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Random Integral Representations for Classes of Limit Distributions Similar to Levy Class L_0

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Summary. For a bounded linear operator Q, on a Banach space E, and a real number β, there are introduced classes, $\mathcal{U}_\beta(Q)$, of some limit distributions such that $\mathcal{U}_0(I)$ coincides with the Lévy class L_0. Elements from $\mathcal{U}_\beta(Q)$ are characterized in terms of convolution equations and as probability distributions of some random integral functionals. The continuity and fixed points of this random mapping is studied. It is shown that fixed points coincide with the class of Q-stable measures.

0. Introduction

The *Lévy class* L_0, called also the class of *self-decomposable measures*, is defined as a class of limit distributions of sequences of the form

$$t_n(\xi_1+\xi_2+\ldots+\xi_n)+x_n, \tag{0.1}$$

where $t_n>0$, $x_n\in\mathbb{R}$, $(\xi_n)_{n\in\mathbb{N}}$ are independent random variables (rv's) and the triangular array $t_n\xi_j$ $(1\leqq j\leqq n\colon n\in\mathbb{N})$ is uniformly infinitesimal. If we replace the infinitesimality assumption by the assumption that $(\xi_n)_{n\in\mathbb{N}}$ are identically distributed then weak limits of (0.1) give the class, $\mathcal{S}$, of stable measures. Finally, the class, ID, of all infinitely divisible distributions coincides with the class of limit distributions of arbitrary uniformly infinitesimal triangular arrays with independent rv's in each row. Elements from ID and $\mathcal{S}$ were characterized in terms of characteristic functions (Fourier transforms) at the very beginning; cf. Loeve (1955), Sect. 22.3, Theorem A, and Sect. 23.4, Theorem B, respectively. Furthermore stable measures were very extensively studied during the last fifty years; cf. Linde (1983), Weron (1984). Characteristic functions for selfdecomposable measures were described by Urbanik (1968) by the extreme-point method (Choquet's Theorem). This method has been applied later on to characterize other limit laws. Other proofs based on a theory of convex functions and an

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extension of measures theorems were found by Sato (1980) and Jurek (1982a), respectively. All of these methods are of an analytic character without any appeal to stochastic methods. Quite recently the technique of random integral representation has been developed. Wolfe (1982) and Jurek-Vervaat (1983) characterized in this manner the class L_0, with completely different proofs. Moreover, the proof of Jurek-Vervaat uses the theory of stochastic processes. From the integral representation we get immediately the characteristic functions of measures from L_0 and the generators of the class L_0. The Jurek-Vervaat result says that a selfdecomposable measure can be viewed as a probability distribution of an integral functional (Laplace transform) of a stochastic process with stationary independent increments. Another way of reading this characterization is that the class L_0 coincides with the class of limit distributions (as $t \to \infty$) of Ornstein-Uhlenbeck type processes, cf. Sato-Yamazato (1984).

It seems that the random integral representation is *the* method of describing classes of limit laws, that it is *the* connection between the theory of limit distributions and the theory of stochastic processes, which were developed quite separately from each other. Since this technique was used with success for many different classes of limit laws, in [8], p. 607 the following hypothesis was stated:

Each class of limit distributions, derived from sequences of independent random variables, is the image of some subset of ID by some mapping defined as a random integral.

In the present paper, we will introduce classes $\mathscr{U}_\beta(Q)$, for $\beta \in \mathbb{R}$ and a bounded linear operator Q on a Banach space E, of limit laws obtained from sequences of independent rv's. Then elements from $\mathscr{U}_\beta(Q)$ are identified by convolution equations in Theorem 1.1. In particular, Corollary 1.1(b) gives $L_0(I) = \mathscr{U}_0(I)$. The main result given in Theorem 1.2 says that for $\beta > 0$,

$$\mu \in \mathscr{U}_\beta(Q) \quad \text{if and only if } \mu = \mathscr{L}\Big(\int_{(0,1)} t^Q \, dY(t^\beta)\Big), \tag{0.2}$$

where $Y(t)$, $0 \leqq t \leqq 1$, is a Banach space valued process with stationary independent increments. Let us note that the classes $\mathscr{U}_{1-\alpha}(I)$ coincide with O'Connor (1979) classes L_α, for $0 < \alpha < 1$, which are defined by some monotonicity property of their Lévy spectral function. The relation (0.2) suggests to consider the following mapping:

$$\mathscr{I}_Q^\beta(\mathscr{L}(Y(1))) := \mathscr{L}\Big(\int_{(0,1)} t^Q \, dY(t^\beta)\Big) \in \mathscr{U}_\beta(Q). \tag{0.3}$$

Theorem 1.3(a) shows that $\mathscr{I}_Q^\beta$ is a homeomorphism between the topological semigroups ID and $\mathscr{U}_\beta(Q)$, for $\beta > 0$. Consequently, Corollary 1.2 gives so-called *generators* for the class $\mathscr{U}_\beta(Q)$, i.e., a set of elements from $\mathscr{U}_\beta(Q)$ that by taking their convolutions and weak limits, we obtain whole class $\mathscr{U}_\beta(Q)$. This is an analogue of the classical fact that the class ID is the smallest closed subsemigroup of $\mathscr{P}$ containing all Poisson measures; cf. Loeve (1955), Sect. 22.1, Theorem B. Finally, Theorem 1.4 shows that Q-stable measures are fixed points of the mappings $\mathcal{J}_Q^\beta$. More precisely, if $\mathcal{S}(Q)$ denotes the class of all Q-stable measures then $\mathcal{J}_Q^\beta(\mathcal{S}(Q))=\mathcal{S}(Q)$.

This paper is organized as follows: Sect. 1 contains notations and all main results. Two auxiliary lemmas are given in Sect. 2 for further references. Sect. 3 gives all proofs, and Sect. 4 contains some references and comments on previous works dealing with random integral representations.

1. Notations and Results

Let E be a real separable Banach space with topological dual E' and the bilinear form $\langle\cdot,\cdot\rangle$ between E' and E. Let $\mathcal{P}(E)$ and $ID(E)$ (or shortly: $\mathcal{P}$ and ID) be topological convolution semigroups of all Borel probability measures and all infinitely divisible ones, respectively, on E. By "$\Rightarrow$" we will denote the *weak convergence* (weak topology) and by "$*$" the *convolution* of measures. Recall that ID is closed (in weak topology) subsemigroup of $\mathcal{P}$ and each $\mu\in ID$ is uniquely determined by a triple: a vector $a\in E$, a Gaussian covariance operator R and a Lévy measure M. More precisely, if $\hat{\mu}$ is the *characteristic functional* (Fourier transform) of μ then for all $y\in E'$

$$(1.1)\qquad \hat{\mu}(y)=\exp\Big[i\langle y,a\rangle-\tfrac{1}{2}\langle y,Ry\rangle+\int_{E\setminus\{0\}}K(y,x)\,M(dx)\Big],$$

where $K(y,x):=e^{i\langle y,x\rangle}-1-i\langle y,x\rangle\,1_B(x)$ for $y\in E'$, $x\in E$ and 1_B is the indicator function of the unit ball in E. In the sequel, we will write $\mu=[a,R,M]$ if $\mu\in ID$ and $\hat{\mu}$ is of the form (1.1). Moreover, for $\mu=[a,R,M]$ and $t\geqq 0$, μ^{*t} is defined as follows: $\mu^{*t}:=[ta,tR,tM]$. Given a Borel measurable mapping f from E into E and a measure μ, we write $f\mu$ for the measure defined by means of the formula

$$(1.2)\qquad (f\mu)(F):=\mu(f^{-1}(F))\quad \text{for all Borel subsets } F \text{ of } E.$$

