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Growth with Regulation in Fluctuating Environments*

II. Intrinsic Lower Bounds to Population Size

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Abstract. Population growth is modelled by means of diffusion processes originating from fluctuation equations of a new type. These equations are obtained in the customary way by inserting random fluctuations into first order non linear differential equations. However, differently from the cases so far considered in the literature, equations possessing two non trivial fixed points are taken into account. The underlying deterministic models depict the regulated growth of a population whose size cannot decrease below some preassigned lower threshold naturally acting as an absorbing boundary. A fairly comprehensive mathematical description of these models is provided.

1 Introduction

A variety of models of population growth in random environment have been proposed in the literature and diffusion models have been constructed by inserting "white noise"-like fluctuations in the growth equations. Thus doing, one of the parameters of the growth equations is viewed as a stochastic process more or less explicitly accounting for the effect of environmental variability (Capocelli and Ricciardi, 1974a, b; Feldman and Roughgarden, 1975; May, 1971, 1973; Montroll, 1972; Nobile and Ricciardi, 1980, 1984; Ricciardi, 1977; Tuckwell, 1974).

A feature shared by the totality of the resulting stochastic processes is their being defined over a finite or infinite interval an end point of which is always the origin (zero-size population). Arbitrarily small population sizes are thus to be expected for decreasing populations unless an artificial "lower threshold" is set at some suitable point of the above mentioned interval to secure an automatic extinction whenever the population size attains such point. In mathematical terms one requires that the lower threshold act as an *absorbing boundary* for the stochastic process modeling the population size.

In order to overcome this not too natural procedure and to provide at the same time a detailed analysis of certain diffusion processes so far not discussed in the literature we shall propose growth equations characterized by *three* equilibrium points. The third equilibrium point identifies with the lower threshold mentioned in the foregoing while the other two equilibria depict the zero population size and the carrying capacity, respectively. By inserting random fluctuations in the growth equations we shall thus be lead to diffusion processes defined over intervals whose lower end points can still be taken as suitable to represent the size of a population. By analogy with the parallel analysis of logistic and logarithmic models (Nobile and Ricciardi, 1980; Nobile et al., 1982) in the sequel we shall also take into account growth equations containing non linearities of logarithmic type.

2 A Deterministic Growth Model

We shall request that the growth law is characterized by the following properties: i) There exists an open interval (β, γ), with β and γ positive real numbers, such that for each initial population size $x_0 \equiv x(0) \in (\beta, \gamma)$ the number $x(t)$ $(t \geqq 0)$ of individuals is an S-shaped function having γ as an horizontal asymptote; ii) $x=\beta$ and $x=\gamma$ are fixed (or equilibrium) points such that the former is unstable and the latter is asymptotically stable; iii) $x=0$ is an asymptotically stable point and iv) $x(t)$ is decreasing in $(0, \beta)$ and in (γ, ∞) with $\lim_{t\to\infty} x(t)=0$ for $0 \leqq x_0 < \beta$ and $\lim_{t\to\infty} x(t)=\gamma$ for $\gamma \leqq x_0 < \infty$. We shall

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finally confine our attention to growth processes that can be described by first order differential equations in the absence of environmental fluctuations.

The above assumptions are quite stringent constraints on the possible resulting growth models. Nevertheless, the class of such models is rather large. In the sequel, by analogy with the logistic case we shall concentrate on the study of a growth process whose intrinsic fertility is a convex ∩ function. More precisely, we shall refer to the following growth law:

$$\frac{dx}{dt}=rx\left(1-\frac{x}{\beta}\right)\left(\frac{x}{\gamma}-1\right) \quad (\gamma>\beta>0,\ r>0),$$
$$x(0)=x_0. \tag{2.1}$$

Subsequently, we shall also consider the case

$$\frac{dx}{dt}=sx\left(1-\frac{\ln x}{b}\right)\left(\frac{\ln x}{c}-1\right) \quad (c>b>0,\ s>0),$$
$$x(0)=x_0 \tag{2.2}$$

whose relation to model (2.1) is reminiscent of that existing between Gompertz and logistic equations (Capocelli and Ricciardi, 1974b).

It is convenient to re-write Eq. (2.1) and (2.2) in the forms:

$$\frac{dx}{dt}=\alpha x(\beta-x)(x-\gamma) \quad (\gamma>\beta>0,\alpha>0),$$
$$x(0)=x_0 \tag{2.3}$$

and

$$\frac{dx}{dt}=ax(b-\ln x)(\ln x-c) \quad (a>0, c>b>0),$$
$$x(0)=x_0, \tag{2.4}$$

where we have set $\alpha\equiv\frac{r}{\beta\gamma}$ and $a\equiv\frac{s}{bc}$. It is then immediately seen that the solution of (2.3) satisfies assumptions i)∼iv). Such solution, $x\equiv x(t)$, is implicitly defined by

$$x^{\gamma-\beta}|x-\gamma|^{\beta}=A|x-\beta|^{\gamma}\exp\{-\alpha\beta\gamma(\gamma-\beta)t\} \tag{2.5}$$

with

$$A\equiv\frac{x_0^{\gamma-\beta}|x_0-\gamma|^{\beta}}{|x_0-\beta|^{\gamma}}. \tag{2.6}$$

Furthermore, it is straightforward to prove that $x(t)$ is convex ∪ in $(0,x_1)\cup(\beta,x_2)\cup(\gamma,\infty)$, where we have set:

$$x_1=\tfrac{1}{3}[\gamma+\beta-(\gamma^2+\beta^2-\beta\gamma)^{1/2}] \quad (0<x_1<\beta),$$
$$x_2=\tfrac{1}{3}[\gamma+\beta+(\gamma^2+\beta^2-\beta\gamma)^{1/2}] \quad (\beta<x_2<\gamma). \tag{2.7}$$

Elsewhere $x(t)$ is convex ∩. Figure 1 shows the behavior of $x(t)$ for various choices of x.

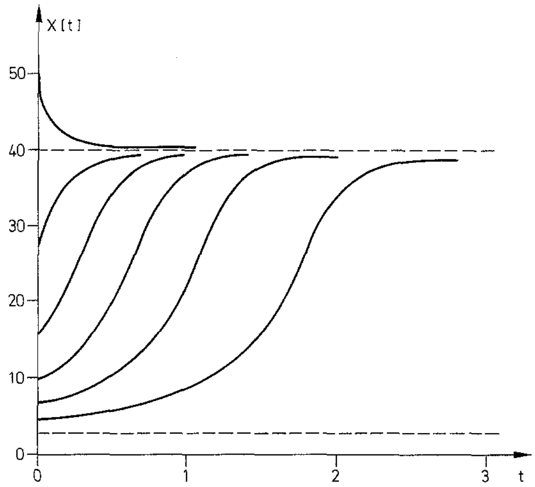

Fig. 1. The solution of Eq. (2.3) is plotted for $\alpha=5\cdot10^{-3}$, $\beta=3$, $\gamma=40$ and for $x_0=5$, 7, 10, 16, 28, 50. Time scale is expressed in arbitrary units

Note that in $(0,\gamma)$ the growth rate has a minimum when the population size equals x_1 whereas it is maximum when the population size is equal to x_2. The times t_M and t_m necessary for the population to attain respectively the maximum and the minimum growth rate, when initially there exist x_0 individuals, are given by

$$t_M=[\alpha\beta\gamma(\gamma-\beta)]^{-1}\ln\left\{\frac{A|x_2-\beta|^{\gamma}}{x_2^{\gamma-\beta}|x_2-\gamma|^{\beta}}\right\},$$
$$t_m=[\alpha\beta\gamma(\gamma-\beta)]^{-1}\ln\left\{\frac{A|x_1-\beta|^{\gamma}}{x_1^{\gamma-\beta}|x_1-\gamma|^{\beta}}\right\} \tag{2.8}$$

with A, x_1, and x_2 defined by (2.6) and (2.7).

It is finally straightforward to derive similar results for model (2.1). While postponing to Sect. 4 the study of model (2.4) we shall now insert "environmental fluctuations" into model (2.3) and provide a somewhat accurate description of the ensuing stochastic processes.

3 Diffusion Models

We shall assume that the intrinsic fertility α in model (2.3) is a stationary normal process $a+\Lambda(t)$ with

$$E[\Lambda(t)]=0,$$
$$E[\Lambda(t_1)\,\Lambda(t_2)]=\sigma^2\delta(t_2-t_1), \quad t_1<t_2, \tag{3.1}$$

where σ^2 denotes the intensity of the white noise $\Lambda(t)$. Such procedure, frequently used for other growth models (see, for instance, Ricciardi, 1977) makes us switch from the study of one population to that of an *ensemble* of macroscopically identical populations sharing the initial size but possessing distinct intrinsic fertilities. Model (2.3) thus leads us to the Stratonovich stochastic differential equation (cf., for instance Stratonovich, 1968):

$$\frac{dX}{dt} = \alpha X(X-\beta)(\gamma - X) + X(X-\beta)(\gamma-X)\Lambda(t)$$
$$(\gamma > \beta > 0,\ \alpha > 0)\,, \tag{3.2}$$

that must be solved under the initial condition $P\{X(0)=x_0\}=1$. As is well known (cf. Stratonovich, loc. cit.), Eq. (3.2) with the assigned initial condition defines temporally homogeneous diffusion processes $X(t)$ having drift $A_1(x)$ and infinitesimal variance $A_2(x)$ given by:

$$A_1(x) = -x(x-\beta)(x-\gamma) \cdot \left\{\alpha - \frac{\sigma^2}{2}[3x^2 - 2x(\beta+\gamma) + \beta\gamma]\right\}$$
$$(\alpha > 0,\ \gamma > \beta > 0)\,, \tag{3.3}$$
$$A_2(x) = \sigma^2 x^2 (x-\beta)^2 (x-\gamma)^2 .$$

Therefore, while disregarding all intervals lying in the negative half axis because of no interest within the present context, the diffusion intervals $I_1 \equiv (0, \beta)$, $I_2 \equiv (\beta, \gamma)$, and $I_3 \equiv (\gamma, \infty)$ must be separately taken into account. Concerning I_1, one can show that its end points are natural boundaries (in the sense of Feller, 1954). A similar conclusion holds also for the end points of I_2. For I_3, instead, while γ is still a natural boundary one can prove that the point at infinity is a *regular* boundary. Hence, for I_1 and I_2 the initial condition alone uniquely specifies the diffusion process. As for I_3, a suitable boundary condition has to be imposed.

