Order **2** (1985), 223–242. 0167–8094/85.15.

The Convexity Lattice of a Poset

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Communicated by D. Kelly

(Received: 10 December 1984; accepted: 17 September 1985)

Abstract. The authors investigate the lattice Co(P) of convex subsets of a general partially ordered set P. In particular, they determine the conditions under which Co(P) and Co(Q) are isomorphic; and give necessary and sufficient conditions on a lattice L so that L is isomorphic to Co(P) for some P.

AMS (MOS) subject classifications (1980). 06A10 (primary); 06A15, 06A23 (secondary).

Key words. Poset, convexity lattice.

1. Introduction

By definition [12], p. 7, a subset S of a poset P is *convex* whenever $a \in S$, $b \in S$ and $a \leqslant b$ imply $[a, b] \subseteq S$. It is obvious that this is a closure property, whence (as in [12], p. 111) the convex subsets of any P form a *complete lattice*, Co(P). Since any singleton $\{a\} \subseteq P$ is convex, this lattice is moreover atomic*, its atoms corresponding one-to-one to the elements $a \in P$. The purpose of this note is to show that the lattices Co(P) so constructed have some interesting and less obvious properties, in particular Co(P) often determines P up to dual isomorphism.

If P is unordered, then obviously $\mathrm{Co}(P) \simeq 2^{|P|}$ is just the (complete) Boolean algebra of all subsets of P. At the opposite extreme, if $P \simeq \mathbf{n}$ is a finite chain, then Co(P) is a *planar* lattice whose diagram was identified in 1908 by A. R. Schweitzer [13]. We have drawn the diagram of Co(5) in Figure 1a. It obviously consists of the upper half of the graph $\mathbf{5}^2$, including its horizontal diameter, everything below this diameter being replaced by $\emptyset$. The diagram for $\mathrm{Co}(\mathbf{2}^2)$ is shown in Figure 1b.

Evidently, a poset P and its dual P have isomorphic Co(P) (see Theorem 5). It is easy to reconstruct the chain **5** from the lattice displayed (see Section 3); more generally, any self-dual chain is determined up to isomorphism, while $\mathbb{N}$ and other infinite chains are determined up to duality by Co(P).

* Here, as in our previous paper [2], we have used the term 'atomic' as defined in [12], p. 196. This is equivalent to Grätzer's 'atomistic' [8], p. 179.

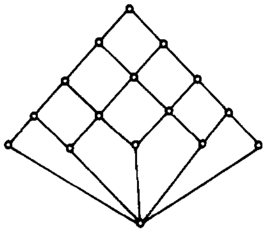

Fig. 1a. Co(5).

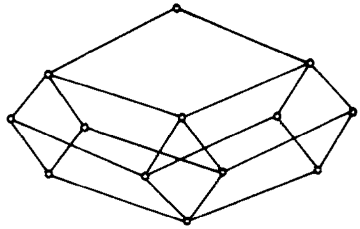

Fig. 1b. Co(2^2).

On the other hand, all posets of height two and the same cardinality n have the same Boolean algebra 2^n of convex subsets as Co(P). Hence, all of the posets in Figure 2 have Co(P) $\simeq 2^5$. Similarly, the nonisomorphic posets displayed in Figures 3a and 3b have isomorphic Co(P).

Fig. 2.

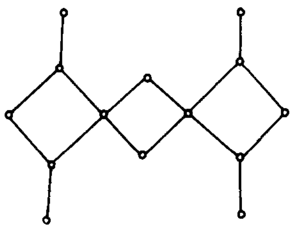

Fig. 3a.

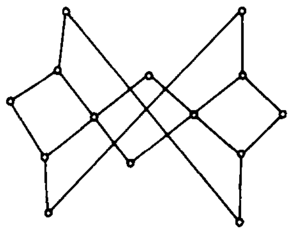

Fig. 3b.

After proving some basic facts about Co(P) in Section 2, we will give a complete discussion of the case that P is a chain in Section 3. Then, in Section 4, we will discuss two reciprocal 'wrapping' and 'unwrapping' algorithms, and give the conditions under which Co(P) and Co(Q) are isomorphic.

In Section 5 we will show that any Co(P) is 'join-semidistributive'. Then we will prove that a given lattice is isomorphic to Co(P) for some P *with least element* o if and only if: (i) it is complete, atomic, join-semidistributive, has Carathéodory rank 2 (see Section 2); (ii) it has an atom o which satisfies $(x \vee y) \wedge o = (x \wedge o) \vee (y \wedge o)$ for every x and y, such that whenever a, b and c are atoms with $o \neq a$, c and $\langle a, b, c\rangle \simeq$ Co(3), then $\langle o, a, c\rangle \simeq$ Co(3); and (iii) its atoms satisfy the condition that $b \leqslant a \vee c$ and $c \leqslant a \vee d$ imply that $c \leqslant b \vee d$. Finally, in Section 6, we prove that a lattice is isomorphic to some Co(P) if and only if it is complete, atomic, join-semidistributive, has Carathéodory rank 2, and satisfies a condition (A) proposed by Altwegg (see Section 6).

2. Elementary Properties

The following two basic properties of Co(P) are easily proved, for any poset P.

LEMMA. *Every* Co(P) *is atomic.*

Proof. Every convex subset of P is the join (in fact the union) of its (convex) one-element subsets, and each of these is an atom of Co(P). □

LEMMA. *The smallest convex subset* $\bar{A} = \bigvee_A a_i$ *containing any set* A *of elements* $\{a_i\}$ *of* P *is the set-union*

$$\emptyset(A) = \bigcup_{A \times A} [a_i, a_j]$$

of the intervals $[a_i, a_j]$ *consisting of all* $x \in P$ *with* $a_i \lessapprox x \lessapprox a_j$.★

Proof. Evidently any convex subset of P that contains A must contain $\emptyset(A)$. Moreover, $\emptyset(A)$ itself is convex since if $x \lessapprox b \lessapprox y$, where $x, y \in \emptyset(A)$, then $x \lessapprox b \lessapprox y$ where $a_i \lessapprox x \lessapprox a_j$, and $a_k \lessapprox y \lessapprox a_n$, whence $a_i \lessapprox x \lessapprox b \lessapprox y \lessapprox a_n$, and $b \in [a_i, a_n] \subseteq \emptyset(A)$. That is, $\emptyset(\emptyset(A)) = \emptyset(A)$, completing the proof. □

The property of Co(P) stated in the lemma can be defined more abstractly.

DEFINITION. An atomic lattice L has *Carathéodory rank* 2 when, given an atom $p \leqslant \bigvee_A a_i$ in L, $p \leqslant a_i \vee a_j$ for *two* suitably chosen atoms in A.

THEOREM 1. *For any poset* P, *the lattice* Co(P) *has Carathéodory rank 2.*

Proof. This follows immediately from the lemma. □

N.B. Co(P) has Carathéodory rank 1 if and only if P has height 1 or less, i.e. if and only if Co(P) is a Boolean algebra.

DEFINITION. An atomic lattice L is *biatomic* when, given an atom $r \leqslant a \vee b$, there are atoms $p \leqslant a$ and $q \leqslant b$ with $r \leqslant p \vee q$.

THEOREM 2. *Any atomic lattice of Carathéodory rank* 2 *is biatomic.*

Proof. Take a and b in L with p an atom under $a \vee b$. Let $a = \bigvee_I a_i$, and $b = \bigvee_J b_j$ with a_i and b_j atoms. If $p \leqslant a_{i_1} \vee a_{i_2}$, then $p \leqslant a$, so $p \leqslant p \vee b_j$ for any $b_j \leqslant b$. Similarly if $p \leqslant b_{j_1} \vee b_{j_2}$, $p \leqslant a_i \vee p$ for any a_i. Otherwise $p \leqslant a_i \vee b_j$ for some i and j and L is biatomic. □

COROLLARY. *Any* Co(P) *is biatomic.*

Affine Convexity. Although the order convexity discussed in this paper should not be confused with the 'affine convexity' discussed in [2–4], the two notions of convexity share several properties.★ Moreover, for any ordered division ring D, Co(D) is the same in both interpretations. However, for $n > 1$, the lattices Co(D^n) defined by *order* betweenness are very different from those defined by *affine* betweenness, as we shall now see.

★ We will use $\lessapprox$ to designate the order relation in P, and $\leqslant$ for that in Co(P).
★ For example, each is algebraic and biatomic. See also Theorem 10 below.

DEFINITION. An atomic lattice L has *Carathéodory rank* n when, given atoms p and $\{p_i\}_{i\in I}$, if $p \leqslant \bigvee_I p_i$, then $p \leqslant p_{i_1} \vee p_{i_2} \vee \cdots \vee p_{i_n}$ for some $i_1, \ldots, i_n \in I$.

LEMMA. *If an atomic lattice L has finite Carathéodory rank, it is algebraic.*

Whereas $\mathrm{Co}(D^n)$ has Carathéodory rank $n + 1$ under affine betweenness (see [6] and [11], p. 103), it has Carathéodory rank 2 under order betweenness, as we have seen.

