# *Who Betrayed Euclid?*
## *(Extract from a letter to the Editor)*

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Some time ago your *Archive* printed a paper on Greek mathematics which, in tone and style as well as in content, fell significantly below the usual standards of that journal. As it has already quite adequately (if perhaps too gently) been refuted there by V. D. WAERDEN and by FREUDENTHAL, there is no need for referring to it by name. My only purpose in this letter is to point out that we have here almost a textbook illustration of the very thesis which the author (let us call him Z) sought to discredit, *viz.*, that it is well to know mathematics before concerning oneself with its history; just as it is well to know Greek before dealing with Greek mathematics.

Z discusses a number of examples from EUCLID; I shall examine only the simplest one, which raises no side-issue; it is taken from EUCLID IX.8. As this consists of parallel statements about squares and about cubes, I may, for brevity, consider only the former. The paper quotes that proposition as follows (in HEATH's literal translation):

"If as many numbers as we please beginning from an unit be in continued proportion, the third from the unit will be a square, as will also those which successively leave out one."

As the proof would show if necessary, the latter clause means the fifth, the seventh, *etc.* "Numbers" (VII, def. 2) means integers other than the unit or "monad". "In continued proportion" (*ἑξῆς ἀνάλογον*; VII, def. 21) means that the ratio of each integer to the next remains the same throughout the sequence. A number is called "a square" (VII, def. 19) if it is equal to some number multiplied with itself.

HEATH also gives an alternative translation of the same statement, equally faithful but in shorthand:

"If $1, a, a_2, a_3, \dots$ be a geometrical progression [*i.e.*, as explained later on, if $1{:}a = a{:}a_2 = a_2{:}a_3 = \cdots$], then $a_2, a_4, a_6, \dots$ are squares".

In EUCLID's proof, the "numbers" in the proposition are denoted by Greek capitals, $A$, $B$, $\Gamma$, $\Delta$, $E$, $Z$; HEATH gives a literal translation of the proof, followed again by a transcript in shorthand, and ends up with the remark:

"The whole result is of course obvious if the geometrical progression is written, with our notation, as $1, a, a^2, a^3, \dots, a^n$".

This gives occasion to Z, after pouring totally unwarranted obloquy upon HEATH, to make this pronouncement:

"If we use modern algebraic symbolism, this ceases altogether to be a proposition and its truthfulness is an immediate and trivial application of the definition of a geometric progression".

When HEATH (imprudently, perhaps) wrote "obvious", he was not writing for laymen. He meant that the result is obvious for one who, having at least learnt school-algebra, will recognize in it the special case $q=2$ of the rule $a^{pq} = (a^p)^q$. He knew that any mathematician would make the distinction (a subtle one to the layman) between the obvious and the trivial. Mathematicians are trained to know the difference between a definition, a notation and a theorem.

Perhaps the *modern* mathematician finds it easier, in this case, to perceive the truth of the matter, because nowadays the exponential notation $x^\alpha$ is used in many situations where $x$, $\alpha$ are not numbers. For instance, the exponent $\alpha$ may be taken from a non-commutative group; some care is then needed in the choice of definitions and notations if the rule $x^{\alpha\beta} = (x^\alpha)^\beta$ is still to hold true. However that may be, one who thinks that the rules governing the use of the exponential notation are trivial must be lacking, not only in mathematical understanding, but also in historical sense. Let him read EUCLID's book IX, then ARCHIMEDES' *Sandreckoner*, then pages 132 to 166 of J. TROPFKE's excellent *Geschichte der Elementar-Mathematik*, volume II. There he will learn that the development of the exponential *notation* and the realization of its *properties* went hand in hand for almost twenty centuries before they reached perfection. If now our notation allows schoolchildren to use the properties of exponentiation without ever being conscious of them, this does them no harm; they may then imagine that this makes those properties "trivial consequences of the definition", but we know better.

To berate HEATH and others for betraying EUCLID when all they do is to use a certain amount of notation to clarify the contents of his writings does not merely indicate a lack of mathematical sense; it argues a deficiency in logic. As everyone knows, words, too, are symbols. The content of a theorem does not change greatly, whether it is expressed in words or in formulas; the choice, as we all know, is mostly a matter of taste and of style. "Euclid's *numbers*", we read in Z's article, "are *given* line-segments, no abstract symbols" (his italics). What are $A$, $B$, $\Gamma$, $\Delta$, $E$, $Z$ in the proof of IX.8, if not symbols?

As to "numbers" being "line-segments", every reader of EUCLID knows how punctilious he is in distinguishing between line-segments (εὐθεῖαι), magnitudes (μεγέθη) and numbers (ἀριθμοί). Where, in IX.8 or indeed in the whole text of books VII, VIII and IX, is there a mention of line-segments? The layman may be misled by the diagrams in the margins; but a mere glance, for instance at the proof of IX.8, will show that the diagram contributes nothing to our understanding of the text, which carries no reference to it. If the unit had been thought of as a unit of length, it would appear in the diagram, but it does not. It is open to question whether such diagrams belong to the "tradition", *i.e.* whether they go back to EUCLID; even if we assume that they do, it is clear to the mathematician's eye that they are no more than a partial visualization of a piece of abstract reasoning. HASSE and his school used diagrams to illustrate the mutual relationships between algebraic number-fields; that did not make their subject into geometry. In EUCLID's books VII, VIII and IX, there is no trace of geometry, nor even of so-called "geometrical algebra". According to our modern classifications, those books are mostly algebra pure and simple (the algebra of the ring of integers); the balance, which is far deeper and more interesting, is pure number-theory. Of course it is more practical to carry out algebraic operations as we do, with the help of our algebraic symbolism, than in words as EUCLID did; just as it is more practical to perform arithmetical operations in the decimal (or, as computers do, in the dyadic) system, rather that as ARCHIMEDES did; this does not affect the substance of the matter. Who, one may ask, has been betraying EUCLID?

One point more deserves touching upon. EUCLID is the first extant mathematical text where the concept of proof is identified with a *gapless* chain of reasoning; this, and for good reasons, is still our view of the matter. Often it compels one to include, so to say for the record, much laborious routine; those who take shortcuts do so at their peril. The trained mathematician has learnt to discern, and indeed to skip, such passages, while the would-be historian concludes (in Z's words) that the writer has had "to toil energetically", little imagining that the poor wretch was just cursing the dullness of his self-inflicted task. It is not always easy, in a given historical context, to distinguish between mere routine and creative reasoning; there can be no worthwhile history of mathematics unless this is done.

To conclude: when a discipline, intermediary in some sense between two already existing ones (say A and B) becomes newly established, this often makes room for the proliferation of parasites, equally ignorant of both A and B, who seek to thrive by intimating to practitioners of A that they do not understand B, and vice versa. We see this happening now, alas, in the history of mathematics. Let us try to stop the disease before it proves fatal.

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*(Received January 6, 1978)*