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A Sperner-Type Theorem

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Abstract. Let $P = P_1 \times P_2 \times \cdots \times P_M$ be the direct product of symmetric chain orders $P_1, P_2, \ldots, P_M$. Let F be a subset of P containing no $l+1$ elements which are identical in $M-1$ components and linearly ordered in the Mth one. Then $\max |F| \leqslant c \cdot M^{1/2} \cdot l \cdot W(P)$, where $W(P)$ is the cardinality of the largest level of P, and c is independent of P, M and l. Infinitely many P show that this result is best possible for every M and l apart from the constant factor c.

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1. Introduction

It is known that the maximal cardinality of an antichain of a symmetric chain order P is $W(P)$, i.e., number of elements in the largest level of P. (For definitions and basic notations see the next section.) Katona [2] discovered a sharpening of this theorem: Let $P = P_1 \times P_2$ be the direct product of the symmetric chain orders P_1, P_2. Suppose that $F \subset P$ does not contain two elements $f_1 < f_2$ such that f_1 and f_2 are equal in one component. Then

$$|F| \leqslant W(P). \tag{1}$$

It was a natural question whether this conclusion held in the case when P was the direct product of M ($M > 2$) symmetric chain orders. Namely, let $P = P_1 \times P_2 \times \cdots \times P_M$ be the direct product of the symmetric chain orders $P_1, P_2, \ldots, P_M$, and let F not contain two elements $f_1 < f_2$ such that f_1 and f_2 are equal in $M-1$ components. Is it true that F satisfies (1)? It turned out to be false even in the case $M = 3$.

On the other hand, if P is a symmetric chain order and $F \subset P$ is a Sperner-k-family, i.e., F contains no $k+1$ elements which are totally ordered, then the cardinality of F does not exceed the number of elements contained in the k largest levels of P.

A natural combination of the above two conditions is the following one:

$$F \subset P \text{ contains no } l+1 \text{ elements}, f_1 < f_2 < \cdots < f_{l+1} \text{ such that } f_1 \text{ and } f_{l+1} \text{ are equal in } M-1 \text{ components.} \tag{2}$$

Griggs [1] proved that condition (2) implies the inequality

$$|F| \leqslant 2^{M-2} \cdot l \cdot W(P).$$

The present author [3] improved this estimate:

$$|F| \leqslant M \cdot l \cdot W(P).$$

The aim of this paper is to prove the following theorem:

THEOREM 1. *Let $P = P_1 \times P_2 \times \cdots \times P_M$ be the direct product of the symmetric chain orders $P_1, P_2, \ldots, P_M$, and suppose that $F \subset P$ satisfies condition* (2). *Then*

$$|F| \leqslant c_1 \cdot M^{1/2} \cdot l \cdot W(P)$$

and this is the best possible upper bound, i.e., *for every M and l there are $P_1, P_2, \ldots, P_M$ symmetric chain orders and an $F \subset P = P_1 \times P_2 \times \cdots \times P_M$ satisfying condition* (2) *such that*

$$|F| \geqslant c_2 \cdot M^{1/2} \cdot l \cdot W(P)$$

where c_1 and c_2 are absolute constants.

2. Definitions and Notations

A poset is called *ranked* if there is a rank function $r: P \to \mathbb{N}$ (nonnegative integers) such that $r(a) = 0$ for some $a \in P$ and, for any $a, b \in P$ such that $a < b$ and there is no c with $a < c < b$ then $r(b) = r(a) + 1$.

A ranked poset is called a *symmetric chain order* (see [2]) if P has a partition $P = C_1 \cup C_2 \cup \cdots \cup C_t$ where each C_i, $C_i = \{a_1, a_2, \ldots, a_{S_i}\}$, is a *symmetric chain* that is

$$r(a_{S_i}) = r(a_{S_i - 1}) + 1 = \cdots = r(a_1) + S_i - 1$$

and

$$r(a_1) + r(a_{S_i}) = \max_{a \in P} r(a) = n.$$

Levels of P are the collections of elements which are identical in rank. $W(P)$ denotes the number of elements of the largest level that is the elements of rank $\lfloor n/2 \rfloor$.