Instead of determining the transition probability density function (p. d. f.) of the process $X(t)$ as solution of a diffusion equation with appropriate boundary and initial conditions in each of the intervals I_1, I_2, and I_3, we remark that the monotone transformation

$$y = h(x) \equiv \frac{1}{\beta\gamma(\gamma-\beta)} \ln\left\{\left|\frac{x-\beta}{x}\right|^{\gamma} \left|\frac{x}{\gamma-x}\right|^{\beta}\right\} \tag{3.4}$$

[decreasing in $(0,\beta)\cup(\gamma,\infty)$ and increasing elsewhere] changes (3.2) into

$$\frac{dY}{dt} = \alpha + \Lambda(t) \qquad (\alpha > 0) \tag{3.5}$$

while the initial condition becomes $P\{Y(0)=y_0\}=1$, with Y and y_0 expressed in terms of X and x_0 via transformation (3.4). On the other hand, the intervals I_1, I_2, and I_3 are mapped into the intervals $J_1 \equiv J_2 \equiv (-\infty, \infty)$ and J_3, respectively, for which the end points $\pm\infty$ are natural boundaries while the end point 0 (corresponding to the right hand point of I_3) is a regular boundary. One can thus easily obtain the transition p.d.f. of $X(t)$ in terms of the transition p.d.f. of a suitably defined Wiener process $Y(t)$. To this purpose we remark that transformation (3.4) is defined for $x \neq \beta$, $x \neq \gamma$, is strictly increasing in (β, γ) and strictly decreasing elsewhere. Therefore, denoting by

$$\varphi(y, t \mid y_0) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(y-y_0-\alpha t)^2}{2\sigma^2 t}\right\} \tag{3.6}$$

the transition p.d.f. of $Y(t)$ in the intervals J_1 and J_2 and making use of (3.4) the transition p.d.f. of $X(t)$ can be written as

$$f(x, t \mid x_0) = \left.\frac{\varphi(y, t \mid y_0)}{\left|\frac{dx}{dy}\right|}\right|_{y=h(x)} \tag{3.7}$$

for all pairs x and x_0 both belonging to J_1 or to J_2. More explicitly, we have:

$$f(x, t \mid x_0) = \frac{1}{\sigma\sqrt{2\pi t}} [x|(x-\beta)(x-\gamma)|]^{-1} \cdot \exp\left\{-\frac{[h(x) - h(x_0) - \alpha t]^2}{2\sigma^2 t}\right\} \tag{3.8}$$

with $h(x)$ defined by (3.4).

It should be remarked that if the interval $I_3 = (\gamma, \infty)$ is taken into account, the procedure used in the foregoing to embody the effects of environmental variability does not lead us to a unique diffusion process, the point at infinity of I_3 being indeed a regular boundary. However, this should be of no surprise since the asymptotic behavior of the functions $\varphi(x) = \alpha x(x-\beta)(\gamma - x)$ and $\psi(x) = \alpha^{-1}\varphi(x)$ appearing on the r.h.s. of (3.2) do not meet the well known sufficient condition for the uniqueness of the solutions of (3.2) (cf. Jazwinski, 1970). Even though the opportunity of extending the validity of model (3.2) to interval I_3 may be questioned, it appears reasonable to specify the ensuing diffusion process by imposing that the point at infinity is a reflecting boundary. This, indeed, insures that any population accidentally becoming infinitely large would not stay unchanged but is brought back to finite sizes due to the finiteness of the carrying capacity. Recalling that the transition p.d.f. $\varphi_r(y, t \mid y_0)$ for a Wiener process in the interval $(0, \infty)$ with a reflecting boundary at the origin is given by (Cox and Miller, 1970; note that some misprints therein present have been corrected)

$$\varphi_r(y, t \mid y_0) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(y-y_0-\alpha t)^2}{2\sigma^2 t}\right\} + \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{4\alpha y_0 t + (y+y_0-\alpha t)^2}{2\sigma^2 t}\right\} - \frac{\alpha}{\sigma^2} \exp\left\{\frac{2\alpha y}{\sigma^2}\right\}\left[1 - \operatorname{erf}\left(\frac{y+y_0+\alpha t}{\sigma\sqrt{2t}}\right)\right], \tag{3.9}$$

where

$$\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x dz\exp(-z^2) \tag{3.10}$$

is the error function, the transition p.d.f. $f_r(x,t|x_0)$ of $X(t)$ in I_3 is finally obtained as:

$$\begin{aligned} f_r(x,t|x_0)=&\frac{[x(x-\beta)(x-\gamma)]^{-1}}{\sigma\sqrt{2\pi t}}\\ &\cdot\left\{\exp\left[-\frac{(h(x)-h(x_0)-\alpha t)^2}{2\sigma^2 t}\right]\right.\\ &+\exp\left[-\frac{4\alpha t h(x_0)+(h(x)+h(x_0)-\alpha t)^2}{2\sigma^2 t}\right]\\ &-\frac{\alpha}{\sigma}\sqrt{2\pi t}\exp\left[\frac{2\alpha h(x)}{\sigma^2}\right]\\ &\left.\cdot\left[1-\mathrm{erf}\left(\frac{h(x)+h(x_0)+\alpha t}{\sigma\sqrt{2t}}\right)\right]\right\}. \end{aligned} \tag{3.11}$$

By means of expressions (3.8) and (3.11) of the transition p.d.f. a detailed description can be achieved for the growth process defined by the stochastic Eq. (3.2). In particular, it is not difficult to see that if the initial population size belongs to the interval $(0,\beta)$, then with probability one asymptotically the population goes extinct in the sense that the probability of having asymptotically a non-zero number of individuals is zero. This mathematically follows from the remark that

$$\lim_{t\to\infty} f(x,t|x_0)=\delta(x),\qquad \forall x_0\in(0,\beta). \tag{3.12}$$

Let us now assume that the initial population size belongs to the interval (β,γ). We then have:

$$\lim_{t\to\infty} f(x,t|x_0)=\delta(x-\gamma),\qquad \forall x_0\in(\beta,\gamma) \tag{3.13}$$

implying that asymptotically the sample paths of the process tend to γ with probability one. For $x_0\in(\gamma,\infty)$, finally, the process' sample paths asymptotically tend to γ and no statistical equilibrium condition takes place:

$$\lim_{t\to\infty} f_r(x,t|x_0)=\delta(x-\gamma),\qquad x_0>\gamma. \tag{3.14}$$

In conclusion, the points 0 and γ (asymptotically stable within the deterministic model) act as attractive boundaries for the diffusion processes defined over the intervals I_1, I_2, and I_3. Instead, the point β act as a repulsive boundary, a reminiscence of its being an unstable equilibrium point within the deterministic model. However, it should be emphasized that these conclusions stem from the assumed positivity of the average intrinsic fertility α. Should this be negative, one would be lead to a diffusion process for which, quite unrealistically, the deterministic carrying capacity γ and the origin would act as repulsive boundaries while the lower threshold β would attract asymptotically all sample paths. Moreover, the process would reach a condition of statistical equilibrium in which for all initial population size x_0 larger than γ with non zero (and time independent) probabilities the sample paths would sweep the entire interval (γ,∞). Hence, with non zero probability any population size lying within the interval (γ,∞) could be attained. From (3.11) in the limit as t goes to infinity the steady state distribution $W(x)$ can be obtained:

$$\begin{aligned} W(x)\equiv&\lim_{t\to\infty} f_r(x,t|x_0)\\ =&\frac{2|\alpha|\sigma^{-2}}{x(x-\beta)(x-\gamma)}\exp\left\{-\frac{2|\alpha|\,h(x)}{\sigma^2}\right\}\\ &(\gamma\leqq x\leqq\infty). \end{aligned} \tag{3.15}$$

Hence it easily follows:

$$\begin{aligned} P\{X(\infty)>\eta\}\equiv&\lim_{t\to\infty} P\{X(t)>\eta\,|\,X(0)=x_0\}\\ =&1-\exp\left\{-\frac{2|\alpha|}{\sigma^2}h(\eta)\right\}, \end{aligned} \tag{3.16}$$

for all $\eta>\gamma$ and with $h(x)$, given by (3.4), monotonically growing to infinity as x approaches γ and monotonically decreasing to zero as x approaches infinity.

