 2)$. It follows that $(x_1, x_2) \nabla (v_1, v_2)$ and $(u_1, u_2) \nabla (y_1, y_2)$ are not valid; furthermore, $L_1 \otimes L_2 = [0, v_1 \oslash v_2] \cup [u_1 \oslash u_2, 1]$. Therefore, by Proposition 6, $L_1 \otimes L_2$ is completely distributive too. This together with Theorem 2 yields the following characterization of tensor products of completely distributive lattices (cf. [3, 12]).

THEOREM 7. *For completely distributive complete lattices L_1 and L_2, the tensor product $L_1 \otimes L_2$ is a free product of L_1 and L_2 in the category of completely distributive complete lattices with complete homomorphisms preserving* 0 *and* 1.

4. Subdirect Decompositions of Tensor Products

In this section we restrict our considerations to doubly founded complete lattices. We call a complete lattice L *doubly founded* if, for every pair of elements $x < y$ in L, $(L \setminus [0, x]) \cap [0, y]$ contains a minimal element and $(L \setminus [y, 1]) \cap [x, 1]$ contains a maximal element. Such minimal and maximal elements are just the $\vee$-irreducible and $\wedge$-irreducible elements of L, respectively, and every element of L is the supremum of $\vee$-irreducible elements and the infimum of $\wedge$-irreducible elements in L. If $\mathbf{J}(L)$ denotes the set of all $\vee$-irreducible elements of L and $\mathbf{M}(L)$ the set of all $\wedge$-irreducible elements of L, then $L \simeq \mathfrak{B}(\mathbf{J}(L), \mathbf{M}(L), \leqslant)$ by the basic theorem of concept lattices in [14], p. 449.

Let us recall that a context (G, M, I) is *reduced* if $\gamma g := (\{g\}'', \{g\}')$ $(g \in G)$ defines a bijection $\gamma: G \to \mathbf{J}(\mathfrak{B}(G, M, I))$ and if $\mu m := (\{m\}', \{m\}'')$ $(m \in M)$ defines a bijection $\mu: M \to \mathbf{M}(\mathfrak{B}(G, M, I))$. We call a context (G, M, I) *doubly founded* if, for each $(g, m) \in G \times M \setminus I$, there exists an $h \in G$ such that γh is minimal in $\{\gamma k \mid k \in \{g\}''$ and $(k, m) \notin I\}$ (notation: $m \searrow h$) and there exists an $n \in M$ such that μn is maximal in $\{\mu p \mid p \in \{m\}''$ and $(g, p) \notin I\}$ (notation: $g \nearrow n$). Clearly, $\mathfrak{B}(G, M, I)$ is doubly founded if (G, M, I) is doubly founded and, for each doubly founded complete lattice L, the context $(\mathbf{J}(L), \mathbf{M}(L), \leqslant)$ is reduced and doubly founded. For this section we make the *general assumption* that all contexts (G, M, I) are reduced and doubly founded.

In [15], it is shown how to study the complete congruence relations of the concept lattice $\mathfrak{B}(G, M, I)$ within the *digraph of weak perspectivities* $(G \dot\cup M, \nearrow \dot\cup \searrow)$. A subset C of $G \dot\cup M$ is said to be *closed* if $g \in C$ and $g \nearrow m$ imply $m \in C$ and if $m \in C$ and $m \searrow g$ imply $g \in C$. Obviously, the closed subsets of the digraph $(G \dot\cup M, \nearrow \dot\cup \searrow)$ form a complete sublattice of the complete lattice of all subsets of $G \dot\cup M$. To each complete congruence relation θ of $\mathfrak{B}(G, M, I)$ we assign a subset $G(\theta) \dot\cup M(\theta)$ of $G \dot\cup M$ defined by

$$G(\theta) := \{g \in G \mid \gamma g \text{ is the smallest element of a } \theta\text{-class}\},$$
$$M(\theta) := \{m \in M \mid \mu m \text{ is the greatest element of a } \theta\text{-class}\}.$$

A slight generalization of Theorem 6 in [15] (proved by the same arguments) is the following theorem.

