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Explicit Matchings in the Middle Levels of the Boolean Lattice

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Abstract. New classes of explicit matchings for the bipartite graph $\mathscr{B}(k)$ consisting of the middle two levels of the Boolean lattice on $2k+1$ elements are constructed and counted. This research is part of an ongoing effort to show that $\mathscr{B}(k)$ is Hamiltonian.

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0. Introduction

Let $\mathscr{B}(k)$ denote the subset of the Boolean lattice $2^{[2k+1]}$ consisting of all elements in levels k and $k+1$, considered as a bipartite graph. In this article we give explicit constructions for a large number of matchings in $\mathscr{B}(k)$. While we believe our results to be interesting for their own sake, we were motivated by the following well known problem: Is $\mathscr{B}(k)$ Hamiltonian for all $k \geqslant 1$? The origins of this problem are unclear; Erdos attributed it to Trotter, Trotter attributed it to Dejter, and Dejter attributed it to Erdos. Dejter now believes the conjecture was first stated in Havel [6]. Dejter and his students [1, 2] have shown this to be the case for $k \leqslant 9$. Any Hamiltonian cycle in a graph of even order is the union of two matchings. One approach to finding a Hamiltonian cycle currently under consideration by Duffus and others [3, 4] is to search for a pair of matchings whose union is a Hamiltonian cycle. In order for this approach to have a reasonable chance of success, it is useful to have a large collection of explicitly described matchings with which to work. One nice set of matchings consists of the lexicographical matchings, which are studied extensively by Duffus and his co-authors in [3] and [4]. Unfortunately, this set is not large enough. It is shown in [4] that two lexicographical matchings never form a Hamiltonian cycle when $k > 1$.

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In this paper, we generalize one explicit construction of lexicographical matchings to produce new matchings, which we will call *lexical* matchings or, more specifically, *i-lexical* matchings. The 0-lexical matchings are just the lexicographical matchings. Like lexicographical matchings, lexical matchings are defined with respect to a fixed ordering of the base set. In Section 1 we define the i-lexical matchings with respect to a fixed ordering and show that we do indeed get matchings. In Section 3 we consider the effect of changing the underlying ordering of the base set. We will show that for $0 \leqslant i < j < k/2$ there are exactly $(2k)!$ i-lexical matchings and no i-lexical matching is a j-lexical matching. If $k > 2$ is even, then there are $(2k)!/2$ $k/2$-lexical matchings. Also, every t-lexical matching is a $(k-t)$-lexical matching.

The *odd graph* $\mathscr{O}(k)$ is the graph whose vertices are the kth level of $2^{[2k+1]}$ and whose edges are pairs of disjoint subsets. It is observed in [3] that any matching in $\mathscr{O}(k)$ can be lifted to a matching in $\mathscr{B}(k)$ and the new matching is not lexicographical. The authors conjecture that for all k, if $\mathscr{O}(k)$ has a matching, then a Hamiltonian cycle in $\mathscr{B}(k)$ can be formed by mating a lexicographical matching with a matching lifted from $\mathscr{O}(k)$. While it was known that $\mathscr{O}(k)$ has matchings whenever $|\mathscr{O}(k)|$ is even [7], explicit matchings apparently were not known. In Section 2 we show that when $k > 0$ is even, any $k/2$-lexical matching can be lowered to a matching in $\mathscr{O}(k)$.

The remainder of this section is devoted to notation and definitions. We assume that n and k are fixed with $n = 2k+1$. Thus, we will usually write $\mathscr{O}$ for $\mathscr{O}(k)$ and $\mathscr{B}$ for $\mathscr{B}(k)$. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. If $S \subset [n]$, then S^c denotes $[n] - S$. The set of k-element subsets of a set S are denoted by $\binom{S}{k}$. When applied to elements of $[n]$, addition and subtraction are modulo n. The set $\{x-k, x-k+1, x-k+2, \ldots, x-1\}$ may be denoted in any of the following ways: $[x-k, x)$, $[x-k, x-1]$, $(x-k-1, x)$, or $(x-k-1, x-1]$. Note that $(x, x) = [n] - \{x\}$. It is convenient to consider a perfect matching of a graph $\mathscr{G}$ to be a function m on the vertices of $\mathscr{G}$ such that x is adjacent to $m(x)$ and $(m \circ m)(x) = x$. If $\mathscr{G}$ is bipartite, we make a further simplification by considering a matching to be a bijection from one part to the other.