4 An Alternative Model

In this section we shall briefly discuss the alternative model (2.2) involving a logarithmic non linearity and shall construct diffusion processes by allowing for random fluctuations of one of the parameters. We find it convenient to refer to the following problem:

$$\begin{aligned} &\frac{dx}{dt}=ax(b-\ln x)(\ln x-c)\,(a>0,c>b>0),\\ &x(0)=x_0 \end{aligned} \tag{4.1}$$

which follows from (2.2) by setting $a=\frac{s}{bc}$. Again three equilibrium points are present: $x=0$ (asymptotically stable), e^b (unstable), and e^c (asymptotically stable). The solution $x(t)$ of (4.1) is increasing in (e^b,e^c) and decreasing elsewhere. Furthermore, setting

$$\begin{aligned} y_1=&\exp\left\{\frac{b+c-2-[(c-b)^2+4]^{1/2}}{2}\right\},\\ y_2=&\exp\left\{\frac{b+c-2+[(c-b)^2+4]^{1/2}}{2}\right\} \end{aligned} \tag{4.2}$$

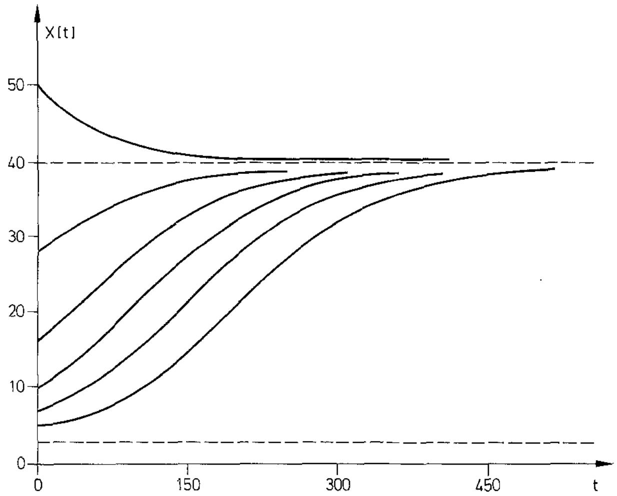

Fig. 2. The solution of Eq. (4.1) is shown for $a=5\cdot 10^{-3}$, $b=1.099$ ($\simeq \ln 3$), $C=3.689$ ($\simeq \ln 40$) and for $x_0=5$, 7, 10, 16, 28, 50. Time scale is expressed in arbitrary units

$(0<y_1<e^b<y_2<e^c)$ the solution of (4.1), $x(t)$, is convex $\cup$ in $(0,y_1)\cup(e^b,y_2)\cup(e^c,\infty)$ whereas it is convex $\cap$ elsewhere. Explicitly, one has:

$$x(t)=\exp\left\{\frac{c(\ln x_0-b)-b(\ln x_0-c)\exp[-at(c-b)]}{\ln x_0-b-\ln(x_0-c)\exp[-at(c-b)]}\right\}. \tag{4.3}$$

Figure 2 shows $x(t)$ for a variety of initial population sizes.

We remark that within the interval $(0,e^c)$ the growth rate has a minimum at y_1 and a maximum at y_2. The times τ_m and τ_M necessary for the population to attain least and largest growth rate are easily obtained from (4.3).

Let us now take into account the environmental variability via the parameterization

$$a\to a+\Lambda(t), \tag{4.4}$$

where $\Lambda(t)$ is the white noise specified by (3.1). Equation (4.1) then becomes the stochastic equation:

$$\begin{aligned}\frac{dX}{dt}&=aX(b-\ln X)(\ln X-c)\\&\quad+X(b-\ln X)(\ln X-c)\Lambda(t),\\ P\{X(0)&=x_0\}=1.\end{aligned} \tag{4.5}$$

We remark that again the sufficient conditions for the uniqueness of the diffusion process $X(t)$ are not satisfied and that drift $A_1(x)$ and infinitesimal variance $A_2(x)$ are given by

$$\begin{aligned}A_1(x)&=x(\ln x-b)(\ln x-c)\Big\{[\ln^2 x-(b+c-2)\ln x\\&\quad-(b+c-bc)]\frac{\sigma^2}{2}-a\Big\},\\ A_2(x)&=\sigma^2x^2(\ln x-b)^2(\ln x-c)^2,\end{aligned} \tag{4.6}$$

respectively. Hence, the diffusion intervals to be considered are $(0,e^b)$, (e^b,e^c), and (e^c,∞) with e^b and e^c natural boundaries and 0 and ∞ regular boundaries.

Again the study of the resulting stochastic processes is greatly simplified by the remark that the transformation

$$y=K(x)\equiv\frac{1}{c-b}\ln\left|\frac{\ln x-b}{\ln x-c}\right| \qquad (c>b>0) \tag{4.7}$$

[which is monotonically increasing in (e^b,e^c) and decreasing elsewhere] changes Eq. (4.5) into that describing a Wiener process having drift a. The above specified intervals are transformed into $(-\infty,0)$, $(-\infty,\infty)$, and $(0,\infty)$, respectively, zero being a regular boundary. To specify the process within the first and the third of these intervals a condition at $x=0$ must be imposed. Clearly such condition should express the absorption of the process at zero within $(-\infty,0)$ and the reflection of the process at zero within the interval $(0,\infty)$.

By making use of transformation (4.7) the transition p.d.f. $f(x,t|x_0)$ of $X(t)$ in the interval (e^b,e^c) can be easily obtained:

$$\begin{aligned}f(x,t|x_0)&=\frac{[x(\ln x-b)(c-\ln x)]^{-1}}{\sigma\sqrt{2\pi t}}\\&\quad\cdot\exp\left\{-\frac{[K(x)-K(x_0)-at]^2}{2\sigma^2 t}\right\}\\&\quad x,x_0\in(e^b,e^c).\end{aligned} \tag{4.8}$$

Within the interval (e^c,∞), instead, one has:

$$\begin{aligned}f_r(x,t|x_0)&=\frac{[x(\ln x-b)(\ln x-c)]^{-1}}{\sigma\sqrt{2\pi t}}\\&\quad\cdot\left\{\exp\left[-\frac{[K(x)-K(x_0)-at]^2}{2\sigma^2 t}\right]\right.\\&\quad+\exp\left[-\frac{4atK(x_0)+[K(x)+K(x_0)-at]^2}{2\sigma^2 t}\right]\\&\quad-\frac{a\sqrt{2\pi t}}{\sigma}\exp\left[\frac{2aK(x)}{\sigma^2}\right]\\&\quad\left.\cdot\left[1-\operatorname{erf}\left[\frac{K(x)+K(x_0)+at}{\sigma\sqrt{2t}}\right]\right]\right\}\quad(x,x_0>e^c),\end{aligned} \tag{4.9}$$

where use of (3.9) has been made.

To determine the transition p.d.f. of $X(t)$ in $(0,e^b)$ we make use of the transition p.d.f. $\varphi_a(y,t|y_0)$ of the Wiener process in $(-\infty,0)$ with an absorbing boundary set at the origin. By resorting to the well known method of images (cf. Cox and Miller, 1970) this function can be proven to be:

$$\varphi_a(y,t|y_0)=\frac{1}{\sigma\sqrt{2\pi t}}\left\{\exp\left[-\frac{(y-y_0-at)^2}{2\sigma^2 t}\right]-\exp\left[-\frac{ay_0}{\sigma^2 t}-\frac{(y+y_0-at)^2}{2\sigma^2 t}\right]\right\}. \quad (4.10)$$

Making use of (3.9) we thus obtain:

$$f_a(x,t|x_0)=\frac{[x(b-\ln x)(c-\ln x)]^{-1}}{\sigma\sqrt{2\pi t}}\cdot\left\{\exp\left[-\frac{(K(x)-K(x_0)-at)^2}{2\sigma^2 t}\right]-\exp\left[-\frac{aK(x_0)}{\sigma^2 t}-\frac{[K(x)+K(x_0)-at]^2}{2\sigma^2 t}\right]\right\}$$

$$(x, x_0 < e^c). \quad (4.11)$$

While skipping for brevity a detailed description of these random growth processes we limit ourselves to mention that the point e^b (an unstable equilibrium point) acts as a repulsive boundary within the stochastic model, whereas the points 0 and e^c (both asymptotically stable equilibrium points) act as attracting boundaries. Furthermore, in the interval $(0, e^b)$ the population is doomed to sure extinction. Indeed, the probability $P(t|x_0)$ of surviving up to the epoque t is given by

$$P(t|x_0)=\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left[\frac{K(x_0)+at}{\sigma\sqrt{2t}}\right]-\frac{1}{2}\exp\left[-\frac{2aK(x_0)}{\sigma^2}\right]\left\{1+\operatorname{erf}\left[\frac{K(x_0)-at}{\sigma\sqrt{2t}}\right]\right\} \quad (4.12)$$

and this decreases to zero as t increases. Similarly to the case of the model of Sect. 3 within the intervals (e^b, e^c) and $(e^c, +\infty)$ no statistical equilibrium takes place since with probability one all sample paths tend to the value e^c. In the first of these two intervals (which is that of interest for the description of growth models) with equal probabilities at each instant t the sample paths lie above and below the point

$$m(t|x_0)=\exp\left\{\frac{b(c-\ln x_0)+c(\ln x_0-b)\exp[(c-b)at]}{c-\ln x_0+(\ln x_0-b)\exp[(c-b)at]}\right\} \quad (4.13)$$

which thus identifies with the median of the distribution.