COROLLARY. *Any* $\mathrm{Co}(P)$ *is algebraic.*

The Anti-exchange Property in $\mathrm{Co}(P)$. Let c be an element of $\mathrm{Co}(P)$ with p and q atoms under c. If an atom r is under $p \vee c$, then $r = p$, $r \leqslant c$, or $r \leqslant p \vee c_1$ for an atom $c_1 \leqslant c$. But this means that $p \lneq r \lneq c_1$ or $c_1 \lneq r \lneq p$ *in P*. Hence, if $q \leqslant p \vee c$, then $c_1 \lneq q \lneq p$, or $p \lneq q \lneq c_1$ in P. If also $p \leqslant q \vee c$, then $c_2 \lneq p \lneq q$ or $p \lneq q \lneq c_2$ for some atom c_2 under c. Thus, either $c_1 \lneq q \lneq p \lneq c_2$, or $c_2 \lneq p \lneq q \lneq c_1$, so p and q are under c, a contradiction. The property described above can be defined in a general atomic lattice as follows.

DEFINITION. An atomic lattice L has the *anti-exchange property* when, for any atoms p and q *not* contained in $c \in L$, $p \vee c = q \vee c$ implies $p = q$.

Edelman [7], p. 292 has noted that the lattice $\mathrm{Co}(P)$ of any finite P has the anti-exchange property. The remarks above generalize his result to arbitrary posets.

THEOREM 3. *Any* $\mathrm{Co}(P)$ *has the anti-exchange property.*

In Section 5 we will show that in biatomic lattices with no infinite chains, the anti-exchange property is equivalent to join-semidistributivity.

Finite Co(P). If P (or, equivalently $\mathrm{Co}(P)$) is finite, then each element is between a maximal and a minimal one. The maximal (as well as the minimal) elements generate a sublattice of $\mathrm{Co}(P)$ isomorphic to the Boolean algebra 2^n (n being the number of maximal elements of P).

The removal of any maximal or minimal element from a convex subset of P gives a convex subset again; hence the maximal and minimal elements can be 'peeled off' in turn, giving a Boolean interval sublattice of $\mathrm{Co}(P)$ (see [7], Thm. 3.3). Finally, we have

THEOREM 4. *For any finite poset P,* $\mathrm{Co}(P)$ *has length* $|P|$, *and the Jordan–Dedekind chain condition holds.*

Proof. The convex sets 'covering' any c in $\mathrm{Co}(P)$ are the $c \vee p$ where p either covers or is covered by some atom $a \leqslant c$. □

In case P is a *lattice*, much more can be said. Besides the order betweenness with which we are concerned, betweenness can be defined in several different ways (see [5], G_1–G_{15}), including lattice betweenness [12], p. 7 in which b is 'lattice-between' a and c whenever b is *order between* $a \wedge c$ and $a \vee c$, and Glivenko's betweenness in which b is 'Glivenko-between' a and c when $(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (b \vee c)$.

THEOREM 5. *For any poset P,* $\mathrm{Co}(P) \simeq \mathrm{Co}(\mathring{P})$.

Proof. Since $a \leq b \leq c$ in P is equivalent to $c \leq b \leq a$ in $\mathring{P}$, the intervals $[a, c]$ in P are equal as sets to the intervals $[c, a]$ in $\mathring{P}$. □

There is nothing to be gained by studying convexity in the more general class of quasi-ordered sets (so-called *quosets*, whose order relation is reflexive and transitive), as the following theorem shows.

THEOREM 6. *Let Q be a quoset, and let θ be the equivalence relation given by letting $x\ \theta\ y$ if and only if $x \leq y$ and $y \leq x$. Then* $\mathrm{Co}(Q) \simeq \mathrm{Co}(Q/\theta)$.

Proof. If $A \in \mathrm{Co}(Q)$, let $A' = \{[x]^\theta : x \in A\}$. Then if $[a]^\theta$, $[c]^\theta \in A'$ with $[a]^\theta \leq [b]^\theta \leq [c]^\theta$ in Q/θ, it is easy to see that $a \leq b \leq c$ in Q; hence, $b \in A$ and $[b] \in A'$ and $A' \in \mathrm{Co}(Q/\theta)$. Conversely one can show that for $B \in \mathrm{Co}(Q/\theta)$, $B = A'$ where $A = \{x : [x]^\theta \in B\}$.

Relations to Ideals. One key to the structure of the atomic lattice $\mathrm{Co}(P)$ is provided by the structure of the lattices $I(P)$ of all *dual* order-ideals of P.* The connection between the lattices $\mathrm{Co}(P)$, $I(P)$, and $I(\mathring{P})$ is provided by the following elementary theorem.

THEOREM 7. *A is a convex subset of P if and only if A is the intersection of an order-ideal and a dual order-ideal.*

Proof. It is easy to show that a convex subset A is of the form $J \cap K$ where J is the set of all $x \leq b$ for some $b \in A$, and K is the set of all y *containing* some $b \in A$. Conversely, since the order-ideals form a Moore family of subsets of P, and any order-ideal or dual order-ideal is convex, the intersection of any order-ideal with any dual order-ideal is convex. □

The following corollary is immediate.

COROLLARY 1. *For any poset P,* $\mathrm{Co}(P)$ *is a meet-epimorphic image of* $I(P) \times I(\mathring{P})$.

Product Decomposition of Co(P). Any partially ordered set has an obvious and well known (see [9], p. 3) unique additive decomposition into its (disjoint) connected components, P_i being a *connected component* of P when $x \in P_i$, and $x \leq y$ *or* $y \leq x$ imply that $y \in P_i$. P is thus the disjoint union (cardinal sum) of its connected components (see [12], p. 55), and A is convex in P if and only if A is the disjoint union of the convex subsets $A_i = A \cap P_i$ of the P_i. Furthermore, the cardinal sum of any collection of posets is a poset which has the given posets as disjoint unions of connected components. Thus we have:

THEOREM 8. *If P is the disjoint union of $\{P_i\}_{i \in I}$, then* $\mathrm{Co}(P) = \Pi_{i \in I} \mathrm{Co}(P_i)$. *Furthermore, the class of all* $\mathrm{Co}(P)$ *is closed under the formation of arbitrary direct products.*

The connected components of P can alternately be viewed as the equivalence classes of the transitive closure of the relation $(\leq \cup \geq)$, which is reflexive and symmetric for any

* This is not to be confused with the lattice of all *lattice* ideals of a given lattice, defined in [12], p. 25 and discussed in [12], p. 113.

poset P. In particular, a one-member connected component is an element of P related to no other element, i.e., it is both a *maximal* and a *minimal* element of P. Considered as a poset, this element has a two-member convexity lattice (the empty set and the singleton) isomorphic to 2. This gives the following corollaries to Theorem 8.

COROLLARY 1. *Let q be both a maximal and a minimal element of poset P. Let $P' = P\backslash\{q\}$. Then* $\mathrm{Co}(P) \simeq \mathrm{Co}(P') \times 2$.

COROLLARY 2. *If* $\mathrm{Co}(P)$ *is (directly) indecomposable, then P has no element which is both maximal and minimal.*

We note that a connected poset P need not have a (directly) indecomposable $\mathrm{Co}(P)$. The posets shown in Figures 4a and 4b have isomorphic convexity lattices, the former being a direct product by Theorem 8, whereas the latter poset is connected. We shall discuss this situation further in Section 4.

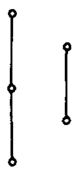

Fig. 4a.

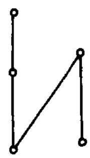

Fig. 4b.

3. Generalized Schweitzer Lattices

The Schweitzer lattices, alluded to in Section 1, are by definition the lattices $\mathrm{Co}(\mathbf{n})$, where $\mathbf{n}$ is a finite chain. Thus, if P is either of the posets drawn in Figure 4, $\mathrm{Co}(P)$ is isomorphic to $\mathrm{Co}(3) \times \mathrm{Co}(2)$, where $\mathrm{Co}(2)$ is simply 2^2. In $\mathrm{Co}(P)$, if p, q, and r are *atoms*, i.e., arbitrary (singleton) *members* of P, then $\langle p, q, r\rangle$ is isomorphic to $\mathrm{Co}(3)$ if and only if p, q and r form a chain in P. Hence, in Figure 5a below, $q \leqslant p \vee r$ in $\mathrm{Co}(3)$, and either $p \nleqslant q \nleqslant r$ or $r \nleqslant q \nleqslant p$ in P.

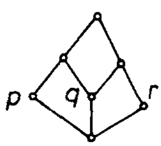

Fig. 5a. Co(3)

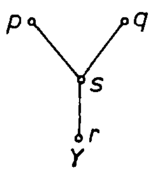

Fig. 5b.

