# *A Study of Maurice Fréchet: I. His Early Work on Point Set Theory and the Theory of Functionals*

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**Contents**



## 1. Introduction

There has been some disagreement about the importance of MAURICE FRÉCHET as a mathematician. My study of FRÉCHET, which I plan to present in three essays on the man and his work, is intended as a contribution to the history of functional analysis, or, to use another term, general analysis. I use the term general analysis for the branch of mathematics that is concerned with the theory of functions that map one abstract set onto another such set, both sets being in abstract spaces endowed with topological structures, so that continuity and other notions used in mathematical analysis can be brought to bear in this general theory of functions. FRÉCHET was the initiator of such a general theory, first given extended consideration in his doctoral thesis, published in 1906. From my study of FRÉCHET I hope to help make possible a well-informed judgment about the historical significance of FRÉCHET's work and his stature as a mathematician.

The most laudatory opinion of Fréchet's work known to me is that expressed by the American mathematician A. D. MICHAL. It is found on page 18 of a book by him, published posthumously in 1958 [A. D. MICHAL].* He wrote that, in his well-considered opinion, FRÉCHET's thesis had had a greater influence on the direction of mathematical thought than any other mathematical work of the twentieth century. FRÉCHET arranged to have the manuscript of MICHAL's book translated into French and published in Paris. In the preface to the book, written by FRÉCHET, are these words: "A. Michal était enthusiaste de son sujet. Il lui attribuait une importance telle qu'elle l'a amené à formuler (p. 18) un jugement excessif que je l'aurais prié d'amender si j'avais connu son texte avant sa mort."

FRÉCHET'S most distinguished advocate, I have no doubt, was JACQUES HADAMARD. In § 6 of this study I have quoted extensively from remarks about FRÉCHET written by HADAMARD for the Paris Académie des Sciences in 1934. Speaking about the boldness of FRÉCHET in creating a completely abstract point set theory, HADAMARD characterized this aspect of FRÉCHET's efforts as without precedent in all that had been published since the work of GALOIS.

I think that it is pretty generally agreed by those qualified to judge that FRÉCHET's thesis *was* an important achievement. *Exactly why* it was important, and in *what way* it was influential, have not as yet been as thoroughly analyzed and discussed from a historical point of view as is warranted. It is the purpose of this first of my planned essays on FRÉCHET to provide a careful analysis and discussion, not only of the thesis, but of other early work of FRÉCHET relating to it, as well as of FRÉCHET'S most significant early work on linear functionals. I have endeavored to identify important influences on FRÉCHET and to point out some specific evidence of the influence of FRÉCHET's early work on other mathematicians.

I have for a long time been familiar with some parts of FRÉCHET's published work. Most of my own mathematical research has been in functional analysis, a subject on which FRÉCHET's work made a significant impact. More recently, beginning in 1977, I have made an intensive study of FRÉCHET's publications, covering practically all of his work that was published before 1930. In the Spring of 1979 I spent three months in Paris, spending most of my days at the Archives of the Académie des Sciences, where the bulk of FRÉCHET's correspondence and other mathematical documents were deposited after his death on June 4, 1973, a few months before his ninety-fifth birthday. These materials had at that time not been fully sorted out, classified, and indexed. However, a young scholar from South America, LUIS CARLOS ARBOLEDA, from the faculty of the Universidad del Valle at Cali, Colombia, was engaged in putting the FRÉCHET papers in good order, while he was at the same time working toward a doctorate in Paris. He subsequently earned his degree in 1980 with a thesis entitled Contributions à l'Étude des Premières Recherches Topologiques (d'après la correspondance et les publications de Maurice Fréchet, 1904–1928).

ARBOLEDA and I spent a good deal of time talking about FRÉCHET and his work. We went together twice to see FRÉCHET's daughter, Mme. HÉLÈNE LEDERER,

* References to the Bibliography are placed in square brackets.

in a suburb of Paris, and talk with her about her father. She was most kind and generous, making available to us for examination a number of important documents belonging to her father that she had not turned over to the Archives. Among the most important of these, for me, at least, were many letters from HADAMARD "to the young FRÉCHET." I refer later to some of the letters shown us by Mme. LEDERER, and quote from them.

ARBOLEDA was very helpful to me, especially in assisting me in locating the letters and documents he thought would be of the greatest interest. Our conversations helped us both, I think, as we shared information, opinions, and conjectures. After I left Paris, ARBOLEDA and I kept in touch. He sent me a draft of his thesis in February of 1980; he incorporated corrections and information from me in the final form of the thesis.

Although my interest, in my study of FRÉCHET, is broader than that of ARBOLEDA in his thesis, there is, of course, considerable overlap in our interests where general topology is concerned. General analysis could not exist without some sort of topological structures in abstract spaces. But even in our treatment of the early work on abstract general topology, my study differs from that in ARBOLEDA'S thesis, for I study and report on some of FRÉCHET'S work in greater detail. I also study the early work of F. RIESZ, in some detail in § 8.

There is an interesting relationship between RIESZ'S work that led him to the so-called RIESZ-FISCHER theorem and FRÉCHET'S work that culminated in the representation theorem for continuous linear functionals on the function space $L^2$. I discuss this relationship in § 9. My detailed study of this, assisted by some letters from RIESZ to FRÉCHET, has enabled me to present an understanding of FRÉCHET'S work on the representation theorem that is entirely new.

The first two chapters of BOURBAKI'S Topologie Générale were published in 1940 [BOURBAKI, 1]. In the Note Historique at the end of the first chapter (on Structures Topologiques), after a discussion of precursors of the development of an axiomatic theory of point sets, one finds these words (on page 83): "Les premières tentatives pour dégager ce qu'il y a de commun aux propriétés des ensembles de points et de fonctions, furent faites par Fréchet et F. Riesz; mais le premier, partant de la notion de limite dénombrable, ne réussit pas à construire un système d'axiomes commode et féconde; ..." As the paragraph continues, credit is given to FRÉCHET for at least (au moins) recognizing the relationship between the principle of BOLZANO-WEIERSTRASS and the BOREL-LEBESGUE theorem, *à propos* of which (it is stated) FRÉCHET introduced the word "compact." Turning to F. RIESZ, BOURBAKI'S Note Historique continues: "Quant à F. Riesz, qui partait de la notion de point d'accumulation (ou plutôt, ce qui revient au même, d'ensemble "derivée"), sa théorie était encore incomplète, et resta d'ailleurs à l'état d'ébauche." I translate "ébauche" as "rough draft," or "sketch." In the following paragraph BOURBAKI opens with this sentence: "Avec Hausdorff commence la topologie générale telle qu'on l'entend aujourd'hui."

When the various *notes historiques* appended to various published works by BOURBAKI were brought together in a book [BOURBAKI, 3], the foregoing material from 1940 was reproduced exactly as originally published. When it came to the attention of FRÉCHET he was greatly concerned. There are evidences of his concern among the FRÉCHET papers in the Archives. There are also evidences of an exchange of views about this between FRÉCHET and some members of the BOURBAKI group. I am not going to go into much detail about this. To try to report on the matter accurately and adequately would require more information than I possess. In brief, it seems that FRÉCHET felt the BOURBAKI account of his role in the beginnings of abstract general topology accorded him less than was his due. His daughter told me that, although her father was in general relatively relaxed about what others thought of him and his work, he was "atteint" (shocked, hurt) by the BOURBAKI Histoire.

In an undated draft of a letter in the Archives (which I judge to have been prepared by FRÉCHET to send to a number of people as part of his reaction to the BOURBAKI Histoire), we find FRÉCHET asserting that it appears to give HAUSDORFF the main recognition for starting topology "as we know it today", but that, in fact, HAUSDORFF presented himself as FRÉCHET'S "continuateur" (follower). (On what basis FRÉCHET attributed this position to HAUSDORFF I do not know; he may have merely inferred it from the scattered references to FRÉCHET in HAUSDORFF'S book of 1914.) Moreover, writes FRÉCHET, "je ne suis pas le seul à être maltraité dans la même citation. Tandis que d'autres travaux de Fréderic Riesz sont abondamment cités ailleurs, ce qu'il a fait en théorie des espaces abstraits est mis au rancart (put on the shelf). Or, tandisque je m'étais d'abord contenté d'étudier le cas de la limite d'une suite dénombrable d'éléments, F. Riesz a été le premier, avant Hausdorff à considérer un cas plus général."

One can clearly detect here FRÉCHET'S annoyance with BOURBAKI. It is perhaps scarcely surprising that FRÉCHET felt some injury to his pride and ego in the degree of recognition accorded to HAUSDORFF by BOURBAKI when compared to what was said of FRÉCHET'S contributions. It is a fact that by the end of the 1920's HAUSDORFF had eclipsed FRÉCHET as a standard source from which to learn about abstract general topology.

Little is known about possible correspondence or personal relation between FRÉCHET and HAUSDORFF. I shall say a little more about this subject in another essay. There are various evidences that FRÉCHET struggled to maintain the interest in, and significance of, a type of topological space more general than HAUSDORFF spaces, and I think FRÉCHET probably had a sense of rivalry with HAUSDORFF. I asked FRÉCHET'S daughter if she could tell me about relations between FRÉCHET and HAUSDORFF. She thought they had never met, but could tell me nothing about her father's opinions of or feelings toward HAUSDORFF. She did say that her father was always able to keep separate his personal feeling toward individuals and his evaluation of the scientific merit of their work.

Even though the foregoing considerations relate in part to things that transpired long after the period in FRÉCHET'S career with which this essay is concerned, they are relevant to my study, because they bear on the issue of the proper place in mathematical history of FRÉCHET'S early work. There are many reasons for thinking that it is this early work of his that is of the greatest historical significance.

In the concluding section of this essay (§ 12), I have more to say about the BOURBAKI Note Historique. Also, as items of significance in my historical study I present two letters written to FRÉCHET late in his life, one from his old friend and colleague ARNAUD DENJOY and one from the distinguished Russian topologist,

PAUL ALEXANDROV, with whom FRÉCHET had an extended correspondence in the 1920's.

I wish to thank Professor JEAN DIEUDONNÉ for his courtesy and helpfulness, both in correspondence and in person when I was in Paris. I am grateful to Professor RENÉ TATON, of the École Pratique des Hautes Études, Centre de Recherches Alexandre Koyré, and to Professor PIERRE DUGAC, of the University of Paris, for helpful courtesies and for conversations about FRÉCHET and historical studies. I appreciate the helpfulness and goodwill of Dr. ARBOLEDA and Mme. LEDERER, whom I have already mentioned; I thank them. The Académie des Sciences was of invaluable assistance in granting me access to the FRÉCHET Papers at the Archives. I am grateful for that and for permission to quote from letters and other documents. I also wish to express my thanks to M. P. BERTHON, the archivist, and his assistant. Finally, I would like to pay tribute to the memory of the late Professor MICHEL LOÈVE, of the University of California, Berkeley, who helped and encouraged me in this study. FRÉCHET was the President of his examining committee for the doctorate in Paris many years ago.

## 2. Fréchet Before 1900

MAURICE FRÉCHET was born September 10, 1878, at Maligny, southeast of Paris. He was the fourth of six children in the family of JACQUES and ZOÉ FRÉCHET, who were Protestants. JACQUES was the Director of a protestant orphanage. The family moved to Paris when the father received appointment there as head of a Protestant school. Then came a disaster when the State secularized instruction and the father lost his livelihood. To meet this crisis Madame FRÉCHET started a boarding house for foreigners. In this way young MAURICE became accustomed to contacts with people who spoke other languages; he showed much interest in foreign countries—an interest he maintained all his life. He always loved to travel. Eventually the senior FRÉCHET found a place in lay teaching and the family fortunes were stabilized at a modest normal level.

When young MAURICE became a student at the Lycée Buffon in Paris a most significant influence came to bear on him, in the person of JACQUES HADAMARD, then in his twenties, who was teaching mathematics in the Lycée in the years 1890–1893. HADAMARD perceived in FRÉCHET a penchant for mathematics; he encouraged the lad's interest and aptitude by actions that went far beyond the usual teacher-student relationship.[1] He gave the boy extra lessons, set problems for him, required written responses, read them and commented back to his young student. This went on to some extent by correspondence even during summer

[1] My account of MAURICE FRÉCHET's early years and his association with JACQUES HADAMARD at the Lycée Buffon is based on an unpublished biographical sketch written about FRÉCHET in October, 1973, by his daughter, HÉLÈNE LEDERER. FRÉCHET had died on June 4 of that year. This biographical sketch was utilized by Professor S. MANDELBROJT in his preparation of the obituary notice about FRÉCHET published in the Comptes Rendus of the Académie des Sciences. See [MANDELBROJT] in the Bibliography.

vacations and after HADAMARD was appointed to a professorship in the University of Bordeaux, a position he held from 1894 to 1897.

