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A Continuous-Mode Model of the Mode Locking in a Pump-Modulated Laser

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Abstract. The active mode-locking process of the multimode laser with an external pump modulation is theoretically investigated in the frequency domain within the framework of the continuous-mode approximation. Intermode interaction and mode-coupling effects, including both AM and FM modulations, are naturally considered in a hierarchical equation of the mode components derived from the multimode Maxwell-Bloch equations. It is reduced to a continuous-mode equation that can be solved analytically in a stationary case, and used to discuss the spectral line shape and the phase dynamics of mode-components as a function of modulation amplitude and detuning of the modulation frequency. We predict a novel oscillation existing below the threshold of the ordinary complete mode-locking: The intensity of the total electric field yields a stable pulse train but its phase varies irregularly in time. This *semi-locked* state is characterized by a *nonlinear chirping*, an asymmetric spectrum, and drifting phases of the field mode-components.

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Active modulation of lasers with a high-Q resonator has attracted a great deal of interest in terms of application to optical communication and mode-locking. Many previous studies have been devoted to the single-mode modulated lasers, e.g., semiconductor lasers with an injection current modulation, and have clarified their modulation characteristics [1]. On the other hand, a multimode laser with a modulated parameter has also been investigated from the practical viewpoint of mode-locking [2] to generate ultrashort optical pulses. In addition, mode-locking and phase-locking phenomena in a multimode laser are of interest from a fundamental point of view. The modulated multimode system is a good example of a nonautonomous dynamical system in terms of the non-equilibrium phase transition and the collective phenomena in dissipative systems. The dynamical response of a low-Q cavity laser to pump modulation has also become an interesting subject of study in terms of bifurcation sequences and optical chaos [3, 4].

In our previous papers [5, 6], we have carried out detailed numerical studies of the pump-modulated multimode laser in both the high-Q and the low-Q [7] cavity cases to clarify its dynamical response to the external modulation. There it was pointed out that the route to the complete mode-locking as a function of modulation amplitude was unexpectedly complicated. The most striking feature is the fact that a "semi-locked" oscillation exists as an intermediate state to the ordinary locking. This semi-locked state exhibits a nonlinear chirped pulse train. Reminiscent of critical phenomena near a nonequilibrium phase-transition (complete mode-locking point), the phase loses its stability and the system displays the semi-locked and the intermittent phenomena [5, 6].

An aim of this paper is to theoretically demonstrate the existence of this "semi-locked" state of oscillation and to clarify its characteristics and parameter-dependences. We are thus concerned with the active mode-locking process in a pump-modulated multimode laser. Our main interest lies in the *spectral structure* of the electric field in the active mode-locking process as a function of the modulation parameters, e.g., the modulation amplitude and the detuning of the modulation frequency. Therefore, we employ a model in the *frequency domain* where a sinusoidal modulation can be taken rigorously into account, while time domain theory[1] is suited to the synchronously pumped mode-locking [2, 10]. Mode-

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[1] This has usually been employed also in the study of passive mode-locking with saturable absorbers [8, 9]

locking with both AM and FM modulations is analyzed theoretically in the frequency domain, particularly for the case where mode-locking is induced by the external pump modulation, analyzed here for the first time. A coupled-mode equation is derived from the multimode Maxwell-Bloch equations with the adiabatic elimination procedures. It is transformed into a continuous-mode equation which is analytically solved in the stationary regime to give a "spectral function" and to clarify the phase dynamics of each mode-component as a function of the modulation parameters. Attention is drawn to the existence of an extra solution describing a novel oscillation just below the ordinary mode-locking threshold, in addition to the ordinary solution corresponding to the completely mode-locked oscillation. This extra state is characterized by an asymmetric spectrum and drifting phases of mode components, in contrast to the completely locked state with a symmetric Gaussian spectrum and fixed stable phases of modes. As a result, the total electric field yields a pulse train whose phase varies in time, being equivalent to the nonlinear phase-chirped pulse train. This "semi-locked" oscillation, which is a kind of critical phenomenon near the nonequilibrium phase transition, is extensively and theoretically investigated to clarify differences from the usual mode-locked state in the frequency domain. Analytical solutions of the model permit more straightforward and broader comparison with experiments.

Section 1 is devoted to the derivation of a relevant equation which describes the mode-locking in a pump-modulated laser with the use of the continuous-mode approximation. This is solved analytically in Sect. 2 allowing a discussion of the asymmetry of the spectrum and the time dependence of the phase. The novel semi-locked state is introduced here. In Sect. 3, we examine in a heuristic manner the modulation detuning effect on locking. The total field dynamics in the semi-locked oscillation is investigated in Sect. 4 to stress the nonlinear chirping of the stable pulses. The applicability of our model and some comments on the experiments are also discussed.