In [2], p. 6, CL1, we noted that *any* three atoms of $\mathrm{Co}(D^n)$ generate either Co(3) or 2^3 as a sublattice. On the other hand, $\mathrm{Co}(Y)$ (Y the poset shown in Figure 5b) does not satisfy this condition since $\langle p, q, r\rangle$ contains a fourth atom s in $\mathrm{Co}(Y)$. Furthermore, $\langle p, s, r\rangle$ and $\langle q, s, r\rangle$ are each isomorphic to Co(3) while $\langle p, q, r, s\rangle$ is *not* isomorphic to Co(4), which would be impossible in $\mathrm{Co}(D^n)$ ([2], p. 6, CL2). In fact, in $\mathrm{Co}(P)$, $\langle p_1, \dots, p_n \rangle$ is isomorphic to $\mathrm{Co}(\mathbf{n})$ exactly when the elements $p_1, \dots, p_n$ are linearly ordered in P.

If C is an arbitrary chain, we know that $I(C)$ and $I(\mathring{C})$ are chains as well. Furthermore, (dual) order-ideals are (dual) lattice ideals in C. Hence, we have a second corollary to Theorem 7 above.

COROLLARY 2 TO THEOREM 7. *If C is chain, then $\mathrm{Co}(C)$ is a meet-epimorphic image of $I(C) \times I(\mathring{C})$, the product of the chain of all ideals of C and that of all ideals of $\mathring{C}$.*

In this case, the *diagram* of $\mathrm{Co}(C)$ is easily visualized (see Figure 6), as a right triangle standing on its hypotenuse with $\emptyset$ adjoined below. The hypotenuse consists of the atoms $[c, c] = \{c\}$, $c \in C$. The left side consists of the *dual* ideals $[a, \infty]$, (a, ∞), etc. where a is an arbitrary 'cut' in C (whose *ideal* completion $\hat{C}$ and Dedekind *order* completion $\bar{C}$ are closely related; see [12], p. 117).

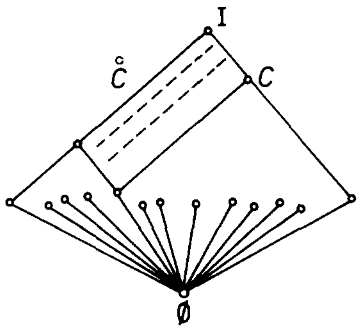

Fig. 6. $\mathrm{Co}(C)$.

THEOREM 9. *If C is a chain, then every proper dual ideal $[[c, c], I]$ of $\mathrm{Co}(C)$ is the direct product $(-\infty, c] \times [c, \infty)$ of two intervals of C.*

The shaded area in the diagram of $\mathrm{Co}(C)$ represents such a dual ideal; clearly $C \backslash \emptyset$ is the union of such proper ideals. The similarity to Co(5) (Figure 1a) is obvious.

When $C = \mathbb{R}$, the pairs $(-\infty, a]$ and (a, ∞) are *complementary* as are $(-\infty, a)$ and $[a, \infty)$. Moreover any complemented element of $\mathrm{Co}(\mathbb{R})$ is an interval of one of the four types described above. Hence, the complemented elements are the elements of C or $\mathring{C}$. For any singleton $\{c\}$, the interval $[\{c\}, I]$ is the product of the chain of dual ideals of the form $(-\infty, b)$ with $b \succ c$, and $[-\infty, b]$ with $b \not\succ c$, and the chain of ideals of the form $[a, \infty)$, $a \not\prec c$ and (a, ∞), $a \prec c$. As abstract chains C and $\mathring{C}$ are isomorphic to $\mathbf{1} \oplus (\mathbb{R} \circ 2)$ and its dual $(\mathbb{R} \circ 2) \oplus \mathbf{1}$ respectively.

4. 'Unwrapping' a Poset

Recall [4], p. 287, that if X is a convex subset of D^n, its extreme points are exactly the atoms p in the lattice of convex subsets of X which satisfy

$$(x \vee y) \wedge p = (x \wedge p) \vee (y \wedge p) \quad \text{for all } x, y. \tag{1}$$

The maximal and minimal elements (if any) in a poset are, in a sense, its 'extreme points', and are crucial in determining when two posets have isomorphic convexity lattices. The 'analogy' between maximal and minimal elements of P and the vertices of a polytope in D^n carries over to $\mathrm{Co}(P)$ as follows.

THEOREM 10. *Let* $\{p\}$ *be an atom of* $\mathrm{Co}(P)$ *(an element of* P*). Then* $\{p\}$ *is maximal or minimal in* P *if and only if* $\{p\}$ *satisfies* (1) *in* $\mathrm{Co}(P)$.

Proof. If, for some x and y, $(x \vee y) \wedge p = 0$, then (1) holds for that x and y. Otherwise, let p be maximal and $(x \vee y) \wedge p = p$. By biatomicity, we have $p \leqslant x_1 \vee y_1$ in $\mathrm{Co}(P)$; hence $p \in \{x_1, y_1\}$ or $x_1 \lessdot p \lessdot y_1$ or dually in P. Thus, $p \lessdot x$ or y and (1) holds. Conversely, if p is neither maximal nor minimal in P, there are elements a and b in P, different from p, with $a \lessdot p \lessdot b$. Hence, in $\mathrm{Co}(P)$, $p = (a \vee b) \wedge p$, but $a \wedge p = b \wedge p = 0$ and (1) fails. □

The notion of *unwrapping* a poset results from the observation that $\mathrm{Co}(P)$ is unaffected by deleting all maximal chains of *two* elements, i.e., deleting any covering relations joining a *maximal* x with a *minimal* y covered by x. The poset in Figure 3b, when unwrapped, becomes that in Figure 3a; similarly for the posets in Figure 4. More formally, we have

THEOREM 11. *Let* $\mathrm{Co}(P)$ *be indecomposable. Let* Q *be the poset whose elements are those in* P*, let* $x \lessdot y$ *in* P *imply* $x \lessdot y$ *in* Q*, and let* $x \lessdot y$ *in* Q *imply that* $x \lessdot y$ *in* P *or that* x *is minimal in* P *and* y *is maximal in* P*. Then* $\mathrm{Co}(P) \simeq \mathrm{Co}(Q)$.

Proof. Let a, $b \leqslant x$ in $\mathrm{Co}(P)$, and let $a \lessdot c \lessdot b$ in Q. If the elements are distinct, then $a \lessdot c \lessdot b$ in P, and $c \leqslant x$, or c is both a maximal and a minimal element in P, a contradiction. Hence, $x \in \mathrm{Co}(Q)$. Conversely, if $x \in \mathrm{Co}(Q)$ and a, $b \leqslant x$ with $a \lessdot c \lessdot b$ in P, then $a \lessdot c \lessdot b$ in Q, hence, $c \leqslant x$ and $x \in \mathrm{Co}(P)$. □

In order to simplify the posets with which we must deal, we define a class of posets essentially formed by unwrapping arbitrary posets in the manner described above.

DEFINITION. A poset P is said to be *coherent* when it is connected and no maximal element of P covers any minimal element of P.

The poset drawn in Figure 3a is coherent; that in Figure 3b is not. Neither of the posets in Figure 4 is coherent; however Figure 4a is the disjoint union of coherent components.

COROLLARY TO THEOREM 11. *Any poset* P *contains coherent subposets* Q_i *such that* $\mathrm{Co}(P) \simeq \Pi\, \mathrm{Co}(Q_i)$.

Proof. The poset ΣQ_i (a disjoint union) is formed by defining $b \not\lessdot m$ in Q_i whenever $b \lessdot m$ in P and $\{b, m\}$ is a maximal chain in P. □

When P is coherent and $x \lessdot y$ in P, there is some z in P so that $\{x, y, z\}$ form a three-element chain and, therefore, $\langle x, y, z\rangle$ is isomorphic to $\mathrm{Co}(3)$ in $\mathrm{Co}(P)$. Using this fact, we say that atoms x and y in an atomic lattice L are *collinear* (and write $x\,\gamma\, y$) if and only if there is an atom z with $\langle x, y, z\rangle$ isomorphic to $\mathrm{Co}(3)$. The relation γ is symmetric, and constructing its transitive closure we obtain an equivalence relation $\overline{\gamma}$, which subdivides P into its $\overline{\gamma}$-connected components in case L is $\mathrm{Co}(P)$. When P is coherent, the relation $\overline{\gamma}$ is degenerate, and there is only one equivalence class, namely all of P. In any coherent poset, starting with an arbitrary element a, we set

$$A_0 = \{a\}$$
$$A_1 = \{x : x\,\gamma\, a\}$$
$$\vdots$$
$$A_n = \{x : x\,\gamma^n\, a\}.$$

Since the transitive closure of γ, $\overline{\gamma}$, is by definition $\bigcup_{i=1}^{\infty} \gamma^i$, it is easy to see

LEMMA. *If P is coherent, then*

$$P = \bigcup_{i=1}^{\infty} A_i.$$

We are now in a position to prove the main theorem of this section, that if P is coherent, then $\mathrm{Co}(P)$ determines P up to duality.