FRÉCHET saved a group of more than twenty letters written to him by HADAMARD during a period (approximately 1890–1899) prior to his entry into the École Normale Supérieure. They are preserved in the posession of his daughter, Mme. HÉLÈNE LEDERER, who allowed me to examine them. A few are dated, but most are not (which was characteristic of HADAMARD). Some have been slightly annotated (by FRÉCHET or someone else, many years later, I think) with surmises of the year or other brief remarks. In the main the letters are part of the tutorial efforts HADAMARD was expending on FRÉCHET, but there are also some expressions of personal concern and advice. The communications of FRÉCHET to HADAMARD have not survived.

In an early letter (probably of 1890) HADAMARD asks FRÉCHET to come to see him (on Av. de Wagram in Paris) at a specified time. He writes that he is happy to know that the boy has taken a first place in physics. "Vous savez que je tiens beaucoup à cela," he wrote.

In 1893 HADAMARD had FRÉCHET writing out and sending him solutions to problems in geometry and algebra. One problem: If $p(x)$ is a polynomial with $p(a) = m$, $p(b) = n$, find the remainder when $p(x)$ is divided by $(x - a)(x - b)$. In one letter HADAMARD asks FRÉCHET to prove PASCAL'S theorem (the one about a hexagon inscribed in an ellipse, I suppose). In a later letter one can see that FRÉCHET has had difficulty with this; HADAMARD responds with corrections and hints and tells FRÉCHET he must always make certain that he has used all of the hypotheses. With one letter there is a torn sheet on which HADAMARD advises FRÉCHET to "travaillez l'allemand; une insuffisance en cette langue vous serait une grande gêne plus tard."

On a postcard of 1895 HADAMARD writes to FRÉCHET "Bravo pour le Concours Générale." In a letter of 1896 HADAMARD assigns his young friend some algebra problems from a book by LAURENT, and asks if he has studied poles and polars. In other letters of about this period we find HADAMARD correcting FRÉCHET'S mistakes on determinants and on division of one polynomial by another; also telling him some things about voltaic cells, about electrostatics, about Newtonian potentials of a system of mass particles, and about friction. In a letter on which the annotation suggests that it was written during the 1897 Easter vacation there is reference to a book on conics by SALMON. HADAMARD writes that FRÉCHET should not suppose that the teaching of mathematics in England is more rigorous than in France. He says it is the other way around—that English texts are rarely rigorous. HADAMARD says that to obtain a definitive form of a theory it is always necessary to go to a French or German book. Then he talks about EUCLID and points out some deficiencies in rigor in what FRÉCHET has sent him. In this same letter HADAMARD writes about the method of successive approximations, saying that the rule for extracting square roots is an application of the method. Proofs about this, he writes, lead into the domain of differential calculus.

On occasion HADAMARD admonished FRÉCHET firmly when he was displeased with what his pupil had written. Evidently FRÉCHET had, in one case, asserted that a certain proposition was easy to prove and then made use of the alleged result. HADAMARD told him that the proposition was, in fact, false, and that he should *never* make such an assertion unless he was actually able to give the proof. In another case HADAMARD was severe with FRÉCHET about a mistake he had made in a geometry problem. One can only conjecture as to the exact nature of FRÉCHET'S fault, which elicited the following response: "... mais vous ne devez en aucune cas faire de déduction fausse, jamais [with double underline!] sous quelque prétexte et en quelque circonstance que ce soit." (The underlining is that of HADAMARD.)

HADAMARD wrote a particularly interesting letter to FRÉCHET in anticipation of his entry into the École Normale Supérieure (which occurred in 1900). The letter is not dated. Here it is in full:

Mon cher ami,

J'ai reçu votre lettre et le problème qui l'accompagne. Je vous avouerai que je n'ai accordé à ce dernier qu'une attention assez distraite, d'abord, parce que j'étais, quand il m'est arrivé, à Rennes, et peu disposé à penser à ces sortes de choses; ensuite pour la raison suivante.

Vous avez dû, pour votre entrée à l'École, vous rompre aux ficelles de la géométrie analytique de mathématiques spéciales. Mais il ne faut pas vous imaginer que la vraie science mathématique ressemble à ce que vous venez d'étudier. Bien au contraire, et ce qu'il y a de plus important pour vous, actuellement, est d'oublier (Hadamard's underlining) le plus possible tout cela: il vous en restera toujours assez.

Vous avez donc raison de dire que cette géométrie analytique n'est pas intéressante: mais il ne faut pas en concluer, pour le moment, que vous devez vous intéresser uniquement à la géométrie pure. C'est parler de ce que vous ignorez. La Géométrie réglée de M. Koenigs appartient, en effet à la science plus relevée: mais l'ouvrage qui peut vous donner l'idée la plus juste de la science, actuellement, est le Cours autographié professé a la Sorbonne par M. Hermite. (J'en ai un exemplaire que je pourrai vous communiquer à la rentrée.)

Donc, pendant le temps qui vous sépare de votre entrée à l'École: ou bien ne faites rien—moyen qui en vaut un autre ou bien lisez le cours de M. Hermite. Mais plus de problèmes d'élémentaires ou de spéciales. Qu'il ne soit plus question de cela. Que la vue du Mont St. Michel vous élève des idées.

Votre bien devoué

J. Hadamard

The close relation between HADAMARD and FRÉCHET endured as long as HADAMARD lived. It was an association for which FRÉCHET was ever grateful. FRÉCHET admired in his teacher the traits of quickness, insight, and fine memory, while often feeling himself to be slow, heedless, forgetful, even stupid.[2] Among FRÉ-

[2] In the biographical sketch of her father by Mme. LEDERER are the following sentences: L'élève admira toujours, outre le génie de ce Maître, la rapidité, le coup d'oeil, la mémoire d'Hadamard. Lui-même, très lent, se sentait souvent déconcerté et très bête.

CHET'S effects are rough notes about his relation with HADAMARD, reaching back to FRÉCHET'S boyhood. There he speaks of HADAMARD as his "spiritual father." In the obituary notice on JACQUES HADAMARD written by FRÉCHET he mentions with appreciation the interest HADAMARD took in him during his schoolboy days. He writes that he was haunted by fear of not being able to answer HADAMARD'S questions. (See the last item by FRÉCHET in the Bibliography.)

## 3. Student Years

FRÉCHET entered the École Normale Supérieure in 1900 after doing his military service. He completed his course there in 1903, passing the agrégation des sciences mathématiques in 1903. Writing about himself in 1933, FRÉCHET said that while at the École Normale Supérieure he hesitated a long time between physics and mathematics, finally choosing the latter because, as things were then organized, one could scarcely separate physics from chemistry and he felt he had no aptitude for the latter.[3]

FRÉCHET began to publish even before his agrégation with two short notes [FRÉCHET, 1, 2].[4] Other minor publications followed in 1903 [FRÉCHET 3, 4], and there were two minor publications in 1904 [FRÉCHET 5, 6].

According to FRÉCHET'S statement on page 29 of the N.T.S. (see note 3), as he approached the beginning of his research he leaned toward higher geometry; he published a paper on the theory of surfaces [FRÉCHET, 12] in 1905. Several serious papers, although their writing was completed subsequent to that of the aforementioned item 12, were published in 1904. One can see even in these early years of his career FRÉCHET'S penchant for publishing quickly and in quantity.

Among the persons whose acquaintance FRÉCHET made while at the École Normale Supérieure were two Americans, EDWIN BIDWELL WILSON, and D. RAY-

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Hadamard pourtant semble ne pas s'être trompé en conseillant à ce lambin, ce distrait de persévérer, et toujours, il suivra l'élève qui lui en garda toute sa vie une profonde reconnaissance.

[3] See page 29 in the item N.T.S. near the end of the listing under FRÉCHET in the Bibliography. This document, important because it presents FRÉCHET'S own testimony about his career and publications, was prepared in connection with his initial candidacy for membership in the Paris Académie des Sciences.

[4] I am including in the bilbiography all of FRÉCHET'S publications through the year 1908, even though I shall be paying attention mainly to his publications on topology, "the functional calculus", and linear functionals in the period 1904–08. My enumeration of FRÉCHET'S publications is chronological. FRÉCHET himself numbered his publications according to a scheme whose rationale is not clear. Because he used this system in his publications when he referred to his own work, I have deemed it useful to indicate FRÉCHET'S own numbering of his publications in my bibliography. FRÉCHET'S numbering coincides with mine for items 1 through 6, but my item 7 is numbered 8 by FRÉCHET. The numbering by FRÉCHET is shown in parentheses, thus: (F8). Because there are pertinent references to some of FRÉCHET'S publications in years after 1908, I have listed such publications as well, retaining for them the numbers appropriate to a comprehensive listing of FRÉCHET'S publications.

MOND CURTISS, both of whom were soon to become prominent in mathematical circles in the United States. CURTISS, who was in Paris at the time, wrote to FRÉCHET on June 3, 1904.[5] It seems that FRÉCHET had translated into French a paper that the American, MAX MASON, wished to have published in France. CURTISS was asking FRÉCHET to translate an addition to the paper. There is a letter of November 6, 1904 from Dr. MASON (who was in New Haven, Conn., U.S.A.) thanking FRÉCHET and telling him that he has sent proof sheets of his article. (It was published in the Journal de Mathématiques, vol. **10**, 1904.) WILSON wrote many letters to FRÉCHET over an extended period of time, most frequently in the periods 1904–08 and 1916–20, but also occasionally in the 1930's and 1940's. It is clear from the correspondence that both CURTISS and WILSON spent some time studying in Paris. For a number of years WILSON travelled to Europe most summers and was in touch with mathematicians there. FRÉCHET'S contacts with Americans were important. They led him to publish quite a bit of his work in the United States and this helped to make his work known to American mathematicians. WILSON was an editor of the Transactions of the American Mathematical Society from 1903 to 1919. Among FRÉCHET'S early publications were three papers in the Transactions, in 1904, 1905 and 1907 [FRÉCHET, 8, 15 and 24].

The earliest letter in the Archives of the Académie de Sciences from WILSON to FRÉCHET was written from Yale on April 19, 1904. He said he had tried, but failed to get in touch with FRÉCHET before leaving Paris, and had heard that FRÉCHET had been forced out of school by illness. However, he had heard, via MAX MASON, that FRÉCHET was still "on earth, not under the grass." The next letter from WILSON, dated August 28, 1904, was written from Munich. It is clear from this letter that WILSON had transmitted a mémoire by FRÉCHET to E. H. MOORE, then the editor-in-chief of the Transactions of the American Mathematical Society. Then on October 10, from Yale, WILSON wrote again (in French) about FRÉCHET'S paper and MOORE. (The paper was evidently [FRÉCHET, 8], the first of three papers on linear operations.) WILSON wrote: "... à vrai dire, M. Moore est absolument ravi de votre jolie petite note," he said that MOORE called the paper a jewel and expressed hope that more like it would come. On January 5, 1905 WILSON wrote that he had received the second paper from FRÉCHET five weeks earlier and that he had already sent FRÉCHET the manuscript and printer's proof. WILSON had made an oral presentation of FRÉCHET'S work to a meeting of the American Mathematical Society a short while previously, also giving a brief exposé of the earlier paper and of the relation of FRÉCHET'S work to that of HADAMARD, BOURLET, and PINCHERLE. Everyone was much interested, he said. He relayed good wishes from E. H. MOORE to FRÉCHET. (There is only one letter that I know of from E. H. MOORE to FRÉCHET from this period; in it MOORE merely acknowledges receipt of a manuscript from FRÉCHET.)

In his letter of January 5 WILSON inquires of FRÉCHET whether he has found a thesis subject. This issue, of a subject for FRÉCHET'S doctoral dissertation, cannot be followed closely in the correspondence that is available in the Archives. There is a letter to FRÉCHET from PAUL LANGEVIN, dated March 30, 1904. It is a response

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[5] Unless otherwise specified, all letters referred to are in the collection of FRÉCHET'S correspondence and papers at the Archives de l'Académie des Sciences in Paris.

to a letter from FRÉCHET in which he had evidently asked LANGEVIN about possible subjects for mathematical research in "les questions électromagnétiques actuelles touchant à la théorie de Lorentz et à la question de l'inertie électromagnétique." LANGEVIN suggests a time for FRÉCHET to come to see him, and in a postscript writes: "Il s'introduit dans les problèmes dont je parlerai des fonctions relatives à des surfaces mobiles et par suite du genre des fonctions de lignes dont vous me parlez; cette concordance m'encourage à croire que vous y pouvez trouver quelqu'intérêt." In a chatty letter dated March 5, 1905, WILSON included this message: Faites la physique. M. Hadamard a raison:" There is nothing to clarify what this means. It may conceivably relate to FRÉCHET'S indecision (mentioned earlier) between physics and mathematics as the subject in which to concentrate his research. It seems plausible to suppose, either that FRÉCHET was still undecided, or that if he had decided, the decision had not yet been communicated to WILSON. Correspondence had been fairly continuous between WILSON and FRÉCHET for nearly a year. The letter of January 5, 1905 was at least the ninth letter from WILSON to FRÉCHET, beginning with the letter of April 19, 1904, and it is evident that FRÉCHET had been writing pretty regularly to WILSON.