In particular, if $\mu,\nu\in\mathcal{P}$ and A, B are bounded linear operators on E, then

$$(1.3)\quad A(\mu*\nu)=A\mu*A\nu;\qquad A(B\mu)=(AB)\mu;\qquad (A\mu)\hat{}\,(y)=\hat{\mu}(A_y^*)\quad \text{for } y\in E',$$

where A^* denotes the adjoint operator of A. (This star should not be confused with the convolution of measures). Moreover, if $\mu,\nu\in ID$, A is a bounded linear operator on E and $t\geqq 0$ then $A\mu, A\nu\in ID$ and we also have.

$$(1.4)\qquad (A(\mu*\nu))^{*t}=A((\mu*\nu)^{*t})=A\mu^{*t}*A\nu^{*t},$$

$$(1.5)\qquad A[a,R,M]=[\tilde{a},ARA^*,AM],$$

where $\tilde{a}:=Aa+\int_{E\setminus\{0\}}(1_B(Ax)-1_B(x))\,Ax\,M(dx)$. For a *random variable* $(rv)\,X$, $\mathcal{L}(X)$ denotes its *probability distribution* and $\mathbb{E}X$ its expected value (Bochner integral). Further, $D_E[a,b]$ denotes the set of E-valued *cadlag* functions on $[a,b]$, i.e., functions that are right-continuous on $[a,b)$ and have left-hand limits on

$(a, b]$. Recall that $D_E[a, b]$ becomes a complete separable metric space under Skorohod metric, cf. Billingsley (1968). In the case $b=\infty$, we similarly define $D_E[a, \infty)$ and for details we refer to Pollard (1984), Chapter VI or to Lindval (1973). Finally for a bounded linear operator A on E, continuously differentiable real-valued function g on $[a, b]$, $D_E[0, \infty)$-valued rv Y and a monotone continuous function τ from $(a, b]$ into $(0, \infty)$ we define

$$\int_{(a,b]} e^{g(t)A}\, dY(\tau(t)) := e^{g(t)A}\, Y(\tau(t))\big|_{t=a}^{t=b} - \int_{(a,b]} (d e^{g(t)A})\, Y(\tau(t)) \tag{1.6}$$
$$= e^{g(b)A}(Y(\tau(b)) - Y(\tau(a))) - \int_{(a,b]} g'(t) A e^{g(t)A}(Y(\tau(t)) - Y(\tau(a)))\, dt.$$

If τ maps the interval $(a, b]$ onto finite interval in $[0, \infty)$ then the integral on the right-hand side there exists path-wise because of Lemma 14.1 in Billingsley (1968). The integrals on $[a, \infty)$ we define as a limit in probability as $b \to \infty$ in (1.6).

From now on, Q is a fixed bounded linear operator on E such that $t^Q := \exp(Q \ln t) \to 0$ (in the operator norm) as $t \to 0$. For a real number β, we say that $\mu \in \mathscr{P}$ belongs to *the class* $\mathscr{U}_\beta(Q)$ if there exists a sequence $(\nu_n) \subseteq ID$ such that

$$\rho_n := n^{-Q}(\nu_1 * \nu_2 * \ldots * \nu_n)^{*n^{-\beta}} \Rightarrow \mu. \tag{1.7}$$

Let us note that this definition is simpler but equivalent to that used in Jurek (1985a, 1985b) for $\beta = 1$.

Of course, $\mu \in ID$, i.e., the classes $\mathscr{U}_\beta(Q)$ are subclasses of the closed semigroup ID. In terms of convolution equations they are characterized as follows.

Theorem 1.1. *Let β be a real number and Q a bounded linear operator on Banach space E such that $t^Q \to 0$ as $t \to 0$. Then a measure $\mu \in ID$ belongs to the class $\mathscr{U}_\beta(Q)$ if and only if for each $0 < c < 1$ there exists a measure $\mu_c \in ID$ such that*

$$\mu = c^Q \mu^{*c^\beta} * \mu_c. \tag{1.8}$$

To explain the title of the present paper, let us recall that *the Lévy class* L_0 (or more general class $L_0(Q)$) is defined as follows: $\mu \in L_0(Q)$ if there exist sequences $(t_n)_{n\in\mathbb{N}}$ of positive real numbers, $(x_n)_{n\in\mathbb{N}}$ of vectors from E and measures $(\nu_n)_{n\in\mathbb{N}}$ such that

$$t_n^Q(\nu_1 * \nu_2 * \ldots * \nu_n) * \delta(x_n) \Rightarrow \mu \tag{1.9}$$

and the triangular array $t_n^Q \nu_j$ $(1 \leqq j \leqq n, n \in \mathbb{N})$ is *uniformly infinitesimal*, i.e., $\lim_{n\to\infty} \max_{1\leqq j\leqq n} (t_n^Q \nu_j)(\|x\| \geqq \varepsilon) = 0$ for $\varepsilon > 0$; cf. Jurek (1983). The following are simple consequences of Theorem 1.1.

Corollary 1.1. (a) *The classes $\mathscr{U}_\beta(Q)$ are closed convolution subsemigroups of ID.*

(b) *If $\alpha \leqq \beta$ then $\mathscr{U}_\alpha(Q) \subseteq \mathscr{U}_\beta(Q)$ and $\mathscr{U}_0(Q) = L_0(Q)$.*

(c) *If a linear bounded operator A commutes with Q then $A\mathscr{U}_\beta(Q) \subseteq \mathscr{U}_\beta(Q)$. In particular, if A is also invertible then $A\mathscr{U}_\beta(Q) = \mathscr{U}_\beta(Q)$.*

(d) *A measure* $\mu=[a, R, M]\in\mathscr{U}_\beta(Q)$ *if and only if for all* $0<c<1$ $R \geqq c^\beta(c^Q R c^{Q^*})$ *and* $M\geqq c^\beta\cdot(c^Q M)$.

(e) *If* $\beta>0$ *then* $\mathscr{U}_\beta(Q)=\mathscr{U}_1(\beta^{-1}Q)$.

(f) *If* $\mu\in\mathscr{U}_\beta(I)$ *and is nondegenerate then* $\beta\geqq -2$.

The next theorem describes the elements from the class $\mathscr{U}_\beta(Q)$ in terms of random integrals. Namely we have

Theorem 1.2. (a) *Let* $\beta>0$ *and* $\tau_\beta(t):=t^\beta$. Then $\mu\in\mathscr{U}_\beta(Q)$ *if and only if there exists a* $D_E[0, 1]$*-valued rv* Y *with stationary independent increments,* $Y(0)=0$ *a.s. such that*

$$\mu=\mathscr{L}\Big(\int_{(0,1)} t^Q\,dY(\tau_\beta(t))\Big)=\mathscr{L}\Big(\int_{(0,1)} t^{\beta^{-1}Q}\,dY(t)\Big).$$

Moreover, for μ_c *in* (1.8) *we have that* $\mu_c:=\mathscr{L}\Big(\int_{[c,1)} t^Q\,dY(\tau_\beta(t))\Big)$.

(b) *A measure* $\mu\in\mathscr{U}_0(Q)$ *if and only if there exists a* $D[0,\infty)$*-valued rv* Y *with stationary independent increments,* $Y(0)=0$ *a.s.,* $\mathbb{E}\log(1+\|Y(1)\|)<\infty$, *such that*

$$\mu=\mathscr{L}\Big(\int_{(0,1)} t^Q\,dY(-\ln t)\Big)=\mathscr{L}\Big(-\int_{(0,\infty)} e^{-tQ}\,dY(t)\Big).$$

Let $[a, R, M]:=\mathscr{L}(Y(1))$ and $[a^{(\beta)}, R^{(\beta)}, M^{(\beta)}]:=\mathscr{L}\Big(\int_{(0,1)} t^Q\,dY(\tau_\beta(t))\Big)$, for $\beta>0$.