5 Another Parameterization of the Growth Equation

In this section we shall consider the model

$$\frac{dx}{dt}=-rx\left(1-\frac{x}{\beta}\right)+\delta x^2\left(1-\frac{x}{\beta}\right),\quad (r>\beta\delta)$$

$$x(0)=x_0 \quad (5.1)$$

obtained from (2.1) by the substitution $\gamma\to r/\delta$. Viewing r as the mean value of the random process $r+\Lambda(t)$ [where $\Lambda(t)$ is the white noise defined in Sect. 3] Eq. (5.1) changes into the following fluctuation equation:

$$\frac{dX}{dt}=-rX\left(1-\frac{X}{\beta}\right)+\delta X^2\left(1-\frac{X}{\beta}\right)-X\left(1-\frac{X}{\beta}\right)\Lambda(t), \quad (5.2)$$

$$P\{X(0)=x_0\}=1.$$

Hence, its solutions $X(t)$ are diffusion processes whose infinitesimal moments are given by:

$$B_1(x)=x\left(1-\frac{x}{\beta}\right)\left[\left(\delta-\frac{\sigma^2}{\beta}\right)x-\left(r-\frac{\sigma^2}{2}\right)\right],$$

$$B_2(x)=\sigma^2 x^2\left(1-\frac{x}{\beta}\right)^2. \quad (5.3)$$

While the point β is a natural boundary, the point at infinity is regular if $2\beta\delta<\sigma^2$ and entrance otherwise. In the sequel we shall refer only to the diffusion interval $I\equiv(\beta,\infty)$ which bears a particular interest within the context of population growth.

Differently from the case of the models discussed in Sects. 3 and 4, $X(t)$ now always admits of a steady state distribution $W(x)$. This can be obtained as:

$$W(x)=\frac{c}{B_2(x)}\exp\left\{2\int_z^x dy\frac{B_1(y)}{B_2(y)}\right\}, \quad (5.4)$$

where $z\in I$ is arbitrary and where the constant c is determined by requiring that the integral of $W(x)$ over I is unity:

$$c=\left\{\int_I dx[B_2(x)]^{-1}\exp\left[2\int_z^x dy\frac{B_1(y)}{B_2(y)}\right]\right\}^{-1}. \quad (5.5)$$

Making use of (5.3), from (5.4) and (5.5) we obtain:

$$W(x)=\beta^{\frac{2\beta\delta}{\sigma^2}+1}\frac{\Gamma\left(\frac{2r}{\sigma^2}\right)}{\Gamma\left(\frac{2\beta\delta}{\sigma^2}+1\right)\Gamma\left(\frac{2r}{\sigma^2}-\frac{2\beta\delta}{\sigma^2}\right)}\cdot x^{-\frac{2r}{\sigma^2}}(x-\beta)^{\frac{2r}{\sigma^2}-\frac{2\beta\delta}{\sigma^2}-1}\quad (r>\beta\delta), \quad (5.6)$$

after imposing that the boundary $x=\infty$ is reflecting if $2\beta\delta<\sigma^2$ (which implies zero flux on the boundary) and by requiring that the flux is vanishing at $x=\infty$ if $2\beta\delta>\sigma^2$.

Indeed, it is easily seen that

$$\exp\left\{2\int_z^x dy\frac{B_1(y)}{B_2(y)}\right\}=K(z)\,(x-\beta)^{\frac{2}{\sigma^2}(r-\beta\delta)+1}x^{-\frac{2r}{\sigma^2}+1}, \quad (5.7)$$

where $K(z)$ denotes a constant. Furthermore, one has:

$$c^{-1}\equiv\int_\beta^\infty dx B_2(x)\exp\left[2\int_z^x dy\frac{B_1(y)}{B_2(y)}\right]=K_1(z)\sigma^{-2}\beta^{1-\frac{2\beta\delta}{\sigma^2}}B\left[1+\frac{2\beta\delta}{\sigma^2},\frac{2}{\sigma^2}(r-\beta\delta)\right], \quad (5.8)$$

where $B(x,y)$ is the Euler beta-function defined as (Gradshteyn and Ryzhik, 1965, No. 8.380):

$$B(x,y)=2\int_0^1 dt\, t^{2x-1}(1-t^2)^{y-1} \qquad (\mathrm{Re}\,x>\mathrm{Re}\,y>0) \tag{5.9}$$

and where use has been made of the identity:

$$B(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{5.10}$$

The steady state distribution expressed by (5.6) can be used to calculate the asymptotic moments of the population. Skipping the straightforward, although rather cumbersome, calculations we limit ourselves to report the results. The mean (asymptotic) value turns out to be

$$E(x)=\frac{r}{\delta} \tag{5.11}$$

thus coinciding with the deterministic carrying capacity. The second order moment is instead

$$E(x^2)=\begin{cases}\dfrac{\beta r}{\delta}\,\dfrac{2r-\sigma^2}{2\beta\delta-\sigma^2}, & \text{if } 2\beta\delta>\sigma^2\\ \infty, & \text{otherways}.\end{cases} \tag{5.12}$$

In general, we can prove that for all integer j if $\frac{\beta\delta}{j-1}>\frac{\sigma^2}{2}$ one has

$$E(x^j)=\beta^j\frac{\Gamma\left(\frac{2r}{\sigma^2}+1\right)\Gamma\left(\frac{2\beta\delta}{\sigma^2}-j+1\right)}{\Gamma\left(\frac{2\beta\delta}{\sigma^2}+1\right)\Gamma\left(\frac{2r}{\sigma^2}-j+1\right)}, \tag{5.13}$$

whereas they are otherways infinite. Making use of (5.11) and (5.12) the variance of the asymptotic population size can be obtained.

To further elucidate the features of the random process under consideration, let denote by $\varepsilon(\beta<\varepsilon<x_0)$ and by $\eta(x_0<\eta<\infty)$ arbitrary values and let $P(\varepsilon|x_0)$ and $P(\eta|x_0)$ denote the first passage time probability of $X(t)$ through the states ε and η, respectively, under the condition that all its sample paths originate at x_0.

Making then use of formulas (4.51)$\sim$(4.54) p. 111 of Ricciardi (1977), after a rather long sequence of calculations one obtains:

$$P(\varepsilon|x_0)=\left(\frac{\varepsilon-\beta}{x_0-\beta}\right)^{1-\frac{2\beta\delta}{\sigma^2}}\cdot\frac{F\left[1-\frac{2r}{\sigma^2};1-\frac{2\beta\delta}{\sigma^2};2\left(1-\frac{\beta\delta}{\sigma^2}\right);-\frac{\beta}{x_0-\beta}\right]}{F\left[1-\frac{2r}{\sigma^2};1-\frac{2\beta\delta}{\sigma^2};2\left(1-\frac{\beta\delta}{\sigma^2}\right);-\frac{\beta}{\varepsilon-\beta}\right]} \tag{5.14}$$

if $\beta<\varepsilon<x_0$ and $2\beta\delta<\sigma^2$ whereas one has $P(\varepsilon|x_0)=1$ when such inequalities are not satisfied.

In (5.14) F is the hypergeometric function

$$F(\alpha;\beta;\gamma;z)=1+\sum_{k=1}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}. \tag{5.15}$$

One can finally prove that $P(\eta|x_0)$ is unity for all $\eta>x_0$.

From the foregoing considerations we conclude that $X(t)$ always attains an equilibrium regime that is independent of the initial state. Furthermore, for each initial state x_0 and for each state $\eta>x_0$ with certainty the process originating in the former will eventually reach the latter.

For the process under consideration it is also possible to obtain an explicit expression for the mean time, $t_1(\eta|x_0)$, necessary to visit the state η for the first time starting from the initial state x_0. Making use of a formula due to Siegert (1951) we indeed have:

$$\begin{aligned} t_1(\eta|x_0)\equiv{}&\int_{x_0}^{\eta}dz\frac{2}{B(z)\,W(z)}\int_{\beta}^{z}dy\,W(y)=\frac{1}{r-\beta\delta}\left\{\ln\frac{\eta-\beta}{x_0-\beta}-\ln\frac{\eta}{x_0}\right.\\ &+\frac{2\beta\delta}{2r+\sigma^2-2\beta\delta}\left[\frac{x_0-\eta}{\beta}+\ln\frac{\eta}{x_0}\right]+\sum_{k=1}^{\infty}\frac{(-2\beta\delta)_k}{(2r+\sigma^2-2\beta\delta)_k}\frac{1}{k+1}\left[\left(\frac{\beta-\eta}{\beta}\right)^k-\left(\frac{\beta-x_0}{\beta}\right)^k\right]\\ &+\sum_{k=2}^{\infty}\frac{(-2\beta\delta)_k}{(2r+\sigma^2-2\beta\delta)_k}\frac{1}{k+1}\\ &\cdot\left[\left(\frac{\beta-\eta}{\beta}\right)^{k+1}F\left(1;k+1;k+2;\frac{\beta-\eta}{\beta}\right)-\left(\frac{\beta-x_0}{\beta}\right)^{k+1}F\left(1,k+1;k+2;\frac{\beta-x_0}{\beta}\right)\right], \end{aligned} \tag{5.16}$$

where F is the hypergeometric function earlier defined.

