J. L. Lagrange's Changing Approach to the Foundations of the Calculus of Variations

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Preface

> ... in the current state of analysis we may regard these discussions [of past mathematics] as useless, for they concern forgotten methods, which have given way to others more simple and more general. However, such discussions may yet retain some interest for those who like to follow step by step the progress of analysis, and to see how simple and general methods are born from particular questions and complicated and indirect procedures.
> J. L. LAGRANGE *Leçons du calcul des fonctions* [1806, 315] (Commenting on the history of the calculus of variations in the 18th century).

This passage, written by JOSEPH LOUIS LAGRANGE at the end of a long and distinguished career, summarizes very well both his view of the history of mathematics as well as the distinctive character of his own extensive contributions to mathematics and mathematical physics. The passion to generalize, to construct simplifying algorithms, principles and methods, constitutes a theme which runs throughout his work in the exact sciences. This tendency is especially evident in his contributions to the calculus of variations, and it comes, therefore, as no surprise to discover that the above passage appears in the course of a discussion devoted to 18th century developments in this subject.

As a result of the surveys of such mathematicians as ROBERT WOODHOUSE [1810], ISAAC TODHUNTER [1861] and HERMAN GOLDSTINE [1980] we possess today a fairly comprehensive outline of LAGRANGE'S main results in the calculus of variations. What has been missing thus far, however, is a study that focusses on LAGRANGE'S approach to the *foundations* of this calculus. The purpose of the following article is to provide such a study. My historical aims are broader and somewhat different than LAGRANGE'S, whose views inevitably reflect those of the eminent practitioner looking back at his own place in the events of the preceding century. I nevertheless hope to convey a sense of the phenomenon of the general emerging from the particular which so stimulated LAGRANGE to include historical surveys in his treatises.

The development of LAGRANGE'S approach to the foundations of the calculus of variations falls naturally into two distinct periods: the first begins with his earliest letters to EULER in the mid-1750's and ends with the publication in 1788 of his treatise the *Méchanique Analitique*; the second is marked by the appearance of his didactic works on the differential, integral and variational calculus at the end of the 18th and beginning of the 19th century. I shall in this study provide an account of these stages, commencing in each case with a description of the origins of LAGRANGE'S research in the work of earlier geometers, followed by a more detailed summary of the research itself. In describing LAGRANGE'S presentation of principles and methods I shall concentrate on the question of how he himself understood the fundamental processes of the calculus of variations. I shall pay special attention to instances where this understanding differs from our own.

A central topic of this study concerns LAGRANGE'S changing derivation of the so-called EULER-LAGRANGE equations. Since the calculus of variations in its classical formulation is somewhat old-fashioned today, I have included a brief survey of the theory involved in the standard derivation of these equations. The purpose of this section is to make this study self-contained by acquainting the reader with the usual elementary concepts and terminology of the subject. Readers familiar with this theory may move directly to Part One.

The Mathematical Theory

I present the elementary theory as it is developed in modern texts. (See, for example, COURANT & HILBERT [1953, 164–274].) Assume f is a twice continuously differentiable function of the three arguments x, y and $y' = dy/dx$. Suppose also that the second derivative y'' is continuous. (The theory can be modified to accomodate various differentiability conditions; the latter played no role at all in the period under consideration in this article and shall, therefore, not be a matter of great concern in what follows.) Consider the definite integral

$$I = \int_a^b f(x, y, y')\, dx. \tag{1}$$

The basic problem of the calculus of variations is to find the functional relation between y and x for which I has a stationary value. This condition is expressed analytically as follows. We consider a family of comparison curves $y = y(x, \alpha)$ parameterized by α. We then define the operator $\delta : \delta(\,) = \partial(\,)/\partial\alpha\,|_{\alpha=0}\, d\alpha$. Thus $\delta y = (\partial y/\partial\alpha)|_{\alpha=0}\, d\alpha$. The condition that I has a stationary value becomes

$$\delta I = \left.\frac{\partial I}{\partial\alpha}\right|_{\alpha=0} d\alpha = 0. \tag{2}$$

An immediate consequence of (2) is the EULER-LAGRANGE equation:

$$\frac{\partial f}{\partial y} - \frac{d\left(\dfrac{\partial f}{\partial y'}\right)}{dx} = 0. \tag{3}$$

To obtain (3) we consider comparison families of the form $y(x, \alpha) = y(x) + \alpha\zeta(x)$, where $\zeta(x)$ is any function of x which possesses a continuous second derivative on $[a, b]$ and for which $\zeta(a) = \zeta(b) = 0$. Through an integration by parts (2) becomes

$$\int_a^b \left[\frac{\partial f}{\partial y} - \frac{d\left(\dfrac{\partial f}{\partial y'}\right)}{dx}\right] \delta y\, dx + \left(\frac{\delta f}{\partial y'}\right) \partial y\,|_a^b = 0$$

or

$$\int_a^b \left[\frac{\partial f}{\partial y} - \frac{d\left(\dfrac{\partial f}{\partial y'}\right)}{dx}\right] \zeta(x)\, dx = 0. \tag{4}$$

(3) follows from (4) by the *fundamental lemma of the calculus of variations*:

Fundamental Lemma: Assume for every function $\zeta(x)$ with continuous second derivative on $[a, b]$ and $\zeta(a) = \zeta(b) = 0$ that the following relation holds:

$$\int_a^b \Phi(x)\,\zeta(x)\,dx = 0,$$

where $\Phi(x)$ is continuous on $[a, b]$. Then $\Phi(x) = 0$ for all x in $[a, b]$.

The proof of the lemma is indirect; we suppose $\Phi(x) > 0$ on some subinterval $[a_1, b_2]$ and obtain a contradiction by considering the function $\zeta(x)$ that is equal to $(x - a_1)^4 (x - b_2)^4$ on $[a_1, b_2]$ and zero elsewhere.

The problem of deriving (3) from (2) is the simplest possible one in the calculus of variations; I shall in this article refer to it as the *elementary problem*. A more general problem is obtained by supposing f is a function of x, y, y' and higher order derivatives $y'', \ldots, y^{(n)}$. In this case the EULER-LAGRANGE equation (3) becomes

$$\text{(5)} \qquad \frac{\partial f}{\partial y} - \frac{d\left(\dfrac{\partial f}{\partial y'}\right)}{dx} + \frac{d^2\left(\dfrac{\partial f}{\partial y''}\right)}{dx^2} - \ldots + (-1)^n \frac{d^n\left(\dfrac{\partial f}{\partial y^{(n)}}\right)}{dx^n} = 0.$$

If f is a function of additional dependent variables and their derivatives up to order n then we obtain an equation similar to (5) for each additional variable.

The theory developed thus far has been non-parametric: one independent and several dependent variables appear in the function f. In the parametric problem we suppose f is a functions of $x, x', x'', \ldots, x^{(n)}, y, y', y'', \ldots, y^{(n)}, z, z', z'', \ldots, z^{(n)}$, where x, y, z are functions of an independent parameter t that does not appear in f. The symbol prime (′) now denotes differentiation with respect to t. (It is usual to assume the variational problem is independent of the choice of parameterization. This condition leads to restrictions on the form of f. I shall not, however, pursue this point further here; see note 6.) Corresponding to the variable x we obtain the EULER-LAGRANGE equation

$$\text{(6)} \qquad \frac{\partial f}{\partial x} - \frac{d\left(\dfrac{\partial f}{\partial x'}\right)}{dt} + \frac{d^2\left(\dfrac{\partial f}{\partial x''}\right)}{dt^2} - \ldots + (-1)^n \frac{d^n\left(\dfrac{\partial f}{\partial x^{(n)}}\right)}{dt^n} = 0,$$

with similar equations for y and z. An important advantage of the parametric approach is its ability to deal with endpoint conditions. Assume for example that $f = f(x, x', y, y', z, z')$ and we permit the second endpoint to move along a curve. In addition to the EULER-LAGRANGE equation we now obtain the end relation

$$\left(\frac{\partial f}{\partial x'}\right)\delta x + \left(\frac{\partial f}{\partial y'}\right)\delta y + \left(\frac{\partial f}{\partial z'}\right)\delta z\,\Big|_b = 0,$$

an equation which expresses what is called the transversality condition.

Let us return to the elementary non-parametric problem. I end by presenting an example which figures prominently in Part Two of this study. The example concerns the case in which $f(x, y, y')$ possesses a primitive function $F(x, y)$ such that $dF/dx = f$. A necessary and sufficient condition for the existence of F is that the EULER-LAGRANGE equation $\partial f/\partial y - d(\partial f/\partial y')/dx = 0$ degenerate into an identity, true for all values of x, y and y'. If $f = f(x, y, y', \ldots, y^{(n)})$ then a primitive $F = F(x, y, y', \ldots, y^{(n-1)})$ will exist if and only if the EULER-LAGRANGE equation (5) reduces to an identity. The necessity and sufficiency of this condition are discussed in Part Two.

(I) Part One

a) Introduction

In June 1754 the eighteen-year-old native of Turin JOSEPH LOUIS LAGRANGE wrote to LEONHARD EULER to announce a minor result he had obtained and, more importantly, to signal to EULER his interest in the latter's extensive work in mechanics and mathematics. One of the subjects mentioned by LAGRANGE concerned EULER's solution to the isoperimetrical problem. This problem was typical of those contained in EULER's classic treatise of 1744 *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes* (Method of finding curved lines which show some maximum or minimum property). In this work the Swiss geometer presented a collection of methods for solving problems in that branch of mathematics that later became known as the calculus of variations. (The name itself was coined by EULER in response to LAGRANGE's early researches.) Although a fundamental advance, the techniques of the *Methodus Inveniendi* required complicated geometrical constructions and reasonings; their application to any given example was arduous, a fact EULER himself acknowledged.

In the year following his first letter to EULER, LAGRANGE carefully studied the *Methodus Inveniendi*. On 12 August 1755 he wrote again to Berlin to describe a method he had devised for solving the problems of that treatise which avoided any appeal to geometry. In three pages LAGRANGE presented his algorithm and showed how it could be applied with "marvelous ease" to three central examples of the *Methodus Inveniendi.*

LAGRANGE'S letter was a remarkable document; it deeply impressed EULER. The two men subsequently corresponded on the subject of LAGRANGE'S new method until October 1756, when the outbreak of the Seven Years' War disrupted for three years communication between Turin and Berlin. Their correspondence prior to this disruption provides a valuable record of how LAGRANGE'S thought evolved during the fifteen months following his second letter. In particular, we are able to trace a change in his approach from a non-parametric to a parametric presentation of the variational method.

In 1760 LAGRANGE submitted two papers to the Turin Academy of Sciences which made public the fruits of his earlier researches: the first is devoted to a parametric development of the mathematical method of variations; the second consists of an extensive application of variational techniques to the principle of least action in dynamics. Both memoirs are helpful in determining how LAGRANGE understood variational processes. The subsequent evolution of this understanding may be traced in a series of treatises on mechanics LAGRANGE composed over the next two and a half decades, a series that culminated with the publication in 1788 of his classic *Méchanique Analitique*. The appearance of this treatise constitutes a natural closing point in the first stage of LAGRANGE'S approach to the foundations of the calculus ot variations.

In the sections which follow I shall examine the development of LAGRANGE'S thought from his early letters to EULER up to the *Méchanique Analitique*. In describing the background in EULER'S work as well as LAGRANGE'S first announcement of his method I have constructed the narrative so that the reader may share in a sense of LAGRANGE'S mental world – his path to the new method and his excitement at the act of discovery – as he composed his second letter to EULER in the spring and summer of 1755. My subsequent account will focus on LAGRANGE'S changing derivation of the fundamental equations of the variational calculus and on the question of what these changes reveal about his conception of this subject.

b) Euler's Methodus Inveniendi (1744)

The background to EULER'S *Methodus Inveniendi* and the work itself have been discussed in CARATHÉODORY (EULER [1744]) and GOLDSTINE [1980].[1] I shall describe in detail two problems treated by EULER that were later presented by LAGRANGE as illustration of the power of his new method: the first is the elementary problem; the second is the elementary problem generalized to include an additional dependent variable and a side condition. I shall concentrate on those features of EULER'S analysis, which, I believe, directly influenced LAGRANGE.

In Proposition III of Chapter Two EULER considers a curve joining the points a and z (see Figure 1). The curve is the geometric representation of an analytical relation between the abscissa x and the ordinate y. The letters M, N, O designate three points of the interval AZ infinitely close together. The letters m, n, o designate the points of the corresponding ordinates Mm, Nn, Oo. EULER sets $AM = x$, $AN = x'$, $AO = x''$ and $Mm = y$, $Nn = y'$, $Oo = y''$. The letter p is defined by the relation $dy = pdx$; hence $p = dy/dx$. EULER presents the relations

$$p = \frac{y' - y}{dx},$$
$$p' = \frac{y'' - y'}{dx}, \tag{1}$$

[1] GOLDSTINE in his account follows CARATHÉODORY. CARATHÉODORY interprets EULER'S analysis in terms of the techniques of the so-called direct methods in the calculus of variations. The resulting description is, I believe, a non-trivial departure from the original treatise; among other things, it obscures features that would have been suitable to have influenced LAGRANGE. Also, I cannot agree with GOLDSTINE when he says $p = (y' - y)/dx$ involves the use of "finite differences" "to approximate" the derivative, or when he uses the symbol Δ to denote $P' - P$. In my opinion such terminology misrepresents EULER'S original analysis.