Then using Lemma 2.2 (ii) and the formula (1.1) we obtain

$$a^{(\beta)}=\int_0^1 t^Q a\,d\tau_\beta(t)+\int_0^1\int_{E\setminus\{0\}}[1_B(t^Q x)-1_B(x)]\,t^Q x M(dx)\,d\tau_\beta(t); \tag{1.10}$$

$$R^{(\beta)}=\int_0^1 t^Q R t^{Q^*}\,d\tau_\beta(t); \tag{1.11}$$

$$M^{(\beta)}(F)=\int_0^1 (t^Q M)(F)\,d\tau_\beta(t)\qquad \text{for Borel subset } F \text{ of } E\setminus\{0\}. \tag{1.12}$$

The above and integral representations suggest to introduce mappings $\mathscr{I}_Q^\beta$ between the semigroup ID and the semigroups $\mathscr{U}_\beta(Q)$. Namely, for $\beta>0$ and $\nu\in ID$ let

$$\mathscr{I}_Q^\beta(\nu):=\mathscr{L}\Big(\int_{(0,1)} t^Q\,dY(\tau_\beta(t))\Big),$$

where Y is a $D_E[0, 1]$-valued rv with stationary independent increments, $Y(0)=0$ a.s. and $\mathscr{L}(Y(1))=\nu$. In other words, if $\nu=[a, R, M]$ then $\mathscr{I}_Q^\beta(\nu)=[a^{(\beta)}, R^{(\beta)}, M^{(\beta)}]$ with $a^{(\beta)}$, $R^{(\beta)}$ and $M^{(\beta)}$ given by (1.10)–(1.12). Similarly, for $\beta=0$, we define the mapping

$$\mathscr{I}_Q^0(\nu):=\mathscr{L}\Big(\int_{(0,1)} t^Q\,dY(-\ln t)\Big)$$

between $ID_{\log}:=\{\nu\in ID\colon \int_E \log(1+\|x\|)\,\nu(dx)<\infty\}$ and $\mathscr{U}_0(Q)$.

Theorem 1.3. (a) *For $\beta>0$, the mapping $\mathscr{I}_Q^\beta$ is an algebraic isomorphism and a homeomorphism between the topological semigroups ID and $\mathscr{U}_\beta(Q)$.*

(b) *The mapping $\mathscr{I}_Q^0$ is an algebraic isomorphism between $ID_{\log}$ and $\mathscr{U}_0(Q)$. Moreover, for ν_n, $\nu\in ID_{\log}$ we have*

$$\mathscr{I}_Q^0(\nu_n)\Rightarrow\mathscr{I}_Q^0(\nu) \quad \text{iff } \nu_n\Rightarrow\nu \quad \text{and} \quad \int_E \log(1+\|x\|)\,\nu_n(dx)\to\int_E \log(1+\|x\|)\,\nu(dx).$$

(c) *The mappings $\mathscr{I}_Q^\beta$, with $\beta\geqq 0$, satisfy also*

(i) $\mathscr{I}_Q^\beta(\nu^{*c})=(\mathscr{I}_Q^\beta(\nu))^{*c}$ *for all $c>0$*

(ii) $V\mathscr{I}_Q^\beta(\nu)=\mathscr{I}_Q^\beta(V\nu)$ *for any bounded linear operator V on E, commuting with Q.*

The class $ID(E)$ of all infinitely divisible measures on E can be described as the smallest closed subsemigroup of $\mathscr{P}(E)$ containing all symmetric Gaussian measures and all shifted compound Poisson measures of the form $[x, 0, \lambda\delta(y)]$, where $x, y\in E$ and $\lambda>0$, cf. Araujo and Giné (1980). Using the mapping $\mathscr{I}_Q^\beta$ we shall find sets of generators for $\mathscr{U}_\beta(Q)$.

Since $t^Q\to 0$ as $t\to 0$ then there are positive constants a and b such that $\|t^Q\|\leqq a t^b$ for all $0<t\leqq 1$, and $\|x\|_Q:=\int_0^1 \|t^Q x\|\,t^{-1}\,dt$ is well-defined norm on E. Let S_Q be the unit sphere in E with respect to the norm $\|\cdot\|_Q$ and for $a>0$ and $z\in S_Q$ and $\beta\geqq 0$ let us define a Borel measure $M_{a,z}$ on $E\setminus\{0\}$ by

$$M_{a,z}(F)=\int_0^a 1_F(t^Q z)\,t^{\beta-1}\,dt.$$

Since $\int\min(1,\|x\|)\,M_{a,z}(dx)<\infty$, $M_{a,z}$ are Lévy measures on E, cf. Araujo and Giné (1980), Chapter III, Theorem 6.3. Let $\mathscr{G}_{\beta,Q}$ be the set consisting of all generalized Poissonian measures $[x, 0, \lambda M_{a,z}]$, $x\in E$, $z\in S_Q$, $\lambda>0$, $a>0$ and all Gaussian measures $[0, R, 0]$ such that $QR+RQ^*+\beta R$ is nonnegative operator.

Corollary 1.2. *For $\beta\geqq 0$, the class $\mathscr{U}_\beta(Q)$ is the smallest closed subsemigroup of $ID(E)$ containing the set $\mathscr{G}_{\beta,Q}$.*

We will say that μ is *Q-stable* if μ is a limit of (1.9), where the uniform infinitesimality assumption is replaced by the requirement that $\nu_n=\nu_1$ for all $n\in\mathbb{N}$. Let $\mathscr{S}(Q)$ denote the class of all Q-stable measures. Then $\mathscr{S}(Q)\subseteq L_0(Q)$ and $\mu\in\mathscr{S}(Q)$ if and only if there is $p>0$ such that for every $t>0$

$$\mu^{*t}=t^{p^{-1}Q}\mu*\delta(z_t) \quad \text{for some } z_t\in E, \tag{1.13}$$

cf. Theorem 3.2 in Jurek (1983). We will refer to p as *an exponent of Q-stable measure.* So, if $\mu=[a, R, M]$ then μ is Q-stable with the exponent p if and only if for all $t>0$

$$\begin{aligned} t\cdot R&=t^{p^{-1}Q}Rt^{p^{-1}Q^*} \quad \text{and} \\ t\cdot M(F)&=(t^{p^{-1}Q}M)(F) \quad \text{for } F\in\mathscr{B}(E\setminus\{0\}). \end{aligned} \tag{1.14}$$

Differentiating the first equality we obtain $R=p^{-1}t^{-1}(Qt^{p^{-1}Q}Rt^{p^{-1}Q^*}+t^{p^{-1}Q}RQ^*t^{p^{-1}Q^*})$ for $t>0$ and hence $R=p^{-1}(QR+RQ^*)$. Conversely, $R=p^{-1}(QR+RQ^*)$ implies

$$t^{p^{-1}Q}Rt^{p^{-1}Q^*}=p^{-1}t^{p^{-1}Q}(QR+RQ^*)\,t^{p^{-1}Q^*}=t\,[d/dt(t^{p^{-1}Q}Rt^{p^{-1}Q^*})]$$

for $t>0$. Hence $t\cdot R=t^{p^{-1}Q}Rt^{p^{-1}Q^*}$ for all $t>0$, i.e., R is a Gaussian covariance operator of Q-stable measure with the exponent p. The second equation in (1.14) one can solve as in Theorem 4.2 in Jurek (1983), using the polar coordinates given by the norm $\|\cdot\|_Q$ introduced in the comment preceding Corollary 1.2; cf. also Remark (b), p. 602, in [6]. We will get that M satisfies the equation (1.4) if and only if there exists finite measure m on the unit sphere S_Q such that

$$(1.15)\qquad M(F)=\int_{S_Q}\int_0^\infty 1_F(t^Q x)\,t^{-(p+1)}\,dt\,m(dx)\qquad \text{for } F\in\mathscr{B}(E\setminus\{0\}).$$

Finally we have that $\mu=[a,R,M]$ is Q-stable with the exponent p if and only if $R=p^{-1}(QR+RQ^*)$ and M is of the form (1.15). Having this characterization of the Q-stable measures (the class $\mathscr{S}(Q)$) we can identify measures of $\mathscr{S}(Q)$ among those of $\mathscr{U}_\beta(Q)$, for $\beta>0$, as follows:

Theorem 1.4. *Let $\beta>0$. If μ is Q-stable with an exponent p then $\mathscr{I}_Q^\beta(\mu)=\mu^{*\beta/(p+\beta)}*\delta(x)$ for some $x\in E$. Conversely, if $\mathscr{I}_Q^\beta(\nu)=\nu^{*c}*\delta(x)$ for some $c>0$ and $x\in E$ then $0<c<1$ and ν is Q-stable measure with the exponent $\beta(1-c)\,c^{-1}$.*

2. Preliminaries

In this section we will prove some auxiliary lemmas needed later in the proofs of the main results.