1. Continuous-Mode Equation

We start with coupled differential equations (multimode Maxwell-Bloch equations) describing the multimode interaction of electric field with a homogeneously broadened active material. The electric field, the atomic polarization, and the population difference are expanded into their cavity longitudinal mode or the corresponding Fourier components, $e_n(t)e^{-in\omega_m t}$, $p_n(t)e^{-in\omega_m t}$, and $d_n(t)e^{-in\omega_m t}$, respectively for $n=0, \pm1, \pm2, \ldots$. This procedure is found in [5, 6] and is justified by employing the "dressed-mode" transform [6, 11]. The normalized form of the equations in the uniform-field approximation [6] is given as

$$\frac{d}{dt} e_n(t) = -Ke_n - in\Delta_m e_n + iKp_n, \tag{1a}$$

$$\frac{d}{dt} p_n(t) = -\gamma_\perp p_n + in\omega_m p_n - i\gamma_\perp \sum_k e_k d_{n-k}, \tag{1b}$$

$$\frac{d}{dt} d_n(t) = -\gamma_\parallel d_n + in\omega_m d_n + \gamma_\parallel (A+1)\delta_{n,0} + \gamma_\parallel \frac{B}{2} \Lambda \delta_{n,1} + i\gamma_\parallel \frac{A}{2} \sum_k [e_k p^*_{k-n} - e^*_k p_{k+n}], \tag{1c}$$

$$d_{-n}(t) = d^*_n(t), \tag{1d}$$

where K, $\gamma_\perp$, and $\gamma_\parallel$ are the effective field decay rate (including transmission losses through the cavity mirrors), the transverse relaxation and the longitudinal relaxation constants, respectively. Pumping is deterministically and sinusoidally modulated with frequency ω_m and a modulation amplitude B around the dc bias $A+1$. The lasing threshold is at $A=0$. Each variable e_n, p_n, and d_n is normalized by its stationary value in the case of no modulation ($B=0$). Detuning of the modulation frequency ω_m from the cavity mode spacing $\Delta\Omega$ is denoted as $\Delta_m \equiv \Delta\Omega - \omega_m$ and its absolute value is assumed to be small. In addition, we further assume for simplicity that the central mode ($n=0$) is just resonant with the atomic transition frequency. The modulation efficiency coefficient Λ is defined in [6] as

$$\Lambda = \frac{1 - R^2 \exp(i\omega_m \Delta t)}{2|\ln R| + i(2\pi - \omega_m \Delta t)}, \tag{2}$$

where R is the mirror reflectivity and Δt is the delay time due to cavity round trip. This coefficient is almost real and unity, $\mathrm{Re}\{\Lambda\} \cong 1$ and $\mathrm{Im}\{\Lambda\} \cong 0$, in the case of (i) the high mirror reflectivity ($R \sim 1$), (ii) short gain medium of length l) in comparison with the cavity length L ($L \gg l$), and (iii) $|\Delta_m|$ is small. A detailed derivation of (1) without using the uniform-field approximation was given in [5, 6] where the "dressed-mode" transform was also introduced.

We confine ourselves to the good-cavity case ($K \ll \gamma_\perp$, $\gamma_\parallel$, $R \sim 1$) where the uniform-field approximation is justified. The polarization p_n can be adiabatically eliminated from the equations. Only nearest-neighbor couplings in the field-mode equation (1a) are employed to proceed with the analytical discussion. This is valid in our *active* and *sinusoidal* modulation scheme because the side-band due to an external modulation is so strong that the higher-order population pulsation terms, d_n ($n = \pm2$, $\pm3, \ldots$), can be neglected. In the passive locking [2, 8, 9] or in the synchronously pumped locking with sharp pump pulses [10], on the other hand, we can no longer employ this truncation and a great many modes should be taken into account. Therefore, the simpler time-domain model is often used in those cases. As a result, a relevant coupled-mode equation [12] is derived in a hierarchical form as

$$\frac{d}{dt} e_n(t) = -Ke_n - in\Delta_m e_n + K\mathcal{D}\left(-n\frac{\omega_m}{\gamma_\perp}\right)[d_0 e_n + d_1 e_{n-1} + d^*_1 e_{n+1}], \tag{3}$$

with $\mathscr{D}(x)\equiv(1+ix)^{-1}$ and $n=0, \pm1, \pm2, \ldots$. Here the main populations, $d_0(t)$ and $d_1(t)$, obey