THEOREM 8. *$\theta \mapsto G(\theta) \dot\cup M(\theta)$ describes an antiisomorphism from the lattice of all complete congruence relations of $\mathfrak{B}(G, M, I)$ onto the lattice of all closed subsets of $(G \dot\cup M, \nearrow \dot\cup \swarrow)$.*

COROLLARY 9. *For a doubly founded complete lattice L the lattice $\mathfrak{C}(L)$ of all complete congruence relations of L is completely distributive.*

Under the antiisomorphism of Theorem 8, the $\wedge$-irreducible elements of $\mathfrak{C}(\mathfrak{B}(G, M, I))$ correspond to the $\cup$-irreducible closed subsets of $(G \dot\cup M, \nearrow \dot\cup \swarrow)$ which are exactly the smallest closed subsets $\langle g \rangle$ containing a given object g. This yields, as in [15], the following subdirect product representation of $\mathfrak{B}(G, M, I)$.

THEOREM 10. $(A, B) \mapsto (A \cap \langle g\rangle,\ B \cap \langle g\rangle)_{g \in G}$ describes an isomorphism from *$\mathfrak{B}(G, M, I)$ onto a subdirect product of the completely subdirectly irreducible concept lattices $\mathfrak{B}(\langle g\rangle \cap G, \langle g\rangle \cap M, I \cap \langle g\rangle^2)$ $(g \in G)$.*

Now, we prepare to analyse such subdirect product representations for tensor products of doubly founded complete lattices. Let us assume that $\mathbb{K}_1 := (G_1, M_1, I_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2)$ are doubly founded reduced contexts; furthermore, let $L_1 := \mathfrak{B}(\mathbb{K}_1)$ and $L_2 := \mathfrak{B}(\mathbb{K}_2)$.

LEMMA 11. $\mathbf{J}(L_1 \otimes L_2) = \mathbf{J}(L_1) \oslash \mathbf{J}(L_2)$ *and* $\mathbf{M}(L_1 \otimes L_2) = \mathbf{M}(L_1) \oslash \mathbf{M}(L_2)$.

Proof. $x_1 \oslash x_2 = \bigvee_{t \in T} y_1^t \oslash y_2^t$ is equivalent to

$$[x_1, 1] \times L_2 \cup L_1 \times [x_2, 1] = \bigcap_{t \in T} ([y_1^t, 1] \times L_2 \cup L_1 \times [y_2^t, 1])$$

which is equivalent to $x_1 = \bigvee_{t \in T} y_1^t$ and $x_2 = \bigvee_{t \in T} y_2^t$. Since $L_1 \oslash L_2$ is supremum-dense in $L_1 \otimes L_2$, the first assertion immediately follows (and dually the second assertion). □

LEMMA 12. *For $g_1 \in G_1, g_2 \in G_2, m_1 \in M_1$, and $m_2 \in M_2$, the following holds:*

$$(g_1, g_2) \nearrow (m_1, m_2) \Leftrightarrow g_1 \nearrow m_1 \quad \text{and} \quad g_2 \nearrow m_2,$$
$$(m_1, m_2) \swarrow (g_1, g_2) \Leftrightarrow m_1 \swarrow g_1 \quad \text{and} \quad m_2 \swarrow g_2.$$

Proof. The assertions are direct consequences of the fact that $(g_1, g_2) \nabla (m_1, m_2)$ is not valid in $\mathbb{K}_1 \times \mathbb{K}_2$ if and only if $(g_1, m_1) \notin I_1$ and $(g_2, m_2) \notin I_2$. □

LEMMA 13. *$\mathbb{K}_1 \times \mathbb{K}_2$ is reduced and doubly founded.*

Proof. Because of

$$\{(g_1, g_2)\}' = \{g_1\}' \times M_2 \cup M_1 \times \{g_2\}' \quad \text{and} \quad \{(m_1, m_2)\}' = \{m_1\}' \times G_2 \cup G_1 \times \{m_2\}',$$

Lemma 11 yields that $\mathbb{K}_1 \times \mathbb{K}_2$ is reduced. From Lemma 12 we obtain that $\mathbb{K}_1 \times \mathbb{K}_2$ is doubly founded. □

LEMMA 14.

$$\langle(g_1, g_2)\rangle \cap G_1 \times G_2 = (\langle g_1\rangle \cap G_1) \times (\langle g_2\rangle \cap G_2)$$

and

$$\langle(g_1, g_2)\rangle \cap M_1 \times M_2 = (\langle g_1\rangle \cap M_1) \times (\langle g_2\rangle \cap M_2)$$

for all $(g_1, g_2) \in G_1 \times G_2$.