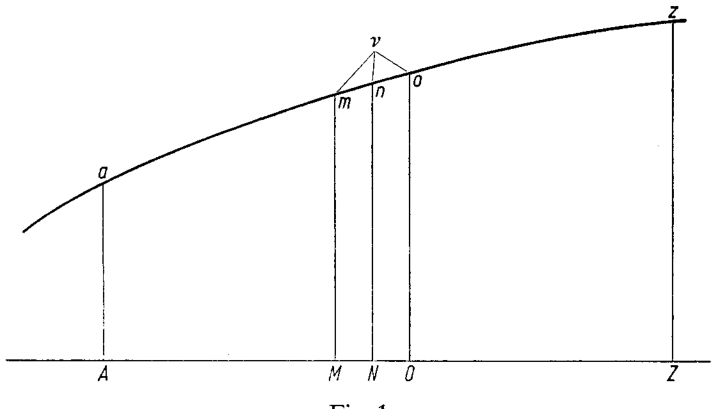

Fig. 1

which give the values of p at x and x' in terms of dx and the infinitesimal differences of the ordinates y, y' and y''.

Suppose now that Z is any function of x, y and p such that the quantity $Z\,dx$ "cannot be integrated". (The meaning of this condition will become clear in II(b).) Consider the (definite) integral $\int Zdx$ corresponding to the abscissa AZ. This integral is equal to

$$\int Z\,dx \text{ (integral from } A \text{ to } M) + Z\,dx + Z'\,dx + \text{etc.},$$

where $Z, Z', \ldots$ denote the values of Z at $x, y, p;\ x', y', p', \ldots$ Suppose $\int Z\,dx$ (A to Z) is a maximum or minimum. Increase the ordinate y' by the infinitely small "particle" nv. The resulting change in $\int Z\,dx$ must then be zero. The only part of this integral that is affected by varying y' is $Z\,dx + Z'\,dx = (Z + Z')\,dx$. Euler writes

$$\begin{aligned} dZ &= M\,dx + N\,dy + P\,dp, \\ dZ' &= M'\,dx + N'\,dy' + P'\,dp'. \end{aligned} \tag{2}$$

He proceeds to interpret the differentials in (2) as the infinitesimal changes in Z, Z', x, y, y', p, p' that result when y' is increased by nv. From (1) we see that dp and dp' equal nv/dx and $(-1)\,nv/dx$. (Euler presents these changes in the form of a table, with the variables in the left column and their increments in the right column.) Hence (2) becomes

$$\begin{aligned} dZ &= P\cdot\frac{nv}{dx}, \\ dZ' &= N'\cdot nv - P'\cdot\frac{nv}{dx}. \end{aligned} \tag{3}$$

Thus the total change in $\int Z\,dx$ equals $(dZ + dZ')\,dx = nv\cdot(P + N'\,dx - P')$. This expression must be equated to zero. Euler sets $P' - P = dP$ and replaces

N' by N. He therefore obtains $0 = N\,dx - dP$ or

$$N - \frac{dP}{dx} = 0, \tag{4}$$

as the final equation for the curve.

Equation (4) is the EULER-LAGRANGE equation for the variational problem. An important feature of EULER's derivation of this equation concerns the shifting role of the symbol d. In (4) and the final step by which (4) is obtained d is used to denote the differential as it was customarily used and understood in 18^{th} century LEIBNIZian analysis. The differential dx is held constant; the differential of any other variable equals the difference of its value at x and its value at an abscissa a distance dx from x. (A historical account of LEIBNIZian analysis is contained in Bos [1974].) By contrast, the differentials dx, dy *etc.* that appear in (2) denote changes in x, y, *etc.* that result when the single ordinate y is increased by the "particle" nv. Hence the "differentials" dy', dp, dp' equal nv, nv/dx, $-nv/dx$; the "differentials" dx, dy, dp'', *etc.* are zero. Note that in both of EULER's uses of the differential there are no limiting processes or finite approximations. The abscissa AZ is divided into infinitely many intervals $MN = NO = \ldots$; the differential dy equals $y' - y$. There is nothing to indicate that EULER even implicitly viewed AZ as the limit of a finite partition, or $dy = y' - y$ as a finite difference approximation. Furthermore, the quantity nv is referred to as a "particle", a choice of language that reflects the discreteness underlying his understanding of infinitesimal processes.

In Chapter Three of the *Methodus Inveniendi* EULER turns to a more complicated class of examples. In Proposition III the problem is to render extreme the integral $\int Z\,dx$, where Z is now a function of x, y, $p = dy/dx$ and a new variable Π.[2] Π is connected to x, y, p by the side condition $\Pi = \int [Z]\,dx$, where $[Z]$ is a function of x, y, p and $\int [Z]\,dx$ is the definite integral of $[Z]$ evaluated from the initial abscissa A to $AM = x$. (The motivation for this problem arises from, among others, examples in which Z depends on x, y, p and the length of path $s = \int_0^x \sqrt{1 + p^2}\,dx$.) Increase y' by nv. The resulting changes dp, dy', dp' equal nv/dx, nv, $-nv/dx$. Let us calculate the corresponding change in Π. We have

$$\begin{aligned} \Pi &= \int [Z]\,dx, \\ \Pi' &= \int [Z]\,dx + [Z]\,dx, \\ \Pi'' &= \int [Z]\,dx + [Z]\,dx + [Z']\,dx, \\ \Pi''' &= \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx, \qquad \text{etc.} \end{aligned} \tag{5}$$

[2] EULER actually assumes that Z is a function of x, y, p and higher order derivatives q, r, ..., where $q\,dx = dp$, $r\,dx = dq$, ... are the relations which define these variables. For simplicity of exposition I describe his analysis for the elementary case. (What I present is exactly his analysis when the partials of Z and $[Z]$ with respect to q, r, ..., namely, Q, $[Q]$, R, $[R]$, ..., are zero and four is subtracted from his superscript numbering scheme.)

Suppose $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp$. The changes in $[Z], [Z'], [Z''], \ldots$ are presented by EULER as follows:

$$d\cdot[Z]\,dx = nv\cdot dx\left(\frac{[P]}{dx}\right),$$

(6)
$$d\cdot[Z']\,dx = nv\cdot dx\left([N'] - \frac{[P']}{dx}\right),$$

$$d\cdot[Z'']\,dx = 0, \qquad \text{etc.}$$

Hence the changes in $\Pi, \Pi', \Pi'', \Pi''', \ldots$ equal

$$d\cdot\Pi = 0,$$

$$d\cdot\Pi' = nv\cdot dx\left(\frac{[P]}{dx}\right),$$

(7)
$$d\cdot\Pi'' = nv\cdot dx\left([N'] - \frac{d[P]}{dx}\right),$$

$$d\cdot\Pi'' = d\cdot\Pi''' = d\cdot\Pi^{(\mathrm{iv})} = \text{etc.},$$

where we have used the fact that $d[P] = [P'] - [P]$.

We now calculate the change in $\int Z\,dx + Z\,dx + Z'\,dx + \text{etc.}$ which results when y' is increased by nv. Suppose $dZ = M\,dx + N\,dy + P\,dp + L\,d\Pi$. The part of the change that arises from the variation of y', p and p' is, as before,

(8)
$$nv\cdot dx\left(N - \frac{dP}{dx}\right).$$

When y' is increased by nv all of the quantities $\Pi, \Pi', \Pi'', \ldots$ are varied. The total change in $\int Z\,dx + Z\,dx + Z'\,dx + \text{etc.}$ due to these variations is

(9)
$$L\,dx\cdot d\Pi + L'\,dx\cdot d\Pi' + L''\,dx\cdot d\Pi'' + \text{etc.}$$

Substituting the values of $d\Pi, d\Pi', d\Pi'', \ldots$ given by (7) into (9) yields

(10)
$$nv\cdot dx(L'[P]) + nv\cdot dx\left([N'] - \frac{d[P]}{dx}\right)(L''\,dx + L'''\,dx + L^{(\mathrm{iv})}\,dx + \text{etc.}).$$

EULER replaces $[L']$ and $[N']$ by $[L]$ and $[N]$ and sets $L''\,dx + L'''\,dx + L^{(\mathrm{iv})}\,dx + \text{etc.}$ equal to $H - \int L\,dx$, where H is the integral of L from A to Z. With these substitutions (10) becomes

(11)
$$nv\cdot dx(H - \textstyle\int L\,dx)\left([N] - \dfrac{d[P]}{dx}\right) + nv\cdot dx(L[P]),$$

which EULER rewrites (using $d(\int L\,dx) = L\,dx$) as

(12)
$$nv\cdot dx\left([N]\,(H - \textstyle\int L\,dx) - \dfrac{d[P]\,(H - \int L\,dx)}{dx}\right).$$

By adding (12) and (8) and equating the resulting expression to zero he obtains the final equation for the problem

$$(13) \qquad 0 = [N]\,(H - \textstyle\int L\,dx) - \dfrac{d[P]\,(H - \int L\,dx)}{dx} + N - \dfrac{dP}{dx}.$$

Note once again the dual role of the symbol d in EULER's derivation of (13). We must distinguish between his employment of d in calculating the change in Z and Π and the more conventional appearance of this symbol in such equations as (13). EULER's notation in Problem III has in fact begun to reflect this dual usage. Thus he sets $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp$ but writes $d \cdot [Z]$ to indicate the change in $[Z]$ that results when y' is increased by nv. (He is, however, not completely consistent—consider equations (7) and (9).) Another interesting feature of his analysis is his replacement of the infinite sum $L''\,dx + L'''\,dx + L^{(\mathrm{iv})}\,dx + \ldots$ by the integral $H - \int L\,dx$. This step is presented as a formal one with no explanation.

Equation (13) is in the modern development of the subject derived by the method of multipliers: the problem it to find the function $y = y(x)$ that renders extreme $\int_a^b Z\,dx$ subject to the condition $d\Pi/dx - [Z] = 0$. This problem leads to the one of rendering extreme the modified integral with the new integrand $Z + \lambda(d\Pi/dx - [Z])$, where λ is a multipler. The EULER-LAGRANGE equation corresponding to Π is $d\lambda/dx = \partial Z/\partial\Pi = L$. By integrating this equation from x to the endpoint and substituting into the equation corresponding to y we obtain (13). It is important to note, however, that although the multiplier method leads to the same result, it is not a correct description of EULER's original procedure, where, for example, the side condition is never considered in differential form. The use of multipliers would seem to involve at least in part ideas not present in the *Methodus Inveniendi*. (The reader may also wish to consult the discussion of this point in GOLDSTINE [1980, 74–76].)

c) Lagrange's letter of 12 August 1755

LAGRANGE's second letter to EULER begins with a brief description of his new method followed by three examples in which it is applied to problems taken from the *Methodus Inveniendi*. The method is based on the addition of the symbol δ to the infinitesimal and integral calculus. LAGRANGE supposes that x is constant with respect to δ, that is, that $\delta x = 0$. δy denotes the corresponding differential of y that occurs in problems of maxima and minima; it is used to distinguish this change in y from the usual differential dy appearing in the same problems. The quantity δFy denotes the increment in Fy (F a function of y) when y is increased by δy. LAGRANGE asserts that $d\delta\,Fy = \delta d\,Fy$, and, more generally, that $d^m\delta\,Fy = \delta d^m\,Fy$; if $Fy = y$ and $m = 1$ we obtain $d\delta y = \delta dy$. (LAGRANGE's only justification for these relations is to refer to a memoir composed by EULER in 1734.[3] The assertion of their validity should, I think, most properly be regarded as an unproven assumption or axiom concerning the formal properties of his new δ-calculus.) He proceeds to lay down the following relations, obtained by integration by parts:

$$\int z\,du = zu - \int u\,dz, \tag{1}$$

$$\int z\,d^2u = z\,du - u\,dz + \int u\,d^2z, \tag{2}$$

$$\int z\,d^3u = z\,d^2u - dz\,du + u\,d^2z - \int u\,d^3z, \tag{3}$$

$$\int u \int z = \int u \times \int z - \int z \int u. \tag{4}$$

LAGRANGE sets $\int u = H$ and $H - \int u = V$, so that (4) becomes

$$\int u \int z = \int Vz. \tag{5}$$

The integrals in (1), (2), and (3) are to be evaluated from an unspecified initial value of x to $x = a$. In the relation $H - \int u = V$, H is the integral of u from the initial value to $x = a$ and $\int u$ is the integral of u from the initial value to $x = x$. $\int u \int z$ in (4) and (5) is obtained by integrating z from the initial value to x and then integrating $u \int z$ from the initial value to a. A similar interpretation holds for $\int z \int u$. $\int u \times \int z$ in (4) equals the product of the integrals of u and z from the initial value of x to $x = a$.