Lemma 2.1 (Jurek (1985a)). *Let ρ_n, $\rho\in ID(E)$, c_n, $c\in\mathbb{R}^+$ and A_n, A be linear bounded operators on E. If $c_n\to c$, $\rho_n\Rightarrow\rho$ and $A_n\to A$ (pointwise) then*

$$A_n\,\rho_n^{*c_n}\Rightarrow A\rho^{*c}.$$

The next lemma deals with random integrals, their characteristic functionals and probability distributions.

Lemma 2.2. (a) *Let τ be a monotone continuous function from $(a,b]$ into $(0,\infty)$ and let Y_1 and Y_2 be $D_E[0,\infty)$-valued rv's with independent increments and $Y_1(0)=Y_2(0)=0$ a.s. If for some positive $s>0$ and all $0\leqq v\leqq w<\infty$*

$$\mathscr{L}(Y_2(w)-Y_2(v))=\mathscr{L}(Y_1(w)-Y_1(v))^{*s}$$

then for any bounded linear operator A and continuously differentiable real-valued function g we have

$$\mathscr{L}\Big(\int_{(a,b]} e^{g(t)A}\,dY_2(\tau(t))\Big)=\mathscr{L}\Big(\int_{(a,b]} e^{g(t)A}\,dY_1(\tau(t))\Big)^{*s},$$

provided one (then both) of these integrals exists.

(b) *If Y_1 has also stationary increments and the integral exists then*

$$\hat{\mathscr{L}}\Big(\int_{(a,b]} e^{g(t)A}\, dY_1(\tau(t))\Big)(y) = \exp \int_{(a,b]} \log \hat{\mathscr{L}}(Y_1(1))(e^{g(t)A^*} y)\, d\tau(t).$$

Proof. Let $a = t_0 < t_1 < \ldots < t_n = b$. Then, by definition, the integrals on the right-hand in (1.6) are approximated, with probability 1, by

$$\sum_{j=1}^{n} (e^{g(t_j)A} - e^{g(t_{j-1})A})\, Y(\tau(t_{j-1})) \tag{2.1}$$

as the width of the partition tends to zero and so are the left-hand integrals in (1.6) by

$$\sum_{j=1}^{n} e^{g(t_j)A} [Y(\tau(t_j)) - Y(\tau(t_{j-1}))]. \tag{2.2}$$

This with (1.4) gives

$$\mathscr{L}\Big[\sum_{j=1}^{n} e^{g(t_j)A}(Y_2(\tau(t_j)) - Y_2(\tau(t_{j-1})))\Big] = \mathscr{L}\Big[\sum_{j=1}^{n} e^{g(t_j)A}(Y_1(\tau(t_j)) - Y_1(\tau(t_{j-1})))\Big]^{*s}$$

which completes the proof of the part (a). Similarly, using the approximation (2.2) we get

$$\log \hat{\mathscr{L}}\Big[\int_{(a,b]} e^{g(t)A}\, dY_1(\tau(t))\Big](y) = \lim_n \sum_{j=1}^{n} (\tau(t_j) - \tau(t_{j-1})) \log \hat{\mathscr{L}}(Y_1(1))(e^{g(t_j)A^*} y),$$

which implies the part (b) and completes the proof.

Remark 2.1. In the sequel we will have integrals over the interval $(0, 1]$ with $g(t) = \log t$, which is not defined at zero. However, the integral in the right-hand side of (1.6) still exists as a limit of sums (2.1). Note that since $t^Q \to 0$, there exists constants $a, b > 0$ such that $\|t^{-1} Q t^Q\| \leq \|Q\|\, a t^{b-1}$ for $0 < t \leq 1$. Also, cf. Remark 1.1 in Jurek (1982b).

3. Proofs

The Proof of Theorem 1.1. Let $\mu \in \mathscr{U}_\beta(Q)$, and choose $1 \leq k_n \leq n$ such that $k_n/n \to c$, where c is a fixed number from the unit interval $(0, 1)$. Then, for ρ_n given by (1.7) we get

$$\rho_n = (k_n/n)^Q \rho_{k_n}^{*(k_n/n)^\beta} * n^{-Q}(\nu_{k_n+1} * \ldots * \nu_n)^{*n^{-\beta}}$$

using (1.3) and (1.4). Since $\rho_n \Rightarrow \mu$, Lemma 2.1 gives that the first factor converges to $c^Q \mu^{*c^\beta}$ and consequently the second one converges, say to $\mu_c \in ID$ which gives (1.8).

Conversely, let $\mu \in ID$ and satisfy (1.8). Let us put $\nu_1 := \mu$ and

$$\nu_k := k^Q \mu_{(k-1)/k}^{*k^\beta} \quad \text{for } k \geqq 2.$$

Using (1.8) for $c=(k-1)/k$ with $k \geqq 2$ we get

$$k^Q \mu = (k-1)^Q \mu^{*((k-1)/k)^\beta} * k^Q \mu_{(k-1)/k}$$

and hence

$$k^Q \mu^{*k^\beta} = (k-1)^Q \mu^{*(k-1)^\beta} * k^Q \mu_{(k-1)/k}^{*k^\beta}.$$

Consequently, we have

$$\hat{\nu}_k(y) = [(k^Q \mu)^\wedge (y)]^{k^\beta} / [((k-1)^Q \mu)^\wedge (y)]^{(k-1)^\beta}, \qquad \text{for } k \geqq 2.$$

which implies

$$\nu_1 * \nu_2 * \ldots * \nu_n = n^Q \mu^{*n^\beta}, \quad \text{i.e.,} \quad \mu \in \mathcal{U}_\beta(Q).$$

The Proof of Corollary 1.1. (a) The semigroup property follows from (1.8) using (1.3) and (1.4). Let $\mu_n \in \mathcal{U}_\beta(Q)$ and $\mu_n \Rightarrow \mu$. Then there are $\mu_{c,n} \in ID$ for $n \in \mathbb{N}$ and $0<c<1$ such that

$$\mu_n = c^Q \mu_n^{*c^\beta} * \mu_{c,n}.$$

Since $\mu_n \Rightarrow \mu \in ID$ then $c^Q \mu_n^{*c^\beta} \Rightarrow c^Q \mu^{*c^\beta}$ and therefore $\mu_{c,n} \Rightarrow \mu_c$ (as $n \to \infty$) for some $\mu_c \in ID$, as $\hat{\mu}_n \neq 0$. Consequently, $\mu \in \mathcal{U}_\beta(Q)$.

(b) For $\alpha \leqq \beta$ and $0<c<1$ we have $c^\beta \leqq c^\alpha$. Hence

$$\mu = c^Q \mu^{*c^\alpha} * \mu_c = c^Q \mu^{*c^\beta} * \mu_c',$$

where $\mu_c' := c^Q \mu^{*(c^\alpha - c^\beta)} * \mu_c$ and therefore $\mathcal{U}_\alpha(Q) \subseteq \mathcal{U}_\beta(Q)$ whenever $\alpha \leqq \beta$. The equality $\mathcal{U}_0(Q) = L_0(Q)$ is a consequence of Theorem 3.1 in [6] applied for a group $\mathbb{U} := \{t^Q : t>0\}$ of bounded linear operators on E.