$$\frac{d}{dt}d_0(t)=-\gamma_{\|}d_0+\gamma_{\|}(A+1) -\gamma_{\|}\frac{A}{2}\sum_\mu\sum_\nu \mathscr{D}\left(\mu\frac{\omega_m}{\gamma_\perp}\right)e_\mu e_\nu^* d_{\mu-\nu}^* -\gamma_{\|}\frac{A}{2}\sum_\mu\sum_\nu \mathscr{D}\left(-\mu\frac{\omega_m}{\gamma_\perp}\right)e_\mu^* e_\nu d_{\mu-\nu}, \tag{4a}$$

$$\frac{d}{dt}d_1(t)=-\gamma_{\|}d_1+i\omega_m d_1+\gamma_{\|}A\frac{B}{2} -\gamma_{\|}\frac{A}{2}\sum_\mu\sum_\nu \mathscr{D}\left((\mu-1)\frac{\omega_m}{\gamma_\perp}\right)e_\mu e_\nu^* d_{\mu-\nu-1}^* -\gamma_{\|}\frac{A}{2}\sum_\mu\sum_\nu \mathscr{D}\left(-(\mu+1)\frac{\omega_m}{\gamma_\perp}\right)e_\mu^* e_\nu d_{\mu-\nu+1}. \tag{4b}$$

Summations are made with $\mu, \nu=0, \pm1, \pm2, \ldots$. With the aid of the adiabatic approximation for d_n (which is also justified in the case of $K\ll\gamma_{\|}$), we evaluate and find that d_0 and d_1 are proportional to the dc pump bias $A+1$ and the modulation amplitude B, respectively:

$$d_0(t)\cong(A+1)\left[1+A\sum_n \mathscr{L}\left(n\frac{\omega_m}{\gamma_\perp}\right)|e_n(t)|^2\right]^{-1}, \tag{5a}$$

$$d_1(t)\cong\frac{A}{2}B\left[1-i\frac{\omega_m}{\gamma_{\|}} +A\sum_n\frac{\gamma_\perp^2-i\gamma_\perp\omega_m}{\gamma_\perp^2+(n^2-1)\omega_m^2-2i\gamma_\perp\omega_m}|e_n(t)|^2\right]^{-1}, \tag{5b}$$

where $\mathscr{L}(x)\equiv(1+x^2)^{-1}$ is a Lorentzian function. Gain saturation and self-phase modulation effects can be taken into account through the denominators of (5a) and (5b). Both d_0 and d_1 depend only on the summation of $|e_n|^2$ with respect to n not on each e_n. Therefore, both are independent of the mode index n. Moreover, they are assumed to be constants because we pay special attention to the stationary case. A remarkable feature is the fact that both AM and FM modulations are induced under the pump modulation. That is, AM and FM terms of the right-hand-side of (3) are explicitly written down respectively as

$$K\mathscr{L}\left(n\frac{\omega_m}{\gamma_\perp}\right)\left[(e_{n+1}+e_{n-1})\operatorname{Re}\{d_1\} +n\frac{\omega_m}{\gamma_\perp}(e_{n+1}-e_{n-1})\operatorname{Im}\{d_1\}\right], \quad \text{for AM}, \tag{6a}$$

$$iK\mathscr{L}\left(n\frac{\omega_m}{\gamma_\perp}\right)\left[n\frac{\omega_m}{\gamma_\perp}(e_{n+1}+e_{n-1})\operatorname{Re}\{d_1\} -(e_{n+1}-e_{n-1})\operatorname{Im}\{d_1\}\right], \quad \text{for FM}. \tag{6b}$$

Therefore, previous analyses [13, 14] confined only to either AM or FM mode-locking are no longer applicable to our problem.

The complex mode field $e_n(t)$ is now divided into its amplitude $\varepsilon_n(t)$ and phase $\theta_n(t)$, i.e., $e_n\equiv\varepsilon_n\exp(i\theta_n)$. According to our numerical results [5, 6] of (1), we confine our discussion to the following situation: (A) $d\varepsilon_n(t)/dt=0$ for all n, and (B) $\Delta_n(t)\equiv\theta_n(t)-\theta_{n-1}(t)=\text{const}\equiv\Delta$ for all n. The former implies stationarity of the mode intensities of the electric field aside from their sinusoidal motion due to $e^{-in\omega_m t}$. The latter is a kind of phase locking of the relative phase difference, but each phase $\theta_n(t)$ can depend on time. The phase of the nth mode is then represented only by the zeroth phase, i.e.,

$$\theta_n(t)=n\Delta+\theta_0(t). \tag{7}$$

This situation described above is general and convenient for discussing the mode-locking process because it includes both AM and FM lockings as special cases.