Let $\mathbf{k} = (k_1, k_2, \ldots, k_s)$ be a vector with nonnegative integer components such that $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_S$, C^{k_i} denotes the k_i element chain, that is $C^{k_i} = \{0, 1, \ldots, k_i - 1\}$ ordered as natural numbers. Then let $Q^{\mathbf{k}} = C^{k_1} \times C^{k_2} \times \cdots \times C^{k_S}$ be an *s-dimensional rectangle.* Levels of $Q^{\mathbf{k}}$ are the collections of elements whose component sums are equal. We will write simply $W(\mathbf{k})$ instead of $W(Q^{\mathbf{k}})$.

3. Proof of Theorem 1

The papers mentioned above [1–3] used the same ideas to prove Sperner-type theorems. Their crucial point was the investigation of the special case of an M-dimensional rectangle. We are able to prove the best statement in this special case:

THEOREM 2. *Let* $\mathbf{k} = (k_1, k_2, \ldots, k_M)$, $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_M$ *and* $F \subset Q^{\mathbf{k}}$ *satisfy condition* (2). *Then*

$$|F| \leqslant c_1 \cdot M^{1/2} \cdot l^* \cdot W(\mathbf{k})$$

where $l^* = \min(l, k_M)$. *On the other hand, there are* $\mathbf{k}$ *and* $F \subset Q^{\mathbf{k}}$ *to any* M *and* l *satisfying condition* (2) *and*

$$|F| \geqslant c_2 \cdot M^{1/2} \cdot l \cdot W(\mathbf{k}).$$

First we show how Theorem 2 implies Theorem 1, by Griggs's [1] and Katona's [2] method. The lower bound in Theorem 1 immediately follows from Theorem 2. For proving the upper bound let us partition each poset P_j, $1 \leqslant j \leqslant M$, into symmetric chains. For each choice of one such chain from each P_j, the product is an M-dimensional rectangle centered at the middle level of P. In this way, a partition of the entire poset P into symmetric rectangles is obtained. In each such rectangle, no $l + 1$ elements of F may be ordered in one component and equal in the remaining $M - 1$ components. Since the middle level of each rectangle is also the middle level of P, it suffices to prove the theorem in the case that P itself is just a rectangle.

Proof of Theorem 2. First we note that $Q^{\mathbf{k}}$ can be represented as a set of vectors $\mathbf{x} = (x_1, \ldots, x_M)$ where the x_i's are integers, $0 \leqslant x_i \leqslant k_i - 1$ and $\mathbf{x} \leqslant \mathbf{y}$ iff $x_i \leqslant y_i$ for every $1 \leqslant i \leqslant M$. Condition (2) reduces to the following one:

F contains no $l + 1$ elements which are equal in $M - 1$ components. (3)

LEMMA 1. *Let* $F \subset Q^{\mathbf{k}}$ *satisfy* (3). *Then*

$$|F| \leqslant k_1 \cdot k_2 \cdot \cdots \cdot k_{M-1} \cdot l^*.$$

Proof. By the condition, for every choice of the first $M - 1$ components we have only l^* possible for the Mth one. □

LEMMA 2. *For any* $\mathbf{k}$ *there is a subset* F *of* $Q^{\mathbf{k}}$ *such that* F *satisfies condition* (3) *and*

$$|F| = k_1 \cdot k_2 \cdot \cdots \cdot k_{M-1} \cdot l^*.$$

Proof. Let F be defined in the following way: Let F_S denote the collection of elements of $Q^{\mathbf{k}}$ whose component sums are congruent to S modulo k_M so

$$F_S = \{\mathbf{f} \in Q^{\mathbf{k}} : \mathbf{f} = (f_1, \ldots, f_M),\ \sum_{i=1}^{M} f_i \equiv S \bmod k_M\}.$$