Proof. By Lemma 12, $\langle g_1\rangle \times \langle g_2\rangle$ is closed in $(G_1 \times G_2 \cup M_1 \times M_2, \nearrow \dot\cup \searrow)$; hence $\langle(g_1, g_2)\rangle \cap G_1 \times G_2 \subseteq (\langle g_1\rangle \cap G_1) \times (\langle g_2\rangle \cap G_2)$. For the proof of the dual inclusion we use the general fact that for each object g in a doubly founded reduced context there is always an attribute m such that $g \nearrow m$ and $m \searrow g$. Let $g_3 \in \langle g_1\rangle \cap G_1$ and $g_4 \in \langle g_2\rangle \cap G_2$; furthermore, let $m_2 \in M_2$ and $m_3 \in M_1$ such that $g_2 \nearrow m_2$, $m_2 \searrow g_2$, $g_3 \nearrow m_3$, and $m_3 \searrow g_3$. There exist paths

$$g_1 \nearrow n_1 \searrow h_1 \nearrow n_2 \searrow \ldots \searrow h_{r-1} \nearrow n_r \searrow g_3 \quad \text{and} \quad g_2 \nearrow p_1 \searrow k_1 \nearrow p_2 \searrow \ldots \searrow k_{s-1} \nearrow p_s \searrow g_4$$

and so, by Lemma 12, we have a path

$$\begin{aligned}&(g_1, g_2) \nearrow (n_1, m_2) \searrow (h_1, g_2) \nearrow (n_2, m_2) \searrow \ldots\\ &\searrow (h_{r-1}, g_2) \nearrow (n_r, m_2) \searrow (g_3, g_2) \nearrow (m_3, p_1) \searrow (g_3, k_1) \nearrow (m_3, p_2) \searrow \ldots\\ &\searrow (g_3, k_{s-1}) \nearrow (m_3, p_s) \searrow (g_3, g_4);\end{aligned}$$

hence $(g_3, g_4) \in \langle(g_1, g_2)\rangle \cap G_1 \times G_2$. This proves the first equality. The second equality dually follows using $\langle(g_1, g_2)\rangle = \langle(m_1, m_2)\rangle$ for some $(m_1, m_2) \in M_1 \times M_2$. □

COROLLARY 15. *$L_1 \otimes L_2$ is completely subdirectly irreducible if and only if L_1 and L_2 are completely subdirectly irreducible.*

As $\mathbb{K}_1 \times \mathbb{K}_2$ is again reduced and doubly founded by Lemma 13, $\mathfrak{B}(\mathbb{K}_1 \times \mathbb{K}_2)$ is a subdirect product of the completely subdirectly irreducible concept lattices $\mathfrak{B}(\langle(g_1, g_2)\rangle \cap G_1 \times G_2, \langle(g_1, g_2)\rangle \cap M_1 \times M_2, \nabla \cap \langle(g_1, g_2)\rangle^2)$ $((g_1, g_2) \in G_1 \times G_2)$ by Theorem 10. This subdirect product representation can be determined by the analogous subdirect product representations of $\mathfrak{B}(\mathbb{K}_1)$ and $\mathfrak{B}(\mathbb{K}_2)$ using the following theorem which is an immediate consequence of Lemma 14 and Theorem 1.