It is interesting to remark that for the case of the growth model (5.1) it is possible to construct and describe a second diffusion process $X(t)$ by expressing the fluctuations of r in terms of a Brownian motion. Indeed, we can write:

$$\begin{aligned} dX(t)={}&\left[-rX\left(1-\frac{X}{\beta}\right)+\delta X^2\left(1-\frac{X}{\beta}\right)\right]dt\\ &-X\left(1-\frac{X}{\beta}\right)dB(t), \end{aligned} \tag{5.17}$$

$$P\{X(0)=x_0\}=1,$$

where $B(t)$ is a Brownian motion having zero mean and correlation function given by:

$$E[B(t)\,B(t')] = \sigma^2 \min(t, t')\,. \tag{5.18}$$

By using Ito's calculus (cf. for instance, Jazwinski, 1970) one can then prove that $X(t)$ is a diffusion process having infinitesimal moments

$$\begin{aligned} B_1'(x) &= x\left(1-\frac{x}{\beta}\right)(\delta x - r)\,,\\ B_2'(x) &= \sigma^2 x^2\left(1-\frac{x}{\beta}\right)^2. \end{aligned} \tag{5.19}$$

The point β is a natural boundary whereas the point at infinity is an entrance boundary. Differently from the previous case, now (cf. Appendix 1) the steady state distribution exists if and only if $r-\beta\delta > \sigma^2/2$ and the asymptotic moment of order j ($j=1,2,\ldots$) exists finite if and only if $j < \frac{2\beta\delta}{\sigma^2}+3$ with the r.h.s. of this inequality being, in turn, less than $\frac{2r}{\sigma^2}+2$ due to the assumption that the steady state distribution exists. Furthermore, the probability for the population to reach asymptotically the state β is zero if $r-\beta\delta > \sigma^2/2$. Finally, one has $P(\varepsilon|x_0)=1$ for all states $\varepsilon < x_0$ and $P(\eta|x_0)$ unity, for all $\eta > x_0$, only if $r-\delta\beta \geqq \sigma^2/2$.

6 A Model with a Logarithmic Singularity

Similarly to what we did in Sect. 5, we now re-write Eq. (2.2) setting in it $f = s/c$ and then change s into the random process $s+\Lambda(t)$. We are thus lead to the fluctuation equation

$$\begin{aligned} \frac{dX}{dt} &= -sX\left(1-\frac{\ln X}{b}\right) + fX\,\ln X\left(1-\frac{\ln X}{b}\right)\\ &\quad - X\left(1-\frac{\ln X}{b}\right)\Lambda(t), \qquad (s > bf) \end{aligned} \tag{6.1}$$

$$P\{X(0)=x_0\} = 1\,.$$

By this procedure we model the growth of the population as a diffusion process characterized by the following drift and infinitesimal variance:

$$\begin{aligned} B_1(x) &= x\left(1-\frac{\ln x}{b}\right)\left[\left(f-\frac{\sigma^2}{2b}\right)\ln x - s + \frac{\sigma^2}{2}\left(1-\frac{1}{b}\right)\right],\\ B_2(x) &= \sigma^2 x^2\left(1-\frac{\ln x}{b}\right)^2. \end{aligned} \tag{6.2}$$

Let us now provide a quick study of the process $X(t)$ with reference to the diffusion interval $I \equiv (e^b, \infty)$. The first important feature of $X(t)$ is the existence of a steady state distribution $W(x)$. Making use of (5.4) and (5.5) one can show (Appendix 2) that:

$$\begin{aligned} W(x) &= \left(\frac{2bf}{\sigma^2}\right)^{\frac{2b}{\sigma^2}(s-bf)} \exp\left(\frac{2b^2 f}{\sigma^2}\right)\cdot\\ &\quad\cdot\left\{\Gamma\left[\frac{2b}{\sigma^2}(s-bf)\right]\right\}^{-1}\\ &\quad\cdot x^{-\left(1+\frac{2bf}{\sigma^2}\right)}(\ln x - b)^{\frac{2b}{\sigma^2}(s-bf)-1}\\ &\quad (e^b < x < \infty,\ s > bf)\,. \end{aligned} \tag{6.3}$$

However, in contrast to the model of Sect. 5 none of the moments of the asymptotic population size exist always. Indeed one has:

$$E(x^j) = \begin{cases} \exp\left[jb + \frac{2b}{\sigma^2}(s-bf)\ln\left(\frac{2bf}{2bf - j\sigma^2}\right)\right], & \text{if } j < \frac{2bf}{\sigma^2}\,,\\ \infty\,, & \text{otherways}\,. \end{cases} \tag{6.4}$$

Furthermore, $X(t)$ is now always recurrent since (Appendix 2):

$$P(\varepsilon|x_0) = 1,\ P(\eta|x_0) = 1\,, \quad \forall x_0, \varepsilon, \eta \in (e^b, \infty)\,. \tag{6.5}$$

For the model of this section the description can be extended in a way to obtain a closed form expression for the moments of the time necessary for the population to attain for the first time any preassigned size S for each fixed initial size. Indeed, the Laplace transform of the first passage time p.d.f. can be calculated so that the above moments at least in principle follow by differentiation of it. For brevity, we shall limit ourselves to determine the mean first passage time from the initial population size to any larger size while disregarding the specification of higher order moments. Some details are given in Appendix 2.

Let us denote by

$$g_\lambda(S|x_0) = \int_0^\infty dt\, e^{-\lambda t} g(S,t|x_0) \tag{6.6}$$

this Laplace transform. Here we have set

$$g(S,t|x_0) = \frac{d}{dt}P\{T \leqq t\} \tag{6.7}$$

with

$$T = \inf\{t : X(t) > S \,|\, X(0) = x_0\}\,. \tag{6.8}$$

Then, for each $\varepsilon < x_0$ we have (Appendix 2):

$$g_\lambda(\varepsilon|x_0)=\left(\frac{\ln x_0-b}{\ln\varepsilon-b}\right)^{A_1}\frac{\Psi\left[A_1,C_1;\frac{2bf}{\sigma^2}(\ln x_0-b)\right]}{\Psi\left[A_1,C_1;\frac{2bf}{\sigma^2}(\ln\varepsilon-b)\right]}. \quad (6.9)$$

Here we have set:

$$\Psi(a,c;x)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a,c;x)+\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\Phi(a-c+1,2-c;x), \quad (6.10)$$

where $\Phi(a,c;x)$ is the Kummer function:

$$\Phi(a,c;x)=1+\sum_{n=1}^{\infty}\frac{a(a+1)\dots(a+n-1)}{c(c+1)\dots(c+n-1)}\frac{x^n}{n!} \quad (6.11)$$

and where A_1 and C_1 (whose integer values should be excluded) are defined as follows:

$$A_1=-\frac{b}{\sigma^2}(s-bf)+\frac{b}{\sigma^2}[(s-bf)^2+2\lambda\sigma^2]^{1/2},$$
$$C_1=1+\frac{2b}{\sigma^2}[(s-bf)^2+2\lambda\sigma^2]^{1/2}. \quad (6.12)$$

For each $\eta > x_0$, instead, one has:

$$g_\lambda(\eta|x_0)=\left(\frac{\ln x_0-b}{\ln\eta-b}\right)^{A_1}\cdot\frac{\Phi\left[A_1,C_1;\frac{2bf}{\sigma^2}(\ln x_0-b)\right]}{\Phi\left[A_1,C_1;\frac{2bf}{\sigma^2}(\ln\eta-b)\right]} \quad (6.13)$$

having again excluded non integer values of C_1.

The mean first passage time $t_1(\eta|x_0)$ can finally be calculated. In Appendix 2 this is obtained in a straightforward way for all $\eta > x_0$ by means of a method due to Siegert (1951). The result is the following:

$$t_1(\eta|x_0)=\frac{b}{s-bf}\left\{\ln\frac{\ln\eta-b}{\ln x_0-b}+\sum_{n=1}^{\infty}\frac{1}{n}\frac{(2bf)^n}{[2b(s-bf)+\sigma^2]\dots[2b(s-bf)+n\sigma^2]}\cdot[(\ln\eta-b)^n-(\ln x_0-b)^n]\right\}. \quad (6.14)$$

The higher order moments $t_n(\eta|x_0)$ may be obtained as:

$$t_n(\eta|x_0)=(-1)^n\frac{d^n g_\lambda(\eta|x_0)}{d\lambda^n}\bigg|_{\lambda=0}, \quad (6.15)$$

with $g_\lambda(\eta|x_0)$ given by (6.9) and (6.13).

Similarly to the case of Sect. 5, a new diffusion process can be constructed by substituting (6.1) with the following stochastic equation:

$$dX(t)=\left[-sX\left(1-\frac{\ln X}{b}\right)+fX\ln X\left(1-\frac{\ln X}{b}\right)\right]dt-X\left(1-\frac{\ln X}{b}\right)dB(t) \quad (s>bf), \quad (6.16)$$

$$P\{X(0)=x_0\}=1.$$

The infinitesimal moments of $X(t)$ are therefore:

$$B_1'(x)=x\left(1-\frac{\ln x}{b}\right)(f\ln x-s),$$
$$B_2'(x)=\sigma^2x^2\left(1-\frac{\ln x}{b}\right)^2. \quad (6.17)$$

Differently from the model earlier discussed, now the steady state distribution exists if and only if $s-bf>\frac{\sigma^2}{2b}$ and the moments of all order $j(j=1,2,\dots)$ tend asymptotically to finite limits if and only if the following inequalities are satisfied:

$$(j-1)\sigma^2<2bf<2s-\frac{\sigma^2}{b}. \quad (6.18)$$

All this can be proved quite similarly to the case of Appendix 1. The analytical results can be immediately obtained by means of the formulas of Appendix 2 with a suitable choice of the parameters. Finally, one can prove that the probability of ultimately reaching the level β is zero if $s-bf>\sigma^2/(2b)$ and that $P(\varepsilon|x_0)=1$ for all $\varepsilon<x_0$ and $P(\eta|x_0)=1$ if and only if $s-bf\geqq\frac{\sigma^2}{2b}$. The expressions of $t_1(\eta|x_0)$ as well as the functions $g_\lambda(\varepsilon|x_0)$, $(\varepsilon<x_0)$, and $g_\lambda(\eta|x_0)$ $(\eta>x_0)$ can be immediately obtained from the formulas of Appendix 2.