Now let us anticipate that the pump modulation couples many modes together. We introduce the continuous-mode approximation [13], $\varepsilon(\omega,t)\equiv\varepsilon_n(t)$, with a continuous variable $\omega\equiv n\omega_m/\gamma_\perp$. The longitudinal mode spacing (nearly equal to the modulation frequency) should be much smaller than the power-broadened gain width. Therefore, $\omega_m/\gamma_\perp$ is smaller than unity. Hereafter $\varepsilon(\omega,t)$ is called the "spectral function" which represents the field amplitude at frequency ω. This procedure is valid when a great many modes oscillate simultaneously and there is little difference between $\varepsilon_n(t)$ and $\varepsilon_{n\pm1}(t)$. In this case, the spectral function $\varepsilon(\omega,t)$ satisfies a diffusion-type equation:

$$\frac{\partial\varepsilon(\omega,t)}{\partial t}=C_2(\omega)\frac{\partial^2\varepsilon}{\partial\omega^2}+C_1(\omega)\frac{\partial\varepsilon}{\partial\omega}+C_0(\omega)\varepsilon, \tag{8}$$

where coefficients C_i are given by

$$C_0=2\mathscr{L}(\omega)\operatorname{Re}\{d_1e^{-i\Delta}\}+\mathscr{L}(\omega)d_0-1, \tag{9a}$$

$$C_1=2\left(\frac{\omega_m}{\gamma_\perp}\right)\omega\mathscr{L}(\omega)\operatorname{Im}\{d_1e^{-i\Delta}\}, \tag{9b}$$

$$C_2=\left(\frac{\omega_m}{\gamma_\perp}\right)^2\mathscr{L}(\omega)\operatorname{Re}\{d_1e^{-i\Delta}\}. \tag{9c}$$

Here we consider the stationary case, $\partial/\partial t=0$; the spectral function $\varepsilon(\omega)$ obeys a homogeneous equation,

$$\frac{d^2\varepsilon(\omega)}{d\omega^2}+\varrho_1\omega\frac{d\varepsilon(\omega)}{d\omega}+\varrho_2(\varrho_3-\omega^2)\varepsilon(\omega)=0, \tag{10}$$

where real coefficients ϱ_1, ϱ_2, and ϱ_3 are defined as

$$\varrho_1=2\frac{\gamma_\perp}{\omega_m}\frac{\operatorname{Im}\{d_1e^{-i\Delta}\}}{\operatorname{Re}\{d_1e^{-i\Delta}\}}, \tag{11a}$$

$$\varrho_2=\left(\frac{\gamma_\perp}{\omega_m}\right)^2\frac{1}{\operatorname{Re}\{d_1e^{-i\Delta}\}}, \tag{11b}$$

$$\varrho_3=d_0-1+2\operatorname{Re}\{d_1e^{-i\Delta}\}. \tag{11c}$$

An important fact here is that coefficients ϱ_1, ϱ_2, and ϱ_3 are all independent of the frequency ω but depend on Δ and pulse energy.

2. The Spectral Function and Phase Dynamics

Assuming the coefficients of (11) to be constants for a heuristic discussion, (10) is solved formally to yield an analytic solution,

$$\varepsilon(\omega)=\frac{1}{D_\lambda(0)}\exp\left(-\frac{\varrho_1}{4}\omega^2\right)D_\lambda(\alpha\omega), \tag{12}$$

where

$$\alpha=(\varrho_1^2+4\varrho_2)^{1/4}, \tag{13a}$$

$$D_\lambda(x)=2^{\lambda/2+1/4}x^{-1/2}W_{\lambda/2+1/4,\,-1/4}(x^2/2), \tag{13b}$$

and W(z) is the Whittaker function. Equation (13b) is Weber's hyperbolic cylinder function and its index is determined as

$$\lambda\equiv\frac{1}{\alpha^2}\left(\varrho_2\varrho_3-\frac{1}{2}\varrho_1\right)-\frac{1}{2}. \tag{14}$$

This index plays a key role in distinguishing between the completely locked and the "semi-locked" states. Here we note that in the case of $\lambda=n\equiv0,1,2,\ldots$,

$$\begin{aligned}D_n(x)&=\exp\left(-\frac{x^2}{4}\right)H_n(x)\\&=(-1)^n\exp\left(\frac{x^2}{4}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2}\right),\end{aligned} \tag{15}$$

where $H_n(x)$ is the Hermite polynomial. The index λ is a monotonically increasing function of the modulation amplitude B, and λ goes to $-\infty$ under weak or no modulation ($B\sim0$) where the continuous mode model may become invalid.