The first problem LAGRANGE considers is to find that relation between x and y which makes Z a maximum or minimum. Z is a function of $x, dx, y, dy, d^2y, \ldots$ LAGRANGE writes

$$\delta Z = N\,\delta y + P\delta\,dy + Q\delta\,d^2y + R\delta\,d^3z + \text{etc.} \tag{6}$$

He asserts that $\delta \int Z = \int \delta Z$ and integrates (6):

$$\delta \int Z = \int N\,\delta y + \int P\,\delta\,dy + \int Q\,\delta\,d^2y + \text{etc.} \tag{7}$$

LAGRANGE next interchanges the d and δ and uses (1), (2), (3) to obtain

$$\delta \int Z = \int N\,\delta y + P\,\delta y - \int dP\,\delta y + Q\,d\,\delta y - dQ\,\delta y + \int d^2Q\,\delta y - \text{etc.} \tag{8}$$

[3] The memoir LAGRANGE refers to is titled "De infinitis curvis eiusdem generis seu methodus inveniendi aequationes pro infinitis curvis eiusdem generis" (EULER [1734]). In this memoir EULER makes use of the following result. Assume z is a function of a and x. Differentiate z holding a constant to obtain $P\,dx$, and differentiate $P\,dx$ holding x constant to obtain $B\,dx\,da$. Next, differentiate z holding x constant to obtain $Q\,da$, and differentiate $Q\,da$ holding x constant to obtain $C\,da\,dx$. Then $B = C$. (In modern notation $\partial^2 z/\partial a\,\partial x = \partial^2 z/\partial x\,\partial a$.) To establish this result EULER considers the three quantities $e = z(x + dx, a)$, $f = z(x, a + da)$ and $g = z(x + dx, a + da)$. He states that $P\,dx = e - z$ and $B\,dx\,da = (g - f) - (e - z)$. ($g - f$ is obtained from $e - z$ by replacing a by $a + da$.) Similarly, $Q\,da = f - z$ and $C\,da\,dx = (g - e) - (f - z)$. Hence $B = C$.

LAGRANGE apparently believed an argument similar to EULER's could be applied to the differential operators d and δ to show that $d\,\delta Fy = \delta\,dFy$. Unfortunately, he provides no details and the result remains in his analysis an unjustified assumption.

Consequently

(9) $$\delta \int Z = \int (N - dP + d^2Q - \text{etc.})\, \delta y + (P - dQ + \text{etc.})\, \delta y + (Q - \text{etc.})\, d\, \delta y + \text{etc.}$$

LAGRANGE supposes that when $x = a$, $\delta y = d\,\delta y = d^2\,\delta y = 0$ *etc.* He (implicitly) assumes these quantities are also zero at the initial value of x. (9) therefore becomes

(10) $$\delta \int Z = \int (N - dP + d^2Q - \text{etc.})\, \delta y.$$

LAGRANGE now states that as a consequence of the "common method of maxima and minima"

(11) $$N - dP + d^2Q - \text{etc.} = 0.$$

In the case where Z contains only x, dx, y, dy (11) reduces to

(12) $$N - dP = 0,$$

which is the EULER-LAGRANGE equation for the elementary problem.

LAGRANGE proceeds to the case in which Z is a function of $x, dx, y, dy, d^2y, \ldots$ and an additional variable π. π is connected to the remaining variables by the side relation $\pi = \int(Z)$, where (Z) is a function of $x, dx, y, dy, d^2y, \ldots$ He writes

$$\delta Z = L\,\delta\pi + N\,\delta y + P\,\delta\, dy + Q\,\delta\, d^2y + \text{etc.},$$

(13) $$\delta(Z) = (N)\,\delta y + (P)\,\delta\, dy + (Q)\,\delta\, d^2y + \text{etc.},$$

$$\delta\pi = \int (N)\,\delta y + \int (P)\,\delta\, dy + \text{etc.}$$

Hence

14) $$\delta \int Z = \int N\,\delta y + \int P\, d\,\delta y + \int Q\, d^2\,\delta y + \text{etc.} + \int L \int (N)\,\delta y + \int L \int (P)\, d\,\delta y + \text{etc.}$$

LAGRANGE sets H equal to the integral of L from the initial value of x to $x = a$. He then writes $H - \int L = V$, where $\int L$ is the integral of L from the initial value to x. We now use the result given by (5) to transform (14):

(15) $$\delta \int Z = \int [N + (N)\,V]\,\delta y + \int [P + (P)\,V]\, d\,\delta y + \int [Q + (Q)\,V]\, d^2\,\delta y + \text{etc.}$$

From (15) LAGRANGE deduces "for the maxima or minima" the equation

(16) $$N + (N)\,V - d[P + (P)\,V] + d^2[Q + (Q)\,V] - \text{etc.} = 0.$$

((16) follows from (15) by the same series of steps by which equation (11) was derived from (7).) If Z and (Z) contain no differentials of y higher than the first, (16) reduces to

(17) $$N + (N)\,V - d[P + (P)\,V] = 0,$$

which is simply the final equation in the second of the two examples from EULER presented in the preceding section.

LAGRANGE continues in the letter by applying his method to another example (Proposition V) from Chapter III of the *Methodus Inveniendi.* He also provides a brief discussion of how his method could be used to determine conditions satisfied by the extremal curve at the endpoints. LAGRANGE ends the letter by noting the applicability of his algorithm to a more general class of problems, namely, the analysis of surfaces that enjoy some extremal property.

Let us turn now to a consideration of the genesis of the results contained in LAGRANGE'S letter. In the preamble LAGRANGE states that his investigation was inspired by EULER'S comment in Proposition 39, Chapter III of the *Methodus Inveniendi*: "A method is therefore desired, free of geometric and linear solutions by which it is evident that, in such investigations of maxima and minima, $-p\,dP$ ought to be written in place of $P\,dp$."[4] (EULER means that if $dZ = N\,dy + P\,dp$ and we write $-p\,dP$ for $P\,dp$ then the equation $N\,dy - p\,dP = 0$ is the one that renders extreme $\int Z\,dx$.) The challenge contained in this passage undoubtedly impressed LAGRANGE. I do not, however, think that mentioning his quotation of the passage provides an adequate explanation of how he arrived at his new method. Instead, such an explanation will be found in a comparative examination of the analyses of the two men. LAGRANGE must have noticed that the the symbol d was being used in two ways in the derivations of the *Methodus Inveniendi.* He came up with the idea of using a new symbol δ to denote the second type of differential. He then experimented with the new symbol in a formal and algebraic way, using such facts, assumed or discovered, as $\delta\,dy = d\,\delta y$, $\delta\,dFy = d\,\delta Fy$ etc. The employment of the δ led him naturally to consider the integral of the variation over the entire length of the interval. Using integration by parts he was able to derive with ease results EULER had obtained only with difficulty. By symbolic manipulation alone LAGRANGE devised an algorithm that revolutionized the study of the calculus of variations.

EULER responded to LAGRANGE'S letter in September 1755 and a lengthy correspondence ensued. These letters shed light on the two men's understanding of the processes of the calculus of variations. They are also interesting as a study of the interaction of two talented but different mathematicians. EULER in his first response draws attention to a fact LAGRANGE had nowhere explicitly mentioned, namely, that his method depends on simultaneously varying all of the ordinates y, not just one, as EULER had done. EULER comments on a further feature of LAGRANGE'S analysis which differs from his own. In the *Methodus Inveniendi* Z is a function of x, y, p, *etc.*, where $p = dy/dx$; in LAGRANGE'S letter Z is a function of x, dx, y, dy, *etc.* Where EULER varies p, LAGRANGE varies dy. Commenting on this difference, EULER states that LAGRANGE is letting dx be unity, and not x, as LAGRANGE had, apparently by a "slip of the pen", written. According to the procedures of the 18^{th} century LEIBNIZian calculus, if y is a function of x, then dx is to be held constant in all calculations. Thus when LAGRANGE said x is constant, he really meant to say dx is constant, the constant

[4] I have used the translation from the Latin of this passage which appears in GOLDSTINE [1980, 111].

being unity. Given this assumption, his results are exactly in accord with those of the *Methodus Inveniendi.*

In fact, EULER was mistaken in suggesting LAGRANGE had made a "slip of the pen". When LAGRANGE said x is constant, he meant δx is zero. As he makes clear in a later letter ([1755b]), the fact that he works with dy and not p simply reflects a more flexible and algebraic approach to the calculus. EULER interpreted integration geometrically as the formal infinite sum of infinitesimal products. Thus, for example, $\int Z\,dx$ equals $Z\,dx + Z'\,dx + \ldots$, and it is therefore important to place $Z\,dx$ in the integrand. LAGRANGE, while not rejecting this interpretation, nevertheless did not feel in practice that it should place any restriction on his use of the calculus. It was for him not a matter of great importance whether one wrote $\int p\,dx$, as EULER would have done, or simply $\int dy$.

A third point EULER raises in his letter concerns the step which takes LAGRANGE from equation (10) to (11). This, of course, is the inference sanctioned today by the fundamental lemma of the calculus of variations. EULER observes that the method of maxima and minima permits us to conclude only that

$$\int (N - dP + d^2Q - \text{etc.})\,\delta y = 0. \tag{18}$$

He seems therefore to be suggesting some further reasoning is needed to arrive at (11), a suggestion clearly connected to his observation that LAGRANGE is in the δ-process simultaneously varying all the ordinates y. Unfortunately, there is a tear in the letter at this place, and EULER turns to some concluding remarks of a general nature.

Before considering LAGRANGE'S response to EULER'S criticism let us examine more closely his reasoning in passing from (10) to (11). LAGRANGE states that (11) follows by the "common method of maxima and minima." He apparently believed that with equation (10) he had reduced the problem in the calculus of variations to one in the common or ordinary differential and integral calculus. According to the rules of this calculus, d and $\int$ are inverse 'operators': $d\int V = \int dV = V$. If Z is a maximum or minimum we clearly have $d\delta \int Z = 0$, or, $d\int \delta Z = 0$. But $\int \delta Z = \int (N - dP + d^2Q - \ldots)\,\delta y$. Hence $d\int \delta Z = 0$ implies, cancelling δy, that $N - dP + d^2Q - \ldots = 0$, which is equation (11).

(LAGRANGE'S reasoning as I have reconstructed it is unsatisfactory. The integral in (10) is a definite one from the initial value of x to $x = a$. The step $d\int V = V$ would need to be valid for $\int V$ equal to the definite integral from the initial value to an arbitrary intermediate value of x. This fact would lead for the variational problem to the conclusion that the quantity δy is identically zero.)

LAGRANGE ([1755b]) agreed with EULER'S criticism of the step (10) to (11). However, the basis of his agreement was not that there was anything deficient in his reasoning, but rather that equation (18) was a more general consequence of (10); (11) itself was in turn simply one of several results one could derive from (18). The interest of the other results presented by LAGRANGE is, however, unclear, a fact noted by EULER himself in his next letter ([1756]). In this letter, EULER advises his young contemporary to exercise caution in applying the δ-method, an apparent reference to LAGRANGE'S penchant for unrestrained algebraic manipulation. As for equation (11), EULER now treats it as a self-evident consequence of the variational analysis. With EULER's letter discussion of the point in their correspondence ends.[5]

d) Lagrange's letters of 20 November 1755 and 5 October 1756

In the year following the communication of his discovery to EULER, LAGRANGE continued to work in the calculus of variations. By October 1756 he had arrived at a revised approach to the subject, an approach that would become the basis for his first published presentation of the variational calculus in 1760. LAGRANGE's methodological shift was inspired by his attempt to apply the δ-method to the principle of least action in dynamics. A letter of EULER to LAGRANGE (24 April 1756) and the Registers of the Berlin Academy (May 1756) reveal that LAGRANGE had submitted a memoir on the principle of least action to the Berlin Academy; the memoir, now lost, received the favorable attention of the Academy's President PIERRE DE MAUPERTUIS. As we shall see in *I(f)*, LAGRANGE would go on to make extensive use of variational techniques in mechanics.

In the letter of 5 October 1756 LAGRANGE takes ideas from his dynamical researches and applies them to the celebrated brachistochrone problem. Assume a bead slides from rest along a smooth stiff wire joining two points in a vertical plane. The brachistochrone problem is to find the shape of the wire that minimizes the time of descent. If we let the initial point be the origin of a rectangular co-ordinate system, the x-axis be directed vertically downward and the x-y plane be the plane of descent, then the time of descent is, up to a constant of proportionality,

$$\int \frac{ds}{\sqrt{x}}, \tag{19}$$

where ds is the differential element of path length and the integral is evaluated from $s = 0$ to its final value. (19) is derived from the proportionality of $v = ds/dt$ to $\sqrt{x}$, the expression for this example of what was known during the 18[th] century as the law of conservation of *vis viva* (later, mechanical energy). The curve that minimizes the time of descent is the cycloid: the curve traced by a point on the perimeter of a circle as it rolls without slipping along a line (in this case, the

[5] The closest statement that I have been able to find in EULER's published treatises concerning the fundamental lemma appears in his [1764b, 187]. EULER was influenced in this memoir by his earlier approach in the *Methodus Inveniendi*. He obtains by LAGRANGE's variational method the necessary condition (*) $\int dx\, \delta y(N - dP/dx + \ldots) = 0$. He interprets the integral in (*) as the sum of all variations $dx\, \delta y(N - dP/dx + \ldots)$ which result when a single value of y is varied. In the latter case (*) reduces to $dx\, \delta y(N - dP/dx + \ldots) = 0$ and the equation $(N - dP/dx + \ldots) = 0$ follows. Since this argument holds for each ordinate y the equation $(N - dP/dx + \ldots) = 0$ is valid throughout the entire interval.

y-axis). Geometers of the period knew that the cycloid is given by the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{x}{a-x}}, \tag{20}$$

where a is a constant equal to the diameter of the generating circle. (For a discussion of equation (20) and the parametric representation of the cycloid in terms of its generating circle see GOLDSTINE [1980, 32–33].)