1. Lexical Matchings

In this section we define the i-lexical matchings with respect to the standard ordering of $[n]$. We begin by introducing some more notation and proving a lemma on which these definitions are based. This lemma seems to be implicit in the work of Feller [5] and Narayana [8]. For subsets R and S of $[n]$, define the S-split of R denoted by R/S, to be $R/S = |R \cap S| - |R \cap S^c|$. For each $x \in S^c$ let $D_S(x)$ denote $\{y \in S^c - \{x\} : [y, x)/S < 0\}$ and $d_S(x)$ denote $|D_S(x)|$. Recall that $n = 2k+1$.

LEMMA 1. *Let* $S \in \binom{[n]}{k}$. *If* x *and* w *are distinct elements of* S^c *then* $D_S(w) \subsetneqq D_S(x)$ *or* $D_S(x) \subsetneqq D_S(w)$.

Proof. Since

$$[w, x)/S + [x, w)/S = [n]/S = -1,$$

exactly one of $[w, x)/S$ and $[x, w)/S$ is negative. Suppose $[w, x)/S$ is negative. We will show that $D_S(w) \subsetneqq D_S(x)$. First note that $w \in D_S(x)$, but $w, x \notin D_S(w)$. Now suppose $y \in D_S(w)$. If $y \in (x, w)$ then

$$[y, x)/S = [y, w)/S + [w, x)/S < 0$$

and so $y \in D_S(x)$. Otherwise, $y \in (w, x)$ and

$$[y, x)/S = [y, w)/S - [x, w)/S < 0$$

and $y \in D_S(x)$. □

COROLLARY 2. *For each* $S \in \binom{[n]}{k}$, d_S *is a well defined bijection from* S^c *to* $\{0, 1, \ldots, k\}$.

Proof. By definition d_S is bounded between 0 and k, and by the lemma the values are distinct. □

For each $S \in \binom{[n]}{k}$, let e_S be the inverse of d_S. The i-lexical matching, M_i, with respect to the standard order on $[n]$, is defined on $\binom{[n]}{k}$ by $M_i(S) = S \cup \{e_S(i)\}$.

THEOREM 4. *For* $i = 0, 1, \ldots, k$, M_i *is a matching in* $\mathscr{B}$.

Proof. Clearly $S \subset M_i(S)$. Thus it suffices to show that M_i is one-to-one. This follows immediately from the next lemma. □

LEMMA 4. *If* S *and* T *are distinct elements of* $\binom{[n]}{k}$ *such that* $S \cup \{x\} = T \cup \{y\}$ *for some* $x, y \in [n]$, *then* $D_S(x) \subsetneqq D_T(y)$ *or* $D_T(y) \subsetneqq D_S(x)$.

Proof. Note that $x \in T - S$, $y \in S - T$, and $R/S = R/T$ if $x, y \in R^c$. Thus

$$(x, y)/T + (y, x)/S = -1$$

and so exactly one of $(x, y)/T$ and $(y, x)/S$ is negative. Say $(y, x)/S$ is negative. We show that $D_T(y) \subsetneqq D_S(x)$. First choose $z \in (y, x) \cap S^c$ such that $[z, x)/S = -1$. Since $x \in T - S$, we have

$$[z, y)/T = [z, x)/S + 1 + (x, y)/T \geqslant 0$$

and $z \in D_S(x) - D_T(y)$. Now suppose that $w \in D_T(y)$. Then $w \neq x$ and $w \neq y$. If $w \in (x, y)$ then

$$[w, x)/S = [w, y)/T + 1 + (y, x)/S < 0$$

and so $w \in D_S(x)$. Otherwise $w \in (y, x)$ and

$$[w, x)/S = [w, y)/T - 1 - (x, y)/T < 0$$

and again $w \in D_S(x)$. □

Notice that the matchings $M_0, M_1, \ldots, M_k$ form a 1-factorization of $\mathscr{B}(k)$.