The phase of each mode, $\theta_n(t)$, is, on the other hand, fully determined only by $\theta_0(t)$ which obeys the equation derived from (3):

$$\begin{aligned}\frac{d\theta_0(t)}{dt}&=-2K\,\mathrm{Im}\{d_1e^{-i\Delta}\}\frac{\omega_m}{\gamma_\perp}\left[\frac{1}{\varepsilon(\omega)}\frac{d\varepsilon(\omega)}{d\omega}\right]_{\omega=0}\\&=\text{constant in time}.\end{aligned} \tag{16}$$

This shows that the phases of all modes drift linearly in time. The time derivative of the phase $\dot{\theta}_0$ gives a uniform frequency shift of each mode. We should notice that these drifting phases of mode components do not correspond to the usual frequency shift of the phase of the *total* field due to linear mode-pulling and -pushing mechanisms. In fact, a nonlinear chirping of the total field is induced, resulting from the asymmetric spectral function, as discussed later. In the case of the continuous mode regime, the imaginary part of d_1 almost vanishes while d_0 and the real part of d_1 are given by

$$d_0=(A+1)\left[1+A\frac{\gamma_\perp}{\omega_m}\int_{-\infty}^{\infty}\frac{\varepsilon^2(\omega)}{1+\omega^2}d\omega\right]^{-1}, \tag{17a}$$

$$\mathrm{Re}\{d_1\}\cong\frac{A}{2}B\left[1+A\frac{\gamma_\perp}{\omega_m}\int_{-\infty}^{\infty}\frac{\varepsilon^2(\omega)}{1+\omega^2}d\omega\right]^{-1}, \tag{17b}$$

$$\mathrm{Im}\{d_1\}\cong0. \tag{17c}$$

Note here that the relative phase difference must satisfy $|\Delta|<\pi/2$ in order to yield a physical solution of (10) that satisfies the natural boundary condition, $\lim_{\omega\to\pm\infty}\varepsilon(\omega)=0$.

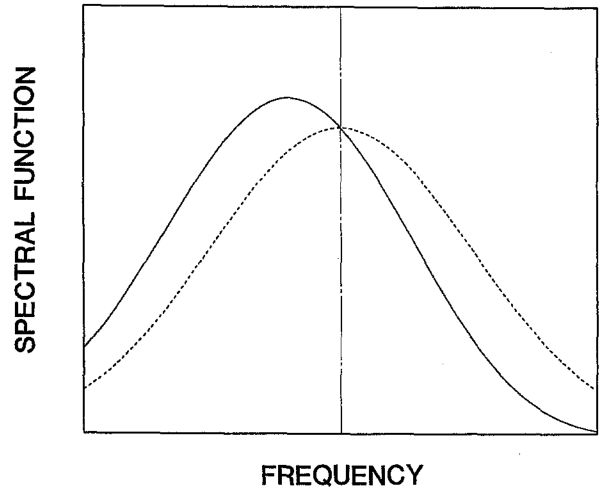

Fig. 1. Schematic plots of the spectral function, $\varepsilon(\omega)$, of the ordinary mode-locked state [dotted curve, corresponding to (18)] and the semi-locked state [full curve, corresponding to the $\lambda<0$ case of (12)]. The dash-dotted line indicates the gain center, $\omega=0$

2.1. Completely Mode-Locked Oscillation

Ordinary mode-locking occurs when the modulation amplitude B ($\propto\mathrm{Re}\{d_1\}$) is sufficiently large. In this case, the nonlinear intermode interaction plays a collective role to emit the maximum energy, which is realized in the $\Delta\sim0$ case.[2] As a result, the index λ is locked to zero and the spectral function $\varepsilon(\omega)$ becomes Gaussian *symmetric* with respect to the center of gain profile $\omega=0$, i.e.,

$$\varepsilon(\omega)=\exp\left(-\frac{\varrho_1+\alpha^2}{4}\omega^2\right). \tag{18}$$

Then (16) becomes zero, i.e., $d\theta_0(t)/dt\equiv0$. As a result of (7), the phases of all modes become constant in time resulting in no frequency shift of the modes, $d\theta_n(t)/dt\equiv0$. This is the ordinary complete mode-locking similar to the AM locking [13].