LAGRANGE'S variational method leads (as we shall see) directly to equation (20). A more general problem is obtained by supposing the second endpoint is not fixed but is free to move along a line. The curve that minimizes the time is in this situation the cycloid which cuts the line normally. LAGRANGE had claimed at the end of his original letter to EULER that he had established this fact by his δ-method; in a letter of 20 November 1755 he provides an explicit demonstration. He apparently felt the demonstration unsatisfactory and set about searching for an alternate method. The search, which was conducted in conjunction with his dynamical investigations, culminated in success in his letter of 5 October 1756.

In his first analysis of the brachistochrone problem, contained in the letter of 20 November 1755, LAGRANGE includes a diagram to illustrate his solution (see Figure 2). The curve AQN is the path of quickest descent to the line BNn. The curve an is an arbitrary comparison curve whose endpoint lies on BNn. AP and PQ are the x and y co-ordinates of a typical point Q on the curve AQN. LAGRANGE assumes the speed of the particle at Q is a function of x and y. I shall present his analysis for the case where the speed is proportional to $\sqrt{x}$. We begin by applying δ to the expression for the time of descent:

$$\delta \int \frac{ds}{\sqrt{x}} = \int \frac{dy\, \delta\, dy}{ds \sqrt{x}} = \int \left(-d\frac{dy}{ds\sqrt{x}}\right) \delta y + \frac{dy}{ds\sqrt{x}} \delta y. \tag{21}$$

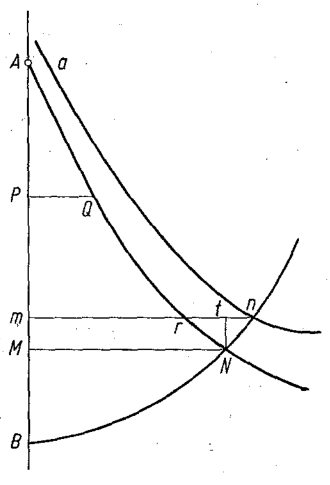

Fig. 2

(21) gives the variation of of $\int(ds/\sqrt{x})$ for a comparison curve whose final point corresponds to the abscissa AM. The final point of the comparison curve an, however, corresponds to the abscissa Am. Hence to find the true "differential" or variation of $\int(ds/\sqrt{x})$ it is necessary to subtract $ds/\sqrt{x}$ from (21):

$$(22) \qquad \delta\int\frac{ds}{\sqrt{x}} = \int\left(-d\frac{dy}{ds\sqrt{x}}\right)\delta y + \frac{dy}{ds\sqrt{x}}\delta y - \frac{ds}{\sqrt{x}}.$$

LAGRANGE now equates (22) to zero and concludes immediately that

$$(23) \qquad d\frac{dy}{ds\sqrt{x}} = 0,$$

a result justified by the remark that "nothing arising from the indeterminate δy enters into [(23)]". (Note that (23) reduces to $dy/ds\sqrt{x} = \text{constant}$, which leads directly to the cycloidal equation (20).) LAGRANGE next sets the part of (22 outside the integral sign equal to zero:

$$\frac{dy}{ds\sqrt{x}}\delta y = \frac{ds}{\sqrt{x}},$$

from which he obtains

$$(24) \qquad dy\,\delta y = ds^2.$$

He writes $dy = rt$, $\delta y = rn$, $ds = rN$. (24) then becomes

$$(25) \qquad (rt)(rn) = (rN)^2,$$

a relation which leads by elementary geometry to the final conclusion that the angle rNn is a right angle.

Equation (22) is the expression, for the integral $\int(ds/\sqrt{x})$, of the variation in the non-parametric variable endpoint problem. LAGRANGE was presumably capable of developing his analysis into a more general theory. However, he soon hit upon an easier method for dealing with this class of problems. In the letter of 5 October 1756 LAGRANGE explains that in his investigation of the principle of least action he found it necessary to vary in the δ-process both the x and y co-ordinates. Such an approach is natural in dynamical problems where x and y are treated as functionally dependent on a third variable, the time. LAGRANGE discovered that there were also advantages in applying the same approach to the usual problems of the calculus of variations. He illustrates this fact using the brachistochrone problem. When both x and y are varied the variation of $\int ds/\sqrt{x}$ becomes

(26)

$$\begin{aligned}\delta\int\frac{ds}{\sqrt{x}} &= \int\left(\frac{dy}{ds\sqrt{x}}\delta\,dy + \frac{dx}{ds\sqrt{x}}\delta\,dx - \frac{ds}{2x\sqrt{x}}\delta x\right)\\ &= \frac{dy}{ds\sqrt{x}}\delta y + \frac{dx}{ds\sqrt{x}}\delta x - \int\left(\left(d\frac{dy}{ds\sqrt{x}}\right)\delta y + \left(d\frac{dx}{ds\sqrt{x}} + \frac{ds}{2x\sqrt{x}}\right)\delta x\right)\\ &= 0.\end{aligned}$$

LAGRANGE equates the coefficients of δx and δy under the integral sign to zero.

He notes that the resulting equations, when reduced, are one and the same:

$$a\,dy^2 = x\,ds^2, \tag{27}$$

where a is a constant of integration. (The equation corresponding to δy leads immediately to (27); the equation corresponding to δx if multiplied by $2\,dx/(ds\sqrt{x})$ and integrated, also yields (27).) LAGRANGE assumes the part outside the integral sign is also zero:

$$\frac{dy}{ds\sqrt{x}}\,\delta y + \frac{dx}{ds\sqrt{x}}\,\delta x = 0. \tag{28}$$

(He sometimes refers to this part as the "constant member" of the expression for the variation given in (26).) If the final point is fixed, then δx and δy are zero and the terms in (28) vanish separately. Assume now that the endpoint is free to move along a line. Let dX and dY be the differential abscissa and ordinate elements of the line at the endpoint. Clearly $\delta x : \delta y = dX : dY$. Hence (28) becomes

$$dy\,dY + dx\,dX = 0, \tag{29}$$

which, LAGRANGE notes, proves that the desired curve cuts the line normally.

LAGRANGE has discovered that the analysis of boundary conditions in problems where the endpoint is variable is facilitated if the curve is represented parametrically.[6] He also noticed in particular examples that when a (two-dimensional) curve is treated parametrically, the two resulting EULER-LAGRANGE equations reduce to the single equation one obtains when the curve is treated non-parametrically. LAGRANGE, however, was unable to verify this fact "a priori", in a completely general manner, and he appeals to EULER for enlightenment on the subject. Unfortunately, with LAGRANGE'S letter communication between Turin and Berlin was disrupted for three years by the Seven-Years' War. In LAGRANGE'S next exposition of the calculus of variations, contained in his paper of 1760, attention would be focussed solely on the parametric problem.

A final point concerning LAGRANGE'S analysis in both letters involves the step from equation (22) to (23) and (26) to (27). This step is not adequately explained. In his original announcement of the variational method the step corresponded to the one sanctioned today by the fundamental lemma; in my earlier discussion I provided a reconstruction for his reasoning. In the brachistochrone

[6] To treat a variational problem parametrically is to assume the solution is independent of the choice of parameterization. A possible source of confusion concerning LAGRANGE'S own development of the subject arises from the fact that the dynamical problem is often regarded as the non-parametric problem *par excellence*: the temporal variable is privleged in the sense that only one configuration of the system is possible for each value of the time. It nevertheless remains true historically that it was dynamical examples that gave LAGRANGE the idea of treating parametrically the usual extremalizing curves of the variational calculus. Certainly, he never discusses in detail the differences between the two approaches. Thus he never draws attention to the restrictions Z must satisfy in the parametric case. (A possible exception is Problem II of [1760a] where he states that Z is an "algebraic" function of the variables x, y, z, dx, dy, dz, ...)

problem with variable endpoints, however, the situation is more complicated because the terms outside the integral sign no longer vanish separately. What precisely LAGRANGE'S reasoning was in this case is unclear and will be the subject of discussion in the next section.

e) Lagrange's first published paper (1760)

LAGRANGE'S first published account of the calculus of variations, contained in the Memoirs of the Turin Academy for 1760–61, has the title "Essai d'une nouvelle méthode pour determiner les maxima et les minima des formules intégrales indéfinies" ([1760a]). LAGRANGE announces at the beginning that the central problem of this branch of mathematics is "to find the very curve for which a given integral expression is a maximum or a minimum in relation to all other curves." In the memoir he develops the subject from the parametric viewpoint enunciated for the brachistochrone problem in his letter of 5 October 1756. The memoir is followed by a much longer one on the application of the variational calculus to the principle of least action in dynamics.

LAGRANGE begins in the memoir by considering the definite integral $\int Z$ of an expression Z. (The adjective "indefinite" in the memoir's title modifies "formulas" and not "integral".) Z is a function of $x, y, z, dx, dy, dz, d^2x, d^2y, d^2z, \ldots,$ and the problem is to find the relation among these variables that maximizes or minimizes $\int Z$. We have first the analytic statement of this condition:

$$\delta \int Z = \int \delta Z = 0, \tag{30}$$

where

$$\begin{aligned} \delta Z &= n\,\delta x + p\,\delta\,dx + q\,\delta\,d^2x + r\,\delta\,d^3x + \ldots \\ &\quad + N\,\delta y + P\,\delta\,dy + Q\,\delta\,d^2y + R\,\delta\,d^3y + \ldots \\ &\quad + \nu\,\delta z + \omega\,\delta\,dz + \chi\,\delta\,d^2\,z + \varrho\,\delta\,d^3z + \ldots. \end{aligned} \tag{31}$$

LAGRANGE substitutes (31) into (30), interchanges the d and δ and integrates by parts:

(32) (A)

$$\begin{aligned} &\int(n - dp + d^2q - d^3r + \ldots)\,\delta x \\ &\quad + \int(N - dP + d^2Q - d^3R + \ldots)\,\delta y \\ &\quad + \int(\nu - d\omega + d^2\chi - d^3\varrho + \ldots)\,\delta z \\ &\quad + (p - dq + d^2r - \ldots)\,\delta x + (q - dr + \ldots)\,d\,\delta x + (r - \ldots)\,d^2\,\delta x + \ldots \\ &\quad + (P - dQ + d^2R - \ldots)\,\delta y + (Q - dR + \ldots)\,d\,\delta y + (R - \ldots)\,d^2\,\delta y + \ldots \\ &\quad + (\omega - d\chi + d^2\varrho - \ldots)\,\delta z + (\chi - d\varrho + \ldots)\,d\,\delta z + (\varrho - \ldots)\,d^2\,\delta z + \ldots = 0. \end{aligned}$$

LAGRANGE presents two consequences of (32). The first is "the indefinite equation":

$$\begin{aligned} &(n - dp + d^2q - d^3r + \dots)\,\delta x \\ (33)\ (B) \qquad &+ (N - dP + d^2Q - d^3R + \dots)\,\delta y \\ &+ (\nu - d\omega + d^2\chi - d^3\varrho + \dots)\,\delta z = 0. \end{aligned}$$

The second is "the determinate equation":

(34) (C)

$$\begin{aligned} &(p - dq + d^2r - \dots)\,\delta x + (q - dr + \dots)\,d\,\delta x + (r - \dots)\,d^2\,\delta x + \dots \\ &+ (P - dQ + d^2R - \dots)\,\delta y + (Q - dR + \dots)\,d\,\delta y + (R - \dots)\,d^2\,\delta y + \dots \\ &+ (\omega - d\chi + d^2\varrho - \dots)\,\delta z + (\chi - d\varrho + \dots)\,d\,\delta z + (\varrho - \dots)\,d^2\,\delta z + \dots = 0. \end{aligned}$$

The expression in (34) is to be evaluated at the beginning and end of the integral $\int Z$ to produce an equation of the form $M' - {}'M = 0$. LAGRANGE notes that when there is no relation among $x, y, z, dx, dy, dz, \dots$ (33) becomes

$$\begin{aligned} &n - dp + d^2q - d^3r + \dots = 0, \\ (35) \qquad &N - dP + d^2Q - d^3R + \dots = 0, \\ &\nu - d\omega + d^2\chi - d^3\varrho + \dots = 0. \end{aligned}$$

When there is a relation among the variables x, y, z, we reduce the latter to the smallest possible number (one or two) that can be independently varied and equate to zero the coefficients of the variations of the reduced variables.

LAGRANGE goes on in the memoir to provide a parametric analysis of both the brachistochrone problem and several examples from Chapter Three of EULER'S *Methodus Inveniendi*. In appendices he presents a detailed study of two problems: the first is to find surfaces of minimum area that are bounded by a given curve and enclose a given volume; the second requires determining polygons of a given number of sides that possess maximum area. (LAGRANGE in this last problem applies the formalism of the differential, integral and variational calculus to the investigation of finite differences and sums.) The reader may consult GOLDSTINE ([1980]) for a detailed description of these results. I shall concentrate on one question: the step which takes LAGRANGE from (30), the assertion that the variation of the definite integral $\int Z$ is zero, to equations (33), (34) and (35).