Actually, the nucleus of FRÉCHET's thesis was taking shape in a series of five notes published in the Comptes Rendus of the Académie: items [FRÉCHET, 9] (November 2, 1904), [FRÉCHET, 16] (January 1, 1905), [FRÉCHET, 17] (February 27, 1905), [FRÉCHET, 18] (March 20, 1905), and [FRÉCHET, 22] (November 27, 1905). Also, there is the closely related longer paper [FRÉCHET, 20], published in America in 1905 (it was received by the editors in May, 1905). Because these papers contain material that appeared in the thesis, which was accepted and published in 1906, it surely seems likely that the choice of thesis topic was made in the first half of 1905.

HADAMARD'S letters to FRÉCHET shed no light on when FRÉCHET made the choice of a thesis topic or what part HADAMARD might have played in FRÉCHET'S decision. Nor is there a great deal in these letters to indicate how much interaction HADAMARD had with FRÉCHET in connection with the latter's research efforts. We know that FRÉCHET was the first of HADAMARD'S doctoral students.[6] Undoubtedly a good deal of the interaction between FRÉCHET and HADAMARD in connection with FRÉCHET'S research for the dissertation was person-to-person, for HADAMARD held a post in Paris starting in 1897. In a postcard from Étaples of September, 1904, HADAMARD told FRÉCHET that he had received his "premier envoi" while passing through Paris on his return from Heidelberg (where there had been an International Congress of Mathematicians). He said he hadn't looked at it yet, but he remarked "Si vous pouvez mettre l'idée principale au lumière, n'y manquez pas." It seems a reasonable conjecture, purely on the basis of timing, that this "premier envoi" from FRÉCHET might have been something related to his first note in the Comptes Rendus [FRÉCHET, 9], which would appear in November of that year. HADAMARD told FRÉCHET that he had seen VOLTERRA at Heidelberg and that the latter had indicated his willingness to receive material that FRÉCHET might send him. This message suggests the possibility that HADAMARD

[6] See the opening paragraph of FRÉCHET'S Notice nécrologique about HADAMARD. It is listed in the Bibliography.

may have talked to VOLTERRA about an intention that FRÉCHET would be working on a general theory of functions having some relation to VOLTERRA's "functions of lines." which he launched in 1887. Two other communications from HADAMARD, relating to FRÉCHET's work in the period 1904—05, will be discussed later in connection with my discussion of FRÉCHET's publications.

There are in the Archives in Paris four letters from HENRI LEBESGUE to FRÉCHET, three written close together in December of 1904 and January of 1905, and one in January of 1906. These letters have been published, along with notes and commentary, in an article by Professor PIERRE DUGAC and myself (see the bibliographical listing under TAYLOR & DUGAC). The letters show that FRÉCHET was studying LEBESGUE's work in 1905 and 1904 and that he benefitted from LEBESGUE's generous replies to questions put to LEBESGUE by FRÉCHET. Some of the substance of a note that FRÉCHET published in the Comptes Rendus of January 2, 1905 [FRÉCHET, 16] entered into the exchange of ideas between LEBESGUE and FRÉCHET. An important consequence of the exchange was the proof of a theorem for which credit must be divided between FRÉCHET and LEBESGUE. This is all explained in the TAYLOR-DUGAC article; I shall say something about it briefly when I discuss FRÉCHET's research.

FRÉCHET's long and close association with ÉMILE BOREL began in 1903, if not sooner. A letter from BOREL to FRÉCHET of date November 6, 1903 opens as follows: Mon cher Fréchet, Je suis très heureux que vous puissiez rédiger mon cours; j'en suis surtout très content pour ses futurs lecteurs. This refers to the role FRÉCHET was to play in connection with BOREL's course of lectures in the winter (November–March) of 1903–04, which through FRÉCHET's labors was turned into a book by BOREL, the main subject of which was the theory of approximation of real continuous functions of a real variable by polynomials (see [BOREL, 2] and [FRÉCHET, 13]). This experience with BOREL clearly was the prime influence in leading FRÉCHET to his work on best approximation to continuous periodic functions by trigonometric polynomials [FRÉCHET, 25, 27].

## 4. Fréchet's First Steps Toward a General Topology

In this section I shall discuss the work of FRÉCHET published in 1904 and 1905 that sketched out a good deal of the groundwork of his thesis (items [FRÉCHET 9, 16, 17, 18, 20, 22] in the bibliography). With the first of these items, published in November, 1904, FRÉCHET launched his work on abstract spaces. However, he did not use the term "abstract space". For many years FRÉCHET avoided this term, referring instead to abstract sets (ensembles abstraits) or abstract classes, reserving the word "space" for classes of elements (or points) determined by a finite or infinite set of coordinate numbers. In this first note [FRÉCHET, 9] he introduces "a certain category $C$ of arbitrary elements (numbers, surfaces, *etc.*) in which one knows how to distinguish distinct elements" and in which one supposes given a definition that assigns a precise meaning to the phrase "the infinite sequence $A_1, A_2, \ldots, A_n, \ldots$ of elements of $C$ has a limit $B$." The definition, otherwise arbitrary, is assumed to be such that (1) if the sequence $\{A_n\}$ has a limit, every sequence $A_{p_1}, A_{p_2}, \ldots$ formed of elements from $\{A_n\}$ with increasing indices

$p_1, p_2, \ldots$ has the same limit as $\{A_n\}$, and (2) if $\{A_n\}$ is given with $A_n = A$ for each $n$, then $\{A_n\}$ has the limit $A$. It is not explicitly stated that the limit of a sequence is assumed to be unique. However, I think we may assume that FRÉCHET has this assumption implicitly in mind, for in his thesis, where he defines a class $(L)$ as a set of elements of arbitrary nature with a notion of sequences of elements that have limits, he imposes exactly the same two assumptions but remarks that the limit of a sequence is "d'ailleurs unique" [FRÉCHET, 23, page 6, top], thus making clear that the uniqueness of limits is not a third, separate, axiom.

Before coming to his consideration of abstract classes FRÉCHET states that it is important in many problems "to know if a quantity that depends on certain elements (points, functions, etc.) effectively attains a minimum in the field (champ) considered." He cites the DIRICHLET principle as one of the most striking justifications of his remark. He then points to the WEIERSTRASS theorem "that each function continuous in a limited interval attains there at least once its maximum," and adds that there would be great interest in extending the WEIERSTRASS theorem in such a way as to be able to deal with the more general problem mentioned earlier. He is thus motivated to consider how he will define continuity for (real) functions (or, as he calls them, functional operations) defined on a class of elements of arbitrary nature. The notion of a category $C$ of elements in which there is a notion of a sequence $\{A_n\}$ having a limit $A$ (subject to the two conditions) then provides FRÉCHET with the means of defining continuous functional operations and appropriate analogues of the concepts of an interval being closed and bounded (limité). First he defines the notion of limit element of a set $E$ in the arbitrary class. An element $A$ is a limit element of $E$ if there is a sequence $\{A_n\}$ of elements of $E$ with $A_1, A_2, \ldots, A_n, \ldots$ all distinct and $\{A_n\}$ having $A$ as limit. The set $E$ is closed if every limit element of $E$ (if any exist) belongs to $E$. Finally, a set $E$ is called *compact* if, whenever $\{E_n\}$ is a sequence of nonempty, closed subsets of $E$ such that $E_{n+1}$ is a subset of $E_n$ for each $n$, there is at least one element that belongs to all of the $E_n$'s.

FRÉCHET'S generalization of the theorem of WEIERSTRASS is then the following: "If $E$ is a closed and compact set in $C$, and if $U$ is a functional operation that is defined and continuous on $E$, then the values $U(A)$ are bounded and $U$ attains an absolute maximum value at some point $A$ of $E$." Continuity means, of course, that $U(A_n) \rightarrow U(A)$ whenever $\{A_n\}$ is a sequence of elements of $E$ having the element $A$ of $E$ as a limit. No proof of the theorem is indicated.

FRÉCHET remarks that, in order to study the properties of compact sets, one can more easily do so by making use of the following result. A set $E$ is compact if and only if every infinite subset of $E$ has at least one limit element in $C$ (but not necessarily in $E$).

In his next note in the Comptes Rendus [FRÉCHET, 16] FRÉCHET makes the observation that it is useful to detach from various theories (the theory of sets of points, of functions of lines, and of functional operations) the features common to them all. In set theory it is important—just as with the notion of power (puissance)—to introduce the notion of limit and limit element without regard to the nature of the elements under consideration. Then, referring back to his previous note in the Comptes Rendus, he points out that there is a question as to whether the derived set of a set $E$ (the set of limit elements of $E$) is necessarily a closed set.

He observes at once that such is not always the case. The counterexample he cites is that in which $E$ is the set of all (real) polynomials in the real variable $x$, in the class of all (real) functions of $f$ defined on a specified interval, with the definition "$\{f_n\}$ has the limit $f$ when $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x$." He does not explain why the derived set of $E$ is not closed, but it is clear from what follows later that he has BAIRE classes in mind and knows that the first BAIRE class is not closed in the foregoing situation.

In looking at the general setting of this example of a case in which a derived set is not closed, FRÉCHET made the following observation (still in [FRÉCHET, 16]). Suppose that $\{f_n\}$ is a sequence of distinct functions with limit $f(x)$ for each $x$, and suppose further that for each $n$ there is a sequence $f_n^{(1)}, f_n^{(2)}, f_n^{(3)}, \ldots$ with $\lim_{p\to\infty} f_n^{(p)}(x) = f_n(x)$ for each $n$ and each $x$. Then it is not necessarily true that there exist two increasing sequences $p_1, p_2, p_3, \ldots$ and $n_1, n_2, n_3, \ldots$ such that $\lim_{i\to\infty} f_{n_i}^{(p_i)}(x) = f(x)$ for each $x$. However, if all of the functions in question are measurable, there do exist increasing sequences $\{p_i\}$ and $\{n_i\}$ such that $\lim_{i\to\infty} f_{n_i}^{(p_i)}(x) = f(x)$ except on a set of measure zero. From this FRÉCHET proceeds to assert (without proof) as a corollary, that each function that is in some BAIRE class is the limit, except on a set of measure zero, of a sequence of polynomials. These ideas are presented more fully and proofs are indicated at the end of Chapter I of FRÉCHET's thesis (pages 15–57 of [FRÉCHET, 23]).

We see here that FRÉCHET has wandered from his theory of abstract sets into the realm of real function theory and has become involved with LEBESGUE's theory of measure. His ideas about sequences of real functions were in the forefront in his exchange of ideas with LEBESGUE in 1904–05, which I mentioned in § 3. These ideas proved to be significant in leading him and LEBESGUE to a proof of the following theorem: Let $f$ be a real function of a real variable defined and measurable on a finite closed interval. Then $f$ is the limit pointwise, except on a set of measure zero, of a sequence of continuous functions (and so, also of a sequence of polynomials). An explication of this on the basis of three letters from LEBESGUE to FRÉCHET is given in the article by TAYLOR & DUGAC cited in the Bibliography.

An undated letter from HADAMARD to FRÉCHET (in the possession of FRÉCHET's daughter HÉLÈNE) is quite evidently related to things FRÉCHET communicated to HADAMARD about a sequence $\{f_n\}$ with limit $f$ and sequences $f_n^{(1)}, f_n^{(2)}, \ldots$ with limit $f_n$. Here is the letter:

Mon cher ami,

Je ne comprends pas très bien la position de la question dont vous me parlez. Supposez que vos $f$ soient des nombres, satisfaisant à vos conditions

$$\lim_{n=\infty} f_n = f, \qquad \lim_{p=\infty} f_{n,p} = f_n.$$

On pourra prendre $f_n$ aussi voisin qu'on voudra de $f$, puis $f_{np}$ aussi voisin qu'on voudra de $f_n$. Donc, etc.

Le raisonnement s'étendra, quelle que soit la signification des $f_{np}$, du moment que la notion de voisinage peut être définie, puisque cette notion est la seule sur laquelle il repose.