Appendix 1

We shall consider the diffusion process having moments

$$\tilde{B}_1(x)=x\left(1-\frac{x}{\beta}\right)(px\theta q)$$
$$\tilde{B}_2(x)=\sigma^2x^2\left(1-\frac{x}{\beta}\right)^2. \quad (A1.1)$$

These reduce to moments (5.19) if we set $p=\delta$ and $q=r$ (with $q>\beta\delta$) whereas they yield (5.3) for $p=\delta-\frac{\sigma^2}{\beta}$ and $q=r-\frac{\sigma^2}{2}$ $\left(\text{with } q>\delta\beta-\frac{\sigma^2}{2}\right)$. Making use of (5.4) and of the normalization condition (5.5) the steady state distribution $W(x)$ can be obtained. With the expressions (A1.1) of the moments from (5.5) we see that $W(x)$ exists only if $q-\beta p>\frac{\sigma^2}{2}$. When this inequality is satisfied one has:

$$W(x)=\beta^{\frac{2b\rho}{\sigma^2}+3}\frac{\Gamma\left(\frac{2q}{\sigma^2}+2\right)}{\Gamma\left(\frac{2p\beta}{\sigma^2}+3\right)\Gamma\left(\frac{2q}{\sigma^2}-\frac{2p\beta}{\sigma^2}-1\right)}x^{-\frac{2q}{\sigma^2}-2}(x-\beta)^{\frac{2q}{\sigma^2}-\frac{2p\beta}{\sigma^2}-2}\qquad(\beta<x<\infty)\,. \tag{A1.2}$$

The asymptotic moments $E(x^j)$ $(j=1,2,\ldots)$ of the process can then be calculated as:

$$E(x^j)=\beta^{\frac{2p\beta}{\sigma^2}+3}\frac{\Gamma\left(\frac{2q}{\sigma^2}+2\right)}{\Gamma\left(\frac{2p\beta}{\sigma^2}+3\right)\Gamma\left(\frac{2q}{\sigma^2}-\frac{2p\beta}{\sigma^2}-1\right)}\int_\beta^\infty dx\, x^{-\frac{2q}{\sigma^2}+j-2}(x-\beta)^{\frac{2q}{\sigma^2}-\frac{2p\beta}{\sigma^2}-2}\,. \tag{A1.3}$$

Recalling (5.9) and (5.10) from (A1.3) we finally obtain:

$$E(x^j)=\beta^j\frac{\Gamma\left(\frac{2q}{\sigma^2}+2\right)\Gamma\left(\frac{2p\beta}{\sigma^2}-j+3\right)}{\Gamma\left(\frac{2p\beta}{\sigma^2}+3\right)\Gamma\left(\frac{2q}{\sigma^2}-j+2\right)},\qquad\left(q-p\beta>\frac{\sigma^2}{2};\,j<\frac{2p\beta}{\sigma^2}+3\right). \tag{A1.4}$$

Similar considerations on the convergence of the involved integrals lead us to the statements of the last sentence of Sect. 5. Indeed if $\frac{2p\beta}{\sigma^2}+1<0$ one has:

$$P(\varepsilon|x_0)=\left(\frac{x_0-\beta}{\varepsilon-\beta}\right)^{\frac{2p\beta}{\sigma^2}+1}\frac{F\left(-\frac{2q}{\sigma^2};-\frac{2p\beta}{\sigma^2}-1;-\frac{2p\beta}{\sigma^2};-\frac{\beta}{x_0-\beta}\right)}{F\left(-\frac{2q}{\sigma^2};-\frac{2p\beta}{\sigma^2}-1;-\frac{2p\beta}{\sigma^2};-\frac{\beta}{\varepsilon-\beta}\right)}, \tag{A1.5}$$

$P(\varepsilon|x_0)$ being otherways unity. Furthermore if $\frac{2q}{\sigma^2}-\frac{2p\beta}{\sigma^2}<1$, it is:

$$P(\eta|x_0)=\left(\frac{x_0-\beta}{\eta-\beta}\right)^{1+\frac{2p\beta}{\sigma^2}-\frac{2q}{\sigma^2}}\frac{F\left(-\frac{2q}{\sigma^2};\frac{2p\beta}{\sigma^2}-\frac{2q}{\sigma^2}+1;\frac{2p\beta}{\sigma^2}-\frac{2q}{\sigma^2}+2;\frac{\beta-x_0}{\beta}\right)}{F\left(-\frac{2q}{\sigma^2};\frac{2p\beta}{\sigma^2}-\frac{2q}{\sigma^2}+1;\frac{2p\beta}{\sigma^2}-\frac{2q}{\sigma^2}+2;\frac{\beta-\eta}{\beta}\right)} \tag{A1.6}$$

with $P(\eta|x_0)=1$ otherways. For completeness we write the expression of the average first passage time for the process having moments (A1.1). Making use of formula (5.16) one has:

$$t_1(\eta|x_0)=\frac{2\beta^2}{\sigma^2}\int_{x_0}^{\eta}dz\,z^{\frac{2q}{\sigma^2}}(z-\beta)^{\frac{2p\beta}{\sigma^2}-\frac{2q}{\sigma^2}}\int_\beta^z dy\,y^{-\frac{2q}{\sigma^2}-2}(y-\beta)^{-\frac{2p\beta}{\sigma^2}+\frac{2q}{\sigma^2}-2}\qquad(\eta>x_0)\,. \tag{A1.7}$$

Therefore, this moment is finite only if $q-\beta p>\sigma^2/2$. For the case of model (5.19) this inequality reads $r-\beta\delta>\sigma^2/2$. From (A1.7) one finally obtains:

$$\begin{aligned}t_1(\eta|x_0)=\frac{2}{2q-2p\beta-\sigma^2}\Bigg\{&\ln\frac{\eta-\beta}{x_0-\beta}+\sum_{k=1}^{\infty}\frac{(-2p\beta-2\sigma^2)_k}{(-2p\beta+2q)_k}\frac{1}{k+1}\left[\left(1-\frac{\eta}{\beta}\right)^k-\left(1-\frac{x_0}{\beta}\right)^k\right]\\&-\ln\frac{\eta}{x_0}+\frac{p\beta+\sigma^2}{q-p\beta}\left[\frac{x_0-\eta}{\beta}+\ln\frac{\eta}{x_0}\right]+\sum_{k=2}^{\infty}\frac{(-2p\beta-2\sigma^2)_k}{(-2p\beta+2q)_k}\frac{1}{k+1}\left[\left(1-\frac{\eta}{\beta}\right)^{k+1}F\left(1;k+1;k+2;1-\frac{\eta}{\beta}\right)\right.\\&\left.-\left(1-\frac{x_0}{\beta}\right)^{k+1}F\left(1;k+1;k+2;1-\frac{x_0}{\beta}\right)\right]\Bigg\},\qquad q-\beta p>\frac{\sigma^2}{2}.\end{aligned} \tag{A1.8}$$

Appendix 2

a) Steady State Distribution

Let us consider the diffusion process of moments

$$\tilde{B}_1(x)=x\left(1-\frac{\ln x}{b}\right)(p\ln x-q),$$
$$\tilde{B}_2(x)=\sigma^2x^2\left(1-\frac{\ln x}{b}\right)^2. \quad (A2.1)$$

This process identifies with that of moments (6.2) for

$$p=f-\frac{\sigma^2}{2b},$$
$$q=s-\frac{\sigma^2}{2}\left(1-\frac{1}{b}\right) \quad (A2.2)$$

whereas it yields that of moments (6.16) when

$$p=f,$$
$$q=s. \quad (A2.3)$$

Note that the condition $s>bf$, associated to models (6.1) and (6.15), becomes $q>bf-\frac{\sigma^2}{2}\left(1-\frac{1}{b}\right)$ when (A2.1) are identified with (6.2) while it yields $q>bf$ when (A2.1) are identified with (6.16). For the process having moments (A2.1) the steady state distribution can be calculated by means of (5.4). One then has:

$$\exp\left[2\int_z^x dy\frac{\tilde{B}_1(y)}{\tilde{B}_2(y)}\right]=c_1x^{-\frac{2bp}{\sigma^2}}|\ln x-b|^{-\frac{2b}{\sigma^2}(bp-q)} \quad (A2.4)$$

where c_1 denotes a constant of integration. Furthermore one has:

$$\left\{\int_{e^b}^{\infty}dx[\tilde{B}_2(x)]^{-1}\exp\left[2\int_z^x dy\frac{\tilde{B}_1(y)}{\tilde{B}_2(y)}\right]\right\}^{-1}=\frac{\sigma^2}{c_1b^2}\left(\frac{2bp}{\sigma^2}+1\right)^{-1+\frac{2b}{\sigma^2}(q-bp)}\exp\left[b\left(\frac{2bp}{\sigma^2}+1\right)\right]\left\{\Gamma\left[\frac{2b}{\sigma^2}(q-bp)-1\right]\right\}^{-1} \quad (A2.5)$$

if $q-bp>\frac{\sigma^2}{2b}$, whereas the integral on the l.h.s. of (A2.5) is divergent if $q-bp\leqq\frac{\sigma^2}{2b}$. According to (5.4) and (5.5) we can thus conclude that for the process of moments (A2.1) the steady state distribution $W(x)$ exists only if $q-bp>\frac{\sigma^2}{2b}$. In this case one has:

$$W(x)=\left(1+\frac{2bp}{\sigma^2}\right)^{\frac{2b}{\sigma^2}(q-bp)-1}\exp\left[b\left(1+\frac{2bp}{\sigma^2}\right)\right]\left\{\Gamma\left[\frac{2b}{\sigma^2}(q-bp)-1\right]\right\}^{-1}x^{-\left(2+\frac{2bp}{\sigma^2}\right)}(\ln x-b)^{\frac{2b}{\sigma^2}(q-bp)-2}. \quad (A2.6)$$