THEOREM 12. *Let P and Q be coherent with $\mathrm{Co}(P) \simeq \mathrm{Co}(Q)$. Then $P \simeq Q$ or $P \simeq \mathring{Q}$.*

Proof. We may assume $\mathrm{Co}(P) = \mathrm{Co}(Q)$ so that the underlying sets of P and Q are the same. We may assume there exist $a \lessdot b \lessdot c$ in P since otherwise P, having only one component, is a singleton, as is Q, and $P \simeq Q$.

Now $a \lessdot b \lessdot c$ in P implies $b \leqslant a \vee c$ in $\mathrm{Co}(P)$, so $b \leqslant a \vee c$ in $\mathrm{Co}(Q)$. Thus, $a \lessdot b \lessdot c$ in Q or $c \lessdot b \lessdot a$ in Q. We may assume the former without loss of generality.

Let A_i^P be the subsets of P as defined above, and let A_i^Q be the analogous subsets of Q. As sets A_1^P and A_1^Q are equal; we will show they are isomorphic as posets. The proof breaks down into several cases:

(I) Let $\langle a, x, y\rangle \simeq \mathrm{Co}(3)$ with $x \leqslant a \vee y$.

Let $a \lessdot x \lessdot y$ in P. If $y \lessdot x \lessdot a$ in Q we have $y \lessdot x \lessdot a \lessdot b$ in Q which implies $a \leqslant b \vee x$ in $\mathrm{Co}(Q)$. But $a \lessdot b \lessdot c$ in P and $a \lessdot x \lessdot y$ in P imply $a \nleqslant b \vee x$ in $\mathrm{Co}(P)$.

Hence $a \lessdot x \lessdot y$ in Q.

Similarly, $y \lessdot x \lessdot a$ in P implies $y \lessdot x \lessdot a$ in Q.

(II) Let $\langle a, x, y\rangle \simeq \mathrm{Co}(3)$ with $a \leqslant x \vee y$.

Let $x \lessdot a \lessdot y$ in P.

If $y \lessdot a \lessdot x$ in Q, then since $x \lessdot a \lessdot b$ in P, we have $a \leqslant b \vee x$ in $\mathrm{Co}(P)$. But $a \lessdot x$ and $a \lessdot b$ in Q, so $a \nleqslant b \vee x$ in $\mathrm{Co}(Q)$.

(III) Suppose $x, y \in A_1$ with $x \lessdot y$ in P. Then there is a z in P with $\langle a, x, z\rangle \simeq \mathrm{Co}(3)$ so a and x are related in P. By (I) and (II) above, a and x are related *in the same way* in Q.

(A) If $a \lessdot x$, then $a \lessdot x \lessdot y$ in P, so by (I) $a \lessdot x \lessdot y$ in Q.
(B) If $a \lessdot a \lessdot y$ in P, then by (II), $x \lessdot a \lessdot y$ in Q.
(C) If $x \lessdot y \lessdot a$ in P, then by (I) $x \lessdot y \lessdot a$ in Q.

Therefore, if $x \lessdot y$ in A_1^P, then $x \lessdot y$ in A_1^Q. The converse can similarly be demonstrated, so A_1^P and A_1^Q are isomorphic as posets.

We now assume A_n^P and A_n^Q are isomorphic as posets. Arguments similar to those used above show that if $\langle b, x, y\rangle \simeq \mathrm{Co}(3)$ with $b \in A_n^P$, then

(I) If $x \leqslant b \vee y$ in $\mathrm{Co}(P)$ and $b \lessdot x \lessdot y$ in P, then $b \lessdot x \lessdot y$ in Q.
(II) If $b \leqslant x \vee y$ in $\mathrm{Co}(P)$ and $x \lessdot b \lessdot y$ in P, then $x \lessdot b \lessdot y$ in Q.

Coherence comes into play during the next part of the argument.

(III) Let z and w be elements of A_{n+1}^P, with $z \lessdot w$ in P. Then there is $b \in A_n$ such that $\langle b, z, c\rangle \simeq \mathrm{Co}(3)$, and $b' \in A_n$ with $\langle b', w, d\rangle \simeq \mathrm{Co}(3)$. Hence, b is related to z and b' is related to w.

(A) If $b \lessdot z$ in P, then $b \lessdot z \lessdot w$ in P, so $b \lessdot z \lessdot w$ in Q.
(B) If $b' \lessdot w$, then $z \lessdot w \lessdot b'$ in P, so $z \lessdot w \lessdot b'$ in Q.
(C) Assume $b \lessdot z$ in P and $b' \lessdot w$ in P.

Since $\langle b, z, c\rangle \simeq \mathrm{Co}(3)$ in $\mathrm{Co}(P)$, and $b \gtrdot z$ in P, then by (I) and (II) $b \gtrdot z$ in Q. Similarly if $b' \lessdot w$ in P, then $b' \lessdot w$ in Q. Thus we have $z \lessdot b$ in Q and $b' \lessdot w$ in Q.

(i) Assume $c \lessdot z$ in P. Then $c \lessdot z \lessdot w$ in P, so $c \lessdot z \lessdot w$ in Q.
(ii) Assume $w \lessdot d$ in P. Then $z \lessdot w \lessdot d$ in P, so $z \lessdot w \lessdot d$ in Q.
(iii) Therefore we must assume $c \gtrdot z$, $b \gtrdot z$, $d \lessdot w$, and $b' \lessdot w$.

If $z \lessdot w$ in P and $z \not\lessdot w$ in Q, then z must be minimal in Q. Otherwise there is an $x \lessdot z$ in Q, and we would therefore have $\langle x, z, b\rangle \simeq \mathrm{Co}(3)$ in $\mathrm{Co}(P)$; therefore $x \lessdot z \lessdot b$ in both P and Q, so $x \lessdot w$ in Q by (i). A similar argument shows that if $z \lessdot w$ in P and $z \not\lessdot w$ in Q, w must be maximal in Q. But by coherence, if $z \lessdot w$ in P with z minimal and w maximal, then there is a y in P with $z \lessdot y \lessdot w$. But this means $\langle z, y, w\rangle \simeq \mathrm{Co}(3)$ in $\mathrm{Co}(P)$ and, therefore, in $\mathrm{Co}(Q)$, so $z \lessdot w$ in Q or $w \lessdot z$ in Q. In the latter case we have $b' \lessdot w \lessdot z$ in Q, so $b' \lessdot w \lessdot z$ in P, a contradiction. Therefore $z \lessdot w$ in Q.

Therefore, by induction, $A_i^P \simeq A_i^Q$ for all i, so it follows from the lemma that $P \simeq Q$. $\square$

In determining when two posets have isomorphic convexity lattices, it is convenient to have the following notation.

DEFINITION. For posets P and Q we say P and Q are *isomedic* (and write $P \sim Q$) when there is a bijection f from P to Q such that $x \lessdot y$ but $f(x) \not\lessdot f(y)$ imply that $\{x, y\}$ is a maximal chain in P; and $f(x) \lessdot f(y)$ but $x \not\lessdot y$ imply that $\{f(x), f(y)\}$ is a maximal chain in Q.

Figures 7a and 7c illustrate posets P and Q which are isomedic respectively to the cardinal sums $P_1 + P_2$ and $Q_1 + \mathring{Q}_2$ of Figures 7b and 7d with $P_1 \sim Q_1$, and $P_2 \sim Q_2$.

The convexity lattices $\mathrm{Co}(P)$ and $\mathrm{Co}(Q)$ are isomorphic; in fact $\mathrm{Co}(P) \simeq (\mathrm{Co}(P_1) \times \mathrm{Co}(P_2) \simeq (\mathrm{Co}(Q_1) \times \mathrm{Co}(Q_2)) \simeq \mathrm{Co}(Q)$, and serve as an illustration of the result to be stated in Theorem 13.

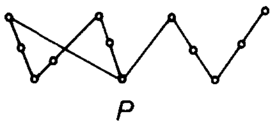

Fig. 7a.

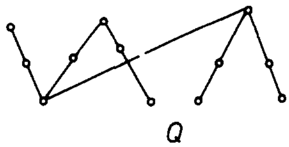

Fig. 7c.

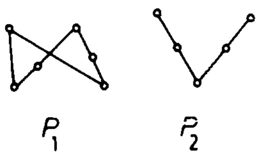

Fig. 7b.

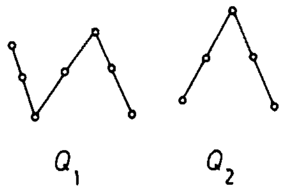

Fig. 7d.

THEOREM 13. *For P and Q posets,* $\mathrm{Co}(P) \simeq \mathrm{Co}(Q)$ *if and only if* $P \sim \Sigma_{i \in I} P_i$; $Q \sim \Sigma_{i \in I} Q_i$; *and for each* $i \in I$, $P_i \simeq Q_i$ or $P_i \simeq \mathring{Q}_i$.