2. Odd Graph Matchings

Let m be a matching in $\mathscr{O}$. In [3] it is observed that m can be lifted to a matching M in $\mathscr{B}$ by setting $M(S) = [n] - m(S)$, for each $S \in \binom{[n]}{k}$. In this section we go the other direction; we show that when k is even, say $k = 2i$, M_i can be lowered to a matching, m_i, in $\mathscr{O}$ by setting $m_i(S) = [n] - M_i(S)$, for each $S \in \binom{[n]}{k}$.

First we introduce some new notation and prove two lemmas, the first of which will be used again in the next section. For each $x \in S^c$, let $D_S^*(x)$ denote $\{y \in S^c - \{x\} : (x, y]/S < 0\}$ and $d_S^*(x)$ denote $|D_S^*(x)|$.

LEMMA 5. *For all $S \in \binom{[n]}{k}$ and all $x \in S^c$, $d_S(x) + d_S^*(x) = k$.*

Proof. It suffices to show that $y \in D_S(x)$ if and only if $y \notin D_S^*(x)$, for all $y \in S^c - \{x\}$. For any $y \in S^c$ we have

$$[y, x)/S + (x, y]/S = (x, x)/S - 1 = -1.$$

Thus, exactly one of $[y, x)/S$ and $(x, y]/S$ is negative. The result follows. □

For $x \in S^c$, let $D_S^c(x)$ denote $\{y \in S : [y, x)/S > 0\}$ and $d_S^c(x)$ denote $|D_S^c(x)|$. Note that $D_S^c(x) = D_T(x)$, where $T = [n] - (S \cup \{x\})$.

LEMMA 6. *For all $S \in \binom{[n]}{k}$ and $x \in S^c$, $d_S^*(x) = d_S^c(x)$.*

Proof. We argue by induction on the complexity of S, where complexity is defined as follows. Suppose S and R are k-sets of $[n] - \{x\}$. We say that S is obtained from R by a *y-switch* if $S = R \cup \{y\} - \{y+1\}$. Let $I = [x-k, x)$. Every k-subset S of $[n] - \{x\}$ can be obtained from I by a series of switches. The complexity, $C(S)$, of S is the least number of switches needed to obtain S from I.

For the base step of the induction we note that $D_I^*(x) = (x, x-k)$ and $D_I^c(x) = I$ and so $d_I^*(x) = d_I^c(x) = k$. Now for the inductive step, let S be obtained from R by a y-switch, where $C(R) < C(S)$. It is easily checked that

$$d_S^*(x) \neq d_R^*(x) \Rightarrow D_S^*(x) = D_R^*(x) - \{y\}$$

and

$$d_S^c(x) \neq d_R^c(x) \Rightarrow D_S^c(x) = D_R^c(x) - \{y+1\}.$$

Thus, by the induction hypothesis that $d_R^*(x) = d_R^c(x)$, it suffices to show that $y \in D_R^*(x) \Leftrightarrow y+1 \in D_R^c(x)$. This follows immediately from the observation that

$$(x, y]/R + [y+1, x)/R = (x, x)/R = 0.$$ □

THEOREM 7. *If $k=2i$ then m_i is a matching in the odd graph $\mathcal{O}(k)$.*

Proof. Clearly $m_i(S)\cap S=\emptyset$. We must show that $(m_i\circ m_i)(S_i)=S$, for all $S\in\binom{[n]}{k}$. This amounts to showing that $d_S(x)=d_T(x)$, where $e_S(i)=x$ and $T=[n]-(S\cup\{x\})$. By Lemmas 5 and 6 we have

$$d_S(x)=i=k-i=d_S^*(x)=d_S^c(x)=d_T(x). \qquad \square$$

3. Orbits of Lexical Matchings

In Section 1, we defined i-lexical matchings with respect to the standard order on $[n]$. We now consider a natural extension of those definitions from two points of view. First, let $\mathbb{A}$ be the group of automorphisms of $\mathscr{B}$ that map k-sets to k-sets. Viewing a matching M as a set of edges, define M to be i-lexical iff there exists an automorphism $\alpha\in\mathbb{A}$ such that for every edge $ST\in\mathscr{B}$, $ST\in M$ iff $\alpha(S)\alpha(T)\in M_i$. If M is viewed as a function from k-sets to $k+1$-sets, then M is i-lexical iff $M=\alpha\cdot M\cdot\alpha^{-1}$. It is shown in [3] that the symmetric group on $[n]$, S_n, is isomorphic to $\mathbb{A}$ by a map that sends σ to α_σ, where $\alpha_\sigma(S)=\{\sigma(s): s\in S\}$, for any subset S of $[n]$. Thus, the following is an alternative description of the i-lexical matchings.