Recalling (A2.2) and (A2.3) we thus conclude that for the process having moments (6.2) the steady state distribution always exists while from (A2.6) expression (6.3) of the text follows. Instead, in the case of the process having moments (6.16) the steady state distribution exists only if it is $s>bf+\frac{\sigma^2}{2b}$. Note the analogy of these results with those pointed out for the first time by May within the context of logistic growth processes (May, 1973; Feldman and Roughgarden, 1975).

b) Asymptotic Moments

The asymptotic moments of any order can be obtained by making use of (A2.6). The result is

$$E(x^j)=M\left(1+\frac{2bp}{\sigma^2}\right)^{\frac{2b}{\sigma^2}(q-bp)-1}\exp\left[b\left(1+\frac{2bp}{\sigma^2}\right)\right]\left\{\Gamma\left[\frac{2b}{\sigma^2}(q-bp)-1\right]\right\}^{-1}, \quad (A2.7)$$

where we have set:

$$M=\int_{e^b}^{\infty}dx\,x^{j-2\left(1+\frac{bp}{\sigma^2}\right)}(\ln x-b)^{\frac{2b}{\sigma^2}(q-bp)-2}. \quad (A2.8)$$

The integral (A2.8) is convergent for $j<\frac{2bp}{\sigma^2}+1$. Hence (cf. Gradshteyn and Ryzhik, 1965, No. 3.382) we obtain:

$$E(x^j)=\exp\left\{bj+\left[\frac{2b}{\sigma^2}(q-bp)-1\right]\ln\frac{2bp+\sigma^2}{2bp-(j-1)\sigma^2}\right\}\quad\left(j<\frac{2bp}{\sigma^2}+1\right), \quad (A2.9)$$

whereas the moment of order j does not exist for $j \geqq \frac{2bp}{\sigma^2}+1$. From (A2.9) expressions (6.4) and the moments for the case of model (6.16) immediately follow.

c) Recurrence of the Process

Let ε be such that $x_0<\varepsilon<e^b$, being otherways arbitrary. Making use of (4.51)~(4.54) of Ricciardi (1977, p. 111) one obtains:

$$P(\varepsilon|x_0)=1-\frac{\int_{\varepsilon}^{x_0} dz\, z^{\frac{2bp}{\sigma^2}}(\ln z-b)^{\frac{2b}{\sigma^2}(bp-q)}}{\int_{\varepsilon}^{\infty} dz\, z^{\frac{2bp}{\sigma^2}}(\ln z-b)^{\frac{2b}{\sigma^2}(bp-q)}}. \tag{A2.10}$$

The integral on the denominator on the r.h.s. of (A2.10) converges only if $p<-\frac{\sigma^2}{2b}$, which cannot occur in our models. Hence, $P(\varepsilon|x_0)=1$. Furthermore, for each η $(\eta>x_0)$ one has:

$$P(\eta|x_0)=1-\frac{\int_{x_0}^{\eta} dz\, z^{\frac{2bp}{\sigma^2}}(\ln z-b)^{\frac{2b}{\sigma^2}(bp-q)}}{\int_{e^b}^{\eta} dz\, z^{\frac{2bp}{\sigma^2}}(\ln z-b)^{\frac{2b}{\sigma^2}(bp-q)}}, \tag{A2.11}$$

with the integral in the denominator being convergent if $q-bp<\frac{\sigma^2}{2b}$ and otherways divergent. Recalling (A2.2) and (A2.3) one thus concludes that $P(\eta|x_0)=1$ for all $\eta>x_0$ in the case of the process of moments (6.2). For the case of the process of moments (6.16) $P(\eta|x_0)$ is unity only if $s-bf \geqq \frac{\sigma^2}{2b}$. When $s-bf<\frac{\sigma^2}{2b}$ one instead obtains (Gradshteyn and Ryzhik, 1965, No. 3.381):

$$P(\eta|x_0)=\frac{\gamma\left[1-\frac{2b}{\sigma^2}(s-bf),\left(1+\frac{2bf}{\sigma^2}\right)(b-\ln x_0)\right]}{\gamma\left[1-\frac{2b}{\sigma^2}(s-bf),\left(1+\frac{2bf}{\sigma^2}\right)(b-\ln \eta)\right]}, \tag{A2.12}$$

where γ denotes the incomplete gamma-function.

d) Determination of $g_\lambda(S|x_0)$

As is well known, $g_\lambda(S|x_0)$ is solution of equation

$$\frac{1}{2}\sigma^2 x_0^2\left(1-\frac{\ln x_0}{b}\right)^2\frac{d^2 g_\lambda}{dx_0^2}+x_0\left(1-\frac{\ln x_0}{b}\right)(p\ln x_0-q)\frac{dg_\lambda}{dx_0}-\lambda g_\lambda=0 \tag{A2.13}$$

whose coefficients are the moments (A2.1). By means of the substitution

$$\begin{aligned} &z=\frac{1}{b}\ln x_0-1,\\ &\omega(z)=z^{-A}g_\lambda(z),\\ &A\equiv\frac{1}{2}\left\{1-\frac{2b}{\sigma^2}(q-bp)+\left[\left(1-\frac{2b}{\sigma^2}(q-bp)\right)^2+\frac{8b^2\lambda}{\sigma^2}\right]^{1/2}\right\}>0, \end{aligned} \tag{A2.14}$$

equation (A2.13) takes the following form:

$$z\frac{d^2\omega}{dz^2}+\left[2A+\frac{2b}{\sigma^2}(q-bp)-\frac{2b}{\sigma^2}\left(\frac{\sigma^2}{2}+bp\right)z\right]\frac{d\omega}{dz}-\frac{2b}{\sigma^2}A\left(\frac{\sigma^2}{2}+bp\right)\omega=0. \tag{A2.15}$$

Setting then

$$\zeta=b\left(1+\frac{2bp}{\sigma^2}\right)z \tag{A2.16}$$

from (A2.15) one obtains:

$$\zeta\frac{d^2\omega}{d\zeta^2}+\left[2A+\frac{2b}{\sigma^2}(q-bp)-\zeta\right]\frac{d\omega}{d\zeta}-A\zeta=0 \tag{A2.17}$$

whose general solution is (Tricomi, 1954):

$$\omega(\zeta)=R\Phi(A,C;\zeta)+Q\zeta^{1-C}\Phi(A-C+1,2-C;\zeta), \tag{A2.18}$$

where

$$C=2A+\frac{2b}{\sigma^2}(q-bp) \tag{A2.19}$$

and where R and Q denote arbitrary constants. Making use of (A2.14) and (A2.16) we thus obtain:

$$\begin{aligned} g_\lambda(x_0)=\left(\frac{\ln x_0}{b}-1\right)^A\Bigg\{R\Phi\left[A,C;b\left(1+\frac{2bp}{\sigma^2}\right)\left(\frac{\ln x_0}{b}-1\right)\right] \\ +Q\left(\frac{\ln x_0}{b}-1\right)^{1-C}\left[b\left(1+\frac{2bp}{\sigma^2}\right)\right]^{1-C}\Phi\left[A-C+1,2-C;b\left(1+\frac{2bp}{\sigma^2}\right)\left(\frac{\ln x_0}{b}-1\right)\right]\Bigg\}. \end{aligned} \tag{A2.20}$$

As customary in this type of problem, constants R and Q can be determined by setting an artificial absorbing boundary at some state h and by looking for the unique solution $g_{\lambda,h}$ of (A2.13) such that

$$\begin{aligned} &\lim_{x_0\to S} g_{\lambda,h}(x_0)=1, \\ &\lim_{x_0\to h} g_{\lambda,h}(x_0)=0. \end{aligned} \tag{A2.21}$$

The desired function $g_\lambda(S|x_0)$ is then obtained as the limit of such solution as h tends to one of the end points of the diffusion interval without passing through S. Note that such limit amounts to removing the artificial barrier h.