Proof. If $\mathrm{Co}(P) \simeq \mathrm{Co}(Q)$ with both lattices indecomposable, if P and Q are coherent, then $P \simeq Q$ or $P \simeq \mathring{Q}$. Otherwise by the corollary to Theorem 11, P and Q can be reduced to coherent posets P' and Q' with $\mathrm{Co}(P) \simeq \mathrm{Co}(P') \simeq \mathrm{Co}(Q') \simeq \mathrm{Co}(Q)$, whence $P' \simeq Q'$ or $P' \simeq \mathring{Q}'$. In the former case $P \sim Q$, and in the latter $P \sim \mathring{Q}$. If $\mathrm{Co}(P)$ and $\mathrm{Co}(Q)$ are decomposable we make the argument above for each factor, and obtain the result. Conversely since $P_i \sim Q_i$ or $P_i \sim \mathring{Q}_i$ implies by Theorem 11 that $\mathrm{Co}(P_i) \simeq \mathrm{Co}(Q_i)$, then if P and Q satisfy the conditions of the theorem, $\mathrm{Co}(P) \simeq \pi \mathrm{Co}(P_i) \simeq \pi \mathrm{Co}(Q_i) = \mathrm{Co}(Q)$. □

COROLLARY. *If L and M are bounded lattices, then* $\mathrm{Co}(L) \simeq \mathrm{Co}(M)$ *exactly when* $L \simeq M$ *or* $L \simeq \mathring{M}$.

5. Semidistributivity and Bounded Posets

Jónsson [10], p. 262, has shown that every sublattice L of a free lattice is *semidistributive* in the sense that:

(SD1) for all a, d in L, the set $\{x \mid a \vee x = d\}$ is meet-closed

and

(SD2) for all a, d in L, the set $\{x \mid a \wedge x = d\}$ is join-closed.

A lattice satisfying (SD1) is called *join-semidistributive;* a lattice satisfying (SD2) is called *meet-semidistributive.*

We shall show that every lattice $\mathrm{Co}(P)$ of convex subsets of a poset is join-semidistributive, or equivalently, if a, b and c are in $\mathrm{Co}(P)$ with $a \vee b = a \vee c$, then $a \vee b = a \vee (b \wedge c)$.

THEOREM 14. *For any poset P, the (biatomic) lattice* Co(P) *is join-semidistributive.*

Proof. Assume that a, b and c are in Co(P), and that $a \vee b = a \vee c$. It is enough to show that $(a \vee b) \wedge (a \vee c) \leqslant a \vee (b \wedge c)$. Taking an atom $x \leqslant (a \vee b) \wedge (a \vee c)$, there are several possibilities:

Since $x \leqslant a \vee b$	and	since $x \leqslant a \vee c$
(1) $a_1 \lesssim x \lesssim b_1$		(i) $a_2 \lesssim x \lesssim c_2$
(2) $b_1 \lesssim x \lesssim a_1$		(ii) $c_2 \lesssim x \lesssim a_2$
(3) $x \leqslant a$		(iii) $x \leqslant a$
(4) $x \leqslant b$		(iv) $x \leqslant c$

$a_i \leqslant a, \quad b_i \leqslant b, \quad c_i \leqslant c.$

These possibilities lead to sixteen possible cases (some of which are quite easily settled). We shall give the details only of cases (1, i) and (4, i), as prototypical of the most complicated possibilities.

(1, i) If $a_1 \lesssim x \lesssim b_1$ and $a_2 \lesssim x \lesssim c_2$, since $b_1 \leqslant a \vee c$, we have either:

- (I) $b_1 \leqslant a$, in which case $x \leqslant a$.
- (II) $b_1 \leqslant c$, in which case $b_1 \leqslant b \wedge c$, so $x \leqslant a \vee (b \wedge c)$.
- (III) $a_3 \lesssim b_1 \lesssim c_3$, where $a_3 \leqslant a$ and $c_3 < c$. Here, since $c_3 \leqslant a \vee b$, there are again several cases:
  - (1) $c_3 \leqslant a$, in which case $x \leqslant a$.
  - (2) $c_3 \leqslant b$, which implies $c_3 \leqslant b \wedge c$, so $x \leqslant a \vee (b \wedge c)$.
  - (3) $a_4 \lesssim c_3 \lesssim b_4$ ($c_3 \leqslant c$ and $b_4 \leqslant b$), in which case $c_3 \leqslant b$ and $x \leqslant a \vee (b \wedge c)$.
  - (4) $b_4 \lesssim c_3 \lesssim a_4$, which implies $x \leqslant a$.
- (IV) $c_3 \lesssim b_1 \lesssim a_3$, which implies $a_1 \lesssim x \lesssim a_3$ and $x \leqslant a$.

(4, i) Let $x \leqslant b$ and $a_1 \lesssim x \lesssim c_1$. Since $c_1 \leqslant a \vee b$, we have

- (I) $c_1 \leqslant a$, whence $x \leqslant a$.
- (II) $c_1 \leqslant b$, so that $c_1 \leqslant b \wedge c$ and $x \leqslant a \vee (b \wedge c)$.
- (III) $a_2 \lesssim c_1 \lesssim b_1$, which implies $c_1 \leqslant b$ and $x \leqslant a \vee (b \wedge c)$.
- (IV) $b_1 \lesssim c_1 \lesssim a_2$, and $x \leqslant a$.

Join-semidistributivity is a property not shared by any lattice Co(D^n) if $n < 2$; in fact, given l, m and k the lines in $\mathbb{R}^2$ shown in Figure 8, $l \vee m = l \vee k = \mathbb{R}^2$, but $l \vee (m \wedge k) = l \vee \mathbf{0}$, which is the open lower half plane with $\mathbf{0}$ adjoined.

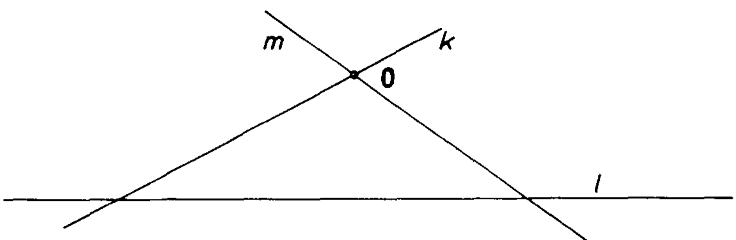

Fig. 8.

However, we can prove the following theorem.

THEOREM 15. *The lattice L of all convex polytopes in D^n (D an ordered division ring) is join-semidistributive.*

Proof. Recall that a *convex polytope* is the convex hull of a *finite* set of points which, after removing any redundant points, we can assume to be its extreme points. These polytopes form a sublattice of the biatomic lattice $\mathrm{Co}(D^n)$. Furthermore, if X and Y are polytopes, every extreme point of $X \vee Y$ is an extreme point of X or an extreme point of Y; since, by biatomicity all points of $X \vee Y$ not in X or Y are interior points of segments $x \vee y$, $x \in X$, $y \in Y$. Neither these, nor the interior points of X or Y can be extreme.

If $X \vee Y = X \vee Z$, then both are $X \vee Y \vee Z$. But the *extreme* points of $X \vee Y = X \vee Z = X \vee Y \vee Z$ must be in X or in Y, *and* in X or in Z. Hence, they must be in X or in $Y \cap Z$ by the distributive law. Since $X \vee Y \vee Z$ is the join of its extreme points, the result follows. □

Join-semidistributivity is connected with the anti-exchange property discussed in Section 2, as shown by the following results.

THEOREM 16. *Any join-semidistributive lattice has the anti-exchange property.*

Proof. If $q \leqslant p \vee a$, $q \wedge a = 0$, and $a \vee p = a \vee q$, then $a \vee q = a \vee (p \wedge q) = a$, so $q \leqslant a$, a contradiction. □

We can obtain a partial converse to Theorem 16, by using this lemma.

LEMMA. *If L has the anti-exchange property, for p and q distinct atoms, $a \vee p = a \vee q$ implies $a \vee p = a \vee (p \wedge q)$ $(= a)$.*

Proof. For $a \vee p = a \vee q$, $p \leqslant a \vee q$, hence $q \nleqslant (a \vee p)$ unless $p \leqslant a$. Similarly, $q \leqslant a$ and $a \vee p = a = a \vee (p \wedge q)$. □

THEOREM 17. *If L is biatomic, satisfies the anti-exchange property, and has no infinite chains, then L is join-semidistributive.*

Proof. Let $a \vee b = a \vee c$ and let b_0 be an atom under b. Then $b_0 \leqslant a \vee c_0$ for some atom $c_0 \leqslant c$; likewise $c_0 \leqslant a \vee b_1$, and we have

$$b_0 \leqslant a \vee c_0 \leqslant a \vee b_1 \leqslant a \vee c_1 \leqslant \cdots \leqslant a \vee c_k \leqslant \cdots.$$

Since L has no infinite chains there is an n with $a \vee c_n = a \vee b_{n+1}$. If $c_n = b_{n+1}$, then $c_n \leqslant b \wedge c$ and $b_0 \leqslant a \vee (b \wedge c)$.