For any $\sigma\in S_n$, let L_σ be the circular ordering $\sigma(1)<_\sigma\sigma(2)<_\sigma\cdots<_\sigma\sigma(n)<_\sigma\sigma(1)$. For any $S\in\binom{[n]}{k}$, $x\in S^c$, and $i=0,1,\ldots,k$, we make the following definitions: $[y,x)^\sigma=\{z: y\leqslant_\sigma z<_\sigma x\}$, $D_S^\sigma(x)=\{y\in S^c:[y,x)^\sigma/S<0\}$; $d_S^\sigma(x)=|D_S^\sigma(x)|$; $e_S^\sigma=(d_S^\sigma)^{-1}$; and $M_i^\sigma(S)=S\cup\{e_S^\sigma(i)\}$. We call M_i^σ the i-lexical matching with respect L_σ. Notice that $\alpha_\sigma\cdot M_i(S)-M_i^\sigma\cdot\alpha_\sigma(S)$. Thus a matching M in $\mathscr{B}$ is i-lexical iff $M=M_i^\sigma$ for some $\sigma\in S_n$.

Let $\mathscr{M}_i$ be the set of i-lexical matchings. In this section we determine the size of the various $\mathscr{M}_i$ and the nature of their intersections. First we make some easy observations. Since $|S_n|=n!$, $|\mathscr{M}_i|\leqslant n!$. For $\sigma,\tau\in S_n$, we call σ a *shift* of τ if for some t, $\sigma(s)=\tau(s+t)$ for all $s\in[n]$. If σ is a shift of τ then $M_i^\sigma=M_i^\tau$. Thus $|\mathscr{M}_i|\leqslant(2k)!$. For $\sigma,\tau\in S_n$, we call σ the *reversal* of τ if $\sigma(s)=\tau(n+1-s)$. Note that if $\sigma(s)=\tau(t-s)$, for some fixed t, then σ is a shift of the reversal of τ. Then using Lemma 5 and the above fact about shifts, $M_i^\sigma=M_{k-1}^\tau$, if σ is a shift of the reversal of τ. Thus $\mathscr{M}_i=\mathscr{M}_{k-i}$ and if $i=k/2$, then $|\mathscr{M}_i|\leqslant(2k)!/2$. We shall show that these are the only equalities among lexical matchings when $k>2$.

The notion of an x-filter is central to our arguments. Let M be a matching in $\mathscr{B}$. The set $S\in\binom{[n]}{k}$ is an *x-vertex* of M if $M(S)-S=\{x\}$. A subset F of $[n]$ is an *x-filter* of M if F intersects every x-vertex of M. We will prove that $\mathscr{M}_i$ and $\mathscr{M}_j$ are disjoint, for $i<j\leqslant k/2$, by showing the size of x-filters in i-lexical matchings depends on i. The next lemma will be used several times to construct a subset $S\in\binom{[n]}{k}$ such that $d_S(x)=i$ and S misses a given set F.

LEMMA 8. *Let* $P \subset [n] - \{x\}$ *and* $w \in [n]$.