2.2. Incompletely Mode-Locked (Semi-Locked) Oscillation

In contrast to the above, when $\mathrm{Re}\{d_1\}$ is slightly smaller than the locking threshold (which will shortly be evaluated), there is another type of solution corresponding to the novel state of oscillation. When the intermode coupling is not strong enough for locking, the phase difference Δ becomes finite ($\Delta\neq0$). As a result, the index λ no longer vanishes and becomes negative; it is not locked to integer $n=0,1,2,\ldots$. The spectral function, $\varepsilon(\omega)$, then becomes *asymmetric*. This is a further solution of (10) in addition to the ordinary one (18). This asymmetry does not result from the linear mode-pulling and pushing in the usual mode-locking theory but is an essentially new result. Figure 1 illustrates the asymmetric spectral function in this state (solid curve) and the symmetric Gaussian

[2] The strict condition $\Delta=0$ is unnecessary for complete locking [6]

spectrum in the ordinary state (dotted curve). According to (16), therefore, the frequency shift $d\theta_0(t)/dt$ becomes nonzero, that is,

$$\frac{d\theta_0(t)}{dt} = -2K \frac{\omega_m}{\gamma_\perp} \operatorname{Im}\{d_1 e^{-i\Delta}\} \frac{\alpha\lambda D_{\lambda-1}(0)}{D_\lambda(0)} = \textit{nonzero} \text{ constant in time}, \quad (19)$$

and the phases of the modes, $\theta_n(t)$, drift linearly in time, keeping the relative phase difference constant. The peak (maximum) of the spectral function $\varepsilon(\omega)$ shifts to the low-energy side of the gain center and its shape is asymmetric with respect to its peak. This is a semi-locked oscillation. This results in the nonlinear chirped pulse train [15] of the total electric field.

3. Detuning Effect on Mode-Locking

Modulation frequency detuning is equivalent to the cavity mismatch which has been studied by many authors in synchronously mode-locked dye lasers [16–20]. Here we seek in a simple manner a necessary condition on Δ_m for the mode-locking: It must satisfy the following relation which is straightforwardly derived from (3):

$$\Delta_m = K \frac{\omega_m}{\gamma_\perp} \mathscr{L}(\omega) d_0 + K\left(\frac{\omega_m}{\gamma_\perp}\right)^3 \mathscr{L}(\omega) \operatorname{Re}\{d_1 e^{-i\Delta}\} \left(\frac{1}{\varepsilon}\frac{d^2\varepsilon}{d\omega^2}\right)_{|\omega| \ll 1} - 2K\omega \operatorname{Im}\{d_1 e^{-i\Delta}\} \left[\mathscr{L}(\omega)\left(\frac{1}{\varepsilon}\frac{d\varepsilon}{d\omega}\right)_{|\omega|\ll 1} - \left(\frac{1}{\varepsilon}\frac{d\varepsilon}{d\omega}\right)_{\omega=0}\right]. \quad (20)$$

This relation is valid in the region of small $|\omega|$. Considering the effective gain region near the atomic frequency (gain center), $|\omega| \ll 1$, the last term of (20) is negligible, and we obtain an inequality for Δ_m for $\operatorname{Re}\{d_1 e^{-i\Delta}\} > 0$ using the convexity of $\varepsilon(\omega)$ at $\omega \sim 0$ for the mode-locked state as

$$\Delta_m \geqq K \frac{\omega_m}{\gamma_\perp}\left[d_0 - \operatorname{Re}\{d_1 e^{-i\Delta}\}\left(\frac{\omega_m}{\gamma_\perp}\right)^2 \left|\left(\frac{1}{\varepsilon}\frac{d^2\varepsilon}{d\omega^2}\right)_{\omega\sim 0}\right|\right]. \quad (21)$$

This is a necessary condition for the mode-locking. This derivation is fully qualitative but heuristic in discussing the asymmetry of the locking condition on Δ_m. Convexity of the spectral function is satisfied within a region of $|\omega| < [2/(\varrho_1 + \alpha^2)]^{1/2}$. The locking threshold is lower in the $\Delta_m > 0$ region than in the region $\Delta_m < 0$. This asymmetric character of the modulation frequency detuning coincides with the numerical calculations [5].

Using (21), we can evaluate the threshold for ordinary mode-locking:

$$(\operatorname{Re}\{d_1\})_{\text{th}} \equiv K \frac{\gamma_\perp}{\omega_m} d_0 \left|\cos\Delta\left(\frac{1}{\varepsilon}\frac{d^2\varepsilon}{d\omega^2}\right)_{\omega\sim 0}\right|^{-1} \cong 2K \frac{\gamma_\perp}{\omega_m} \frac{d_0}{(\varrho_1 + \alpha^2)\cos\Delta}. \quad (22)$$

That is, mode-locking occurs when $\operatorname{Re}\{d_1\} > (\operatorname{Re}\{d_1\})_{\text{th}}$. Here $\operatorname{Re}\{d_1\}$ is proportional to the modulation amplitude B [see (17b)]. Therefore, the threshold value of the modulation amplitude can be determined from this relation.