Otherwise by the lemma, $a \vee c_n = a \vee b_{n+1} = a \vee (c_n \wedge b_{n+1}) = a$ and $b_0 \leqslant a$. Thus, $b \leqslant a \vee (b \wedge c)$, hence, $a \vee b = a \vee (b \wedge c)$. □

Join-semidistributivity, together with two technical conditions, enable us to characterize the lattices $\mathrm{Co}(P)$, where P is a poset with smallest element o.

THEOREM 18. *A complete atomic lattice L is isomorphic to* $\mathrm{Co}(P)$ *for some poset P with smallest element o if and only if*

(i) *L is join-semidistributive,*
(ii) *L has Carathéodory rank* 2,

(iii) *There is an atom o satisfying* (1) [*See Section* 4] *such that, whenever a, b, c are atoms of L with $o \neq a, c$ and $\langle a, b, c\rangle \simeq \mathrm{Co}(3)$, then $\langle o, a, c\rangle \simeq \mathrm{Co}(3)$.*

(iv) *If a, b, c and d are atoms of L with $b \leqslant a \vee c$ and $c \leqslant a \vee d$, then $c \leqslant b \vee d$.*

Proof. We have already seen that $\mathrm{Co}(P)$ always satisfies (i) and (ii). If P has a smallest element o, and if $\langle a, b, c\rangle$ is isomorphic to $\mathrm{Co}(3)$, then we can assume without loss of generality that $a \preccurlyeq c$, so $o \preccurlyeq a \preccurlyeq c$, and $\langle o, a, c\rangle \simeq \mathrm{Co}(3)$, and (iii) holds. If $b \leqslant a \vee c$ in $\mathrm{Co}(P)$ and $c \leqslant a \vee d$, then if $a \preccurlyeq b \preccurlyeq c$, we must have $a \preccurlyeq c \preccurlyeq d$, whence $b \preccurlyeq c \preccurlyeq d$, so $c \leqslant b \vee d$. If $c \preccurlyeq b \preccurlyeq a$, then $d \preccurlyeq c \preccurlyeq a$, so $d \preccurlyeq c \preccurlyeq b$, and $c \leqslant b \vee d$.

Conversely, if we assume (i), ..., (iv) hold in a lattice L, we let P be the set of atoms in L and define $\preccurlyeq$ on P by $x \preccurlyeq y$ if and only if $x \leqslant o \vee y$ in L. Then $x \preccurlyeq x$ since $x \leqslant o \vee x$ for every x, so $\preccurlyeq$ is reflexive. If $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x \leqslant o \vee y$, and $y \leqslant o \vee x$, so $o \vee y = o \vee x$, and by join-semidistributivity, this equals $o \vee (x \wedge y)$, so $x = y$, and $\preccurlyeq$ is anti-symmetric. If $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \leqslant o \vee y \leqslant o \vee z$, so $x \preccurlyeq z$, and P is a poset under $\preccurlyeq$.

Since $o \leqslant o \vee x$ for every x, o is the smallest element of P.

If B is a subset of P satisfying the condition

$$p \leqslant \bigvee_{b \in B} \{b\} \text{ implies } p \in B \tag{$*$}$$

we will show that $B \in \mathrm{Co}(P)$. For this, we take $b_1, b_2 \in B$ and x such that $b_1 \preccurlyeq x \preccurlyeq b_2$. We have $b_1 \leqslant o \vee x$, and $x \leqslant o \vee b_2$, so by (iv), $x \leqslant b_1 \vee b_2$. Hence, $x \in B$, and $B \in \mathrm{Co}(P)$. Finally, we show that if B is in $\mathrm{Co}(P)$, then B satisfies ($*$). Here we assume that p is an atom under $\bigvee_{b_i \in B} b_i$. This implies by (ii) that $p \leqslant b_1 \vee b_2$ for some b_i, whence $\langle b_1, p, b_2\rangle \simeq \mathrm{Co}(3)$, so by (iii) $\langle o, b_1, b_2\rangle \simeq \mathrm{Co}(3)$. By (1) we cannot have $o \leqslant b_1 \vee b_2$, therefore $b_1 \leqslant o \vee b_2$, or $b_2 \leqslant o \vee b_1$. Assuming the former, since $p \leqslant b_1 \vee b_2$, $o \vee p \leqslant o \vee b_1 \vee b_2 = o \vee b_2$, so $p \quad b_2$. Since $b_1 \leqslant b_2 \vee o$, (iv) implies that $b_1 \leqslant o \vee p$. Hence, $b_1 \preccurlyeq p \preccurlyeq b_2$, so $p \in B$, and ($*$) holds. □

6. The Altwegg Condition: Characterization of Co(P)

In order to characterize the lattices $\mathrm{Co}(P)$, we need to be able to define an *order relation* on the set of *atoms* of lattices satisfying appropriate conditions. We start with the notion of *related atoms.*

In general, three atoms x, y and z of a lattice L are called *independent* [12], p. 86 (14), when

$$x \wedge (y \vee z) = y \wedge (x \vee z) = z \wedge (x \vee y) = 0,$$

i.e., when none of them is contained in the join of the other two. We now define two (distinct) atoms x and y of L to be *related* when some triple $\{x, y, z\}$ of atoms of L is *not* independent, i.e., when there is an atom $z \neq x, y$ with $z \leqslant x \vee y$, $y \leqslant x \vee z$, or $x \leqslant y \vee z$.

We note that no x is related to itself, and that if $L = \mathrm{Co}(P)$, with $\{x, y\}$ a maximal chain in P, then x and y are not related in the sense above. (Cf. Theorem 11 and its corollary.)

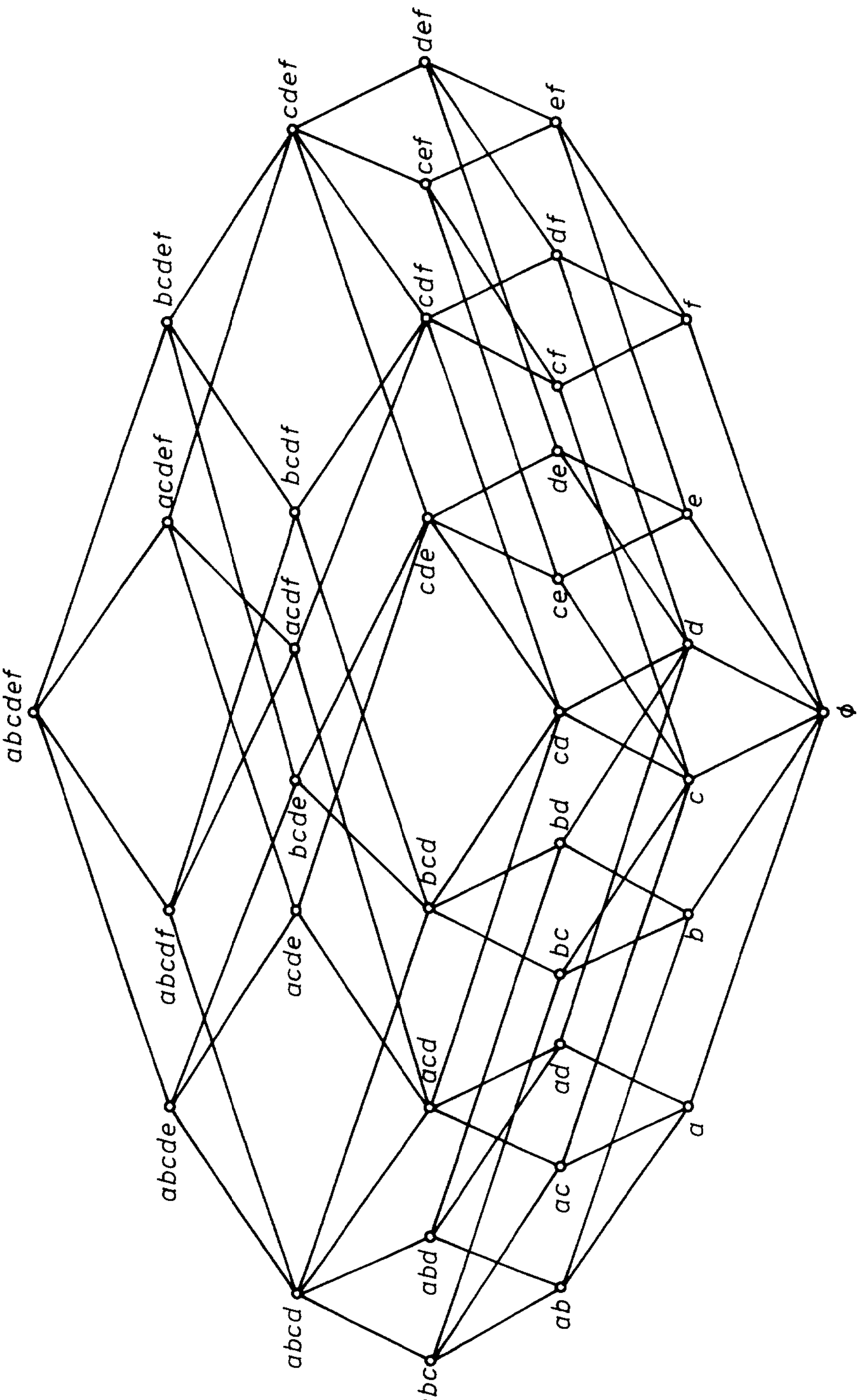
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Fig. 9.