(a) *If* $(w, x)/P \geqslant 2$, *then there exists* $y \in P \cap (w, x)$ *such that* $Q = P - \{y\}$ *satisfies*

$$D_Q(x) \cap (w, x) = D_P(x) \cap (w, x).$$

(b) *If* $(w, x)/P \geqslant 1$, *then there exists* $z \in P \cap (w, x)$ *and* $u \in (w, x)$ *such that* $R = P - \{z\}$ *satisfies*

$$D_R(x) \cap (w, x) = (D_P(x) \cap (w, x)) \cup \{u\}.$$

Proof. (a) Choose $y \in P \cap (w, x)$ to minimize $|(w, y)|$. For $u \in (w, x)$, note that $[u, x)/Q = [u, x)/P$, if $u \in (y, x)$, and $[u, x)/Q = [u, x)/P - 2$ if $u \in (w, y]$. In the latter case, because $(w, y) \subset (w, x) - P$ and $(w, x)/P \geqslant 2$, we have $[u, x)/P \geqslant 2$ for all $u \in (w, y]$. Hence $[u, x)/Q \geqslant 0$ for all $u \in (w, y)$.

(b) Choose $u \in (w, x)$ with $[u, x)/P = 1$ so as to minimize $|(w, u])|$. To see that such an integer exists note that (i) $(w, x)/P \geqslant 1$, (ii) $|[v, x)/P - [v+1, x)/P| = 1$, for all $v \in (w, x)$, and (iii) $[x-1, x)/P \leqslant 1$. Choose $z \in P \cap [u, x)$ to minimize $|[u, z]|$. Then for $v \in (w, x)$, $[v, x)/R$ is $[v, x)/P - 2$ if $z \in [v, x)$ and $[v, x)/P$ otherwise. It follows from the choice of u and z that (b) holds. □

When w has been specified and P satisfies the appropriate hypothesis of Lemma 8, let $q(P)$ denote Q and $r(P)$ denote R. We are now prepared to prove the main results of this section.

LEMMA 9. *If* M *is i-lexical, where* $k > 0$ *and* $i \leqslant k/2$, *then for every* $x \in [n]$ *the size of the smallest x-filter of* M *is* $i + 1$.

Proof. Without loss of generality, assume $M = M_i$. First note that the set $I = [x - (i+1), x)$ is an x-filter of M, since $I \cap S = \emptyset$ implies $I \subset D_S(x)$ and thus $d_S(x) > i$.

Next, suppose F is a subset of $[n] - \{x\}$ of size i. We show that F is not an x-filter of M by constructing an x-vertex of M, which misses F. Let $P = F^c - \{x\}$. Note that $0 \leqslant d_P(x) \leqslant i \leqslant k - i$. We use Lemma 8, with $w = x$, to construct a descending sequence of sets $P_0, P_1, \ldots, P_{k-i}$, such that $P_0 = P$, $P_{j+1} = r(P_j)$, for $j < i - d_P(x)$, and $P_{j+1} = q(P_j)$, for $j \geqslant i - d_P(x)$. Then $d_{P_{k-i}}(x) = i$ and $|P_{k-i}| = k$, but $F \cap P_{k-i} = \emptyset$, as desired. □

THEOREM 10. *If* $0 \leqslant i < j \leqslant k/2$, *then* $\mathscr{M}_i \cap \mathscr{M}_j = \emptyset$.

Proof. Suppose $M \in \mathscr{M}_i$. Using Lemma 9, M has an x-filter of size j, and, therefore, is not in $\mathscr{M}_j$. □

For $i \leqslant k/2$ and $\sigma \in S_n$, let $\mathscr{W}_i^{\sigma}(x)$ denote the set $\{F : F$ is a x-filter of M_i^{σ} of size $i+1\}$. For each $x \in [n]$ we can recover $\mathscr{W}_i^{\sigma}(x)$ from M_i^{σ}. We shall see that we can nearly recover σ from $\{(i, \mathscr{W}_i^{\sigma}(x)) : x \in [n]\}$.

LEMMA 11. *Suppose* $i < k/2$ *and* $\sigma \in S_n$. *Then for every* $x \in [n]$, $\bigcap \mathscr{W}_i^{\sigma}(x) = \{\sigma(s-1)\}$, *where* $s = \sigma^{-1}(x)$.