(c) If $AQ = QA$ then $Ac^Q = c^Q A$ for all $c>0$ and the inclusion follows from (1.8). If A is invertible then A^{-1} commutes with Q and $A^{-1}\mathcal{U}_\beta(Q) \subseteq \mathcal{U}_\beta(Q)$, i.e., $\mathcal{U}_\beta(Q) \subseteq A\mathcal{U}_\beta(Q) \subseteq \mathcal{U}_\beta(Q)$.

(d) Let $\mu = [a, R, M]$ and $\mu_c = [a_c, R_c, M_c]$. Using (1.5) we see that (1.8) is equivalent to $a = c^\beta \tilde{a} + a_c$, $R = c^\beta (c^Q R c^{Q*}) + R_c$, $M = c^\beta \cdot (c^Q M) + M_c$ for all $0<c<1$. Hence $R \geqq c^\beta (c^Q R c^{Q*})$ and $M \geqq c^\beta \cdot (c^Q M)$ for all $0<c<1$. Conversely, if R is a Gaussian covariance operator satisfying the above inequality, then $R - c^\beta (c^Q R c^{Q*})$ is also covariance operator. Similarly, if M is a Lévy measure, satisfying inequality $M \geqq c^\beta \cdot (c^Q M)$ on $E \setminus \{0\}$, then so is $M - c^\beta \cdot (c^Q M)$. Finally, for $\mu = [a, R, M]$ we obtain the factorization (1.8) for all $0<c<1$.

(e) Note that for $\beta>0$, $\lim_{t \to 0} t^{\beta^{-1}Q} = 0$ and replacing c^β by t in Theorem 1.1 we have $\mu = t^{\beta^{-1}Q} \mu^{*t} * \mu_t$ for all $0<t<1$. Thus $\mathcal{U}_\beta(Q) = \mathcal{U}_1(\beta^{-1} Q)$.

(f) Suppose that μ has non-zero Gaussian part, i.e., $\langle y, Ry \rangle > 0$ for some $y \in E'$. Then from (d) we get $c^{2+\beta} \langle y, Ry \rangle \leqq \langle y, Ry \rangle$ for all $0<c<1$. Hence we obtain $\beta \geqq -2$. If μ has non-zero Poisson part, i.e., $M \neq 0$, then there is non-zero $y \in E'$ such that $M(\{x: |\langle y, x\rangle| \leqq 1\}) > 0$. From (d) we get for $0 < c < 1$

$$0 < c^{2+\beta} \int_{\{x: |\langle y, x\rangle| \leqq 1\}} \langle y, x\rangle^2 M(dx) \leqq c^\beta \int_{\{x: |\langle y, cx\rangle| \leqq 1\}} \langle y, cx\rangle^2 M(dx)$$
$$\leqq \int_{\{x: |\langle y, x\rangle| \leqq 1\}} \langle y, x\rangle^2 M(dx) < \infty,$$

which implies $\beta \geqq -2$.

Proof of Theorem 1.2 (a). Because of Corollary 1.1(e) we could restrict ourselves to the class $\mathscr{U}_1(Q)$ instead of $\mathscr{U}_\beta(Q)$. Since it does not simplify the proof we consider the general case. For $\tau_\beta(t) := t^\beta$, with $\beta > 0$, the integral $\int_{(0,1)} t^Q dY(\tau_\beta(t))$ exists for all $D_E[0, 1]$-valued rv's Y because of Lemma 14.1 in Billingsley (1968) and Remark 2.1. Furthermore, using Lemma 2.2(a), we have

$$\mathscr{L}\Big(\int_{(0,1)} t^Q dY(\tau_\beta(t))\Big) = \mathscr{L}\Big(\int_{(0,c)} t^Q dY(\tau_\beta(t))\Big) * \mathscr{L}\Big(\int_{[c,1)} t^Q dY(\tau_\beta(t))\Big)$$
$$= c^Q \mathscr{L}\Big(\int_{(0,1)} s^Q dY(\tau_\beta(s))\Big)^{*c^\beta} * \mathscr{L}\Big(\int_{[c,1)} t^Q dY(\tau_\beta(t))\Big)$$

for all $0 < c < 1$, i.e., $\mu = \mathscr{L}\Big(\int_{(0,1)} t^Q dY(\tau_\beta(t))\Big) \in \mathscr{U}_\beta(Q)$.

Conversely, let $\mu \in \mathscr{U}_\beta(Q)$. Then, by Theorem 1.1,

$$\mu = e^{-tQ} \mu^{*e^{-\beta t}} * \mu_t, \quad \text{for all } t > 0. \tag{3.1}$$

Applying Lemma 2.1 with the fact that $\hat{\mu} \neq 0$ we obtain

$$\mu_t \Rightarrow \delta(0) \text{ as } t \to 0 \quad \text{and} \quad \mu_t \Rightarrow \mu \text{ as } t \to \infty. \tag{3.2}$$

Let $(\tilde{Z}_t)_{t \geqq 0}$ be E-valued random function with independent increments such that $\tilde{Z}_0 = 0$ a.s. and

$$\mathscr{L}(\tilde{Z}_{t+h} - \tilde{Z}_t) := e^{-tQ} \mu_h^{*e^{-\beta t}}, \quad \text{for } t, h \geqq 0, \tag{3.3}$$

so in particular,

$$\mathscr{L}(\tilde{Z}_h) := \mu_h. \tag{3.4}$$

To see the consistency, we must show that $\mathscr{L}(\tilde{Z}_{t+h}) = \mu_{t+h}$, if $\tilde{Z}_{t+h} - \tilde{Z}_t$ and $\tilde{Z}_t$ are independent with the distributions given by (3.3) and (3.4). From (3.1) it follows that for $t, s > 0$

$$\mu = e^{-(t+s)Q} \mu^{*e^{-\beta(t+s)}} * \mu_{t+s} = e^{-tQ}(e^{-sQ} \mu^{*e^{-\beta s}} * \mu_s)^{*e^{-\beta t}} * \mu_t$$
$$= e^{-(t+s)Q} \mu^{*e^{-\beta(t+s)}} * e^{-tQ} \mu_s^{*e^{-\beta t}} * \mu_t$$

and consequently

$$\mu_{t+s} = e^{-tQ} \mu_s^{*e^{-\beta t}} * \mu_t.$$

Hence

$$\mathscr{L}(\tilde{Z}_{t+h}) = \mathscr{L}(\tilde{Z}_{t+h} - \tilde{Z}_t) * \mathscr{L}(\tilde{Z}_t) = e^{-tQ} \mu_h^{*e^{-\beta t}} * \mu_t = \mu_{t+h}$$

as was to be proved. Finally as in Application A1.2 in Jurek-Vervaat (1983) we conclude that there exists a $D_E[0, \infty)$-valued rv Z with the same finite-dimensional distributions as a random function $\tilde{Z}$, thus with independent increments and

(3.5) $$\mathscr{L}(Z(t)) = \mu_t \quad \text{for } t > 0.$$

Furthermore, from (3.3) we get

(3.6) $$\mathscr{L}[e^{hQ}(Z(w+h) - Z(v+h))] = e^{-vQ}(\mu_{w-v})^{*e^{-\beta(v+h)}} = (e^{-vQ}(\mu_{w-v})^{*e^{-\beta v}})^{*e^{-\beta h}} = \mathscr{L}(Z(w) - Z(v))^{*e^{-\beta h}}$$

for $0 \leqq v \leqq w$ and $0 \leqq v + h$.