We first consider the case $S\equiv\eta>x_0$. We then take $h<\eta$ and pass to the limit as h tends to e^b. Recalling (A2.20) and making use of (A2.21) we obtain:

$$\begin{aligned} R&=\frac{\left(\frac{\ln\eta}{b}-1\right)^{-A}}{\varphi_1(\eta)-\left(\frac{\ln\eta}{b}-1\right)^{1-C}\left(\frac{\ln h}{b}-1\right)^{C-1}\varphi_2(\eta)\frac{\varphi_1(h)}{\varphi_2(h)}}, \\ Q&=\frac{\left(\frac{\ln\eta}{b}-1\right)^{-A}\left[b\left(1+\frac{2bp}{\sigma^2}\right)\right]^{C-1}}{\left(\frac{\ln\eta}{b}-1\right)^{1-C}\varphi_2(\eta)-\varphi_1(\eta)\left(\frac{\ln h}{b}-1\right)^{1-C}\frac{\varphi_2(h)}{\varphi_1(h)}}, \end{aligned} \tag{A2.22}$$

where

$$\begin{aligned} \varphi_1(w)&=\Phi\left[A,C;b\left(1+\frac{2bp}{\sigma^2}\right)\left(\frac{\ln w}{b}-1\right)\right], \\ \varphi_2(w)&=\Phi\left[A-C+1,2-C;b\left(1+\frac{2bp}{\sigma^2}\right)\left(\frac{\ln w}{b}-1\right)\right]. \end{aligned} \tag{A2.23}$$

Taking the limit of (A2.22) as $h\to e^b$ we then get:

$$\begin{aligned} \lim_{h\to e^b} R&=\theta(C-1)\,[\varphi_1(\eta)]^{-1}\left(\frac{\ln\eta}{b}-1\right)^{-A}, \\ \lim_{h\to e^b} Q&=\theta(1-C)\,[\varphi_2(\eta)]^{-1}\frac{\left(\frac{\ln\eta}{b}-1\right)^{C-A-1}}{\left[b\left(1+\frac{2bp}{\sigma^2}\right)\right]^{1-C}}, \end{aligned} \tag{A2.24}$$

where $\theta(\cdot)$ denotes the Heaviside unit step function. From (A2.20) and (A2.24) we finally obtain:

$$g_\lambda(\eta|x_0)=\begin{cases} \dfrac{\varphi_2(x_0)}{\varphi_2(\eta)}, & C<1, \\ \left(\dfrac{\ln x_0-b}{\ln\eta-b}\right)^A\dfrac{\varphi_1(x_0)}{\varphi_1(\eta)}, & C>1. \end{cases} \tag{A2.25}$$

In the case of model (6.2), for which it is always $C<1$, expression (A2.25) yields (6.13) of the text.

A similar calculation leads to the expression of $g_\lambda(\varepsilon|x_0)$ with $e^b<\varepsilon<x_0$. Now, the conditions to be imposed are:

$$\begin{aligned} &g_\lambda(\varepsilon|\varepsilon)=1, \\ &g_\lambda(\varepsilon|h)=0. \end{aligned} \tag{A2.26}$$

Thus doing, constants R and Q appearing in (A2.20) are determined as:

$$R=\left(\frac{\ln\varepsilon}{b}-1\right)^{-A}\left\{\varphi_1(\varepsilon)-\left(\frac{\ln\varepsilon}{b}-1\right)^{1-C}\left(\frac{\ln h}{b}-1\right)^{C-1}\varphi_2(\varepsilon)\frac{\varphi_1(h)}{\varphi_2(h)}\right\}^{-1},$$

$$Q=\left(\frac{\ln\varepsilon}{b}-1\right)^{-A}\left[b\left(1-\frac{2bp}{\sigma^2}\right)\right]^{C-1}\left\{\left(\frac{\ln\varepsilon}{b}-1\right)^{1-C}\varphi_2(\varepsilon)-\varphi_1(\varepsilon)\left(\frac{\ln h}{b}-1\right)^{1-C}\frac{\varphi_2(h)}{\varphi_1(h)}\right\}^{-1}. \tag{A2.27}$$

To calculate the limit of (A2.27) as $h\to\infty$ it is convenient to express the functions φ in terms of function Ψ defined by (6.10). We then have:

$$\left(\frac{\ln h}{b}-1\right)^{1-C}\frac{\varphi_2(h)}{\varphi_1(h)}=\frac{\Gamma(A)}{\Gamma(C-1)}\left[b\left(1+\frac{2bp}{\sigma^2}\right)\right]^{C-1}\cdot\left\{\frac{\Psi\left[A,C;b\left(1+\frac{2bp}{\sigma^2}\right)\left(\frac{\ln h}{b}-1\right)\right]}{\varphi_1(h)}-\frac{\Gamma(1-C)}{\Gamma(A-C+1)}\right\} \tag{A2.28}$$

whose limit as $h\to\infty$ is easily obtained since for $A>0$ φ_1 diverges while Ψ goes to zero.

In the limit as $h\to\infty$ we thus finally obtain

$$g_\lambda(\varepsilon|x_0)=\left(\frac{\ln x_0-b}{\ln\varepsilon-b}\right)^A\frac{\Psi\left[A,C;b\left(1+\frac{2bp}{\sigma^2}\right)\left(\frac{\ln x_0}{b}-1\right)\right]}{\Psi\left[A,C;b\left(1+\frac{2bp}{\sigma^2}\right)\left(\frac{\ln\varepsilon}{b}-1\right)\right]} \tag{A2.29}$$

which yields (6.9) of the text as a particular case.

From (A2.25) and (A2.29) one then easily concludes that for the process having moments (6.16) it is $P(\varepsilon|x_0)=P(\eta|x_0)=1$ for all ε and η such that $e^b<\varepsilon<x_0$, $x_0<\eta<\infty$.

e) Determination of $t_1(\eta|x_0)$

Even though the mean time necessary for the process to attain for the first time the state η starting from $x_0(\eta>x_0)$ can be obtained by calculating the derivative with respect to λ of the function (A2.25) for $\lambda=0$, we shall determine it more simply by making use of Siegert's formula:

$$t_1(\eta|x_0)=\int_{x_0}^{\eta}dz\frac{2}{\tilde{B}_2(z)\,W(z)}\int_{e^b}^{z}dy\,W(y), \tag{A2.30}$$

with $W(x)$ given by (A2.6). Hence:

$$t_1(\eta|x_0)=\frac{2b^2}{\sigma^2}\int_{x_0}^{\eta}dz\,z^{\frac{2bp}{\sigma^2}}(\ln z-b)^{\frac{2b}{\sigma^2}(bp-q)}\,U(z), \tag{A2.31}$$

where

$$U(z)\equiv\int_b^{\ln z}dy\exp\left[-\left(\frac{2bp}{\sigma^2}+1\right)y\right](y-b)^{-\frac{2b}{\sigma^2}(bp-q)-2}$$

$$=\exp\left[-b\left(1+\frac{2bp}{\sigma^2}\right)\right]\left(1+\frac{2bp}{\sigma^2}\right)^{1+\frac{2b}{\sigma^2}(bp-q)}\gamma\left[\frac{2b}{\sigma^2}(q-bp)-1,(\ln z-b)\left(1+\frac{2bp}{\sigma^2}\right)\right] \tag{A2.32}$$

if $q-bp>\sigma^2/(2b)$ (if, instead, $q-bp\leqq\sigma^2/(2b)$ the function U is divergent and thus $t_1=\infty$). On the r.h.s. of (A2.32) γ is the incomplete gamma-function. From (A2.32) one easily gets:

$$t_1(\eta|x_0)=\frac{2b^2}{\sigma^2}\int_\tau^\theta dz\,e^z z^{-\frac{2b}{\sigma^2}(q-bp)}\gamma\left[\frac{2b}{\sigma^2}(q-bp)-1;z\right]\qquad\left(q-bp>\frac{\sigma^2}{2b}\right), \tag{A2.33}$$

where we have set:

$$\tau=(\ln x_0-b)\left(1+\frac{2bp}{\sigma^2}\right),$$

$$\theta=(\ln\eta-b)\left(1+\frac{2bp}{\sigma^2}\right). \tag{A2.34}$$

By expressing the incomplete γ-function in terms of Kummer's function (6.11) via the well known identity

$$\gamma(\zeta, x)=\frac{x^\zeta}{\zeta}e^{-x}\Phi(1,1+\zeta;x) \tag{A2.35}$$

after a term-by-term integration we finally obtain:

$$t_1(\eta|x_0)=\frac{2b^2}{\sigma^2}\left[\frac{2b}{\sigma^2}(q-bp)-1\right]^{-1}\left\{\ln\frac{\ln\eta-b}{\ln x_0-b}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(1+\frac{2bp}{\sigma^2}\right)^n}{\left[\frac{2b}{\sigma^2}(q-bp)\right]\left[\frac{2b}{\sigma^2}(q-bp)+1\right]\cdots\left[\frac{2b}{\sigma^2}(q-bp)+n-1\right]}[(\ln\eta-b)^n-(\ln x_0-b)^n]\right\}\quad\left(q-bp>\frac{\sigma^2}{2b}\right). \tag{A2.36}$$

From (A2.36) formula (6.14) of the text immediately follows while for the case of the process having moments (6.16) formula (A2.36) yields:

$$t_1(\eta|x_0)=\frac{2b^2}{\sigma^2}\left[\frac{2b}{\sigma^2}(s-bf)-1\right]^{-1}\left\{\ln\frac{\ln\eta-b}{\ln x_0-b}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(1+\frac{2bf}{\sigma^2}\right)^n}{\left[\frac{2b}{\sigma^2}(s-bf)\right]\left[\frac{2b}{\sigma^2}(s-bf)+1\right]\cdots\left[\frac{2b}{\sigma^2}(s-bf)+n-1\right]}[(\ln\eta-b)^n-(\ln x_0-b)^n]\right\}\quad\left(s-bf>\frac{\sigma^2}{2b}\right). \tag{A2.37}$$

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