THEOREM 16.

$$\begin{aligned}&\mathfrak{B}(\langle g_1, g_2\rangle \cap G_1 \times G_2, \langle(g_1, g_2)\rangle \cap M_1 \times M_2, \nabla \cap \langle(g_1, g_2)\rangle^2)\\ &\quad \simeq \mathfrak{B}(\langle g_1\rangle \cap G_1, \langle g_1\rangle \cap M_1, I_1 \cap \langle g_1\rangle^2) \otimes \mathfrak{B}(\langle g_2\rangle \cap G_2, \langle g_2\rangle \cap M_2, I_2 \cap \langle g_2\rangle^2)\end{aligned}$$

for all $(g_1, g_2) \in G_1 \times G_2$.

COROLLARY 17. *The completely subdirectly irreducible factors of $L_1 \otimes L_2$ are up to isomorphism the tensor products $S_1 \otimes S_2$ where S_1 and S_2 are the completely subdirectly irreducible factors of L_1 and L_2, respectively.*

For the study of arbitrary subdirect product representations of $L_1 \otimes L_2$, we clarify how the lattice $\mathfrak{C}(L_1 \otimes L_2)$ of all complete congruence relations of $L_1 \otimes L_2$ can be determined by $\mathfrak{C}(L_1)$ and $\mathfrak{C}(L_2)$ (cf. [5]).

THEOREM 18. $\mathfrak{C}(L_1 \otimes L_2) \simeq \mathfrak{C}(L_1) \otimes \mathfrak{C}(L_2)$

Proof. By Theorem 10 and Lemma 14, the ordered set $\mathbf{M}(\mathfrak{C}(L_1 \otimes L_2))$ is isomorphic to $\mathbf{M}(\mathfrak{C}(L_1)) \times \mathbf{M}(\mathfrak{C}(L_2))$ which is isomorphic to $\mathbf{M}(\mathfrak{C}(L_1) \otimes \mathfrak{C}(L_2))$. Now, we use that $\mathfrak{C}(L_1 \otimes L_2)$ and $\mathfrak{C}(L_1) \otimes \mathfrak{C}(L_2)$ are completely distributive by Lemma 13, Corollary 9, and Theorem 7, and that every element of these lattices is the meet of $\wedge$-irreducible elements. Then we obtain the asserted isomorphism from the proof of Theorem 2 in [9].

5. Measurement and Tensor Products

In [14], a conceptual approach to the area of measurement is introduced by defining a *scale* to be a context $\mathbb{S} := (G_{\mathbb{S}}, M_{\mathbb{S}}, I_{\mathbb{S}})$ which can be said to be well known. An *$\mathbb{S}$-measure* of a context (G, M, I) is a mapping σ from G into $G_{\mathbb{S}}$ such that $\sigma^{-1}A$ is an extent of (G, M, I) for every extent A of $\mathbb{S}$; σ is called *full* if σ^{-1} induces an isomorphism from $\mathfrak{B}(\sigma G, M_{\mathbb{S}}, I_{\mathbb{S}} \cap \sigma G \times M_{\mathbb{S}})$ onto $\mathfrak{B}(G, M, I)$. Full $\mathbb{S}$-measures may be analysed lattice-theoretically which is the substance of the following proposition [14], p. 461.

PROPOSITION 19. *Let $\mathbb{S} := (G_{\mathbb{S}}, M_{\mathbb{S}}, I_{\mathbb{S}})$ be a context such that $\{g\}' = \{h\}'$ implies $g = h$ for all $g, h \in G_{\mathbb{S}}$. For a full $\mathbb{S}$-measure σ of a context (G, M, I), let $\bar{\sigma}(A, B) := ((\sigma A)'', (\sigma A)')$ for all $(A, B) \in \mathfrak{B}(G, M, I)$. Then $\sigma \mapsto \bar{\sigma}$ describes a bijection from the set of all full $\mathbb{S}$-measures of (G, M, I) onto the set of all embeddings ι of $\mathfrak{B}(G, M, I)$ into $\mathfrak{B}(\mathbb{S})$ with the property that for each $g \in G$ there is an $h \in G_{\mathbb{S}}$ with $\iota(\{g\}'', \{g\}') = (\{h\}'', \{h\}')$ ($\iota 0 \neq 0$ is allowed).*