Il n'est donc pas immediatement valable dans le cas le plus général; mais pour le cas de fonctions il est applicable sans modification. L'uniformité de la convergence entrainerait que, pour toute loi suivante laquelle $n$ et $p$ augmenteraient indéfiniment, $f_{n,p}$ tendrait vers $f$.

Il n'y a donc, encore une fois, pas de question pour les fonctions, ni pour tout autre cas ou ce limite dériverait de ce voisinage.

Feriez vous bien de partir, en général, de la notion de voisinage et non de celle de limite? Ceci vous regarde, de même que la recherche de cas où, le raisonnement précédent n'étant plus valable, sa conclusion serait en défaut.

J'ai reçu votre envoi complémentaire et vous en remercie.

M. Painlevé passe ses vacances à Loctudy, pres Pont l'Abbé (Finistère). Je ne pense pas qu'il l'ait quitté encore.

Je suis heureux que vous passiez de bonnes vacances. Nous sommes aussi très bien ici. Faites comme le nègre, et croyez moi

Bien à vous

J. Hadamard

In attempting to interpret the significance of this letter we first observe that the location of PAINLEVÉ on vacation (at Loctudy) is on the coast at the southern part of the western tip of the peninsula of Brittany. This indicates that the letter was written during the summer vacation, probably toward its end, for HADAMARD conjectures that PAINLEVÉ has not yet left Loctudy. The reference to PAINLEVÉ is probably related to the fact that it was PAINLEVÉ who presented to the Académie the group of notes by FRÉCHET that were published on the Comptes Rendus in 1904–05. The mathematical subject matter of the letter identifies it with the second of the notes [FRÉCHET, 16], which was of date January 2, 1905. The most intriguing thing about HADAMARD'S letter is his raising with FRÉCHET the question whether he would not do better to use the notion of neighborhood (voisinage) rather than of limit. This could well have been the push that started FRÉCHET on a different approach to abstract point set topology, ultimately leading him to the idea of an écart, which is presented in his fourth note to the Académie [FRÉCHET, 18], in March of 1905.

In his third note in the Comptes Rendus [FRÉCHET, 17], of date February 27, 1905, FRÉCHET considers a concrete illustration of his general theory by introducing the class of all real sequences $\{a_n\}$, each sequence being an element. If $A = \{a_k\}$ and $A_n = \{a_k^{(n)}\}$, he defines $A$ to be the limit of $\{A_n\}$ if and only if $\lim_{n\to\infty} a_k^{(n)} = a_k$ for each $k$. This class he denotes by $E_\infty$. (Later, in his thesis, he denotes it by $E_\omega$.) FRÉCHET observes that in $E_\infty$ every derived set is closed and that a set is compact if and only if there exists a sequence $\{M_k\}$ of positive numbers such that $|a_k| < M_k$ for each $k$ and each $A = \{a_k\}$ in the set.

In the fourth note in the Comptes Rendus [FRÉCHET, 18], of date March 20, 1905, FRÉCHET returns to the abstract point of view. This time, however, he makes the notion of a sequence $\{A_n\}$ having a limit $A$ depend on an idea of numerical measurement of the nearness of one element to another. He opens with a remark to the effect that WEIERSTRASS, in his lectures on the calculus of variations, made use of what FRÉCHET describes as the "voisinage de deux courbes infiniment voisins." FRÉCHET asserts that he will show the interest there is in extending this notion, under the name écart, to the case of two arbitrary elements. His procedure is to assume that, corresponding to two arbitrary elements $A$, $B$ in a specified class of elements of arbitrary nature, there is a real number, denoted by $(A, B)$. He calls this the écart of $A$ and $B$. He assumes that (1) $(A, B) \geqq 0$ and that (2) $(A, B) = 0$ if and only if $A = B$; also that (3) if $(A, C)$ and $(B, C)$ are infinitely small, so is $(A, B)$. It is not explicitly stated that $(A, B) = (B, A)$, but I think this is intended. The meaning of (3) is not made precise here, but it is made precise in FRÉCHET's thesis (to be discussed in § 5). Finally, FRÉCHET states that a sequence $\{A_n\}$ will be said to have limit $A$ provided that $\lim_{n\to\infty} (A_n, A) = 0$. With this framework of assumptions FRÉCHET asserts that every derived set is closed.

One dictionary meaning of the French word "écart" is "difference," or "deviation." FRÉCHET's use of the word is probably based on terminology used by CAMILLE JORDAN, who called the expression

$$|x - a| + |y - b|$$

the écart of two planar points $(x, y)$ and $(a, b)$. This is a measure of the amount by which the two points are separated that serves much the same purpose as the Euclidean distance

$$[(x - a)^2 + (y - b)^2]^{\frac{1}{2}}.$$

Using the écart, FRÉCHET defines what it means for a functional operation to be uniformly continuous on a set and states that a functional operation that is continuous on a closed and compact set is uniformly continuous there. He also defines the notion of uniform continuity for a family of functional operations, noting that the notion of equicontinuity for ordinary functions goes back to ASCOLI in 1884 (see the Bibliography). He mentions ARZELÀ in this connection, as well. Then he states the following theorem. Let $P$ be a set that is compact and perfect (*i.e.* $P$ coincides with its derived set). Suppose there is a denumerable set $D$ such that every element of $P$ is either in $D$ or the derived set of $D$. Let $G$ be a family of continuous functional operations defined on $P$. Then, in each infinite subset of $G$ there exists an infinite sequence of functional operations that converges to a limit operation uniformly on $P$ if and only if the members of $G$ are equicontinuous and uniformly bounded on $P$. FRÉCHET states that ASCOLI obtained a theorem analogous to this (for ordinary functions). He also refers to works of ARZELÀ (see the Bibliography). For historical notes about the roles of ASCOLI and ARZELÀ see page 392 in [DUNFORD & SCHWARTZ].

Next, FRÉCHET mentions that in most of the situations in classical analysis in which convergent sequences of functions are used, the condition for the type of convergence involved is expressible with the aid of some écart. FRÉCHET also observes that the convergence he defined in $E_\infty$ can be expressed by means of the écart

$$(A, B) = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{|a_k - b_k|}{1 + |a_k - b_k|}.$$

In this same note FRÉCHET mentions briefly how to define an écart in the family of all continuous curves in space of three dimensions. Finally, he remarks that his generalization of the theorem of WEIERSTRASS (in [FRÉCHET, 9]) can be regarded as a generalization of a theorem of ARZELÀ in which the latter considered what FRÉCHET would call a continuous functional operation on a class of curves [ARZELÀ, 1, 2].

There are two more publications by FRÉCHET in 1905 that contribute directly to his dissertation. They are both on the same subject: the introduction and use of an écart in the study of the class whose elements are continuous curves in 3-space. The first of these [FRÉCHET, 20] reached the editors of the Transactions of the American Mathematical Society in May, 1905. The second [FRÉCHET, 22] is a note of date 27 November, 1905 in the Comptes Rendus. Continuous curves arise from triples of real continuous functions defined on a finite closed interval of real numbers, but FRÉCHET regards a curve itself as a particular kind of linearly ordered point set in 3-space, the order arising from the fact that the curve can be represented parametrically in a variety of equivalent ways. With a suitable definition of the écart $(C_1, C_2)$ of two curves $C_1$, $C_2$, FRÉCHET shows that for any three curves there is a valid triangular inequality

$$(C_1, C_3) \leqq (C_1, C_2) + (C_2, C_3).$$

This inequality is sufficient to show that the third condition required of an écart (in [FRÉCHET, 18]) is satisfied. He also shows that a necessary and sufficient condition that $\lim_{n\to\infty} (C_n, C) = 0$ is that, no matter how $C$ is parametrized, say in the form

$$x = f(t), \quad y = g(t), \quad z = h(t) \quad a \leqq t \leqq b,$$

there exist parametrizations of $C_n$ $(n = 1, 2, \ldots)$ by triples of functions $(f_n, g_n, h_n)$ with the same parameter $t$, such that $f_n$, $g_n$, $h_n$ converge to $f$, $g$, $h$ respectively, uniformly with respect to $t$ on $[a, b]$. Here FRÉCHET has a concrete example of an abstract class with an écart. He shows that a sequence $\{C_n\}$ is such that $(C_n, C_m)$ approaches 0 when $m$ and $n$ both become infinite if and only if the sequence has a limit $C$ in the class of curves. The principal result of the paper is a theorem which gives a necessary and sufficient condition for a set $E$ of curves to be compact. It is stated in a form that does not involve any particular parametrizations of the members of $E$. To every set $E$ of curves is assigned an "index of compactness," which is a nonnegative number that is, by definition, 0 for finite sets, and which, for infinite sets, is arrived at by a succession of limits superior and limits inferior. The definition requires consideration of all possible infinite subsets $H$ of $E$ and all possible sequences $\{C_n\}$ from $H$. The theorem is that $E$ is compact if and only if the index of compactness is 0. FRÉCHET comments that ASCOLI was the first to give a condition that a family of curves have the property that FRÉCHET calls compactness. He also refers to ARZELÀ. FRÉCHET stresses that his own formulation of a necessary and sufficient condition for compactness is different (and one senses that he feels it is in a way superior) because it does not use parametric representations, whereas ASCOLI and ARZELÀ formulate conditions that depend upon particular parametrizations. One may observe, however, that the ASCOLI-

ARZELÀ conditions are convenient to apply in actual usage, whereas FRÉCHET's criterion is not.

In the last of the publications to be considered in this section, [FRÉCHET, 22], FRÉCHET goes further in developing topological ideas and results in the class of curves with écart defined as in the previous paragraph. He asserts that in any perfect set $P$ of curves there exists a denumerable subset whose derived set is $P$. He introduces the notion of an element of condensation of a set of elements by an adaptation to his situation of the notion originally defined by LINDELÖF. FRÉCHET's definition is that $A$ is an element of condensation of $E$ if it is a limit element of $E$ which remains a limit element of every set obtained from $E$ by omitting from it a denumerable subset. Concerning a set $E$ of curves that is compact and nondenumerable he asserts: The set of elements of condensation of $E$ is a perfect set. The curves in $E$ that are not elements of condensation is a denumerable set. So here he is thinking of a generalization of the CANTOR-BENDIXSON theorem.

Next, FRÉCHET introduces the notion of a curve $C$ being strictly interior (intérieure au sens étroit) to a set $E$ of curves if $C$ is in $E$ but is not a limit element of the complement of $E$. Then he states a theorem that is obviously related to the famous covering theorem known as the HEINE-BOREL or BOREL-LEBESGUE theorem: In order that a compact set $E$ (of curves) be closed it is necessary and sufficient that, whenever $G$ is a family of sets $I$ (of curves) such that each element of $E$ is strictly interior to at least one of the $I$'s, there exists a finite set of the $I$'s forming a family having the same property as $G$.

Finally, FRÉCHET states a sort of converse of his generalization of the WEIERSTRASS theorem: If a set $E$ of curves is such that every continuous functional operation on $E$ is bounded and attains an absolute maximum at some point of $E$, then $E$ must be closed and compact.

Before going on (in the next section) to a discussion of FRÉCHET'S thesis, let us pause to consider what may be said about the originality of FRÉCHET'S contribution of the concept of compactness and his demonstration of its relation to the generalizations of the WEIERSTRASS theorem. Without question FRÉCHET was the originator (in 1904) of the word "compact" and of the formulation of the generalized WEIERSTRASS theorem for real functions defined on a set in an abstract class in which there is a topology based on postulates. The only question there can be about his originality is: how much did he owe to ASCOLI and ARZELÀ? I suppose it is unlikely that we shall ever know whether FRÉCHET knew the works of ASCOLI and ARZELÀ in 1904 when he was working out the material for his first note in the Comptes Rendus. I think it is unlikely that he got his ideas for that first note from knowledge of the writings of ASCOLI and ARZELÀ. His first mention of the two Italians is in his note of March 20, 1905 in the Comptes Rendus. Most likely, I believe, FRÉCHET'S attention was called to the works of ASCOLI and ARZELÀ by HADAMARD, who realized that, in the special concrete situation of functions of curves (funzioni di linee), FRÉCHET'S very general result about the maximum of a functional operation had been dealt with fifteen years earlier by ARZELÀ, using ideas first introduced by ASCOLI. The essence of a straightforward proof of the WEIERSTRASS theorem, using convergent sequences, is to be able to select a convergent subsequence from an arbitrary sequence of elements in the domain of definition of the function whose maximum is sought. ARZELÀ saw that this selection of a convergent subsequence would always be possible if the curves on which the function of curves was defined were equicontinuous and uniformly bounded. We do not know what method FRÉCHET envisaged for his proof of the generalized WEIERSTRASS theorem when he first announced it. However, there is reason, from what he wrote later in his thesis, to suppose that his first formulation of a topology for an abstract class was motivated by the desire to postulate some simple conditions that would enable him to arrive at certain preconceived results. This could have caused him to decide on the use of the notion of convergent sequences as the notion to be axiomatized, and, concomitantly, to choose his definition of compactness so that it would enable him to formulate and prove the generalized theorem of WEIERSTRASS.