Let us define

(3.7) $$\tilde{Y}(t) := \int_{(0,t]} e^{sQ} dZ(s), \quad \text{for } t \geqq 0.$$

Then $\tilde{Y}$ is a $D_E[0, \infty)$-valued rv with independent increments as so is Z. Moreover, if $0 \leqq s < t$ and $s + h \geqq 0$ then from (3.6) and Lemma 2.2(a) we obtain

(3.8) $$\mathscr{L}(\tilde{Y}(t+h) - \tilde{Y}(s+h)) = \mathscr{L}\Big(\int_{(s,t]} e^{wQ} e^{hQ} dZ(w+h)\Big) = \mathscr{L}\Big(\int_{(s,t]} e^{wQ} dZ(w)\Big)^{*e^{-\beta h}} = \mathscr{L}(\tilde{Y}(t) - \tilde{Y}(s))^{*e^{-\beta h}}.$$

Furthermore let us define $Y_1(t) := -\tilde{Y}(-\beta^{-1} \log t)$ for $0 < t \leqq 1$. Then Y_1 has independent increments and from (3.8) we obtain

(3.9) $$\mathscr{L}(Y_1(ct) - Y_1(cs)) = \mathscr{L}(Y_1(t) - Y_1(s))^{*c}$$

for s, t, cs and ct from $(0, 1]$. Hence we get

$$\mathscr{L}(Y_1(2^{-1}) - Y_1(2^{-n})) = \mathscr{L}\left(\sum_{j=1}^{n-1} (Y_1(2^{-j}) - Y_1(2^{-j-1}))\right) = \mathscr{L}(Y_1(1) - Y_1(2^{-1}))^{*(2^{-1} + \ldots + 2^{-n+1})} \Rightarrow \mathscr{L}(Y_1(1) - Y_1(2^{-1})) = \mathscr{L}(-Y_1(2^{-1}))$$

as $n \to \infty$. Therefore, by independent increments property, $Y_1(2^{-1}) - Y_1(2^{-n})$ is convergent in probability. Thus so is $\{Y_1(2^{-n})\}$ and denote its limit by ξ. (In fact, $\xi = 2Y_1(2^{-1})$). For $a > 0$ and $k \in \mathbb{N}$ such that $2^{-k} a < 1$ the equality (3.9) implies

$$\mathscr{L}(Y_1(2^{-n-k}) - Y_1(2^{-n-k} a)) = \mathscr{L}(Y_1(2^{-k}) - Y_1(2^{-k} a))^{*2^{-n}} \Rightarrow \delta_0,$$

as $n\to\infty$. Consequently, $Y_1(2^{-n-k})-Y_1(2^{-n-k}a)\to 0$ in probability and therefore also

(3.10) $$Y_1(2^{-n}a)\to\xi \quad \text{in probability, for each } a>0.$$

Since by (3.9), $\mathscr{L}(Y_1(t)-Y_1(2^{-n}t)=\mathscr{L}(Y_1(1)-Y_1(2^{-n}))^{*t}=\mathscr{L}(-Y_1(2^{-n}))^{*t}$ from (3.10) we infer that $\mathscr{L}(Y_1(t)-\xi)=\mathscr{L}(-\xi)^{*t}$ for $0<t\leqq 1$. Since for $0<s<t\leqq 1$

$$Y_1(t)-\xi=(Y_1(t)-Y_1(s))+(Y_1(s)-\xi)$$

is a sum of two independent variables we get $\mathscr{L}(Y_1(t)-Y_1(s))=\mathscr{L}(-\xi)^{*(t-s)}$, i.e., Y_1 has stationary increments. Finally, letting $Y(0):=0$ and $Y(t):=Y_1(t)-\xi)$ for $0<t\leqq 1$ we get $D_E[0,1]$-valued rv with stationary independent increments.

To conclude the proof note that, by definitions of $\tilde{Y}$, Y_1 and Y we have

$$Z(t)=\int_{(0,t]} e^{-sQ}\,dY(s)=-\int_{(0,t]} e^{-sQ}\,dY_1(e^{-\beta s})=-\int_{(0,t]} e^{-sQ}\,dY(e^{-\beta s})$$

and by (3.2) and (3.5)

$$\mu=\lim_{t\to\infty}\mathscr{L}(Z(t))=\mathscr{L}\Big(-\int_{(0,\infty)} e^{-sQ}\,dY(e^{-\beta s})\Big)=\mathscr{L}\Big(\int_{(0,1)} t^{Q}\,dY(\tau_\beta(t))\Big);$$

which completes the proof of the part (a) of Theorem 1.2.

Proof of Theorem 1.2(b). Since $\mathscr{U}_0(Q)=L_0(Q)$, from Jurek (1982b) we have

$$\mu\in\mathscr{U}_0(Q) \text{ iff } \mu=\mathscr{L}\Big(\int_{(0,\infty)} e^{-tQ}\,dY(t)\Big)=\mathscr{L}\Big(-\int_{(0,1)} s^{Q}\,dY(-\ln s)\Big),$$

where Y is a $D_E[0,\infty)$-valued rv with independent and stationary increments such that $\mathbb{E}\log(1+\|Y(1)\|)<\infty$ and $Y(0)=0$ a.s. Finally, taking $A:=-I$ in (c) of the Corollary 1.1 we conclude the proof of the part (b) of Theorem 1.2.

Proof of Theorem 1.3(a). For fixed $y\in E'$ let us put

$$g_y(s):=\log(\mathscr{I}_Q^\beta(v))^\wedge(s^{Q^*}y) \qquad \text{for } s\geqq 0.$$

Then from Lemma 2.2(b) we obtain

$$g_y(s)=\beta s^{-\beta}\int_0^s \log\hat{v}(r^{Q^*}y)\,r^{\beta-1}\,dr$$

and hence

$$\left.\frac{dg_y(s)}{ds}\right|_{s=1}=-\beta\log(\mathscr{I}_Q^\beta(v))^\wedge(y)+\beta\log\hat{v}(y).$$

Consequently

$$\hat{v}(y)=(\mathscr{I}_Q^\beta(v))^\wedge(y)\exp\left[\beta^{-1}\frac{d}{ds}(\log(\mathscr{I}_Q^\beta(v))^\wedge(s^{Q^*}y))|_{s=1}\right]$$

and this implies that $\mathscr{I}_Q^\beta$ is one-to-one. From Theorem 1.2(a) we have that $\mathscr{I}_\beta^Q$ maps onto $\mathscr{U}_\beta(Q)$. Let $v_1, v_2\in ID$ and Y_1, Y_2 are independent $D_E[0,1]$-valued

rv's with stationary independent increments such that $Y_1(0)=Y_2(0)=0$ a.s. and $\mathscr{L}(Y_j(1))=\nu_j$ for $j=1, 2$. Then

$$\mathscr{I}_Q^\beta(\nu_1)*\mathscr{I}_Q^\beta(\nu_2)=\mathscr{L}\Big[\int_{(0,1)} t^Q\,dY_1(\tau_\beta(t))+\int_{(0,1)} t^Q\,dY_2(\tau_\beta(t))\Big]$$
$$=\mathscr{L}\Big[\int_{(0,1)} t^Q\,d((Y_1+Y_2)(\tau_\beta(t)))\Big]=\mathscr{I}_Q^\beta(\nu_1*\nu_2),$$

which shows that $\mathscr{I}_Q^\beta$ is a homomorphism. It remains to establish the continuity of $\mathscr{I}_Q^\beta$ and its inverse. Suppose ν_n, $\nu\in ID$ and $\nu_n\Rightarrow\nu$. Let us choose $D_E[0,1]$-valued rv's Y_n and Y with stationary independent increments such that

$$Y_n(0)=Y(0)=0 \text{ a.s.},\quad \mathscr{L}(Y_n(1))=\nu_n \quad\text{and}\quad \mathscr{L}(Y(1))=\nu.$$