It seems natural to call the smallest number n such that there exists a full $\mathbb{S}^n$-measure from (G, M, I) into the nth power $\mathbb{S} \times \mathbb{S} \times \cdots \times \mathbb{S}$, the *$\mathbb{S}$-dimension* of (G, M, I). In this paper we only discuss the $\mathbb{S}$-dimension for the case that $\mathbb{S}$ is a *Guttman scale*, i.e., $\mathfrak{B}(\mathbb{S})$ is a chain (cf. [4], chap. 5). Let us define the *Guttman dimension* of a context to be its smallest $\mathbb{S}$-dimension where $\mathbb{S}$ is a Guttman scale. An essential difference from order measurement as it is discussed in [14], lies in the fact that the existence of a full $\mathbb{S}^n$-measure may have important consequences for the measured context.

THEOREM 20. *Let $\mathbb{S} := (C, C, <)$ for a finite chain C and let σ be a full $\mathbb{S}^n$-measure of the context (G, M, I). Then (G, M, I) is isomorphic to $(P, P, \not\leq)$ for some finite ordered set P and $\mathfrak{B}(G, M, I)$ is isomorphic to the finite distributive lattice $\mathbb{2}^P$ of all order-preserving maps from P into $\mathbb{2}$.*

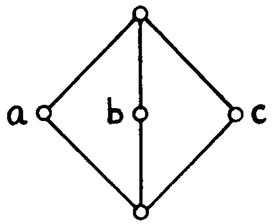
Proof. $\mathfrak{B}(\mathbb{S}^n)$ is a finite distributive lattice by Theorem 7 and $\mathbf{J}(\mathfrak{B}(\mathbb{S}^n)) = C \oslash C \oslash \cdots \oslash C$ (n times) by Lemma 11. If $\sigma g = (c_1, \ldots, c_n)$ then $\bar{\sigma}(\{g'', \{g\}') = c_1 \oslash \cdots \oslash c_n \in \mathbf{J}(\mathfrak{B}(\mathbb{S}^n))$. By Proposition 19, it follows that $\mathfrak{B}(G, M, I)$ is a finite distributive lattice. If we define $P := \mathbf{J}(\mathfrak{B}(G, M, I))^d$, we obtain the assertions from well known results about finite distributive lattices. As $\mathbf{J}(\mathfrak{B}(\mathbb{S}^n)) \simeq C^n$, we have the following corollary. □

COROLLARY 21. *For a finite-ordered set P, the Guttman dimension of* $(P, P, \not\leq)$ *equals the order dimension of P.*

For the general study of full $\mathbb{S}^n$-measures, lattice-theoretical characterizations of the tensor products $\mathfrak{B}(\mathbb{S}) \otimes \cdots \otimes \mathfrak{B}(\mathbb{S})$ have to be worked out. Here we only show what the Hasse diagram looks like for the concept lattice of the square of the nominal scale with three values.