Equations (33), (34) and (35) are the EULER-LAGRANGE equations and endpoint relations for the general parametric variable endpoint problem. I briefly review their modern derivation. Assume for simplicity that x, y, and z are independent. Equation (32) is valid for the subset of curves with fixed endpoints, where the terms outside the integral sign in (32) are zero. We therefore apply the fundamental lemma and obtain (35). (35) remains valid when we expand the class of curves to the original set. Hence the integral terms in (32) are zero, and (34) follows immediately. We would not today present a relation of the form (33). Rather, when x, y, z are not independent, we would first reduce the variables in (32) and derive the reduced counterpart to (35).

The modern derivation of (34) and (35) was developed in the second half of the 19th century. Before then, researchers in the calculus of variations presented a diverse set of arguments to justify the passage from equation (32) to (33), (34) and (35). It is interesting how many of these arguments can be traced back to LAGRANGE'S various treatises. (A survey of these developments leading up to the 20th century is provided in HUKE [1931].) What I wish to emphasize here is that the modern derivation involving the fundamental lemma, is *not* a correct interpretation of LAGRANGE'S procedure in the memoir of 1760. The modern derivation involves concepts, ideas and underlying structure absent in his presentation of the variational calculus; it would be a historical error to describe his analysis in terms of this theory.[7]

What then was LAGRANGE'S reasoning? Notice first that (33) and (34) are presented conjointly as consequences of (32). Furthermore, LAGRANGE uses the adjectives "indefinite" and "determinate" respectively to describe (33) and (34). The reconstruction I shall now provide for his reasoning is taken from a treatise published in 1810 by one of LAGRANGE'S close admirerers, the English mathematician ROBERT WOODHOUSE. Consider the following passage from WOODHOUSE'S *A Treatise on Isoperimetrical Problems and the Calculus of Variations* (modified inessentially to fit LAGRANGE'S presentation):

> For the formula [(32)] is composed of two parts: one, affected by the integral sign, expresses the sum of all the separate variations throughout the whole extent of the curve or integrated quantity; the other part, independent of the integral sign, is affected only by the variations at the extreme points, and therefore cannot by any combination with the other (which by changing δx, δy, δz, ... may be varied at will) form a sum equal to nothing. Hence since $\delta \int Z$ must $= 0$; each part separately, the one under the integral sign $\int$, the other not affected by it, must $= 0$.
>
> [1810, 90]

(Although this passage comes from the chapter devoted to LAGRANGE'S development of the calculus of variations, WOODHOUSE makes no reference to LAGRANGE'S original derivation; his treatise is not historical in this sense.) WOODHOUSE'S argument reappears in the writings of 19th century authors (see, for example, CARLL [1890, 41–42]). A version of it was enthusiastically advanced by ERNST MACH in his classic *The Science of Mechanics*; *A Critical and Historical Account of its Development* ([1883, 537]). It is, I contend, the most plausible reconstruction of the reasoning underlying LAGRANGE'S original derivation.

I have devoted space to LAGRANGE'S reasoning because I believe an appreciation of the patterns of inference involved in his derivations is essential to an historical understanding of his work. As we shall see in the next section, such

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[7] In my article ([1983]) I incorrectly interpret LAGRANGE'S derivation of the dynamical equations from the principle of least action in terms of the fundamental lemma. This interpretative error in no way affects the other conclusions of that article. See our discussion in I(f).

an appreciation is also of substantive concern for evaluating his contributions over the next three decades to the foundations of the calculus of variations.

f) Lagrange's treatises on mechanics 1760–1788

LAGRANGE'S contributions to the foundations of mechanics can be traced in his memoir on least action ([1760b]), in two memoirs he composed on the subject on the Moon's libration for prize competitions of the Paris Academy of Sciences ([1764], [1780]), and in the *Méchanique Analitique* of 1788. Each of these treatises provides a general method for deriving the equations of motion of an arbitrary system of bodies: in the memoir of 1760, from the principle of least action; in the remaining three treatises, from a generalization of the principle of virtual work. I have explored this subject in detail elsewhere ([1983]) from the viewpoint of the history of dynamics. I shall here concentrate on the question of what LAGRANGE'S foundational researches in mechanics reveal about his evolving conception of the calculus of variations.

To understand LAGRANGE'S use of the variational calculus in the dynamical memoir of 1760 it is sufficient to examine his derivation of the equations of motion of a single particle. He begins with the principle of least action

$$\delta \int v\, ds = 0, \tag{35}$$

which asserts that the 'action' $\int v\, ds$ is a minimum. (I assume the particle has unit mass; v is the speed and $ds = v\, dt$.) By the law of conservation of *vis viva* we have

$$\tfrac{1}{2} v^2 = \text{constant} - \int (P\, dp + Q\, dq + \ldots), \tag{36}$$

where $P, Q, \ldots$ are the forces acting on the particle and $p, q, \ldots$ are the particle's distances from the force centers. LAGRANGE takes the variation of each side of (36) and obtains, after some reductions:

$$v\, \delta v = -P\, \delta p - Q\, \delta q - \ldots \tag{37}$$

Combining (35) and (37) he eventually derives an equation of the form

$$\int (d\dot{\vec{r}} + \vec{\nabla} V\, dt) \cdot \delta \vec{r} + \dot{\vec{r}} \cdot \delta \vec{r} = 0, \tag{38}$$

where I have used vector notation to facilitate the description of his procedure. ($\vec{r} = (x, y, z)$ is the position vector of the particle, $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity, $\delta\vec{r} = (\delta x, \delta y, \delta z)$ is the variation of $\vec{r}$, and $\vec{\nabla} V = (\partial V/\partial x, \partial V/\partial y, \partial V/\partial z)$ is the gradient of the potential V. Details of how (38) follows from (35) and (37) are provided in FRASER [1983].) Since (38) is valid "whatever values one supposes for the differences $\delta x, \delta y, \delta z$", LAGRANGE obtains as the final result the three equations

$$\begin{aligned} d\dot{x} + (\partial V/\partial x)\, dt &= 0, \\ d\dot{y} + (\partial V/\partial y)\, dt &= 0, \\ d\dot{z} + (\partial V/\partial z)\, dt &= 0. \end{aligned} \tag{39}$$

He subsequently remarks that we may suppose the endpoints are "given in position", *i.e.*, fixed, so that $\delta x = \delta y = \delta z = 0$ at the initial and final configurations. Hence each of the terms $\dot{x}\,\delta x$, $\dot{y}\,\delta y$ and $\dot{z}\,\delta z$ in the part of (38) outside of the integral sign is zero.

In passing from (38) to (39) LAGRANGE is following the procedure established in the preceding mathematical memoir. Unfortunately, the situation here is not quite analogous to the earlier one. LAGRANGE is supposing that the integrand in (38) is zero; then, because the variations δx, δy, and δz are arbitrary, he concludes that their coefficients are zero. However, δx, δy and δz are not in this case arbitrary: they must satisfy the energy relation (37). This relation combined with the other conditions of the problem (in particular, the (unmentioned) condition that the time is not varied in the δ-process) will in general limit the class of variations and render illegitimate the inference from (38) to (39). LAGRANGE'S derivation falters at its final step.

Despite this difficulty LAGRANGE is nevertheless able to obtain the correct equations of motion. I have explained this fact elsewhere ([1983]) by showing that his procedure reduces in practice to the one associated today with 'HAMILTON'S principle'. Interesting though this fact may be for the history of mechanics, it is not of central concern here. For our purposes, LAGRANGE'S variational treatment of the particle's motion illustrates how wrong it would be to interpret the step (38) to (39) in terms of the fundamental lemma. Thus, for example, LAGRANGE turns to the endpoint conditions only after he has derived (39). In addition, he is clearly not concerned with the kind of examination of the class of comparison arcs that would be necessitated by the use of this lemma.

In the memoir of 1764 on lunar libration LAGRANGE replaces the principle of least action by the general principle of virtual velocities as the fundamental axiom of mechanics. (In later mechanics the term 'virtual work' would replace 'virtual velocities'. LAGRANGE'S axiom is often referred to as 'D'ALEMBERT'S principle' in today's texts.) This principle may be written

$$\Sigma m\ddot{\vec{r}} \cdot \delta\vec{r} = \Sigma\vec{F} \cdot \delta\vec{r}, \tag{40}$$

where m, $\vec{r}$, $\ddot{\vec{r}}$, $\delta\vec{r}$, $\vec{F}$ are the mass, position vector, acceleration, virtual displacesment and external force for a typical particle of the system. LAGRANGE expresses the work function in terms of the variation of the potential so that (40) becomes

$$\Sigma m\ddot{\vec{r}} \cdot \delta\vec{r} + \delta V = 0. \tag{41}$$

He provides a general demonstration that $\Sigma m\ddot{\vec{r}} \cdot \delta\vec{r}$ may be expressed in the form $\Sigma(d(\partial T/\partial \dot{q}_i)/dt - \partial T/\partial q_i)$, where T is one half the *vis viva*, *i.e*, the kinetic energy, and q_i are independent co-ordinates which completely describe the system. (41) therefore yields the Lagrangian dynamical equations of motion.

I shall describe LAGRANGE'S procedure for the case of a single particle with one degree of freedom. We first express $\frac{1}{2}v^2$ in terms of the generalized co-ordinate q and the time derivative $\dot{q}$ of q:

$$\tfrac{1}{2}v^2 = f(q, \dot{q}). \tag{42}$$

Taking the variation of each side of (42) we obtain

$$v\,\delta v = \dot{\vec{r}} \cdot \delta\dot{\vec{r}} = \left(\frac{\partial f}{\partial q}\right)\partial q + \left(\frac{\partial f}{\partial \dot{q}}\right)\delta\dot{q}. \tag{43}$$

LAGRANGE integrates (43) by parts:

$$\dot{\vec{r}} \cdot \delta\vec{r} - \int \ddot{\vec{r}} \cdot \delta\vec{r} = \left(\frac{\partial f}{\partial \dot{q}}\right) \partial q - \int \left(\frac{d\left(\frac{\partial f}{\partial \dot{q}}\right)}{dt} - \frac{\partial f}{\partial q}\right) \delta q. \tag{44}$$

He proceeds to argue that (44) "must be identical and consequently it is necessary that the algebraic part of the first member be equal to the algebraic part of the second, and the integral part be equal to the integral part." Hence "removing the integral sign" LAGRANGE concludes that

$$\ddot{\vec{r}} \cdot \delta\vec{r} = \left(\frac{d\left(\frac{\partial f}{\partial \dot{q}}\right)}{dt} - \frac{\partial f}{\partial q}\right)\delta q. \tag{45}$$

In passing from (44) to (45) LAGRANGE seems to be saying that the arbitrariness of the virtual displacements (treated analytically as variations) and the extended nature of the integration process together imply that the parts of (44) inside and outside the integral signs cannot depend on each other. His reasoning here is similar to that involved in the step from (32) to (33) in the preceding mathematical memoir. However, in the earlier analysis equation (32) at least signified the fact that the variation of a definite, assignable quantity is zero. In equation (44), by contrast, integration is used as a transformational device to express the quantity $\ddot{\vec{r}} \cdot \delta\vec{r}$ in terms of the new variable q. (If LAGRANGE knew of the fundamental lemma he could have easily converted the demonstration into one acceptable today. It would consist of deriving HAMILTON's principle from the principle of virtual work and then using the fundamental lemma to obtain the equations of motion from HAMILTON's principle.)

LAGRANGE soon discovered an alternate derivation of the dynamical equations that did not involve the use of integration. In his second memoir of 1780 on lunar libration he replaces the above argument by one based on the following relation:

$$\ddot{\vec{r}} \cdot \delta\vec{r} = \frac{d(\dot{\vec{r}} \cdot \delta\vec{r})}{dt} - \tfrac{1}{2}\,\delta v^2. \tag{46}$$

(46) is obtained commuting d and δ, a fact LAGRANGE now refers to as the "fundamental principle of the calculus of variations." The derivation itself is more or less standard today and it would be unnecessary to describe it (see GOLDSTEIN [1950, 16–18]).