In any event, even if FRÉCHET *did* know the ARZELÀ result in 1904, he still deserves credit for his singling out his notion of compactness for use in an abstract postulational setting.

## 5. The Doctoral Thesis, 1906

It is important to examine FRÉCHET'S thesis carefully, for very likely it had more impact on the world of mathematics than anything else FRÉCHET ever wrote.

FRÉCHET placed the date April 2, 1906 at the end of the published form of his thesis [FRÉCHET, 23]. The thesis consists of an introduction plus two parts of approximately equal lengths: I Introduction de la notion de limite dans les ensembles abstraites; II Applications de la théorie générale.

Before discussing the main features of the thesis it is well to consider briefly the terms "functional operation" (opération fonctionelle) and "the functional calculus" (le calcul fonctionnel). The word "fonctionnelle" seems to have been originated by HADAMARD *as an adjective* in the term "opération fonctionnelle". In his paper of 1903 [HADAMARD, 2], HADAMARD uses this latter term for a numerically valued function defined on a class of functions. The first systematic discussion of functions of this kind was that published by VOLTERRA in 1887 in five notes [VOLTERRA, 1, 2] using the terminologies "functions that depend on other functions" and "functions of lines" (*i.e.* of curves). FRÉCHET, on pages 1 and 4 of his thesis, defines a functional operation as a numerically valued function defined on a class of elements of arbitrary nature; he cites numbers, points, functions, lines, surfaces, *etc.* as examples of the kinds of elements to which his considerations may be applied. Then he states that the object of the functional calculus is the systematic study of functional operations.

HADAMARD did not use the word fonctionnelle as a noun in 1903. FRÉCHET uses the word as a noun in one section of his thesis (59, page 38); he asserts, inaccurately, that HADAMARD, in his paper of 1903 designated an operation acting on continuous functions as a fonctionnelle. This is simply a slip on FRÉCHET'S part. I think the slip is understandable and easily accounted for, because there is evidence that HADAMARD made the specific proposal to FRÉCHET that the word fonctionnelle be used as a noun. The evidence is in a letter to FRÉCHET from HADAMARD (one of the letters shown to me by FRÉCHET'S daughter); the letter is undated, but from its contents we can infer that it was written at some time in 1904 or 1905. Here is the letter:

Mon cher ami,

Ci-joint votre Mémoire qui me parait intéressant. Vous avez bien fait de donner un exemple.

Page 188, pourquoi ne pas dire qu'il s'agit de l'analyse de la forme paramétrique de Weierstrass?

Peut-être feriez vous bien de mentioner que cette forme est celle que je vous ai exposé au College de France. Je crois avoir été le premier à l'indiquer pour les intégrales multiples.

Je me suis decidé à appeler *fonctionnelle* les "fonctions de fonctions" ou "fonctions de lignes". $U$ est fonctionnelle de $\phi(x)$. Je vous propose à tout hasard cette dénomination.

Bien à vous
J. Hadamard

The mémoire here under discussion by HADAMARD is FRÉCHET'S paper [FRÉCHET, 14] on the method of HAMILTON-JACOBI. HADAMARD must have been reading from page proofs sent him by FRÉCHET, for in the paper as published there is a footnote on page 188 that reads as follows:

"Cette forme (3) est l'analogue de la forme paramétrique employé par Weierstrass dans le cas d'une intégrale simple. L'extension au cas d'une intégrale multiple a été realisé pour la première fois sous une forme pratique par M. Hadamard dans son cours au Collège de France."

FRÉCHET'S paper bears the date October, 1904. It appears in volume 11 (1905) of the Annali di Matematici. It seems likely that HADAMARD saw the page proofs late in 1904 or early in 1905 and that FRÉCHET added the footnote when he sent the page proofs back to the editor or printer. The paper itself was not connected with FRÉCHET'S work on abstract spaces. It dealt with a topic in the calculus of variations and related partial differential equations. It was an extension of some work of VOLTERRA.

Part I of FRÉCHET'S thesis (pages 1–33, slightly less than half of the entire work) is essentially about the rudiments of point set topology in abstract sets, with some discussion of continuous functional operations, individually, or in families. FRÉCHET doesn't use the word "topology;" instead he speaks about "l'étude des ensembles abstraits." He refrains in the thesis and for a number of years afterward from the free use of ordinary geometric terminology in connection with abstract sets. He does not call them spaces and he does not call their elements points. He is nevertheless intent on building up a theory that closely resembles the point set theory of that time, for sets on the line or in Cartesian space of higher (finite) dimension. The crucial notion, for FRÉCHET, is that of a limit element of a set in an abstract class, and in the thesis he always bases the definition of a limit element on the notion of the limit of a sequence of elements. Initially this latter notion is introduced (as in his very first note in the Comptes Rendus, which I have discussed in § 4) as an undefined notion subject to two postulates. In the thesis it is made clear that limits of sequences are assumed to be unique. An abstract class with sequential limits merely governed by these postulates and nothing more is called "une classe (*L*)," which I shall render into English as "an *L*-class". The letter *L* is not a symbol for the class itself.

FRÉCHET finds that he cannot go very far with a theory built up for *L*-classes with nothing else in the way of structure to work with. Most of Part I of the thesis is devoted to what FRÉCHET calls *V*-classes, which I shall describe more fully later. A *V*-class is a particular kind of *L*-class. The *V* stands for voisinage (neighborhood); in a *V*-class the notion of a sequence $\{A_n\}$ having a limit $A$ is defined by using a numerical measure of the nearness of $A_n$ to $A$. This numerical measure is governed by certain axioms.

Early in his thesis (on page 4) FRÉCHET states that the best known and most important results in point set theory (la théorie des ensembles) are those that are based on the notion of the limit of a sequence of elements. He is referring, of course, to the theory when the elements are either real numbers, thought of as points on a line, or points in a plane, or points in *n*-space. He then observes that the notion of a limit is always defined *after* having specified the nature of the elements being considered. Then, shifting his attention, he refers to the study of various concrete examples of groups, in each of which the group operation of composition of two elements is defined after taking account of the nature of the elements. (He mentions groups of motions, groups of substitutions, transformations, *etc.*) It is at this point that FRÉCHET begins to prepare the way for his abstract approach to point set theory by discussing the virtues of abstract group theory.

Speaking of group theory, FRÉCHET writes (on page 5): "On ne peut arriver à une théorie commune qu'en *s'abstenant* de donner une définition générale de ce mode de composition, mais en recherchant les conditions communes aux définitions particulières et en ne retenant que celles qui étaient indépendantes de la nature des éléments considerés." Here FRÉCHET cites a work by DESÉGUIER (see Bibliography). A few lines later FRÉCHET writes: "... dans la théorie des groupes le mode de composition est supposé défini *à l'avance* dans chaque cas particulier; mais on ignore volontairement cette définition pour ne retenir que certaines conditions générales qu'elle remplit mais qui ne la déterminent pas." He goes on to say (I paraphrase): Sometimes proofs are made more difficult because one is deprived of a concrete representation. But what is lost in that way is compensated for by eliminating the need to repeat the same reasoning many times in different forms. One also gains in seeing more clearly what is truly essential. FRÉCHET then asserts that he is going to try to do, for the functional calculus, and in particular for the theory of abstract sets, the same thing that has been done for abstract group theory.

FRÉCHET evidently was impressed by something said by BOREL in his first book, for in a footnote on page 5 of the thesis FRÉCHET quotes part of what BOREL said concerning his own definition of measure. BOREL wrote that he recognized that a definition of measure could not be useful unless it had certain properties, and that he had posited these properties *a priori*, using them to define the class of sets to be regarded as measurable. The fundamental idea, said BOREL, was to introduce the essential properties of the sets, that is to say, those properties that are indispensable for the reasoning that is to follow [BOREL, 1, page 48]. FRÉCHET's choice of the notion of an $L$-class as the initial basis for an abstract theory of sets may well have an act in which he was consciously guided by BOREL's words.

In the second of the two chapters that comprise Part I of his thesis, FRÉCHET abandons axiomatically defined $L$-classes as the basis for his general theory. On pages 17 and 18 he explains why he felt it necessary to impose further restrictions in order to overcome the disadvantages of the great generality of $L$-classes. (1) He wanted to maintain the abstract axiomatic method. (2) He wanted his theory to apply to those classes of elements that occur most frequently in the applications. (I think it may be concluded safely that the classes in question were, for him, those dealt with in Part II of the thesis.) (3) He wanted his theory to include sought-for generalizations of theorems about point sets on the line and real continuous functions of a real variable. One specific thing FRÉCHET wanted was a theory in which derived sets are closed. He says that he considered imposing as an axiom a requirement on the concept of convergent sequences that would lead to this desired end.[7] Instead, however, he decided on the course of action that he had used in his fourth note in the Comptes Rendus. He considers an abstract set on which there is defined a real binary function of pairs $A$, $B$ of elements. The value of the function is denoted by $(A, B)$. It is assumed that $(A, B) = (B, A) \geqq 0$, that $(A, B) = 0$ if and only if $A = B$, and that there exists a real function $f(\varepsilon)$ of the positive real variable $\varepsilon$, such that $f(\varepsilon) > 0$, and $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$, and such that $(A, C) \leqq f(\varepsilon)$ whenever $(A, B) \leqq \varepsilon$ and $(B, C) \leqq \varepsilon$. In this situation the class is called a $V$-class. FRÉCHET called it "une classe $(V)$." FRÉCHET says that the number $(A, B)$ is called the neighborhood (voisinage) of $A$ and $B$.

In view of the common usage today, in abstract topology, of the notion of neighborhoods as families of sets subjected to certain axioms, FRÉCHET's usage in calling the number $(A, B)$ a neighborhood seems odd. In his note [FRÉCHET, 18] in the Comptes Rendus FRÉCHET had used the name écart for $(A, B)$. I think his eventual decision to use the name voisinage in this connection is probably the result of HADAMARD's influence, as perhaps expressed initially in the letter from HADAMARD to FRÉCHET that I have quoted in §4.

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[7] The axiom that FRÉCHET had in mind was this:

If $\{A_n\}$ has the limit $A$ and if the sequences $\{A_1^{(p)}\}, \{A_2^{(p)}\}, \ldots, \{A_n^{(p)}\}, \ldots$ are such that, for each fixed $n$, $\{A_n^{(p)}\}$ has the limit $A_n$, then there exist sequences $n_1, n_2, \ldots$ and $p_1, p_2, \ldots$ such that the sequence $\{A_{n_k}^{(p_k)}\}$ (with running index $k$) has the limit $A$.

FRÉCHET mentions that this axiom is fulfilled in the case of many definitions of a sequential limit, particularly when the convergence is uniform. The reference here to uniform convergence corresponds very closely to the mention made of uniform convergence in the letter of HADAMARD to FRÉCHET that is quoted in § 4. Here we have an evidence of consultation between HADAMARD and FRÉCHET during the writing of FRÉCHET's thesis.

A $V$-class becomes a special sort of $L$-class by means of the definition that a sequence $\{A_n\}$ of elements in the $V$-class has the element $A$ as limit provided that $(A_n, A) \to 0$ as $n \to \infty$. This definition guarantees that limits are unique. It also leads to the consequence that derived sets are closed.

With one exception all of the theorems in Chapter II of FRÉCHET'S thesis are theorems in which the basic underlying structure is an arbitrary $V$-class. The one exception is a theorem in § 51. In sections 49, 50 and 51 FRÉCHET is concerned with what he calls "une class ($E$)." I shall use the terminology "$E$-class" for such a class. An $E$-class is a special kind of $V$-class in which the axiom that, if $(A, B) \leqq \varepsilon$ and $(B, C) \leqq \varepsilon$, then $(A, C) \leqq f(\varepsilon)$, is replaced by the triangularity axiom asserting that $(A, C) \leqq (A, B) + (B, C)$ for all triples $A, B, C$ of elements.

It is clear that an $E$-class is a $V$-class. One can take $f(\varepsilon) = 2\varepsilon$. When dealing with an $E$-class FRÉCHET calls $(A, B)$ the écart of $A$ and $B$, thus modifying his previous usage of the term écart in his fourth note in the Comptes Rendus.