EXAMPLE. The atomic lattice shown in Figure 9 is $\mathrm{Co}(P_6)$ where

$$P_6 = (1+1) \oplus (1+1) \oplus (1+1) \simeq 3 \circ (1+1)$$

as shown in Figure 10.

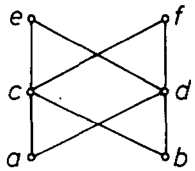

Fig. 10. P_6.

To draw $\mathrm{Co}(P_6)$ we can assign to the elements a, b, c, d, e, f the vectors of height 1 and x-components $-3, -1.5, -0.5, 0.5, 1.5, 3$; and to any 'convex' set $S \subset P_6$ of cardinality m (*except* $\{bcde\}$ and $\{acdf\}$) the vector $(\Sigma_S\, x_i, m) = (\Sigma_S\, x_i, |S|)$. The sets $\{bcde\}$ and $\{acdf\}$ are assigned $(-0.5, 4)$ and $(0.5, 4)$ respectively. In the drawing of $\mathrm{Co}(P_6)$ we have labelled each element by listing the atoms under it. There are six atoms and eleven elements of rank 2; hence there are four joins of pairs of atoms which contain a third atom. These are $a \vee e$, $a \vee f$, $b \vee e$, and $b \vee f$. Since c and d are beneath each of these joins, the following pairs of atoms of L are related: (a, c), (a, d), (a, e), (c, e), (c, f), (d, e), (d, f), (a, f), (b, c), (b, d), (b, e), and (b, f).

DEFINITION. A *fence* in an atomic lattice is a sequence of atoms $p_0 p_1 \cdots p_n$ such that p_i is related to p_{i+1} for $i = 0, \ldots, n-1$. A fence is a *zig-zag* when $p_i \wedge (p_{i-1} \vee p_{i+1}) = 0$ for $i = 1, \ldots, n-1$.

CONTINUATION OF THE EXAMPLE. In the lattice shown in Figure 9, *acebdfbd* is a fence (each adjacent pair is related) but it is not a zig-zag since $c \leqslant a \vee e$ and $d \leqslant b \vee f$. However, dropping c and d gives us a zig-zag, *aebfbd*.

EXAMPLE. If P is the poset drawn in Figure 11, then *in the atomic lattice* $\mathrm{Co}(P)$, *abcdefghij* and *efgifjfi* are fences while *acegij* and *acdceghgijf* are zig-zags.

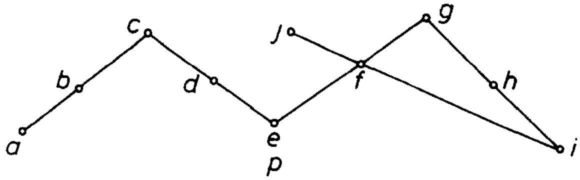

Fig. 11.

In view of the definition of related atoms, adjacent terms in any fence must be distinct. Intuitively, the order relation in P reverses between each successive pair of any zig-zag in $\mathrm{Co}(P)$; for example the zig-zag *acdceghgijf* depicted above related to the ordering

$a \lesssim c \gtrsim d \lesssim c \gtrsim e \lesssim g \gtrsim h \lesssim g \gtrsim i \lesssim j \gtrsim f$ in the poset P. This suggests the final condition we will use to characterize the $\mathrm{Co}(P)$, which we will call condition (A), suggested by Altwegg's axiom Z_6 ([1], p. 150).

DEFINITION. An atomic lattice satisfies the condition (A), if it has no zig-zag of the form $p_0 p_1 \cdots p_{2n} p_0 p_1$.

Condition (A) says that there can be no zig-zag with total number of terms *odd,* whose first and last pair are identical. In the poset of Figure 11, *eghgijfeg* has first and last pair (eg) identical, and the total number of terms is nine, but $f \leqslant j \vee g$ in $\mathrm{Co}(P)$, hence, the sequence is not a zig-zag. However *eghgijfgeg* is a zig-zag with first and last pair identical, this time having 10 terms.

The fence *abcdefghij* in the example of Figure 10 was *reduced* (by dropping b, d, f, and h as we dropped c and d in the first example) to make the zig-zag *acegij,* still starting with a and ending with j. It is easy to see that this can always be done, as the following lemma shows.

LEMMA. *Any fence* $p_0 \ldots p_n$ *can be reduced to a zig-zag* $p_0 p_{1'} \ldots p_{k'} = p_n$ *where* $1' < 2' < \cdots < k' = n$.

Proof. If $p_1 \leqslant p_0 \vee p_2$, we eliminate p_1. Then p_0 is still comparable with p_2, so $p_0 p_2 \ldots p_n$ is a fence. If $p_2 \leqslant p_0 \vee p_3$, remove p_2. Otherwise if $p_3 \leqslant p_2 \vee p_4$, remove p_3. Upon continuing the process, we finally arrive at a zig-zag of the form $p_0 p_{1'} \ldots p_{k'} = p_n$. □

Fences can also be used to decompose some atom lattices into direct products.

NOTATION. Let L be an atomic lattice, with a and b fixed related atoms of L. Then we denote by P_{ab} $\{x \mid x$ an atom of L and there is a fence $abp_1 \ldots p_n x\}$. Denote by P'_{ab} the set of atoms not in P_{ab}.

THEOREM 19. *Let L be an atomic lattice with Carathéodory rank* 2. *Let a and b be fixed related atoms of L. Then*

(i) *For q an atom of L, if* $q \leqslant \bigvee_{x \in P_{ab}} x = c$, *then* $q \in P_{ab}$.

(ii) *For q an atom of L, if* $q \leqslant \bigvee_{x \in P'_{ab}} x = d$, *then* $q \in P'_{ab}$.

(iii) $L \simeq [0, c] \times [0, d]$.

Proof. (i) For $q \leqslant \bigvee_{x \in P_{ab}} x$, $q \leqslant x_1 \vee x_2$ for some $x_i \in P_{ab}$. Assuming $q \neq x_i$, we have $abp_1 \ldots p_n x_1$ a fence, and therefore $abp_1 \ldots p_n x_1 q$ a fence for some p_i, so $q \in P_{ab}$.

The proof of (ii) is similar to that of (i).

(iii) The mapping $(e, f) \to e \vee f$ from $[0, c] \times [0, d]$ to L is onto since any element y in L may be written as $(y \wedge c) \vee (y \wedge d)$, because any atom y_1 under y is either under $y \wedge c$ or $y \wedge d$. Hence, $y \leqslant (y \wedge c) \vee (y \wedge d)$, but the reverse inequality holds in any lattice, and $y = (y \wedge c) \vee (y \wedge d)$. □

The mapping is one-to-one. For if $e, g \leqslant c$, with $e \neq g$, let $p \leqslant e$ and $p \nleqslant g$ for some atom p. Then for any $f \leqslant d$, $p \leqslant e \vee f$, and $p \nleqslant g \vee f$; otherwise $p \leqslant g_1 \vee f_1$ where g_1

and f_1 are atoms under g and f respectively, and since p, q_1 and f_1 must be distinct, g_1 and f_1 are related. But since $g_1 \in P_{ab}$, this means $f_1 \in P_{ab}$, which contradicts (ii). Thus, $e \neq g \leqslant c$ implies $e \vee f \neq g \vee f$ for all $f \leqslant d$, and the argument can be extended to show that if $(e, f) \neq (g, h)$ then $e \vee f \neq g \vee h$.

The mapping clearly preserves joins, and if $e, g \leqslant c$ with $f \leqslant d$, $(e \wedge g) \vee f \leqslant (e \vee f) \wedge (g \vee f)$. An atom $p \leqslant (e \vee f) \wedge (g \vee f)$ must be under some $e_1 \vee f_1$, and hence must equal e_1 or f_1 as in the paragraph above, and must also be under $g_2 \vee f_2$, again equaling either g_2 or f_2. But p cannot be equal both to e_1 and f_2 or to both g_2 and f_1, hence, $p \leqslant e \wedge g$ or $p \leqslant f$, so $p \leqslant (e \wedge g) \vee f$, and the mapping preserves meets.