Proof. Without loss of generality assume σ is the identity and write $\mathscr{W}_i(x)$

for $\mathscr{W}_i^\sigma(x)$. For $t = 2, 3, \ldots, i+2$, let $F_t = [x-(i+2), x) - \{x-t\}$. Since $d_{F_t^c}(x) > i$, each F_t is an x-filter of M_i. Thus, $\bigcap \mathscr{W}_i(x) \subset \{x-1\}$. Suppose F is an $(i+1)$-subset of $[n] - \{x\}$, such that $x - 1 \notin F$. We must show that F is not an x-filter. To this end we construct an x-vertex of M_i which misses F. Let $P = F^c - \{x\}$. Since $x - 1 \notin F$, and $|F| = i+1$, $0 \leqslant d_P(x) \leqslant i \leqslant k-(i+1)$. We use Lemma 8, with $x = x$, to construct a descending sequence of sets $P_0, P_1, \ldots, P_{k-(i+1)}$ such that $P_0 = P$, $P_{j+1} = r(P_j)$ if $j < i - d_P(x)$, and $P_{j+1} = q(P_j)$ otherwise. Then $d_{P_{k-(i+1)}}(x) = i$ and $|P_{k-(i+1)}| = k$, but $F \cap P_{k-(i+1)} = \emptyset$, as desired. □

THEOREM 12. *If* $i < k/2$ *then* $|\mathscr{M}_i| = (2k)!$.

Proof. Since τ has exactly n shifts, it suffices to show that $M_i^\sigma = M_i^\tau$ if and only if σ is a shift of τ. By our previous comments, $M_i^\sigma = M_i^\tau$ if σ is a shift of τ. On the other hand suppose $M_i^\sigma = M_i^\tau$. Choose t so that $\sigma(n) = \tau(t)$. Using Lemma 11 we have $\{\sigma(-1)\} = \{\tau(t-1)\}$. Thus, $\sigma(-1) = \tau(t-1)$. Repeated use of Lemma 11 yields $\sigma(-2) = \tau(t-2)$, $\sigma(-3) = \tau(t-3), \ldots, \sigma(1) = \tau(1+t)$. Thus, σ is a shift of τ. □

LEMMA 13. *Suppose* $i = k/2$ *and* $k > 2$. *Then for all* $\sigma \in S_n$ *and every* $x \in [n]$, $\{\sigma(s-1), \sigma(s+1)\}$ *is the unique pair that intersects every filter in* $\mathscr{W}_i^\sigma(x)$, *where* $s = \sigma^{-1}(x)$.

Proof. Without loss of generality assume that σ is the identity. For $t = 2, 3, \ldots, i+2$, let

$$F_t = [x-(i+2), x) - \{x-t\} \text{ and } F_t^* = (x, x+(i+2)] - \{x+t\}.$$

Each F_t is in $\mathscr{W}_i(x)$, since $d_{F_t^c}(x) > i$. Each F_t^* is in $\mathscr{W}_i(x)$, since $d^*_{(F_t^*)^c}(x) > i$ and $d_P^*(x) = d_P(x)$ for any x-vertex P. Clearly $\bigcap\{F_t : t = 2, 3, \ldots, i+2\} = \{x-1\}$ and $\bigcap\{F_t^* : t = 2, 3, \ldots, i+2\} = \{x+1\}$. Since $k > 2$, we have $i+2 \leqslant k$, and thus $F_t \cap F_u^* = \emptyset$ for all t and u. We have shown no pair other than $\{x-1, x+1\}$ intersects all filters in $\mathscr{W}_i(x)$. To show this pair does intersect all filters in $\mathscr{W}_i(x)$, suppose that F is a $(i+1)$-subset of $[n] - \{x-1, x, x+1\}$. We must show that there exists an x-vertex of M_i which misses F. Let $P = F^c - \{x\}$. Note that $0 \leqslant d_P(x) \leqslant i$, since $x - 1 \notin F$ and $|F| = i+1$. We can use Lemma 8 $i-1$ times to obtain a set of size $2i = k$. If $d_P(x) \geqslant 1$, proceed as in the proof of Lemma 11. Otherwise let t be the greatest integer less than n such that $v = x - t$ is in F. Apply Lemma 8 with $w = v$ to construct a descending sequence of sets $P_0, P_1, \ldots, P_s$ as follows. Let $P_0 = P$. If $(v, x)/P_j \geqslant 1$, set $P_{j+1} = r(P_j)$; otherwise set $s = j$ and stop. Note that $x - 1 = x + 2k$ is in P, and so $|(v, x) \cap P| \leqslant (3i-2)$. Also $|(v, x) \cap F| = i$. Thus, $(v, x)/P \leqslant 2i - 2$. Since $(v, x)/P_{j+1} = (v, x)/P_j$, we have $s \leqslant i - 1$. Note that $v \in D_{P_s}(x)$ and thus $d_{P_s}(x) = s + 1$. Thus, $i - 1 - s$ more applications of Lemma 8 with $w = x$ will provide us with the desired x-vertex. □