The exceptional proposition, for which FRÉCHET finds it necessary to use the triangularity axiom, is the following (page 31 of the thesis): If in an $E$-class a set $G$ is such that every functional operation defined and continuous on $G$ is (1) bounded on $G$, and (2) attains its least upper bound at some element of $G$, then $G$ is a closed and compact set. FRÉCHET surmises on page 50 that this theorem is true for $V$-classes in general. This conjecture was subsequently proved correct by H. HAHN [HAHN, 1] in 1908. A number of years later, in 1917, E. W. CHITTENDEN (see bibliography) rendered the further study of $V$-classes as distinguished from $E$-classes of negligible interest by showing that in a given $V$-class one can introduce a distance function (that is, an écart) making the class into an $E$-class in such a way that the notion of limit element derived from the voisinage originally given in the $V$-class is the same as the notion of limit element derived from the écart. Thus, the notions of closed set, compact set, continuous functional, and so on, are the same, whether one bases them on the original "voisinage" or the new "écart." CHITTENDEN'S construction of the écart utilized in a fundamental way the technique devised by HAHN in proving the correctness of FRÉCHET'S surmise.

An $E$-class is precisely what we now call a metric space, with $(A, B)$ the distance between $A$ and $B$. The name metric space was originated by F. HAUSDORFF, who introduced the name "metrischer Raum" (page 211 in [HAUSDORFF]) and the axioms without crediting them to FRÉCHET. It is clear from HAUSDORFF'S book that he was familiar with FRÉCHET'S thesis, for he does refer to FRÉCHET in connection with certain topics.

It seems a bit strange that FRÉCHET never gives a concrete example of a $V$-class with a voisinage that does not satisfy the triangle inequality.

I come now to consideration of some of the details in Part I of the thesis. In Chapter I, devoted to $L$-classes, FRÉCHET defines the derived set $E'$ of a set $E$ (as the set of all limit elements of $E$), and then defines the concepts "closed set" and "perfect set" in terms of the derived set exactly as in the point set theory of his day for sets on the real number line. For a set $E$ he defines an element $A$ as intérieur à $E$ au sens étroit[8] when $A$ is in $E$ and is not a limit element of the com-

[8] In using the phrase "au sens étroit" FRÉCHET was apparently being influenced by a contemporary usage in which to say that a point $x$ on the real number line is interior

plement of $E$. In reporting on FRÉCHET'S work I shall henceforth drop the phrase "in the narrow sense" (au sens étroit), and simply call $A$ an interior element of $E$ (or say that $A$ is interior to $E$) when $A \in E$ and $A \notin (E^{\sim})'$, where $E^{\sim}$ is the complement of $E$. FRÉCHET defined $A$ to be an element of condensation of a set $E$ if $A$ is a limit element of $E$ which continues to be a limit element of each set obtained from $E$ by suppressing a denumerable set of elements of $E$. This definition is not the same as that of LINDELÖF, but is equivalent to LINDELÖF'S definition when applied to sets on the line. A set is called condensed if for every one of its nondenumerable subsets there is at least one element of condensation of the subset.

For his definition of compactness FRÉCHET adopts, not his original definition of 1904, but the alternative criterion for compactness, given in his first note in the Comptes Rendus: a set $E$ is called compact if every infinite subset of $E$ has a limit element (in the $L$-class, but not necessarily in $E$). He then calls a set *extremal* if it is closed and compact. I shall avoid the use of the adjective extremal, using instead the two adjectives closed and compact. I do this because FRÉCHET'S use of the adjective extremal proved to be ephemeral.

FRÉCHET'S generalization of the WEIERSTRASS theorem, namely, that a functional operation which is defined and continuous on a closed and compact set is bounded and attains its least upper bound, is presented in the thesis as a corollary of a theorem in which there is no mention of continuity. Here is that theorem: Let $U$ be a functional operation defined on a closed and compact set $E$. Then there exists at least one element $A$ of $E$ such that the least upper bound (finite or infinite) of $U$ on $E$ is the same as the least upper bound of $U$ on each subset $K$ of $E$ for which $A$ is in $K$ and not a limit element of $E - K$. FRÉCHET says of this theorem that it was first announced by WEIERSTRASS for the case when $E$ is a set of points (*i.e.* points in Euclidean space) and then by ARZELÀ (on page 347 of [ARZELÀ, 1] for sets of curves. This reference to ARZELÀ is curious and misleading, for ARZELÀ never used the term "compact;" he was considering a family of curves represented by functions that were uniformly bounded and equicontinuous, and hence possessed (when a suitable notion of limit elements was employed) of the property that FRÉCHET calls compactness. It seems doubtfull to me that FRÉCHET had all this in mind when he first announced his generalization of the WEIERSTRASS theorem. I suspect that the way the matter is presented in the thesis is a second effort, resulting from FRÉCHET'S having read ARZELÀ'S work subsequent to 1904.

For $V$-classes FRÉCHET introduces the notion of separability: a $V$-class is separable if it contains a denumerable set whose derived set is the entire class. By this definition a separable $V$-class is infinite and perfect. (In a later publication, in 1921 (see page 421 of [FRÉCHET, 75]), FRÉCHET modified the definition of separability of a class to require merely that there be an at most denumerably infinite set which, together with its derived set, constitutes the entire class.)

For $V$-classes FRÉCHET also introduces the notion of completeness, *although he does not use this word for the notion.* He says of a sequence $\{A_n\}$ in a $V$-class that it satisfies the condition of CAUCHY when to each $\varepsilon > 0$ corresponds an

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to an interval in the narrow sense meant that $x$ belonged to the interval but was not at either end. See page 117 in ZORETTI'S article listed in the Bibliography.

$n$ such that $(A_n, A_{n+p}) < \varepsilon$ for $p = 1, 2, 3, \ldots$ . He then says of a $V$-class that it admits a generalization of CAUCHY'S theorem if each sequence that satisfies the condition of CAUCHY has a limit in the class. To shorten the exposition I shall use modern terminology and call such a class complete.[9] In a number of theorems FRÉCHET deals with $V$-classes that are complete and separable. He calls them *normal*. This terminology has not survived; in later developments of abstract topology the word normal is given an entirely different meaning.

In FRÉCHET'S development of the theory of $V$-classes he formulated and proved some theorems that constituted very important progress in the nascent discipline of what came to be called general topology. Greater generality for some of these theorems was attained later in the work of other mathematicians, but FRÉCHET deserves to be recognized for his demonstration that significant results could be obtained in an abstract treatment.

Underlying the theorems to which I refer are two distinctly separate ideas. One of them is the idea of representing a set (which may be the whole $V$-class) as the union of a family of sets, each of which is arbitrarily small, in a sense to be made clear. (What we now call a *union* of sets was called a *sum* in those days. I shall use the modern term in my exposition.) The interest here attaches to the issue of how numerous a family is required in a particular situation. FRÉCHET got this idea from a suggestion made by HADAMARD in a publication whose broader significance for FRÉCHET'S work I shall discuss in § 6. The other is that of a *covering* of a given set by a family of sets. (A family $\mathscr{M}$ of sets M is called a *covering*[10] of a set E if each element of E is interior to some set M from $\mathscr{M}$. One can also say that $\mathscr{M}$ covers E. Here interest attaches to the question: If $\mathscr{M}$ covers E, how small a number of members of $\mathscr{M}$ is sufficient to cover E?

FRÉCHET showed in his thesis (§ 40, page 25) that for an arbitrarily small positive $\varepsilon$, any separable $V$-class can be regarded as the union of the members of a denumerable family $\mathscr{K}(\varepsilon)$ of sets $\mathrm{K}(\varepsilon)$ with the property that, if $A$, $B$ are two elements in the same $\mathrm{K}(\varepsilon)$, the *voisinage* $(A, B)$ is less than $\varepsilon$. For convenience of exposition I shall describe this property by saying that the sets $\mathrm{K}(\varepsilon)$ are $\varepsilon$-small. FRÉCHET also observes that the situation can be arranged so the family $\mathscr{K}(\varepsilon)$ is a covering of the $V$-class, that is, that each element of the class is interior to some $\mathrm{K}(\varepsilon)$.

In § 25 (page 25) of the thesis FRÉCHET states the following important theorem: In a normal $V$-class a set E is compact if and only if for each $\varepsilon > 0$ it is the union

[9] In his book of 1914, on page 315, HAUSDORFF calls a metric space vollständig if every sequence satisfying CAUCHY'S condition is convergent. He mentions that this concept was recognized and used by FRÉCHET. When FRÉCHET himself came (later) to use the word complete he gave it a different meaning. He called a metric space complete if, among all the distance functions that are compatible with the pre-existing notion of limit elements, there is at least one distance function with respect to which every sequence that satisfies CAUCHY'S condition is convergent. See page 341 of [FRÉCHET, 75]. KURATOWSKI calls this kind of completeness "completeness in the topological sense." See the footnote on page 198 of KURATOWSKI'S book listed in the Bibliography.

[10] The word "covering" is not used by FRÉCHET, but he makes use of the concept without giving it a name. My introduction of the word, in accordance with modern usage, is to simplify the exposition.

of a finite family $\mathscr{K}(\varepsilon)$ of $\varepsilon$-small sets $K(\varepsilon)$. (The phraseology here is mine, not FRÉCHET'S.) The property by which FRÉCHET here characterizes compact sets in a normal $V$-class later came to be known as the property of *total boundedness*.[11] Still later, after FRÉCHET'S property of compactness had come to be called relative compactness, a totally bounded set was called a *precompact* set by some authors.

FRÉCHET proves the "if" part of his theorem without using separability of the $V$-class. He does use completeness, however, and it is essential. For the "only if" part he doesn't use completness but he does use separability. He also uses his generalization of the BOREL covering theorem, which he had proved earlier in the thesis (§ 36, page 22) in the following form: Let E be a closed and compact set in a $V$-class, and let $\mathscr{M}$ be a denumerable covering of E. Then $\mathscr{M}$ contains a finite number of member sets that also form a covering of E. FRÉCHET asserts, correctly, but without giving the proof, which is not trivial, that if E is compact (and closed), then the union of E and E′ is compact (as well as closed). Essential use is made of the truth of this fact, which FRÉCHET proved in a slightly different context in a later paper, in 1910 (see § 6, page 4 in [FRÉCHET, 39]). FRÉCHET'S use of separability of the $V$-class is not really essential for the truth of the "only if" part of the total boundedness theorem; it plays a role merely because of the particular method of proof he chose. In fact, it is easy to show quite directly that if E is a compact set in an arbitrary $V$-class, then for every $\varepsilon > 0$, E is contained in the union of a finite number of $\varepsilon$-small sets and, as a consequence of this, that E contains an at most denumerable set D such that E is contained in the union of D and D′. FRÉCHET demonstrated these things (in [FRÉCHET, 39]) for a metric space, but the arguments can be applied in a $V$-class.

Next in the thesis (§ 26, page 26) comes FRÉCHET'S abstract version of the BOREL-LEBESGUE (or HEINE-BOREL) theorem, and a converse of it: Let E be a set in a normal $V$-class. Then, in order that in every (not necessarily denumerable) covering $\mathscr{M}$ of E there be a finite number of member sets M also forming a covering of E, it is necessary and sufficient that E be closed and compact.

FRÉCHET was interested in generalizing the theorem of CANTOR and BENDIXSON which asserts (for point sets on the line) that every closed set is the union of two disjoint sets, one of them perfect and the other scattered (*i.e.* containing no nonempty subset that is dense-in-itself). The decomposition is unique. Either constituent set may be empty, and the scattered set is at most denumerable. FRÉCHET observed (page 19) that if E is a condensed set in a $V$-class, the set P of its elements of condensation is perfect (it may be empty, of course). Moreover, the set $E - P$ is at most denumerable. Later (page 27) FRÉCHET obtains the result that in a normal $V$-class any nondenumerable set is condensed. Then he concludes that in a normal $V$-class a closed set is the sum of two disjoint sets, one of them perfect, the other at most denumerable.

The second half of FRÉCHET's thesis is devoted to illustrations of his abstract theory as applied to concrete examples. Apart from the obvious examples of the

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[11] In his book of 1914 HAUSDORFF calls a set E in a metric space total beschränkt if for each positive $\varrho$, E is contained in the union of a finite number of spheres of radius $\varrho$ (see page 311).

real line and Euclidean or Cartesian $n$-space there are four concrete examples of $E$-classes, each of which is normal (that is, complete and separable). Here are the examples.

A. The class of real functions defined and continuous on a finite closed interval $J$ of the real line, with écart defined by

$$(f, g) = \text{the maximum of } |f(t) - g(t)| \text{ for } t \text{ in } J.$$

B. The class $E_\omega$ of points $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$, where the coordinates $x_i$ and $y_i$ are real numbers and the écart is

$$(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

This is the same as the $E_\infty$ of [FRÉCHET, 17].