LAGRANGE founds his classic *Méchanique Analitique* of 1788 on the general principle of virtual work. Beginning with this principle he provides two derivations of the Lagrangian equations of motion ([1788, 216–223] = [Œ*uvres* 11 (1888), 325–331]). The first is taken directly from the memoir of 1780; the second is a version of the derivation of the memoir of 1764. I shall describe the second derivation. Assume Φ is a function of the variables $x, y, z, \ldots$ and their derivatives $\dot{x}, \dot{y}, \dot{z}, \ldots$ (I have modernized slightly LAGRANGE's notation, using '...' in place of his '&c', and replacing his differentials dx, dy, dz by $\dot{x}, \dot{y}, \dot{z}$. In addition, I present his derivation for the case where Φ contains no differentials of x, y and z higher than the first.) Consider the expression

$$\left(\frac{\partial\Phi}{\partial x} - \frac{d\left(\frac{\partial\Phi}{\partial\dot{x}}\right)}{dt}\right)\delta x + \left(\frac{\partial\Phi}{\partial y} - \frac{d\left(\frac{\partial\Phi}{\partial\dot{y}}\right)}{dt}\right)\delta y + \left(\frac{\partial\Phi}{\partial z} - \frac{d\left(\frac{\partial\Phi}{\partial\dot{z}}\right)}{dt}\right)\delta z + \ldots$$

which LAGRANGE writes in the form

$$A\,\delta x + B\,\delta y + C\,\delta z + \ldots \tag{47}$$

LAGRANGE wishes to show that (47) remains invariant under transformation of coordinates. Thus if we express $x, y, z, \ldots$ in terms of a new set of variables $\xi, \psi, \varphi, \ldots$, (47) becomes

$$A'\,\delta\xi + B'\,\delta\psi + C'\,\delta\varphi + \ldots, \tag{48}$$

where

$$A' = \left(\frac{\partial\Phi}{\partial\xi} - \frac{d\left(\frac{\partial\Phi}{\partial\dot{\xi}}\right)}{dt}\right),\quad B' = \left(\frac{\partial\Phi}{\partial\psi} - \frac{d\left(\frac{\partial\Phi}{\partial\dot{\psi}}\right)}{dt}\right),\quad C' = \left(\frac{\partial\Phi}{\partial\Phi} - \frac{d\left(\frac{\partial\Phi}{\partial\dot{\Phi}}\right)}{dt}\right), \ldots.$$

LAGRANGE's demonstration of this fact begins by expressing $\delta\Phi$ in terms of the two sets of variables and their variations and equating the resulting expressions. The equation thus formed is integrated by parts to yield

$$\int(A\,\delta x + B\,\delta y + C\,\delta z + \ldots) + Z = \int(A'\,\delta\xi + B'\,\delta\psi + C'\,\delta\varphi + \ldots) + Z', \tag{49}$$

where

$$Z = \left(\frac{\partial\Phi}{\partial\dot{x}}\right)\delta x + \left(\frac{\partial\Phi}{\partial\dot{y}}\right)\delta y + \left(\frac{\partial\Phi}{\delta\dot{z}}\right)\delta z + \ldots$$

and

$$Z' = \left(\frac{\partial\Phi}{\partial\dot{\xi}}\right)\delta\xi + \left(\frac{\partial\Phi}{\delta\dot{\psi}}\right)\delta\psi + \left(\frac{\partial\Phi}{\partial\dot{\varphi}}\right)\delta\varphi + \ldots.$$

LAGRANGE differentiates each side of (49) and transposes terms:

$$A\,\delta x + B\,\delta y + C\,\delta z + \ldots - A'\,\delta\xi - B'\,\delta\psi - C'\,\delta\varphi - \ldots = dZ' - dZ, \tag{50}$$

a result that "must be identical and valid whatever the variations or differences indicated by the letter δ." He proceeds as follows

> Thus since the second member of [(50)] is an exact differential relative to the characteristic d, the first member must be one also relative to the same characteristic, independent of the characteristic δ; which is not possible because the terms of the first member contain simply the variations δx, δy, $\delta z, \ldots, \delta\xi$, $\delta\psi$, $\delta\varphi, \ldots$, and nowhere the differentials of these variables.

Whence it follows that for [(50)] to be valid the two members must be zero separately; which gives the two identical equations

$$[(51)] \qquad A\,\delta x + B\,\delta y + C\,\delta z + \ldots = A'\,\delta\xi + B'\,\delta\psi + C'\,\delta\varphi + \ldots$$
$$dZ = dZ'$$

which may be useful on different occasions ([1788, 223] = [*Œuvres* 11 (1888), 330–331]).

LAGRANGE illustrates the usefulness of (51) by applying it to the dynamical case where $\Phi = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and $A = \ddot{x}$, $B = \ddot{y}$, $C = \ddot{z}$, thereby deriving the Lagrangian equations corresponding to the co-ordinates $\xi, \psi, \varphi, \ldots$

LAGRANGE'S derivation of equation (51) is a curious one. Although it seems to be a version of the argument from the memoir of 1764, there is an important difference. In the earlier treatise LAGRANGE appeared to be reasoning from the extended nature of the integration process when he passed from (44) to (45). In the present derivation, by contrast, integration plays an inessential role: LAGRANGE could have arrived at (50) simply by using properties of the differential d and rearranging terms. As for the final step from (50) to (51), it seems to me to be an unusual but valid way of arriving at the desired result.

Although LAGRANGE'S use of integration to derive (51) is avoidable, it is by no means clear that he himself appreciated or recognized this fact. From the memoir of 1760 through the two memoirs on libration to the *Méchanique Analitique*, one can discern an increasing tendency on his part to view the calculus of variations in terms of its formal properties. Thus in this last work he sets forth the following two "fundamental principles" of the subject: the first asserts that d and δ commute; the second is the analytic device of integration by parts. The elevation of the latter to a central place indicates the importance LAGRANGE had come to attach to the purely algorithmic role of integration in his variational method.

(II) Part Two

a) Introduction

The decade 1780 to 1790 was a critical one in LAGRANGE'S career. We know from a letter ([1782]) he wrote to LAPLACE that he had essentially completed his masterpiece, the *Méchanique Analitique*, by 1782, an event that reportedly left him exhausted. In 1783 the three most important people in his adult life died: his wife and cousin, VITTORIA CONTI, whom he had married in 1767; D'ALEMBERT, his closest friend; and EULER, his first mentor. His position as director of the mathematical class of the Berlin Academy became uncertain in 1786 as a result of the changed circumstances following the death of FREDERICK the Great. In 1787, responding to a generous offer from the French government, LAGRANGE travelled to Paris to become a pensionnaire of the Academy of Sciences. He remained there until his death in 1813.

LAGRANGE'S decision to move to Paris would have important consequences for his future approach to the foundations of the differential, integral and variational calculus. As a result of the French Revolution a new school, the École Polytechnique, was established in Paris in 1795 to train engineers. LAGRANGE was called upon to deliver lectures on mathematics and mechanics at the École. Out of these lectures grew his two major didactic works: *Théorie des fonctions analytiques* ([1797]) and *Leçons sur le calcul des fonctions* ([1806]).[8] In these treatises he presents his celebrated attempt to base the differential and integral calculus on a theory of TAYLOR power series. In addition, for the first time he explicity addresses the question of the foundations of the calculus of variations.

LAGRANGE'S interest in foundations reflected a broader concern for rigor that had developed at the end of the 18th century within European mathematics. The reasons for this concern and LAGRANGE'S own contributions have been discussed in the literature (see GRABINER [1981]). In the variational calculus his research centered on two topics: an analytical definition of the variation; a general demonstration that the EULER-LAGRANGE equations follow from the vanishing of the first variation. LAGRANGE'S explicit treatment of these topics constitutes a departure from his earlier work, where his understanding of fundamental processes can be inferred only indirectly in the course of investigations devoted to the solution of specific problems. His later approach also reflects, we shall see, the distinctive style that characterizes the didactic works composed during his Paris period.

b) The Eighteenth Century Background

The origins of LAGRANGE'S later theory may be traced in 18th century investigations devoted to establishing criteria for when differential expressions are integrable. Assume f is a function of x. y and $y' = dy/dx$. Geometers of the period were concerned with the question of when there exists a primitive function $F = F(x, y)$ such that, for all $y = y(x)$, $dF/dx = f$. As I noted in my initial

[8] LAGRANGE'S *Mécanique Analytique* is also included in the section of his *Oeuvres* titled "Ouvrages Didactiques". In addition, there is evidence that his early transition from the principle of least action to the principle of virtual velocities, the fundamental axiom of the *Méchanique Analitique*, was influenced by his teaching duties at the Turin Artillery School in the late 1750's and early 1760's. (For details on this subject see FRASER [1983].) I nevertheless believe there are reasons for distinguishing this treatise from the didactic works composed when he was in Paris. The *Méchanique Analitique* was written during LAGRANGE'S Berlin period, when his professional responsibilities were exclusively for research. In addition, it seems extremely unlikely that his lectures to artillery students at Turin were sufficiently advanced to have included a derivation of the Lagrangian equations of motion. (Indeed, these equations can only be located with difficulty in his memoir of 1764 on libration, a memoir that was itself an advanced monograph for specialists.)

discussion of the mathematical theory, a necessary and sufficient condition for the existence of F is that the EULER-LAGRANGE equation

$$(1) \qquad \frac{\partial f}{\partial y} - \frac{d\left(\frac{\partial f}{\partial y'}\right)}{dx} = 0,$$

be an identity in x, y and y'. In the general case f will be a function of $x, y, y', \ldots, y^{(n)}$, F will be a function of $x, y, y', \ldots y^{(n-1)}$, and condition (1) becomes

$$(2) \qquad \frac{\partial f}{\partial y} - \frac{d\left(\frac{\partial f}{\partial y'}\right)}{dx} + \ldots + (-1)^n \frac{d^n\left(\frac{\partial f}{\partial y^{(n)}}\right)}{dx^n} \equiv 0.$$

The fact that the identity (1) is a necessary and sufficient condition for the existence of a primitive may be shown directly, with no reference to the calculus of variations. This fact, however, was not well understood during the period under consideration here; the whole matter became clarified only much later (see the account in II(d)). During the 18th century, discussion of the condition tended to occur in expositions of the variational calculus. Indeed, geometers of the period came to believe that there was a profound connection between integrability and the foundations of the calculus of variations.

In the *Methodus Inveniendi* (1744) EULER is careful to specify that the quantity $Z\,dx$ in the integral $\int Z\,dx$ of a typical variational problem "cannot be integrated". Discussion of the point forms the subject of the opening proposition of Chapter I of the treatise ([1744, 16–17]). EULER observes here that if $\int Z\,dx$ is a maximum or minimum then Z is integrable only when a specific relation between x and y is assumed. What he means is that there can be no primitive of Z which works for all $y = y(x)$. If such a primitive existed, $\int Z\,dx$ would depend only on the values of x and y at the endpoints, and would, therefore, be the same for all curves which coincide with the actual curve at these points. In this, the degenerate case, $\int Z\,dx$ would not be a maximum or minimum, a conclusion which contradicts the initial assumption.

Twenty-five years later, in 1770, EULER returned to the question of integrability in an appendix on the calculus of variations to the third volume of his *Institutiones Calculi Integralis* ([1770, 410–411]). In Theorem 3 § 92 he considers an expression V involving x, y and $p = dy/dx$. He wishes to show that a necessary and sufficient condition that $V\,dx$ be integrable, independent of any relation between x and y, is the identity

$$(2) \qquad N - \frac{dP}{dx} \equiv 0,$$

where N and P are defined by the relation $dV = M\,dx + N\,dy + P\,dp$. The demonstration is verbal, rather sketchy, and involves ideas from the calculus of variations. If (2) holds, the variation of $\int V\,dx$ will not depend on any intermediate values of x any y. EULER is appealing here to the fact that in his earlier result

$$\delta \int V\, dx = \int \left(N - \frac{dP}{dx}\right) \delta y\, dx + \delta y\, P, \tag{3}$$

the integral term on the right side will be zero. He concludes from this fact that a primitive must exist. Conversely, let us assume the existence of a primitive. The variation of this primitive will clearly depend only on the values of x, y and δy at the endpoints. EULER implicitly assumes in this situation that the variation must vanish. Hence the integral term on the right side of (3) is zero. Equation (2) is presented as a self-evident consequence of this last fact.

EULER'S use of ideas from the calculus of variations to show that (2) is a sufficient condition for the existence of a primitive is unconvincing. Indeed, 18^{th} century geometers were never able to provide a satisfactory demonstration of this result. His argument for the necessity of (2) is plausible but still involves the variational calculus. In 1765 the French geometer the Marquis DE CONDORCET had provided a direct demonstration of necessity in his treatise *Du Calcul Intégral.* In Problem I of Part One CONDORCET considers a function V of x, y, dx, dy. He supposes dx is held constant and sets $p = dx$, $p' = dy$. Assume there exists a function B such that $dB = V$. We then have

$$V = dB = \frac{dB}{dx} p + \frac{dB}{dy} p'. \tag{4}$$

CONDORCET writes

$$dV = Np + N'p' + P'\, dp', \tag{5}$$

and differentiates (4)

$$dV = d\left(\frac{dB}{dx}\right) p + d\left(\frac{dB}{dy}\right) p' + \frac{dB}{dy}\, dp'. \tag{6}$$

By equating the coefficients of p' and dp' in (5) and (6) he obtains

$$N - dP' = 0, \tag{7}$$

which is the desired necessary condition.

CONDORCET'S demonstration is essentially the proof we would present today, with one important qualification: he has not shown why he is able to equate the coefficients of p' in (5) and (6). To justify this step he needs to use the fact (as he would have expressed it) that $d\, dB/dy\, dx = d\, dB/dx\, dy$. (Using partial derivatives we see that $N' = \partial V/\partial y = \partial((\partial B/\partial x)\, p + (\partial B/\partial y)\, p')/\partial y = (\partial^2 B/\partial y\, \partial x)\, p + (\partial^2 B/\partial y^2)\, p'$. On the other hand $d(\partial B/\partial y) = (\partial^2 B/\partial x\, \partial y)\, p + (\partial^2 B/\partial y^2)\, p'$. Since $\partial^2 B/\partial x\, \partial y = \partial^2 B/\partial y\, \partial x$ we have $N' = d(\partial B/\partial y)$.) I have, to be fair, only presented his analysis for the elementary case, and it is possible the more general treatment which appears in the original treatise had the effect of obscuring CONDORCET'S understanding of details.