THEOREM 14. *Suppose* $i = k/2$. *If* $k = 2$ *then* $|\mathscr{M}_i| = 6$. *If* $k > 2$ *then* $|\mathscr{M}_i| = (2k)!/2$.

Proof. The case $k=2$ follows from inspection. In the case $k>2$, it suffices to show that $M_i^\sigma = M_i^\tau$ if and only if σ is a shift of τ or a shift of the reversal of τ, since there are exactly $2n$ such permutations. It follows from our earlier remarks that if σ is a shift or a shift of the reversal of τ, then $M_i^\sigma = M_i^\tau$. Now suppose that $M_i^\sigma = M_i^\tau$. Then $\bigcap \mathscr{W}_i^\sigma(x) = \bigcap \mathscr{W}_i^\tau(x)$, for all x. Suppose $\sigma(n) = \tau(t)$. Then $\{\sigma(n-1), \sigma(1)\} = \{\tau(t-1\}, \tau(t+1)\}$, by Lemma 13. If $\sigma(1) = \tau(1+t)$, then again by Lemma 13, $\{\sigma(n), \sigma(2)\} = \{\tau(t), \tau(2+t)\}$. So $\sigma(2) = \tau(2+t)\}$. Continuing in this manner, we see that σ is a shift of τ. Otherwise $\sigma(1) = \tau(t-1)$. Using Lemma 13 again, $\{\sigma(n), \sigma(2)\} = \{\tau(t-2), \tau(t)\}$. Thus, $\sigma(2) = \tau(t-2)$. Continuing in this manner we see that σ is a shift of the reversal of τ. □

4. Concluding Remarks and Problems

We have considerably enlarged the catalogue of explicit matchings in $\mathscr{B}$. In this section we consider the usefulness of this catalogue with respect to producing Hamiltonian cycles in $\mathscr{B}(k)$ for the first few values of k. In particular, we will see that it is not yet large enough to accomplish the goal for all k. If $k=1$ or $k=2$, then $M_0 \cup M_1$ is a Hamiltonian cycle. If $k=3$ then $M_1 \cup M_1^\sigma$ is a Hamiltonian cycle, where σ is the permutation (2, 4, 6, 1, 3, 5, 7). The case $k=4$ is more complicated. A computer search shows that there is no pair of lexical matchings that mates to form a Hamiltonian cycle. However, a Hamiltonian cycle can be formed by mating the lexicographical matching M_0 with a matching M lifted from a matching m in $\mathscr{O}$; the matching M is closely related to M_1. Let $f(S) = [n] - M_1(S)$. Notice that $f(S)$ is adjacent to S in $\mathscr{O}$. Taking all edges of the form $\{S, f(S)\}$ partitions the vertices of $\mathscr{O}$ into five even cycles. Each of these even cycles gives rise to two matchings of its vertices. We can obtain 32 matchings of $\mathscr{O}$ by combining one matching from each of the cycles. One of these matchings is m. By Lemmas 5 and 6, M, the lifting of m to $\mathscr{B}$, has half of its edges in M_1 and half of its edges in M_3. Using a different approach Duffus, Hanlon, and Roth [3] found the same cycle.

While the results listed above are seductive, they are not entirely convincing. We have no general explanation for these phenomena. In the hope of clarifying the situation, we list two test problems.

PROBLEM 1. It is known that if $k \neq 2^i - 1$, then $|\mathscr{O}(k)|$ is even and $\mathscr{O}(k)$ has a matching. Find general classes of explicit matchings. □

PROBLEM 2. Show that for sufficiently large k, two matchings from $\mathscr{M}_i$ cannot be combined to form a Hamiltonian cycle. □

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