4. Discussion: Dynamics of the Total Electric Field

The total (normalized) electric field $E_{\text{total}}(t)$, which means the envelope function of the electric field of a pulse, is given by the inverse Fourier transform of the spectral function $\varepsilon(\omega)$ as

$$E_{\text{total}}(t) \propto \exp(i\dot{\theta}_0 t) \int_{-\infty}^{\infty} \exp\left(-\frac{\varrho_1}{4}\omega^2\right) D_\lambda(\alpha\omega) \times \exp(-i\omega t)\exp(i\tilde{\Delta}\omega)d\omega, \quad (23)$$

where $\tilde{\Delta} = \Delta\gamma_\perp/\omega_m$. The phase of the total field, $\Theta(t) \equiv \arg E_{\text{total}}(t)$ is

$$\Theta(t) = \arctan \frac{\operatorname{Im}\left\{e^{i\dot{\theta}_0 t} \int_{-\infty}^{\infty} \varepsilon(\omega) e^{i(\tilde{\Delta}-\omega)t} d\omega\right\}}{\operatorname{Re}\left\{e^{i\dot{\theta}_0 t} \int_{-\infty}^{\infty} \varepsilon(\omega) e^{i(\tilde{\Delta}-\omega)t} d\omega\right\}} \quad (24a)$$

$$= \Theta_0 + \Theta_1 t + \Theta_2 t^2 + \Theta_3 t^3 + \ldots. \quad (24b)$$

In the complete mode-locking [$\lambda = 0$, $\dot{\theta}_0 = 0$, $\Delta \cong 0$, and $\varepsilon(\omega) = \varepsilon(-\omega)$], the phase of the total field becomes constant by cancelling out the phase $e^{-i\omega t}$ due to the symmetry of $\varepsilon(\omega)$ to show a coherent pulse train with a constant phase (co chirping), i.e., $\Theta_n = 0$ for $n \geqq 1$. The asymmetric spectrum $\varepsilon(\omega) \neq \varepsilon(-\omega)$ in the novel oscillation state ($\lambda < 0$), on the other hand, does not cancel the mode phases, resulting in a stable optical pulse train with time-varying phase (no chirping), i.e., $\Theta_n = 0$ for $n \geqq 1$. The asymmetric This is a semi-locked oscillation. Qualitatively speaking, in our $\lambda < 0$ case, the down-chirping is induced in the front part of the pulse and the lagging part is up-chirped [6]. Although the mode amplitude is constant in time (but asymmetric), i.e., $d\varepsilon_n/dt = 0$, the mode phases are not constant but their relative phase difference is identical and constant in time. This state may generally be observable in pump-modulated lasers by controlling the modulation strength in a careful manner. Note here that the "asymmetry" and the "peak shift" of the spectrum should be clearly distinguished. In fact, nonlinear frequency chirping results from the asymmetry of the mode spectrum not from the shift due to the usual mode-pulling and pushing. A general relation between the asymmetric spectral function (12) and the up-/down-chirping (linear chirping) of the total electric field (i.e., $\Theta_2 > 0$ or < 0) will be discussed elsewhere.

Attention should be paid in applying our results to the synchronously pumped mode-locked laser (SPML) where pump laser pulses are very narrow and δ-function-like in time [10]. In such a case, many modes couple with one another, and a theoretical treatment may become more difficult because the nearest-neighbor coupling [as used in (3)] is insufficient. A sinusoidal modulation scheme is suitable for the theoretical analysis. However, the qualitative nature of SPML may be similar to that of the sinusoidally modulated system studied in this paper. The novel semi-locked oscillation, the asymmetric spectrum, and the frequency detuning effect are all universal features of the pump modulated multimode laser.

An experimental study of this semi-locked oscillation is now in progress and will be reported in the near future. In a real experimental system, other nonlinear-dispersive dynamical effects, e.g., self-phase modulation (SPM), are often important because they may prevent or mask the novel phase dynamics proposed in this paper [21]. However, the SPM effect become dominant only for very high light intensity (e.g., intense light pulses with a high peak value) operated far above the threshold to ordinary mode-locking, whereas the semi-locked state exists just below the locking threshold. Actually some experimental results suggesting the novel semi-locked dynamics have already been obtained. We hope that this novel operation can be realized and detected in a careful experiment. When comparing with experimental results, one should nevertheless pay careful attention to the effects of SPM in the dye, the inhomogeneous broadening of the gain, and intracavity dispersive media on the (semi) mode-locked spectrum.