Then, by Theorem VI.5.5 of Gihman and Skorohod (1974), we obtain $\mathscr{L}(Y_n)\Rightarrow\mathscr{L}(Y)$ in $D_E[0,1]$. Since the functional

$$\phi(y)=\int_0^1 t^Q\,dy(\tau_\beta(t)):=y(1)-\int_0^1 d(t^Q)\,y(\tau_\beta(t)),\qquad y\in D_E[0,1],$$

is continuous in the Skorohod topology (cf. Billingsley (1968), p. 121), the Continuous Mapping Theorem (cf. Billingsley (1968), Theorem 5.1) gives

$$\mathscr{I}_Q^\beta(\nu_n)=\mathscr{L}(\phi(Y_n))\Rightarrow\mathscr{L}(\phi(Y))=\mathscr{I}_Q^\beta(\nu)$$

which shows the continuity of $\mathscr{I}_Q^\beta$. Conversely, let $\mathscr{I}_Q^\beta(\nu_n)\Rightarrow\mathscr{I}_Q^\beta(\nu)$, Y_n, Y be $D_E[0,\infty)$-valued rv's with stationary independent increments and $\mathscr{L}(Y_n(1)=\nu_n$, $\mathscr{L}(Y(1))=\nu$. Let us define

$$Z_n(t):=\int_{(0,t]} s^Q\,dY_n(\tau_\beta(s)),\quad Z(t):=\int_{(0,t]} s^Q\,dY(\tau_\beta(s))\qquad\text{for } t\geqq 0.$$

Then, Z_n and Z are $D_E[0,\infty)$-valued rv's with independent increments $\mathscr{L}(Z_n(1))\Rightarrow\mathscr{L}(Z(1))$. Furthermore

$$\mathscr{L}(Z_n(t))=\mathscr{L}\Big(t^Q\int_{(0,1]} s^Q\,dY(\tau_\beta(t)\,\tau_\beta(s))\Big)=t^Q\mathscr{L}(Z_n(1))^{*t^\beta}\qquad\text{for } t>0,$$

and this with Lemma 2.1 gives

$$\mathscr{L}(Z_n(t_n))\Rightarrow\mathscr{L}(Z(t))\qquad\text{whenever } t_n\to t \text{ in } \mathbb{R}^+.$$

Hence, for $t_n>s_n$ and $t_n\to t$, $s_n\to s$ we get

$$\mathscr{L}(Z_n(t_n)-Z_n(s_n))\Rightarrow\mathscr{L}(Z(t)-Z(s))\qquad\text{as } n\to\infty.$$

So, the finite-dimensional distributions of Z_n converge to the ones of Z. Finally this gives

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\sup_{0\leqq s<t<s+h\leqq a} P\{\|Z_n(t)-Z_n(s)\|>\varepsilon\}=0$$

for positive a and ε. Therefore, Theorem VI.5.5 in Gihman-Skorohod (1974) with Lindvall (1973) imply that $\mathscr{L}(Z_n) \Rightarrow \mathscr{L}(Z)$ in $D_E[0, \infty)$. As before the functional

$$\psi(z) = \int_{(1, 2^{1/\beta}]} t^{-Q} dz(t) := 2^{-\beta^{-1}Q} z(2^{1/\beta}) - z(1) - \int_1^{2^{1/\beta}} (d(t^{-Q}))\, z(t), \qquad z \in D_E[0, \infty)$$

is continuous (in the Skorohod topology) at z if z is continuous at $t=1$ and $2^{\beta^{-1}}$. Appealing once again to the Continuous Mapping Theorem we obtain

$$\begin{aligned} \mathscr{L}(Y_n(1)) &= \mathscr{L}(Y_n(2) - Y_n(1)) = \mathscr{L}\Big(\int_{(1, 2^{1/\beta}]} t^{-Q} dZ_n(t)\Big) = \mathscr{L}(\psi(Z_n)) \\ &\Rightarrow \mathscr{L}(\psi(Z)) = \mathscr{L}\Big(\int_{(1, 2^{1/\beta}]} t^{-Q} dZ(t)\Big) = \mathscr{L}(Y(1)), \end{aligned}$$

which shows the continuity of the inverse of $\mathscr{I}_Q^\beta$.

Proof of Theorem 1.3 (b). The algebraic properties of $\mathscr{I}_Q^0$ are proved in Jurek (1982b) and the topological ones in Jurek and Rosinski (1985). Let us note only that the weak convergence in $ID_{\log}$ has to be strengthened by requirement that $\int_E \log(1 + \|x\|)\, \nu_n(dx) \to \int_E \log(1 + \|x\|)\, \nu(dx)$.

Proof of Theorem 1.3 (c). Both properties easily follow from the integral representations given in Theorem 1.2.

Proof of Corollary 1.2. The class $ID(E)$ can be viewed as the smallest closed subsemigroup of $\mathscr{P}(E)$ containing all symmetric Gaussian measures and all Poissonian measures $[x, 0, \lambda\delta(y)]$ with $x, y \in E$ and $\lambda > 0$, cf. Araujo and Giné (1980), Theorem 4.7 in Chapter III. Because of the continuity of the mappings $\mathscr{I}_Q^\beta$, $\beta > 0$, cf. Theorem 1.3, it is enough to find how the generators of ID are transformed by the mappings $\mathscr{I}_Q^\beta$. At first, let us note that $[0, R, 0] \in \mathscr{U}_\beta(Q)$ if and only if $\langle y, Ry \rangle \geqq \langle y, e^{-tQ_1} R e^{-tQ_1^*} y \rangle$ for all $y \in E'$, all $t \geqq 0$, where $Q_1 := Q + (\beta/2) I$, cf. Corollary 1.1 (d). Hence we get

$$\begin{aligned} \langle y, Ry \rangle \geqq \langle y, e^{-sQ_1} R e^{-sQ_1^*} y \rangle &= \langle e^{-sQ_1^*} y, R e^{-sQ_1^*} y \rangle \\ &\geqq \langle y, e^{-(s+t)Q_1} R e^{-(s+t)Q_1^*} y \rangle \geqq 0, \end{aligned}$$

for all $t, s \geqq 0$. Thus the functions $f_y(t) := \langle y, e^{-tQ_1} R e^{-tQ_1^*} y \rangle$, $t \geqq 0$, are monotone decreasing, nonnegative and $f_y(0) = \langle y, Ry \rangle$. Consequently, $\frac{d}{dt}(f_y(0) - f_y(t)) = \frac{d}{dt} \langle y, (R - e^{-tQ_1} R e^{-tQ_1^*})\, y \rangle = \langle y, e^{-tQ_1}(Q_1 R + R Q_1^*)\, e^{-tQ_1^*} y \rangle \geqq 0$ for all $t \geqq 0$ and $y \in E'$, which implies that $Q_1 R + R Q_1^*$ is nonnegative operator. Conversely, if $Q_1 R + R Q_1^*$ is nonnegative, then we have

$$0 \leqq \langle e^{-tQ_1^*} y, (Q_1 R + R Q_1^*)\, e^{-tQ_1^*} y \rangle = -\frac{d}{dt} \langle y, e^{-tQ_1} R e^{-tQ_1^*} y \rangle = -\frac{d}{dt} f_y(t)$$

for $y\in E'$ and $t\geqq 0$. Therefore $f_y(t)$ is monotone decreasing which gives $\langle y, Ry\rangle \geqq \langle y, e^{-tQ_1} Re^{-tQ_1^*} y\rangle$, i.e., $R\geqq e^{-tQ_1} Re^{-tQ_1^*}$. So, we have proven that

$$R\geqq e^{-tQ_1} Re^{-tQ_1^*} \quad \text{for } t\geqq 0 \text{ iff } Q_1 R+RQ_1^*\geqq 0,$$