Then take

$$F = F_0 \cup F_1 \cup \cdots \cup F_{l^*-1}.$$

It is clear that $|F| = k_1 \cdot k_2 \cdot \cdots \cdot k_{M-1} \cdot l^*$. Let us indirectly suppose that there are $l + 1$ elements of F which are equal in $M - 1$ components. Obviously, there are two of them, say $\mathbf{f}$ and $\mathbf{g}$, whose component sums are congruent modulo k_M. Then $d = \Sigma_{i=1}^{M} f_i - \Sigma_{i=1}^{M} g_i \equiv 0 \bmod k_M$ holds and $\Sigma_{i=1}^{M} f_i - \Sigma_{i=1}^{M} g_i = f_j - g_j$ for some $1 \leqslant j \leqslant M$, implies

$-k_M < d < k_M$ and $d = 0$. Hence, $\mathbf{f} = \mathbf{g}$ follows which is a contradiction. □

The next lemma says that the hypercube maximizes the ratio of $k_1 \cdot k_2 \cdot \cdots \cdot k_{M-1}$ over the size of the largest level.

LEMMA 3. *Let* $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_t < k_{t+1} \leqslant \cdots \leqslant k_M$ *and*

$$\tilde{k}_i = \begin{cases} k_i & \text{if } i \neq t, \\ k_i + 1 & \text{if } i = t. \end{cases}$$

Then

$$\frac{k_1 \cdot k_2 \cdot \cdots \cdot k_{M-1}}{W(\mathbf{k})} \leqslant \frac{\tilde{k}_1 \cdot \tilde{k}_2 \cdot \cdots \cdot \tilde{k}_{M-1}}{W(\tilde{\mathbf{k}})}.$$

Proof. We will prove the equivalent inequality

$$W(\tilde{\mathbf{k}}) \leqslant \frac{k_t + 1}{k_t} \cdot W(\mathbf{k}).$$

When we fix the tth component in $Q^{\mathbf{k}}$ or $Q^{\tilde{\mathbf{k}}}$ we obtain symmetric chain orders which are isomorphic to the same $M-1$-dimensional rectangle. The inequality follows immediately from the fact that $W(\tilde{\mathbf{k}})$ is equal to the sum of the cardinalities of the $k_t + 1$ largest levels in this rectangle and $W(\mathbf{k})$ is equal to the sum of the cardinalities of the k_t largest levels in the $M-1$-dimensional rectangle. □

Now, we only have to determine the cardinality of the middle level of an M-dimensional cube. It is known [4] that

$$W(\underbrace{C^k \times \cdots \times C^k}_{M\text{-times}}) = \frac{k^M}{(2\pi M)^{1/2} \cdot \sigma} \cdot \exp\left[\frac{-\left(\left\lfloor\frac{M(k-1)}{2}\right\rfloor - \frac{M(k-1)}{2}\right)^2}{2\sigma^2 M}\right] + \\ + O\left(\frac{k^{M-1}}{M}\right)$$

where

$$\sigma^2 = \frac{1}{k}\sum_{j=0}^{k-1} j^2 - \left(\frac{1}{k}\sum_{j=0}^{k-1} j\right)^2 = \frac{k^2-1}{12}.$$

This shows that

(i) $\tilde{C}_1 \dfrac{k^{M-1}}{M^{1/2}} \leqslant W(C^k \times \cdots \times C^k)$

(ii) $\tilde{C}_2 \dfrac{k^{M-1}}{M^{1/2}} \geqslant W(C^k \times \cdots \times C^k)$.

Lemmas 1 and 3 and (i) imply the upper bound because

$$\frac{|F|}{W(P)} \leqslant \frac{k_M^{M-1} \cdot l^*}{W(C^{k_M} \times \cdots \times C^{k_M})} \leqslant \frac{k_M^{M-1} \cdot l^*}{\tilde{C}_1 \cdot k_M^{M-1} \cdot M^{-1/2}} = C_1 \cdot M^{1/2} \cdot l^*.$$

Lemma 2 and (ii) imply the lower bound in a similar way, hence the proof of Theorem 2 is completed. □

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