	a	b	c
a	x		
b		x	
c			x

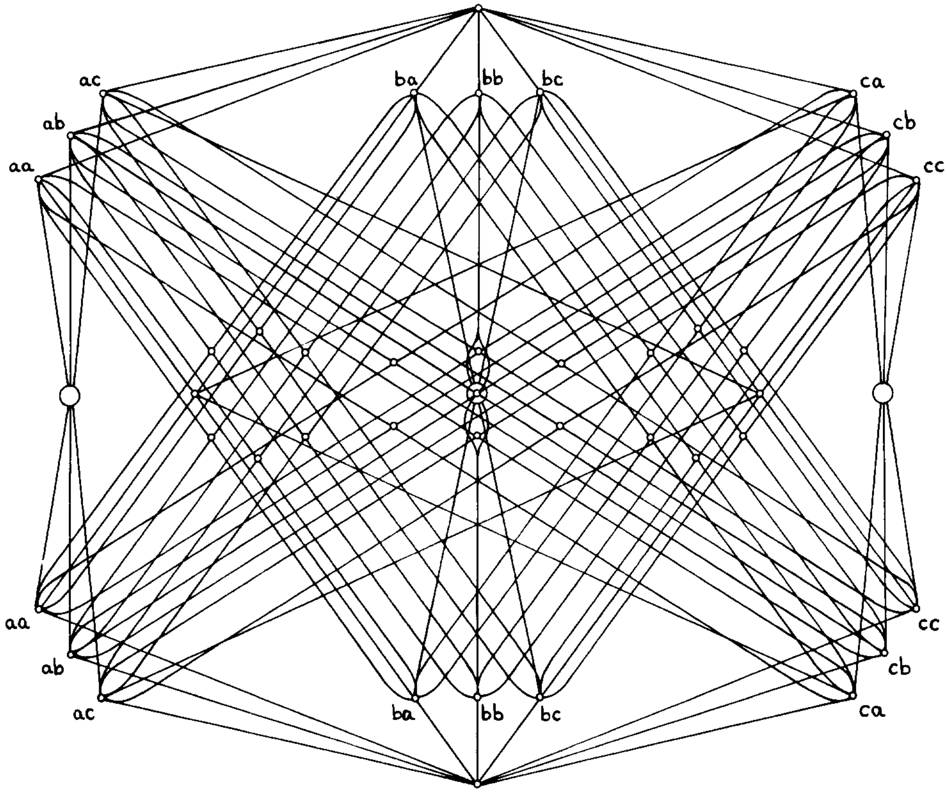
The nominal scale $\mathbb{N}_3$ with 3 values



$\mathfrak{B}(\mathbb{N}_3)$

	aa	ab	ac	ba	bb	bc	ca	cb	cc
aa	x	x	x	x			x		
ab	x	x	x		x			x	
ac	x	x	x			x			x
ba	x			x	x	x	x		
bb		x		x	x	x		x	
bc			x	x	x	x			x
ca	x			x			x	x	x
cb		x			x		x	x	x
cc			x			x	x	x	x

$\mathbb{N}_3 \times \mathbb{N}_3$



$\mathfrak{B}(\mathbb{N}_3 \times \mathbb{N}_3) \simeq \mathfrak{B}(\mathbb{N}_3) \otimes \mathfrak{B}(\mathbb{N}_3)$

References

1. B. Banaschewski and E. Nelson (1976) Tensor products and bimorphisms, *Canad. Math. Bull.* 19, 385–402.
2. H.-J. Bandelt (1980) The tensor product of continuous lattices, *Math. Z.* **172**, 89–96.
3. H.-J. Bandelt (1983) Coproducts of bounded (α, β)-distributive lattices, *Alg. Universalis* 17, 92–100.
4. M. Barbut and B. Monjardet (1970) *Ordre et classification, algèbre et combinatoire*, II, Hachette, Paris.
5. G. Gratzer, H. Lakser, and R. Quackenbush (1981) The structure of tensor products of semilattices with zero, *Trans. Amer. Math. Soc.* **267**, 503–515.
6. G. Kalmbach (1976) Extension of topological homology theories to partially ordered sets, *J. reine angew. Math.* **280**, 134–156.
7. D. G. Mowat (1968) A Galois problem for mappings, PhD Thesis, Univ. of Waterloo.
8. E. Nelson (1976) Galois connections as left adjoint maps, *Comment. Math. Univ. Carolinae* 17, 523–541.
9. G. N. Raney (1952) Completely distributive complete lattices, *Proc. Amer. Math. Soc.* **3**, 677–680.
10. G. N. Raney (1960) Tight Galois connections and complete distributivity, *Trans. Amer. Math. Soc.* 97, 418–426.
11. Z. Shmuely (1974) The structure of Galois connections, *Pac. J. Math.* **54**, 209–225.
12. Z. Shmuely (1979) The tensor product of distributive lattices, *Alg. Universalis* **9**, 281–296.

13. A. G. Waterman (1966) Colloquium lecture at McMaster University.
14. R. Wille (1982) Restructuring lattice theory: an approach based on hierarchies of concepts, in: *Ordered Sets* (ed. I. Rival), Reidel, Dordrecht, Boston, pp. 445–470.
15. R. Wille (1983) Subdirect decomposition of concept lattices, *Alg. Universalis* **17**, 275–287.