which gives the Gaussian generators from $\mathscr{G}_{\beta,Q}$. Let $a\in E\setminus\{0\}$ and $\lambda>0$. Then $a=s^Q u$ for some $s>0$ and $u\in S_Q$ as the function $\Phi_Q\colon S_Q\times(0,\infty)\to E\setminus\{0\}$ given $\Phi_Q(z,t):=t^Q z$ is a homeomorphism, cf. Jurek (1984), Proposition 2. Since $(\lambda\cdot M)^{(\beta)}=\lambda\cdot M^{(\beta)}$, cf. (1.12), it is enough to find $(\delta(s^Q u))^{(\beta)}$. Let $F:=\{r^Q z\colon z\in A, r\in B\}$, where $A\in\mathscr{B}(S_Q)$ and $B\in\mathscr{B}(0,\infty)$ $s^Q F=\{r^Q z\colon z\in A, r\in sB\}$ and from (1.12) we get

$$\begin{aligned}(\delta(s^Q u))^{(\beta)}(F)&=\beta\int_0^1 \delta(s^Q u)\{r^Q z\colon z\in A, r\in t^{-1}B\}\, t^{\beta-1}\,dt\\ &=\beta 1_A(u)\int_0^1 1_{t^{-1}B}(s)\, t^{\beta-1}\,dt=\beta s^{-\beta}\delta(u)(A)\int_0^s 1_B(r)\, r^{\beta-1}\,dr\\ &=\beta s^{-\beta}\int_{S_Q}\int_0^s 1_A(x)\,1_B(r)\, r^{\beta-1}\,dr\,\delta(u)(dx)\\ &=\beta s^{-\beta}\int_{S_Q}\int_0^s 1_F(r^Q x)\, r^{\beta-1}\,dr\,\delta(u)(dx)\\ &=\beta s^{-\beta}\int_0^s 1_F(r^Q u)\, r^{\beta-1}\,dr\\ &=\beta s^{-\beta} M_{s,u}(F).\end{aligned}$$

Since Φ_Q is a homeomorphism this equality extends to all $F\in\mathscr{B}(E\setminus\{0\})$, which proves $(\delta(a))^{(\beta)}=\lambda M_{s,u}$ for some λ, s positive and $u\in S_Q$.

The case $\beta=0$ one can prove similarly, but we have to strengthen the weak topology on $ID_{\log}$; cf. Theorem 1.3(b). The complete proof is given in Jurek-Rosinski (1985).

Proof of Theorem 1.4. Let $\mu=[a,R,M]$ be Q-stable with an exponent $p>0$. Then $R=p^{-1}(QR+RQ^*)$, cf. the discussion following (1.14), and using (1.11) we obtain

$$\begin{aligned}R^{(\beta)}&=\beta p^{-1}\int_0^1 t^Q(QR+RQ^*)\,t^{Q^*}t^{-1}t^\beta\,dt=\beta p^{-1}\int_0^1 \frac{d}{dt}(t^Q R t^{Q^*})\,t^\beta\,dt\\ &=\beta p^{-1}R-\beta p^{-1}\int_0^1 t^Q R t^{Q^*}\,d\tau_\beta(t)=\beta p^{-1}R-\beta p^{-1}R^{(\beta)},\end{aligned}$$

i.e., $R^{(\beta)}=\beta/(p+\beta)R$. Further, since M is of the form (1.15) then (1.12) gives

$$\begin{aligned}M^{(\beta)}(F)&=\beta\int_0^1\int_{S_Q}\int_0^\infty 1_F((rt)^Q x)\,r^{-(p+1)}\,dr\,m(dx)\,t^{\beta-1}\,dt\\ &=\beta\int_{S_Q}\int_0^1\int_0^\infty 1_F(s^Q x)\,s^{-(p+1)}\,ds\,t^{p+\beta-1}\,dt\,m(dx)=\beta/(p+\beta)\,M(F)\end{aligned}$$

for all $F\in\mathscr{B}(E\setminus\{0\})$. Consequently, $\mathscr{I}_Q^\beta(\mu)=\mu^{*\beta/(p+\beta)}*\delta(z)$, for some $z\in E$, if μ is Q-stable with an exponent p.

Conversely, let $\nu=[b, T, N]$ and $\mathscr{I}_Q^\beta(\nu)=\nu^{*c}*\delta(z)$ for some $c>0$ and $z\in E$. Then (1.11) and (1.12) implies

$$(3.12)\qquad cT=\beta\int_0^1 t^Q T t^{Q^*} t^{\beta-1}\,dt \quad\text{and}\quad c\cdot N(F)=\beta\int_0^1 N(t^{-Q}F)\,t^{\beta-1}\,dt.$$

Hence we get

$$QT+TQ^*=c^{-1}\beta\int_0^1 t^Q(QT+TQ^*)\,t^{Q^*}t^{\beta-1}\,dt=c^{-1}\beta\int_0^1\frac{d}{dt}(t^Q T t^{Q^*})\,t^\beta\,dt$$

$$=c^{-1}\beta\left\{T-\beta\int_0^1 t^Q T t^{Q^*} t^{\beta-1}\,dt\right\}=c^{-1}\beta\{T-cT\}=(1-c)\,c^{-1}\beta T$$

i.e., T is a Gaussian covariance operator of Q-stable measures with the exponent $(1-c)\,c^{-1}\beta$ whenever $c<1$.

To solve the equation for N, let us introduce so-called *Q-Lévy spectral function L_N*, as follows

$$L_N(A;s):=N\{r^Q u: u\in A\quad\text{and}\quad r>s\}$$

where $s>0$ and A is a Borel subset of the unit sphere S_Q. Then from (3.12) we get

$$cL_N(A;s)=\beta s^\beta\int_s^\infty L_N(A;r)\,r^{-(\beta+1)}\,dr$$

and hence the following differential equation

$$cs\frac{d}{ds}L_N(A;s)=\beta(c-1)\,L_N(A;s)\qquad\text{for almost all } s>0.$$

Therefore putting $p:=\beta(1-c)/c$, and $m_1(A):=L_N(A;1)$ we conclude that

$$L_N(A;s)=m_1(A)\,s^{-p},$$

where m_1 is a finite measure on S_Q and $p>0$, i.e., $0<c<1$. Consequently

$$N(F)=\int_{S_Q}\int_0^\infty 1_F(t^Q u)\,t^{-(p+1)}\,dt\,m(du),\qquad F\in\mathscr{B}(E\setminus\{0\}),$$

where $m:=pm_1$ is a finite measure on S_Q. So, N is a Lévy measure corresponding to Q-stable measure with the exponent $p=\beta(1-c)/c$.

4. Comments

(a) The class $\mathcal{U}_1(Q)$ has been examined in Jurek (1985b). The proof of Theorem 1.2, for $\beta=1$ is obtained by a sequence of equivalent statements, cf. [9], Theorem 2.3, and none of these has a proof like the proof of Theorem 1.2(a). In the particular case $Q=I$, relations between $\mathcal{U}_1(I)$ and $L_0(I)$ were investigated in Jurek (1985a). Elements from $\mathcal{U}_1(I)$ were defined by an inequality involving Lévy measures and some nonlinear mapping (cf. [8], p. 596) and then characterized as limit distributions. Moreover only the continuity of $\mathcal{I}_I^1$ is established there. Finally, O'Connor (1979) introduced a family of classes L_α on $\mathbb{R}$. They are defined by some monotonicity property of their Lévy spectral functions. From his Theorem 1 and our Lemma 2.2(ii) we infer that his class L_α, $0<\alpha<1$, coincides with the class $\mathcal{U}_{1-\alpha}(I)$.

(b) For $\beta<0$, the integral $\int_{(0,1)} t^Q \, dY(\tau_\beta(t))$ involves whole trajectories of $Y(s)$, for $1<s<\infty$, and existence of such integrals may depend on some moment conditions. This and other questions are going to be discussed in a separate paper.

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