CONDORCET notes that if $V = A\, dx + B\, dy$ then (7) reduces to the known condition for exact differentiability: $dB/dx = dA/dy$. He further comments that there is an accord between his investigation and the work of EULER and LAGRANGE in the calculus of variations, an accord "founded on an analytical identity between the two questions". Unfortunately, his explanation of this "analytical identity" reduces to little more than the observation of the formal similarity provided for the two cases by the presence of the EULER-LAGRANGE equation.

CONDORCET'S *Du Calcul Intégral* is written at a lower level than the treatises of the great mathematicians of the period. Nevertheless, by treating integrability independently of the calculus of variations, and suggesting the resulting theory may be fundamentally connected to that subject, CONDORCET provided the inspiration for LAGRANGE'S later research.

c) The New Foundations

In the *Théorie des fonctions analytiques* and *Leçons sur le calcul des fonctions* (hereafter referred to as *Théorie* and *Leçons*) LAGRANGE attempts using a theory of TAYLOR power series to provide a basis for the calculus which avoids consideration of infinitesimal quantities. His program is strongly algebraic, with a relative lack of concern for questions of convergence and uniqueness; it has been discussed in detail in the literature (see GRABINER [1981] and OVAERT [1976]). A sense for LAGRANGE'S approach, as well as an understanding of the immediate background to his work on integrability and the calculus of variations, may be obtained through a brief description of the notations he introduces in the two treatises. If f is a function of x then the "first derived function" or derivative of f with respect to x is defined to be the coefficient of i in the TAYLOR series expansion of $f(x+i)$. LAGRANGE uses the notation $f'(x)$ to indicate this derivative. Higher order derivatives are defined to be the coefficients of $i^2/2$, $i^2/6$, $i^4/24$, ... in this expansion and are denoted $f''(x), f'''(x), f^{(4)}(x)$, ... Suppose now that f is an expression involving x, y, z, where y and z are functions of x. I indicate the correspondence between LAGRANGE'S notations and the ones we employ today:

$$f'(x) = \frac{\partial f}{\partial x}, \quad [f(x, y, z)]' = \frac{df}{dx},$$

$$f'(y) = \frac{\partial f}{\partial y}, \quad [f'(y)]' = \frac{d\left(\dfrac{\partial f}{\partial y}\right)}{dx},$$

$$f''(y) = \frac{\partial^2 f}{\partial y^2}, \quad f''(y, z) = \frac{\partial^2 f}{\partial y\,\partial z}.^{9}$$

These quantities, like the derived functions of f, are defined by their place in a TAYLOR power series. Thus $f'(y)$ and $f'(z)$ are the coefficients of i and j in the TAYLOR expansion of $f(x, y+i, z+j)$; $f''(y, z)$ is the coefficient of ij, etc.

[9] LAGRANGE is not entirely consistent in the two treatises in his use of notation; I have described the latter as it is employed in his presentation of the theory of integrability and the calculus of variations.

LAGRANGE'S choice of notation, involving primes, commas and parentheses, reflects his basic goal: to free the calculus from all reliance on infinitely small entities, represented by the symbols df, dx and dy, and thereby to avoid the geometric interpretations and logical difficulties that were characteristic of traditional presentations of the subject.

i) Théorie des fonctions analytiques (1797)

In § 170–§ 183 of the *Théorie* LAGRANGE considers a function f of x, y, y', y'', ..., where y is a function of x and y', y'', ... are the derived functions of y with respect to x. He wishes to show a necessary and sufficient condition for the existence of a primitive of f, independent of any relation between x and y, is that the EULER-LAGRANGE equation,

$$f'(y) - [f'(y')]' + [f''(y'')]'' - \ldots = 0, \tag{8}$$

be an identity. I shall describe his analysis for the elementary case in which $f = f(x, y, y')$ and (8) becomes

$$f'(y) - [f'(y')]' = 0. \tag{9}$$

He begins by replacing y by $y + \omega$, where ω is any function of x, and expands $f(x, y + \omega, y' + \omega')$ in a series

$$f(x, y + \omega, y' + \omega') = f(x, y, y') + P + Q + R + \ldots, \tag{10}$$

where $P = \omega f'(y) + \omega' f'(y')$, $Q = \frac{1}{2}\omega^2 f''(y) + \omega\omega' f''(y, y') + \frac{1}{2}\omega'^2 f''(y')$, $R =$ part of the expansion containing third degree terms in ω and ω', and so on. Suppose now that $f(x, y, y')$ has a primitive. Then $f(x, y + \omega, y' + \omega')$ will have one also, as will $f(x, y + \omega, y' + \omega') - f(x, y, y') = P + Q + R + \ldots$ LAGRANGE states "it is easy to see that each of the quantities P, Q, R, *etc.* must separately be a prime function [*i.e.*, the derived function of a primitive], since these quantities are composed of different dimensions of the indeterminate ω and its derived function ω', it being impossible, by the nature of derived functions, that the primitive functions of P, Q, R, *etc.* be mutually dependent." In particular, $P = \omega f'(y) + \omega' f'(y')$ has a primitive, which, LAGRANGE says, will be of the form $\alpha + \omega\beta$, where α and β are functions of x. Thus $\alpha' + \omega\beta' + \omega'\beta = \omega f'(y) + \omega' f'(y')$, from which we obtain the relations $\alpha' = 0$, $\beta = f'(y')$, $\beta' = f'(y)$. Equation (9) is an immediate consequence of these last relations.

Suppose conversely that (9) holds independently of any relation between x and y. Clearly then $P = \omega f'(y) + \omega' f'(y')$ has a primitive, namely $\omega f'(y')$. But if P has a primitive so will the function obtained by replacing y by $y + \omega$ in P. (In today's notation this function equals $\omega\, \partial f/\partial y(x, y + \omega, y' + \omega') + \omega'\, \partial f/\partial y'(x, y + \omega, y' + \omega')$.) If we expand this function in a TAYLOR series the parts of the resulting expansion consisting of dimensionally homogeneous terms in ω and ω' will each separately have a primitive. The series equals $P + 2Q + \ldots$, where $2Q = \omega^2 f''(y) + 2\omega\omega' f''(y, y') + \omega'^2 f''(y')$ is the part of the expansion of dimension two in ω and ω'. Hence $2Q$ and therefore Q has a primitive. We may in turn apply the same procedure to Q to obtain a series of the form

$Q + 3R + \dots$, thereby ensuring the existence of a primitive for R. In this way LAGRANGE establishes the existence of a primitive for each of the parts P, Q, R, *etc.* in (10). Hence $f(x, y + \omega, y' + \omega') - f(x, y, y')$ has a primitive. LAGRANGE sets $y = -\omega$ in this last relation and notes that $f(x, 0, 0)$, a function of the single variable x, has a primitive. He therefore arrives at the desired final result, the existence of a primitive for the original function $f(x, y, y')$.

LAGRANGE'S study of integrability in the *Théorie* is closely connected with his derivation in that treatise of the EULER-LAGRANGE equations in the calculus of variations. His treatment of this last subject, however, is unsatisfactory, and I shall describe it only briefly. The problem is now to find the particular function $y = y(x)$ which renders the primitive, *i.e.*, integral, of $f(x, y, y')$ evaluated between definite limits a maximum or minimum. LAGRANGE argues correctly that a necessary condition for this to be the case is that the primitive of the quantity $\omega f'(y) + \omega' f'(y')$ (ω a small function of x) be zero. (Clearly for small ω the primitive of $\omega f'(y) + \omega' f'(y')$ dominates in the primitive of the expansion of $f(x, y + \omega, y' + \omega') - f(x, y, y')$. Hence for small ω the primitive of $\omega f'(y) + \omega' f'(y')$ must be always positive or always negative. Let $\omega = i\alpha$, where α is a function of x and i is a small constant. The primitive of $\omega f'(y) + \omega' f'(y')$ will be multiplied by i, and since i can be positive or negative, this primitive must equal zero.) LAGRANGE sets this primitive equal to $\alpha + \omega\beta$ and concludes (as he did for the case I described earlier involving the existence of a general primitive) that $f'(y) - [f'(y')]' = 0$. In this last step he appears to be evaluating the primitive between the initial limit and an arbitrary intermediate value of x. However, his procedure here is, unlike the earlier case involving the existence of a general primitive, completely illusory. Thus he has not shown why he is able to set the primitive equal to $\alpha + \beta\omega$, a step that can be justified only by assuming in advance the validity of the EULER-LAGRANGE equation for the function y of x that solves the variational problem. (LAGRANGE is, however, more successful in the *Théorie* in his investigation of the second variation and the so-called LEGENDRE condition. See GOLDSTINE [1980, 145–147].)

ii) Leçons sur le calcul des fonctions (1806)

In the *Leçons* LAGRANGE separates his investigation of integrability from his treatment of the calculus of variations, expands his presentation of each of these subjects, and discusses the connection between them. In the twenty-first lesson he provides a new proof that (8) is a necessary condition for the existence of a primitive and uses this proof as a model in the following lesson for his derivation of the EULER-LAGRANGE equations in the variational calculus. This subject is now systematically developed using the power series techniques he had employed in the differential and integral calculus.

To show that the identity of the relation $f'(y) - [f'(y')]' = 0$ is a necessary condition for the existence of a primitive for $f(x, y, y')$, LAGRANGE again expands $f(x, y + \omega, y' + \omega')$ in a series and uses the (assumed) fact that the part of this series containing linear terms in ω and ω', $\omega f'(y) + \omega' f'(y')$, must also have a primitive. He writes $\omega f'(y) + \omega' f'(y') = (N - P')\,\omega + (P\omega)'$, where $N = f'(y)$

and $P = f'(y')$. Since $(P\omega)'$ has a primitive, $(N - P')\,\omega$ must have one also. Because ω is arbitrary, LAGRANGE says, "it is easy to convince oneself" that $N - P'$ or $f'(y) - [f'(y')]'$ is (identically) equal to zero. (LAGRANGE'S reasoning is apparently this. He has shown that $(N - P')\,\omega$ has a primitive U. If U is not identically zero, it must contain ω; hence its derivative will contain a term of the form $(\partial U/\partial\omega)\,\omega'$. Clearly, however, $\partial U/\partial\omega$ must equal zero and so U cannot contain ω, a contradiction. Thus U is identically zero.)

LAGRANGE'S proof of necessity, like his earlier demonstration in the *Théorie*, depends on the assumption that if $f(x, y, y')$ has a primitive then the linear part of the expansion of $f(x, y + \omega, y' + \omega')$ must also have a primitive. His proof of sufficiency, which also requires this assumption, is simply a more detailed version of the argument that had appeared in the *Théorie*. In the *Leçons* LAGRANGE provides some discussion of specific examples. Thus if $f(x, y, y')$ satisfies (9) it will be of the form $\Psi(x, y) + y'\Phi(x, y)$, where Ψ and Φ satisfy the condition $\Psi'(y) = \Phi'(x)$. More generally, if $f = f(x, y, y', \ldots, y^{(n)})$ satisfies (8) then it will be of the form $\Psi(x, y, y', \ldots, y^{(n-1)}) + y^{(n)}\,\Phi(x, y, y', \ldots, y^{(n-1)})$. LAGRANGE continues by examining the conditions f will satisfy in the general case and ends with the following comments:

> The problem we have just resolved concerning the equations of condition, which must hold in order that a given function of several variables and their derivatives have a primitive function independent of any relation between these variables, has an intimate connection with another more important problem, which has exercised the geometers for nearly a century. It is the famous problem of isoperimeters, which, taken in all its generality, consists in finding the equations which must hold between variables, in order that the unknown primitive function of a given function of these variables and their derivatives be a maximum or minimum.
>
> The same forms of equations resolve the two problems, but with this difference, that, in the first, the equations must hold identically, and be verified alone, whereas in the second problem, they become equations among the variables necessary for the existence of the maximum or minimum.
>
> [1806, 383]

To this statement LAGRANGE adds the remark that he will show in his subsequent investigation of the calculus of variations the reason for the identity of results in the two problems. He prepares this investigation by closing the lesson with a selective survey of 18th century developments in the variational calculus.

LAGRANGE opens the twenty-second lesson with the observation that the traditional method of variations, "founded on the use of the combination of the characteristics d and δ", requires at base the consideration of infinitely small quantities. He proposes to extend his algebraic program to the calculus of variations by providing an alternate development of this method which avoids infinitesimals. He does so using a definition for the variation first introduced by EULER ([1771]) in a memoir presented to the St. Petersburg Academy of Sciences. EULER'S definition is essentially the modern one. Let $y = X(x)$ be a function of x. Consider a class of comparison functions $y = X(x, t)$ parameterized by the variable t; $y = X(x)$ corresponds to $t = 0$. ($X(x, t)$ might equal $X(x) + tY(x)$, where $Y(x)$ is some function of x.) EULER defines the variation δy of y to be $(dy/dt)\,dt$, where (dy/dt) is to be evaluated at $t = 0$. In modern notation we would write $\delta y = (\partial y/\partial t)|_{t=0}\,dt$. If Z is an expression involving x, y and the derivatives of y with respect to x, the variation of Z is defined to be the (partial) derivative of Z with respect to t evaluated at $t = 0$ multiplied by dt.