COROLLARY. *If L is an indecomposable atomic lattice of Carathéodory rank* 2, *then given a and b related atoms, any atom p is in P_{ab}.*

We now consider general indecomposable atomic lattices of Carathéodory rank 2 which satisfy (A). On the atoms of such a lattice, we (1) define an order relation $\lessdot$; and (2) show that the atoms form a partially ordered set P under $\lessdot$, we finally show that the original lattice is isomorphic to $\mathrm{Co}(P)$.

THEOREM 20. *Let L be an indecomposable atomic lattice of Carathéodory rank* 2 *which satisfies (A). For any two atoms p and q, define $p \lessdot q$ to mean that either* (i) $p = q$, *or* (ii) *there is a zig-zag $abp_1, \ldots, p_{2n}pq$, where a and b are fixed related atoms. Then $(P, \lessdot)$ is a poset.*

Proof. We first show that $\lessdot$ is well-defined by assuming there are zig-zags $abp_1 \ldots p_{2n}pq$ and $abq_1 \ldots q_{(2k+1)}pq$. Then $abp_1 \ldots p_{2n}pqpq_{(2k+1)} \ldots q_1bab$ is a zig-zag whose first and last pair are identical, and which contains $2n + 2k + 1 + 8$ terms, an odd number. This contradicts (A), so we have shown that if there is a zig-zag $abp_1 \ldots p_{2n}pq$, every zig-zag beginning with ab and ending with pq has an even number of terms.

The relation $\lessdot$ was defined to be reflexive. If $p \neq q$, and $p \lessdot q$, then there is a zig-zag $abp_1 \ldots p_{2n}pq$, hence, $abp_1 \ldots p_{2n}pqp$ is a zig-zag with an odd number of terms, so there is no zig-zag starting with ab and ending with qp having an even number of terms. Hence, $q \lessdot p$ fails and $\lessdot$ is antisymmetric.

If $p \lessdot q$ and $q \lessdot r$, we have zig-zags $abp_1 \ldots p_{2n}pq$ and $abq_1 \ldots q_{2k}qr$. Then $abp_1 \ldots p_{2n}pqrqq_{2k} \ldots q_1bab$ is a fence with $2n + 2k + 9$ terms, an odd number. By (A) this fence is not a zig-zag, but $abp_1 \ldots p_{2n}pq$ and $rqq_{2k} \ldots q_1bab$ is a zig-zag, hence, $pqrq$ is not a zig-zag. But $r \wedge (q \vee q) = 0$, hence, $q \leqslant p \vee r$. Thus, $abp_1 \ldots p_{2n}pr$ is a zig-zag and $p \lessdot r$. □

COROLLARY 1. $a \lessdot b$.

Proof. $ababab$ is a zig-zag. □

COROLLARY 2. *If $p \lessdot q$ and $q \lessdot r$ with p, q, r distinct, then $q \leqslant p \vee r$.*

Proof. This was proved in the last paragraph of the proof of Theorem 20, where we showed that $\lessdot$ is transitive. □

To complete the characterization of $\mathrm{Co}(P)$ we must assume join-semidistributivity. This will be used in proving a converse to Corollary 2 above.

LEMMA. *Let L be an atomic join-semidistributive lattice satisfying (A). Then for a, b, p and q distinct atoms of L,*

(i) $a \leqslant b \vee q$ implies $b \wedge (a \vee q) = 0$,
(ii) $p \leqslant a \vee q$ and $a \leqslant b \vee q$ imply $a \leqslant p \vee b$.

Proof. (i) If $a \leqslant b \vee q$ and $b \leqslant a \vee q$, then $a \vee q = b \vee q$, whence, by join-semidistributivity, $a \vee q = (a \wedge b) \vee q = q$, a contradiction.

(ii) Since $a \leqslant b \vee q$, $b \wedge (a \vee q) = 0$, and $q \wedge (b \vee a) = 0$ by (i). Since $p \leqslant q \vee a$, $a \wedge (q \vee p) = 0$. Thus, $abqapab$ is a zig-zag with seven terms, a contradiction of (A), unless $a \leqslant p \vee b$. □

THEOREM 21. *Let L be an indecomposable atomic join-semidistributive lattice of Carathéodory rank 2 satisfying (A). Let p, q and r be distinct atoms of L with $q \leqslant p \vee r$. Then either $p \preccurlyeq q \preccurlyeq r$ in P or $r \preccurlyeq q \preccurlyeq p$ in P.*

Proof. By the corollary to Theorem 19, $p \in P_{ab}$, hence, there is a fence of the form $abp_1 \ldots p_n pq$. We now reduce the fence as much as possible by eliminating some or all of the p_i. If all the p_i can be eliminated, then we have a fence $abpq$. If $b \leqslant a \vee p$, $abapq$ is a fence, and it is a zig-zag unless $p \leqslant a \vee q$. In this case $abaqp$ is a fence which is a zig-zag unless $a \leqslant b \vee q$. But then we have $p \leqslant a \vee q$ and $a \leqslant b \vee q$, so by (ii) of the lemma above, $a \leqslant p \vee b$, a contradiction. Hence, we can assume that there is a zig-zag of the form $abp_1 \ldots p_n pq$. If n is even we have p q. Suppose $abq_1 \ldots q_{(2k+1)} pr$ is a zig-zag. Then $abq_1 \ldots q_{(2k+1)} prpq$ is a fence, and since $r \wedge (p \vee p) = p \wedge (r \vee q) = 0$, $abq_1 \ldots q_{(2k+1)} prpq$ is a zig-zag with $2k + 7$ terms, a contradiction. Thus, $p \preccurlyeq r$.

If $abs_1 \ldots s_{(2t+1)} qr$ is a zig-zag, since $r \wedge (p \vee q) = 0$, $abs_1 \ldots s_{(2t+1)} qrp$ is a zig-zag with $2t + 6$ terms, which contradicts $p \preccurlyeq r$. Thus $q \preccurlyeq r$.

Thus we have shown that $q \leqslant p \vee r$ implies $p \preccurlyeq q$ or $q \preccurlyeq p$. In the former case we have shown that $p \preccurlyeq q \preccurlyeq r$. The latter case can similarly be shown to imply that $r \preccurlyeq q \preccurlyeq p$.

Our final results characterize the lattices $\mathrm{Co}(P)$.

THEOREM 22. *A complete atomic indecomposable lattice L is isomorphic to* $\mathrm{Co}(P)$ *for some poset P if and only if L is join-semidistributive, has Carathéodory rank 2, and satisfies (A).*

Proof. We have already shown that any $\mathrm{Co}(P)$ is join-semidistributive, of Carathéodory rank 2 and satisfies (A).

If L satisfies the conditions above, we have seen that the atoms of L form a poset under $\preccurlyeq$. Let $A \in \mathrm{Co}(P)$. We will show that A is exactly the set of atoms under the lattice element $\bigvee_{x \in A} x$. If p is an atom under $\bigvee_{x \in A} x$, then there are x_1 and x_2 in A with $p \leqslant x_1 \vee x_2$. Then by Theorem 21, $x_1 \preccurlyeq p \preccurlyeq x_2$ in P, or dually. In either case $p \in A$. Conversely if c is an element of L, and B is the set of atoms under c, we will show that $B \in \mathrm{Co}(P)$. To do this we take p and r in B with $p \preccurlyeq q \preccurlyeq r$. Then by Corollary 2 to Theorem 20 we have $q \leqslant p \vee r \leqslant c$. Hence, $q \in B$ and $B \in \mathrm{Co}(P)$. □

Since the properties listed in Theorem 22 are preserved under the formation of direct products and conversely; and since the class of all lattices $\mathrm{Co}(P)$ is closed under the formation of direct products (by Theorem 8), we have:

THEOREM 23. *A lattice L is isomorphic to* Co(P) *for some P if and only if L is complete, atomic join-semidistributive, satisfies* (A), *and has Carathéodory rank* 2.

ILLUSTRATION. We return to the example shown in Figure 9, and show that it is Co(P_6).

Since a and c are related, we can assume $a \lessdot c$. Since *acbdbd* is a zig-zag, we have $b \lessdot d$. The following other zig-zags give the relationships in parentheses: *acbece* $(c \lessdot e)$, *acbfdf* $(d \lessdot f)$, *acbdbc* $(b \lessdot c)$, *acbdad* $(a \lessdot d)$, *acbfcf* $(c \lessdot f)$, *acbede* $(d \lessdot e)$. By transitivity, both a and b are under e and f; hence, P is isomorphic to P_6 as shown in Figure 10.

Note that removal of the maximal and minimal elements of P leaves only the convex set $\{c, d\}$. Hence $[\{c, d\}, P_6]_{\mathrm{Co}(P_6)}$ is isomorphic to 2^4 (see Figure 9). There are also several copies of Co(2^2) to be found in Co(P_6); for example the sub-poset of P_6 containing b, c, d and e is isomorphic to $\mathbf{2}^2$, and the interval $[\emptyset, \{b, c, d, e\}]$ in Co(P_6) can be seen to be isomorphic to the lattice in Figure 1b.

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