C. The class (FRÉCHET gives it no name) of all complex functions $f$ of the complex variables $z$ that are defined and holomorphic in the interior of a fixed plane region $A$ (FRÉCHET calls it an "aire") whose boundary consists of one or more contours. Let $\{A_n\}$ be a sequence of bounded regions (aires) such that each $A_n$ is contained in the interior of $A_{n+1}$ and of $A$ and such that any given bounded region in the interior of $A$ is in the interior of $A_n$ when $n$ is sufficiently large. The écart of two functions $f$, $g$ in the class is defined as

$$(f, g) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{M_n(f, g)}{1 + M_n(f, g)},$$

where $M_n(f, g)$ is the maximum of $|f(z) - g(z)|$ when $z$ is in the closure of $A_n$.

D. Here the class is the family of curves in Euclidean 3-space with écart defined as in the paper [FRÉCHET, 20] discussed in § 4.

In dealing with these examples FRÉCHET describes the way in which compactness of a set of elements can be characterized in a manner that relates more directly than does the mere definition of compactness to the characteristics of the set of elements in the concrete situation. For instance, in Example C, a set of the holomorphic functions is compact if and only if the functions are uniformly bounded on each closed and bounded set lying in the interior of $A$.

It is worth noting that the thesis contains no example of a concrete $E$-class whose elements are taken from classes of functions that are measurable or measurable and summable in the LEBESGUE sense. Nor is there mention of the class whose elements are real sequences $\{x_n\}$ such that $\sum_1^\infty x_n^2$ is convergent.

A strong influence on FRÉCHET's work in the thesis undoubtedly came from HADAMARD. It was through him that FRÉCHET became interested in the calculus of variations and in functional operations on the class of functions continuous on a finite closed interval. (In this connection see § 9.) There is clear evidence, given by FRÉCHET himself, of the influence of HADAMARD deriving from a brief communication presented by HADAMARD at the first International Congress of Mathematicians held in Zurich in 1897 ([HADAMARD, 1]). The general thrust of HADAMARD'S remarks in this communication is that it would be worthwhile to make a study of sets composed of functions. Such sets, he says, might have properties quite different from sets of numbers or of points in $n$-dimensional space. He asserts that the set-theoretic study of classes of functions would no doubt play a fundamental role in the theory of partial differential equations of mathematical physics. For example, such studies would lead to the provision of a firm basis for the well-known arguments in which the solutions of such equations are made to depend on questions about a minimum.[12] (This, clearly, refers to such things as the DIRICHLET principle.) HADAMARD then says that such questions are intimately bound up with the nature of the domain in which the minimum is sought. He observes that there are special difficulties in the calculus of variations, where the existence or nonexistence of a minimum can depend on what orders of derivatives (first, second, *etc.*) of the unknown function occurs in the expression under the integral sign.

As just one suggestion of the type of problem which would deserve early consideration in a set theory of function classes, HADAMARD proposes the following: Consider a given class of functions, as for instance the class of real functions continuous on the closed interval (0, 1) with specified values at 0 and 1. Divide the class into subsets such that any two functions interior to the same subset are at a distance (in the sense of WEIERSTRASS) apart less than a preassigned positive number $\delta$. Then study the properties of this family of subsets, and, as the first order of business, consider the cardinal number (puissance) of this family.

FRÉCHET cites HADAMARD'S Congress paper in the bibliography of his thesis. His very first note in the Comptes Rendus of 1904, in which he introduces the notion of compactness and gives a generalization of the WEIERSTRASS theorem about attainment of minimum values, goes in the direction suggested by HADAMARD, but actually overshoots HADAMARD'S idea by making a leap to total abstraction.

FRÉCHET took up quite explicitly HADAMARD'S suggestion about considering subdivisions of a class into subsets of small diameter. On page 25 of his thesis FRÉCHET refers directly to language used by HADAMARD in his Congress paper (see the final paragraph of this paper, on page 202) in speaking about what I have called "$\varepsilon$-small sets."

It is interesting to see what FRÉCHET himself had to say, almost thirty years later, about his progression into abstraction as a method of dealing with things in analysis that interested him. In his N.T.S. (see note 3), in which he summarized in 1933 his credentials as a candidate for election to the Académie des Sciences,

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[12] HADAMARD'S exact words: Mais c'est principalement dans la théorie des équations aux derivées partielles de la physique mathématique que les études de cette espèce joueraient, sans nul doute, un role fondamental. Pour n'en citer qu'un example c'est grâce à ces recherches qu'on arriverait à donner un fondement solide aux raisonnements bien connus qui ramènent la définition des intégrales de ces équations à des questions de minimum.

we find FRÉCHET's account of how *at that time* he recalled the beginnings of his work in functional analysis (l'analyse fonctionnelle) and general analysis (l'analyse générale). He writes, on page 61 of the N.T.S., that be hecame acquainted with the ideas of VOLTERRA through the teaching of HADAMARD, and in seeking to generalize these ideas and broaden their field of applicability, he was led to the notion of functions of an abstract variable. In speaking about functionals on various types of domains of definition (classes of curves, surfaces, or functions), FRÉCHET writes (on page 62): "... it was clear in advance that there would be a great many common properties and that it would be tedious to demonstrate them in succession. To succeed in proving one of these properties for functionals of elements of various sorts it would be necessary to have definitions of the limit of a sequence of elements couched in terms independent of the nature of these elements." On page 71 FRÉCHET writes: "Mais, à la même époque j'avais déjà commencé à envisager des variables plus générales que celles de M. Volterra. En essayant en 1904 d'étendre aux fonctionnelles de M. Volterra les propriétes des fonctions continues, je m'étais aperçu que la connaissance de la nature des variables envisageés—ligne ou fonction—n'était que de peu importance, qu'il suffisait d'imposer à la définition de la convergence d'une suite d'éléments des conditions simples très générales. Dans la note (11)[13] de cette meme année, je mettais en oeuvre pour la première fois ce point de vue et je jetais ainsi les bases d'une Science que j'ai appelée plus tard l'Analyse générale selon une dénomination empruntée au regretté E. H. Moore."

On page 28 of the N.T.S. FRÉCHET defines l'analyse fonctionnelle as the study of numerical functions of a variable of arbitrary nature, and l'analyse générale as the study of functions in which both the independent and dependent variable are of arbitrary character (that is, abstract.) Neither of these appellations was used by FRÉCHET in the years 1904–1906. According to FRÉCHET's book on abstract spaces [FRÉCHET, 132] the name analyse fonctionnelle was the invention of PAUL LÉVY in his book with that title [LÉVY, 1]. E. H. MOORE's first published work using the term "general analysis" was a paper given at the Fourth International Congress of Mathematicians [MOORE].

## 6. Discussion of the Significance and Impact of Fréchet's Thesis

In this section I comment on the historical significance of FRÉCHET's thesis. I also present some information about the effects produced by FRÉCHET's thesis on mathematicians who were living at the time.

It seems to me that the most significant general aspects of FRÉCHET's thesis are two in number. The first is his initiation of abstract point set topology and his development of the abstract theory to an extent that demonstrated conclusively the feasibility of such a procedure. It was FRÉCHET who decisively opened the way. The second is his display of a field of application of the general abstract

[13] The number 11 here is a reference to the publication [FRÉCHET, 9], which is (F11) in FRÉCHET's system of numbering.

theory to classes of functions and curves. His examples of application were few and the results he harvested in his thesis were of limited extent. Nevertheless, he called attention to a new point of view in analysis: a perspective on classes of functions as metric spaces.

The importance of FRÉCHET'S achievement, as a matter of history, is not lessened by the fact that neither $L$-classes nor $V$-classes survived as conceptions of permanent interest. What was significant, historically, was the fact that an abstract point set topology was shown to be feasible and fruitful and that it applied significantly to realms other than point set theory in Euclidean space of 1 or $n$ dimensions. The fact that later developments in topology brought to the forefront topological spaces more general than metric spaces but also more fruitful and satisfactory than $L$-classes, is of great significance for the later history of topology and for other parts of modern mathematics. This fact, however, does not justify the eclipsing of FRÉCHET'S early work in the presentation of a history of topology.

In constructing a general abstract topology FRÉCHET introduced a number of important concepts whose emergence was dependent on the process of abstraction: namely, his notions of compactness, separability, and completeness. The present-day meaning of compactness is not exactly the same as FRÉCHET'S compactness, but is closely related to it. One of FRÉCHET'S significant achievements is his discovery, in his abstract theory, of the essential interrelatedness of the general idea that is present in the BOLZANO-WEIERSTRASS theorem of ordinary point set theory and the general idea that is present in the BOREL and BOREL-LEBESGUE covering theorems on the line or in $n$-space. Another specific significant detail in FRÉCHET'S thesis is his discovery of the linkage, under suitable general conditions, between the concept of a set being compact and the concept of its having the property that came to be known as total boundedness. (In § 5 I mentioned FRÉCHET'S theorem about this in a normal $V$-class.)

The publication of FRÉCHET'S thesis was a stimulus with tangible effects. In a note in the Paris Comptes Rendus of that same year (1906) [F. RIESZ, 1] F. RIESZ cited FRÉCHET'S thesis and made significant use of the notion of a separable $E$-class in connection with a new and very simple proof of a theorem originally due to E. SCHMIDT, to the effect that, under suitable conditions, a system of mutually orthogonal functions is at most denumerably infinite. RIESZ also made appreciative comments about FRÉCHET'S abstract theory in other work that he published in 1906, 1907, and 1908. In this work RIESZ launched a different approach to an abstract general topology; I shall discuss it in § 8. FRÉCHET made important use of RIESZ'S ideas in his own later work.

In 1908 ARTHUR SCHOENFLIES published his second report [SCHOENFLIES] on point set theory for the Jahresbericht der Deutschen Mathematiker Vereinigung. (The first report had appeared in 1900.) SCHOENFLIES devoted a large part of one of his chapters to the ideas and results that are in FRÉCHET'S thesis. I shall discuss this part of the report of SCHOENFLIES as well as his correspondence with FRÉCHET in § 7.

In 1908 HANS HAHN'S paper, which I mentioned in § 5, showed that he had read FRÉCHET'S thesis with care and found it a stimulus to his powers of analysis. In connection with this I observe that there is a letter from HADAMARD to FRÉCHET (among those shown to me by FRÉCHET'S daughter, undated, but evidently written in 1909, because in it he states that he has just learned that FRÉCHET is installed at Rennes, where FRÉCHET took up an appointment in 1909) in which HADAMARD congratulates FRÉCHET for having his work followed up on by HAHN. HADAMARD writes: "Je vous félicite d'avoir Hahn pour continuateur." In his book on real functions [HAHN, 2] HAHN makes several references to FRÉCHET as the originator of various ideas (metric spaces, compactness, completeness, separability). On page 47 HAHN writes that FRÉCHET was the first to occupy himself with the theory of point sets in abstract spaces.

In the Jahresbericht über die Fortschritte der Mathematik, volume 37, pages 348–349 (published in 1909 but carrying reports on publications of 1906) the review of FRÉCHET'S thesis contains the following statement: This thesis, presented to the Faculty of Sciences in Paris, which is extensive, significant, and filled with new theorems, is a systematic unification and further development of the numerous investigations published by the author in 1904 and 1905.[14] The writer of this review (identified only by the initials GZ) refers to the aforementioned report of SCHOENFLIES, saying that one can find there discussion of the significance of the thesis for the recent development of set theory.

It is known, of course, that FELIX HAUSDORFF was acquainted with FRÉCHET'S thesis. This is made clear by the references to it in HAUSDORFF'S book of 1914. It is not known, however, to what extent FRÉCHET'S work influenced HAUSDORFF. In a later publication I intend to discuss possible influences on HAUSDORFF and the roles of HAUSDORFF and FRÉCHET in the developments in general topology from the years of the First World War to the period right after that war. It is known, for example, that the work of both FRÉCHET and HAUSDORFF was of great interest to the pair of young Russians, PAUL ALEXANDROFF and PAUL URYSOHN.[15]

FRÉCHET'S thesis and other early work was noticed and followed with interest in America. It is an interesting fact that four of the students who wrote doctoral dissertations under E. H. MOORE at the University of Chicago became interested in the work of FRÉCHET and engaged in research that led to publications relating to FRÉCHET'S thesis. The four were: E. W. CHITTENDEN, T. H. HILDEBRANDT, A. D. PITCHER, and R. E. ROOT. In the cases of HILDEBRANDT and ROOT their theses involved work related to that of FRÉCHET. In the cases of CHITTENDEN and PITCHER their FRÉCHET-related publications were separate from their theses. MOORE'S own work on general analysis seems to have been inspired especially by HILBERT'S work on integral equations, and shows no particular influence from FRÉCHET, although MOORE was aware of and interested in FRÉCHET'S work.