In this paper, we have not discussed a supermode solution [13, 22, 23], which corresponds to the solution (12) with positive integers $\lambda = n = 1, 2, 3, \ldots$. This is also characteristic feature of the multimode system but its applicability and origins are not yet clear.

In the next step of this work, we will employ the self-consistent scheme in order to solve exactly the continuous-mode equation (10) including the SPM and the nonlinear gain saturation effects. Considering the coefficients ϱ_i as a function of the pulse energy, the problem becomes strongly nonlinear. The stability of these states [24] and a cooperative *dynamical* mechanism of intermode interaction [25] are beyond the scope of this paper and remain to be discussed. This paper presents a *heuristic* and *qualitative* study and makes predictions about the pump-modulated laser dynamics. The main results of this paper can be summarized as follows.

(i) We present for the first time, for the case of pump modulation, a frequency domain model of the active mode-locking in the multimode laser which includes both AM and FM interactions between the mode-components.

(ii) Applying the continuous-mode approximation to the model, the spectral function and the phase dynamics of the mode-components are analytically examined. A novel state of oscillation (called a semi-locked oscillation) is found to exist below the threshold of the modulation amplitude for ordinary mode-locking. This state is characterized by an asymmetric spectrum and by drifting phases of the mode-components.

(iii) Although the total light *intensity* yields a stable pulse train, its *phase* shows nonlinear chirping in contrast to the chirpless pulse train of the completely locked oscillation. This novel semi-locked oscillation is a kind of critical phenomenon near the phase transition point. We hope that this phenomenon will provide further motivation for exploring a new operational regime of lasers.

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References

1. See for example, K.Y. Lau, A. Yariv: *Semiconductors and Semimetals*, ed. by W.T. Tsang (Academic, New York 1985) Vol. 22, Chap. 2, and references cited therein
2. See for example, G.H.C. New: Rep. Prog. Phys. **46**, 877 (1983) and references cited therein
3. T. Ogawa: Phys. Rev. A **37**, 4286 (1988)
4. T. Ogawa: Jpn. J. Appl. Phys. **27**, 2292 (1988)
5. T. Ogawa, E. Hanamura: Opt. Commun. **61**, 49 (1987)
6. T. Ogawa, E. Hanamura: Appl. Phys. B **43**, 139 (1987)
7. More detailed results for the low-Q cavity case are given in T. Ogawa: Appl. Phys. B **49**, 397 (1989)
8. H.A. Haus: J. Appl. Phys. **46**, 3049 (1975)
9. H.A. Haus: IEEE J. QE-**11**, 736 (1975)
10. D.L. MacFarlane, L.W. Casperson: J. Opt. Soc. Am. B **6**, 292, 1175 (1989)
11. V. Benza, L.A. Lugiato: Z. Phys. B **35**, 383 (1979)
12. An exact stationary solution of this recurrence equation can be obtained in the form of a continued fraction
13. H. Haken, M. Pauthier: IEEE J. QE-**4**, 454 (1968)
14. S.E. Harris, O.P. McDuff: IEEE J. QE-**1**, 245 (1965)
15. A *linear* chirped solution in a time-domain model was given by M. Morin, M. Piche, R. Tremblay: Opt. Commun. **68**, 213 (1988)
16. A. Scavannec: Opt. Commun. **17**, 14 (1976)
17. J. Herrmann, U. Motschmann: Opt. Commun. **40**, 379 (1982)
18. J.M. Catherall, G.H.C. New: IEEE J. QE-**22**, 1593 (1986)
19. F. Minami, K. Era: Opt. Commun. **56**, 46 (1985)
20. M. Yamashino, K. Yamada, T. Sato: IEEE J. QE-**18**, 95 (1982)
21. Group velocity dispersion (GVD) can be compensated by prisms inserted in the cavity. Because self-phase modulation (SPM) may be one of the principal obstacles for detecting the semi-locked oscillation, it should be given special attention in experiments. The spectrum in the SPM regime is distinguishable from that of our semi-locked operation in terms of its fringe-like structure. Its chirping characteristics are also different
22. Xiaoyi Bao: J. Opt. Soc. Am. B **6**, 1370 (1989)
23. H.A. Haus: IEEE J. QE-**11**, 323 (1975)
24. H.A. Haus, P.-T. Ho: IEEE J. QE-**15**, 1258 (1979)
25. The connection to the conventional Adler theory is shown, e.g., in H.A. Haus, H. Statz, I.W. Smith: IEEE J. QE-**21**, 78 (1985)