LAGRANGE takes EULER's definition and expresses it in terms of his own power series presentation of the calculus. Suppose $y = \varphi(x)$ is a function of x. LAGRANGE considers a comparison class of functions $y = \varphi(x, i)$, where $y = \varphi(x)$ corresponds to $i = 0$, and expands $\varphi(x, i)$ in a series: $y + i\dot{y} + (i^2/2)\,\ddot{y} + (i^3/3 \cdot 2)\,\dddot{y} + \ldots$ (The dot is used to distinguish differentiation with respect to i from differentiation with respect to x, the latter being denoted by a prime.) The quantities $y, \dot{y}, \ddot{y}, \ldots$ in this expansion are to be evaluated at $i = 0$. The variation of y is defined to be the value of $\dot{y}$ at $i = 0$. If U is an expression involving x, y and the derivatives of y with respect to x, U will become a function of i when $y = \varphi(x)$ is replaced by the comparison function $y = \varphi(x, i)$. LAGRANGE expands U in a series, $U + \dot{U} + (1/2)\,\ddot{U} + (1/3 \cdot 2)\,\dddot{U} + \ldots$, and defines the first, second, third, *etc.*, variation of U to be the quantity $\dot{U}, \ddot{U}, \dddot{U}$, *etc.*, evaluated at $i = 0$.

Suppose now that $V = f(x, y, y')$ and we wish to find the function $y = y(x)$ which renders the primitive of V evaluated between definite limits a maximum or minimum. LAGRANGE argues, as he had in the *Théorie*, that a necessary condition for this to be the case is that the primitive of $\dot{V}$ evaluated between these limits be zero for all possible values of $\dot{y}$ (possible, that is, within the (unmentioned) restriction that $\dot{y}$ be small). He notes that $\dot{V} = N\dot{y} + P\dot{y}'$, where $N = f'(y)$ and $P = f'(y')$ and rewrites $\dot{V}$ as follows: $\dot{V} = (N - P')\,\dot{y} + (P\dot{y})'$. He observes that $(P\dot{y})'$ has a primitive (namely, $P\dot{y}$) regardless of the value of $\dot{y}$. He continues as follows:

> ... by contrast, [the expression $(N - P')\,\dot{y}$], being multiplied by $\dot{y}$, cannot have a primitive function unless we give particular values to the variation $\dot{y}$: thus, as this variation must remain indeterminate it will be impossible for the primitive of $\dot{V}$ to be zero, unless the [expression $(N - P')\,\dot{y}$] disappears, which will give the equation independent of $\dot{y}$
>
> $$N - P' + Q'' - R''' + \ldots = 0.$$
>
> This equation contains the necessary relation between the variables x and y for the existence of the maximum or minimum and is what we shall call the general equation of the maximum or minimum.
>
> [1806, 404][10]

By modelling this derivation after the earlier one, in which the relation $N - P' \equiv 0$ was shown to be the consequence of the existence of a primitive, LAGRANGE

[10] LAGRANGE presents the derivation for the general case, in which V is a function of higher-order derivatives $y'', y'', \ldots$. Thus in the original treatise he obtains the equation $N - P' + Q'' - R''' + \ldots = 0$. I have described his analysis for the elementary case in which $Q = R = \ldots = 0$.

believed he had established the reason for the identity of results in the two cases. He refers in this connection to the 18th-century work of EULER and CONDORCET and concludes with the comment that his own analysis "leaves nothing to be desired" on the subject.

LAGRANGE continues in the lesson by considering a range of special topics in the calculus of variations; problems with variable endpoints, the parametric problem, problems with more than one independent variable, *etc.* It is worth emphasizing that his treatment in the *Théorie* and *Leçons* is almost entirely non-parametric. In particular, the parametric and variable endpoint problems which had occupied such an important place in his early development of the variational calculus here play no role at all. It is rather to the theory of integrability that LAGRANGE turns for guidance in presenting the foundations of this subject.

d) An assessment

Three points concerning LAGRANGE'S analysis described in the preceding section require assessment: the first is his definition of the variation; the second is his derivation in the *Leçons* of the EULER-LAGRANGE equation in the calculus of variations; the third is his demonstration in the *Théorie* that the identical vanishing of this equation is a sufficient condition for the existence of a primitive.

LAGRANGE'S definition of the variation in terms of coefficients in a TAYLOR power series suffers from the drawback associated with his power series approach to the differential and integral calculus: he has not provided a theory to deal with questions of existence, convergence and uniqueness. The construction of such a theory would require the formulation of a new basis for the calculus, a basis that would need to be prior to and more fundamental than the series approach itself.

LAGRANGE'S derivation of the equation $N - P' = 0$ in the calculus of variations should, I think, be regarded as an interesting failure. He assumes that since the definite integral of the first variation $\dot{V}$ of V is zero there must be a primitive for $\dot{V}$ which works for all variations $\dot{y}$ of y. In particular, the term $(N - P')\,\dot{y}$ in the expression for $\dot{V}$ must have a primitive and this can happen only if $N - P' = 0$. It is, however, simply not true that a general primitive of $\dot{V}$ must exist. Although $\dot{y}$ is arbitrary, y will be a particular function of x; the conditions of the variational problem in no way require the existence of such a primitive.

LAGRANGE'S derivation is nevertheless interesting because of the motivation underlying it. LAGRANGE was impressed by the appearance of the same equation in the calculus of variations and the theory of integrability. It is important to note that for him (unlike EULER) the two subjects were separate. We see today that if a primitive $F = F(x, y)$ of $f(x, y, y')$ exists then the integral $I = \int_a^b f(x, y, y')\,dx$ has the same value for all functions y of x with prescribed values at the endpoints. Thus I has a stationary value and the equation $N - P' = 0$ follows for all $y = y(x)$ from the variational analysis. (This was, in fact, the argument advanced by EULER in 1770 which I described in II(b).) For LAGRANGE by contrast, the variational problem made sense only when I has a maximum or minimum; indeed, he needed this condition to show the first variation of I is zero. In his development of the two subjects the appearance of the same equation was explained by the structural similarity between its derivation in each case.

LAGRANGE'S demonstration that the identical vanishing of $N - P'$ is a sufficient condition for the existence of a primitive is more difficult to evaluate. The argument depends on two assumptions: that the expression $f(x, y + \omega, y' + \omega')$ may for arbitrary functions ω be expanded in a series about x, y, y'; that if $f(x, y, y')$ possesses a primitive then so will each part in the expansion of $f(x, y + \omega, y' + \omega')$ containing dimensionally homogeneous terms in ω and ω'. The second assumption seems especially to be one which requires more explanation. The whole demonstration strikes me as an analytical *tour de force*, as an example of LAGRANGE'S remarkable adroitness at algebraic deduction. (LAGRANGE'S proof of necessity in the *Leçons* also depends on the above two assumptions. He had constructed this proof so that it would serve as a model for his subsequent derivation of the EULER-LAGRANGE equation in the calculus of variations. In later mathematics necessity was established using the method of either CONDORCET or EULER described in Section II(b).)

Whatever the merits of LAGRANGE'S demonstration, it served to inspire later researchers in the theory of integration. These researches are surveyed in TODHUNTER [1861, 505–530]. Two approaches to the problem of establishing sufficiency developed. The first, involving ideas from the calculus of variations, was based on the proof presented by EULER in 1770 in the *Institutiones Calculi Integralis*, described earlier in II(b). The second followed LAGRANGE in attempting to prove sufficiency directly.

EULER had noted that if $N - P' \equiv 0$ then the variation $\delta \int_a^b f(x, y, y')\,dx$ will depend only on the values of x, y, y' and δy at the endpoints. He concluded from this that a primitive must exist, apparently on the grounds that the integral $\int_a^b f(x, y, y')\,dx$ will, by varying b, depend only on the values of x and y at the upper endpoint and must, therefore, be a specifiable function of these variables. EULER'S demonstration was resuscitated in the nineteenth century by BERTRAND ([1841]) and has enjoyed an occasional popularity since, appearing most recently in COURANT & HILBERT [1953, 194]. It would seem to be a rather unsatisfactory way of proving the desired result.

Attempts at establishing sufficiency directly in the 19$^{\text{th}}$ century were many and varied. I shall describe an unexceptional demonstration due to CAMILLE JORDAN ([1896, 479–480]) which appeared at the end of the century in a textbook for students at the *École Polytechnique*. Before doing so I must comment briefly on my own presentation of LAGRANGE'S theory. For expository reasons I have described his analysis for the elementary problem, in which f is a function of x, y and y'. In the original treatise he tends to work with the function $f = f(x, y, y', \ldots, y^{(n)})$; $n = 1$ is treated as a special case. By approaching the question of integrability at this level of generality LAGRANGE may have missed a simple proof of sufficiency. He knew in the elementary problem that the relation $N - P' \equiv 0$ implies f to be of the form $\Psi(x, y) + y'\Phi(x, y)$, where Ψ and Φ satisfy the condition $\Psi'(y) = \Phi'(x)$, *i.e.*, $\partial\Psi/\partial y = \partial\Phi/\partial x$. ALEXIS CLAIRAUT had shown as early as 1739 that this condition is sufficient for f to be an integrable function of x and y. Indeed, CLAIRAUT showed how to integrate f by reducing the problem to one in the theory of a single ordinary differential equation. (A discussion of CLAIRAUT's analysis appears in KATZ [1981].) It is surprising that LAGRANGE nowhere refers to CLAIRAUT's result. In any case, CAMILLE JORDAN showed that the identity

$$\frac{\partial f}{\partial y} - \frac{d\left(\dfrac{\partial f}{\partial y'}\right)}{dx} + \frac{d^2\left(\dfrac{\partial f}{\partial y''}\right)}{dx^2} - \dots + (-1)^n \frac{d^n\left(\dfrac{\partial f}{\partial y^{(n)}}\right)}{dx^n} \equiv 0 \tag{11}$$

is a sufficient condition for the existence of a primitive for $f(x, y, y', \dots, y^{(n)})$ using a generalization by induction of CLAIRAUT's idea. If f satisfies (11) identically then it follows that the coefficient of $y^{(2n)}$ in (11), $\partial^2 f/\partial y^{(n)}\,\partial y^{(n)}$, is zero. Hence f is of the form $\Psi(x, y, y', \dots, y^{(n-1)}) + y^{(n)}\Phi(x, y, y', \dots, y^{(n-1)})$. Define the function U as follows:

$$U = \int_0^{y^{(n-1)}} \Phi(x, y, y', \dots, y^{(n-1)})\, dy^{(n-1)}. \tag{12}$$

Then $\partial U/\partial y^{(n-1)} = \Phi$ and so we have

$$\frac{dU}{dx} = \frac{\partial U}{\partial x} + \left(\frac{\partial U}{\partial y}\right) y' + \dots + \left(\frac{\partial U}{\partial y^{(n-2)}}\right) y^{(n-1)} + \Phi y^{(n)}. \tag{13}$$

Let $f_1 = f - (dU/dx)$. Then f_1 is a function of $x, y, y', \dots, y^{(n-1)}$. Because dU/dx has the primitive U it must satisfy an equation of the form (11). (We are assuming necessity here; as I mentioned earlier, this may be shown using the method of either EULER or CONDORCET described in II(b).) In addition, f by assumption satisfies (11). Hence f_1 satisfies (11). Since f_1 is a function of $x, y, y', \dots, y^{(n-1)}$ we conclude by the induction hypothesis that f_1 has a primitive. (For $n = 1$, f_1 is a function of x alone and the result follows from the theory of one-variable differential equations.) We therefore obtain the final result, the existence of a primitive for f itself.

Conclusion

In studying the work of LAGRANGE we have before us a career spanning six decades. We are able to see changes and developments in LAGRANGE's approach to analysis arising from considerations connected to the mathematics itself as well as to the wider circumstances of his professional life. LAGRANGE's contributions to the calculus of variations, although only a small part of his overall work in the exact sciences, provide such a case study. In the first stage his motivation stemmed from a desire to construct a general theory applicable with ease to a range of particular examples. Questions of rigor are treated only incidentally in the course of obtaining definite results. In the second stage, by contrast, the demands of pedagogy and an emerging concern for rigor during the period prompted a self-conscious exploration of the foundations of the variational calculus. At each stage LAGRANGE'S strongly formal and algebraic style is manifested in its own distinctive way.

LAGRANGE'S passage from the first to the second stage might be characterized more broadly as a transition from a context of discovery to a context of justification. This transition is reflected in the intended audience for his work in each case: in the first, a very small group of geometers at the forefront of research in mathematics; in the second, a much larger class of students eager to learn from an eminent practitioner. It would be interesting to determine whether one can as a general principle in the history of mathematics distinguish two approaches to rigor, arising respectively from research and teaching. The account of LAGRANGE'S work presented in this article may be regarded as a study in support of such an investigation.

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