Another American, E. R. HEDRICK, then at the University of Missouri, who took his Ph. D. at Göttingen and was at the École Normale Supérieure briefly in 1901, was stimulated by FRÉCHET'S work to undertake an investigation of

[14] The exact German text is: „Die umfangreich und bedeutsame, eine Fülle neuer Sätze enthaltende Arbeit die der Verfasser der Faculté des Sciences zu Paris als Thèse vorgelegt hat, bildet eine systematische Zusammenfassung und Weiterführung der zahlreichen Einzeluntersuchungen die der Verfasser in den Jahren 1904 und 1905 an verschiedenen Stellen veröffentlicht hat, und über die an diesem Orte berichtet worden ist."

[15] See the paper [ARBOLEDA, 1] listed in the Bibliography.

abstract point set theory on a level apparently more general than that of FRÉCHET'S $E$-classes. He published his results in 1911. See the listing of HEDRICK in the Bibliography.

An appreciation of FRÉCHET'S pioneering work in general topology was expressed many years later by HADAMARD, in a secret report presenting the case for FRÉCHET'S election to the Académie des Sciences in 1934. (A vacancy in the roster of the Section de Géometrie was caused in 1933 by the death of PAINLEVÉ. The candidate actually elected to replace PAINLEVÉ was GASTON JULIA, not FRÉCHET. The membership of the section then comprised E. BOREL, E. CARTAN, E. GOURSAT, J. HADAMARD, JULIA and H. LEBESGUE.) Here is an excerpt from HADAMARD'S report, which is dated 26 février, 1934.

"Mai la nouvelle discipline (that is, the functional calculus) lui doit surtout une initiative d'une rare puissance et d'une rare audace. La veritable difficulté du Calcul fonctionnel réside dans la nature des éléments sur lesquels il porte. Les fonctions sont des êtres infiniment plus complexes et plus différentes les un les autres que ne le sont les nombres; il est ainsi, en conséquence de leurs relations mutuelles. L'espace fonctionnel, c'est de structure infiniment plus compliqueé que celle de l'espace ordinaire, et, en tout cas, notablement différente d'elle. Les difficultés du Calcul fonctionnel sont, des lors pour nous ce que seraient celles de l'analyse ordinaire si nous n'avions aucune notion du continu, et c'est pur cela que ces difficultés sont très grandes.

"Or, on sait que l'étude du continu est, depuis un peu plus d'un demi-siècle, poursuivi dans deux voies profondement différentes. L'analyse n'a nullement abandonné celle que nous ont légué Newton et Leibniz, mais elle s'est engagé aussi dans celle qu' a ouvert Georges Cantor avec sa Théorie des Ensembles des points. Les travaux de M. Paul Lévy relèvent de la première conception et transportent au domaine fonctionnel plusieurs notions essentielles empruntées de l'analyse classique. M. Fréchet s'est, au contraire, rendu compte qu'il était nécessaire d'appliquer au même domaine la Théorie des Ensembles.

"Mais il a été beaucoup plus loin encore. Du moment qu'on abandonne les ensembles ponctuels à la considération desquels s'était borné Cantor, convient-il de s'astreindre à étudier uniquement les ensembles de fonctions? Le très grand mérite de M. Fréchet est d'avoir compris qu'il n' y a aucune raison pour cela. Nous ne sommes plus aidés par cette intuition du continu qu'a guidé Cantor et ses continuateurs même dans les conclusions le plus parodoxales auxquelles ils ont abouti; du moment qu'il est ainsi, il n'y a aucune avantage, tout au moins jusqu'à un nouvel ordre, à supposer que les éléments sur lesquels nous raisonnerons et que nous grouperons en ensembles soient des fonctions plutôt qu'autre chose. M. Fréchet nous a appris a raissoner sur des ensembles entièrement abstracts, c'est à dire composés d'éléments sur lesquels on ne fait, tout au moins en commençant, aucune hypothèse. Il va d'un coup à l'extrême généralité, une généralité qui, par definition, ne pourra jamais être depassée. Il n'y fait que, moyennant ces seules restrictions, on puisse aller tres loin dans l'étude ainsi enterprise."

Then, after some words about E. H. MOORE and his work, HADAMARD continues:

"... En fait, l'audace deployée, l'effort d'abstraction accompli par M. Fréchet sont, à ce qu'il nous semble, sans précédent dans tout ce qui a été publié depuis l'oeuvre de Galois.

"De telles hardisses se justifient par le succès. M. Fréchet doit une partie du sien à la lumineuse netteté avec laquelle les principes sont dégagés. Sa conception, contrairement à celle de Moore, moins claire, moins lapidaire, a fait ses preuves d'une façon inattendue et inespérée."

FRÉCHET's reputation grew abroad sooner that it did in France. His first teaching position after receiving the doctorate was at a lycée in Besançon, in 1907. Then he had university positions at Nantes (1908), Rennes (1909), and Poitiers, to which he was officially attached from 1910 until the end of the First World War. He was soon mobilized into the army in 1914 and remained in active service until the end of the war. He was with the armies in the field, much of the time as an interpreter with the British. In 1919 he was selected to go to Strasbourg to help reorganize the university there. He became a professor in the Faculté de Sciences and Director of the Mathematical Institute of the University. Finally, in 1928, he was called to Paris. Ultimate recognition by election to the Académie of Sciences was slow in coming, however. The members of the Section de Géometrie (limited to six at that time) were long-lived. PAUL MONTEL was elected to replace GOURSAT in 1937; A. DENJOY was elected to replace LEBESGUE in 1942; R. GARNIER was elected to replace CARTAN in 1952; and FRÉCHET was elected to replace BOREL in 1956. By that time he was seventy-seven years old.

## 7. Schoenflies and Fréchet

ARTHUR SCHOENFLIES was, in 1907, a professor of mathematics at the university in Königsberg. During much of that year he and FRÉCHET were in correspondence. In the Archives of the Académie des Sciences in Paris are sixteen pieces of correspondence (mostly letters, a few cards) from SCHOENFLIES to FRÉCHET. He wrote in German; I presume that FRÉCHET replied in French. As far as I know, FRÉCHET's letters to SCHOENFLIES have not survived. SCHOENFLIES had received a copy of FRÉCHET's thesis sometime prior to February 12, 1907. By November of 1907 SCHOENFLIES had completed reading the proofs of his second report for the Jahresbericht der Deutschen Mathematiker Vereinigung, covering the extensive developments in general set theory and point set theory since his first report (published in 1900). The parts of the second report that are relevant to a historical study of the work of FRÉCHET are contained in Chapter III (On the general theory of point sets) and Chapter VII (Sets of curves and function space).[16]

I shall begin by discussing some aspects of the second report of SCHOENFLIES. Obviously it drew the thesis of FRÉCHET to the attention of a rather large mathematical audience in Europe, doubtless reaching a far greater number of people than would have of their own accord read the thesis in the Rendiconti del Circolo Matematico de Palermo, where it was published.

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[16] The titles of the chapters in German are III Allgemeine Theorie der Punktmengen, and VII Die Kurvenmengen und der Funktionalraum.

I think it is clear from the report that the greatest interest of SCHOENFLIES in FRÉCHET'S work was on those parts of it that yielded theorems close to or identical with the general results of point set theory on the line or in space of $n$-dimensions. SCHOENFLIES had a tendency to think of point set theory as a part of the geometry of his day, and this seems to have cut him off, to a considerable extent, from appreciation of the relevance of FRÉCHET'S work to the kinds of analysis that HADAMARD had in mind when he made his presentation to the International Congress in Zurich in 1897, about the possible applications of an extension of ordinary point set theory to a point set theory of classes of functions. This attitude of SCHOENFLIES shows up, for instance, on page 73, where he writes that the principal advance (Hauptfortschritt) made in the theory of point sets is in the extension of the theory to $R_\infty$. When SCHOENFLIES speaks of FRÉCHET'S $R_\infty$ he means what FRÉCHET denoted by $E_\infty$ in one of his notes in the Comptes Rendus in 1905 [FRÉCHET, 17] and later, in his thesis, denoted by $E_\omega$. At this juncture SCHOENFLIES says that in ordinary point set theory three things are basic: the definition of a limit element of a set, the BOLZANO-WEIERSTRASS theorem, and the theorem that a derived set is closed. And, says SCHOENFLIES, all these things can be extended to $R_\infty$. With them all the theorems of the CANTOR theory remain valid, in particular, the "principal theorem" (das Haupttheorem), by which he means the CANTOR-BENDIXSON theorem.

That SCHOENFLIES was more interested in the preservation of the known results of point set theory than in a broadening of the theory to embrace different phenomena is shown by his remarks about the significance of FRÉCHET'S work. He asserts (on page 73 of the report) that the significance of FRÉCHET'S investigations lies in the fact that he was obliged to take into account the bases for each of the individual theorems as well as their mutual relationships. In this way was yielded up the knowledge of those particular assumptions under which the theorems of point set theory will permit of extension to other infinite sets, in particular to function space.[17]

The first available letter from SCHOENFLIES to FRÉCHET is dated February 12, 1907. In it he expressed great interest in FRÉCHET'S thesis and asked several questions, among them the following: In a $V$-class is the voisinage (SCHOENFLIES calls it a Distanzwert) $(A, B)$ a continuous function of $B$? That is, does $(B, B_n) \to 0$ imply that $(A, B_n) \to (A, B)$? SCHOENFLIES says that if this is so, he thinks he can improve on FRÉCHET'S proof of the BOREL covering theorem in § 36 of the thesis. FRÉCHET evidently responded to this question, for on page 288–289 of his report SCHOENFLIES describes a counterexample, for the knowledge of which he thanks FRÉCHET, to show that $(A, B)$ need not be a continuous function of $B$ in all cases. His description of the example is defective, but the situation can be repaired to provide a genuine counterexample. In fact, one can have a situation in which $0 < (A, B) < r$, $(B, B_n) \to 0$, $(A, B_n) > r$, and $(A, B_n) \to r$, thus showing also

[17] Here is the statement of SCHOENFLIES in his German: Die allgemeine Bedeutung der Fréchetschen Untersuchungen liegt besonders darin, dass sie ihn nötigten, die Grundlagen der einzelnen Sätze, sowie ihr gegenseitiges Verhältnis in Betracht zu ziehen. Auf diese Weise ergab sich die Kenntnis derjenigen besonderen Voraussetzungen, unter denen die Sätze der Punktmengentheorie eine Ausdehnung auf andere unendliche Mengen gestatten, insbesondere auf den Funktionalraum.

that $B$ can be an element of the set of elements $P$ for which $(A, P) < r$ and yet not be an interior element of the set.

There must be a letter missing between the letter of February 12 and that of March 3, for in the latter letter SCHOENFLIES is trying to mollify FRÉCHET's feelings about something he took amiss in a letter from SCHOENFLIES. Evidently SCHOENFLIES expressed some views about geometry, and what he had said about FRÉCHET's discussion (in the thesis) of an $E$-class of curves had upset FRÉCHET. One can get some clues to the source of the trouble from the remarks about geometry by SCHOENFLIES in the introduction to his report as well as from the letter of March 3. In the letter he writes that he would be very sorry if his words (in the missing letter, I presume) had been interpreted as conveying a flavor of disdain on his part. Far from that, he assured FRÉCHET. He had read the thesis with care and was working a considerable part of it into his report. "Nonetheless," he says, "I cannot regard your results on sets of curves—and that is what we are talking about—as geometric results. To be sure, it all depends on what one understands by geometry." He then goes on to explain a bit his rather firmly held views about a proper distinction between geometry and the use of analysis in dealing with geometric matters. He thinks that in FRÉCHET's work the analytic aspect has almost completely eliminated the geometry. However, he says that FRÉCHET's way of defining the distance between two curves pleases him. He closes the letter with repeated assurance of his great interest in FRÉCHET's work and with assertion that in the thesis he sees essential progress in set theory.

In spite of the fact that SCHOENFLIES clearly was interested in FRÉCHET's work, his point of view toward it seems to have been severely constrained by his tendency to regard the BOLZANO-WEIERSTRASS principle and the BOREL or HEINE-BOREL covering principle as being central to a satisfactory theory. He didn't care for FRÉCHET's adjective "compact." On page 281 he suggested an alternative word: "lückenlos" (without gaps), or, as another possibility, "abschliessbar" (closable). He had mentioned the latter terminology to FRÉCHET in a letter dated October 27. He also wondered why FRÉCHET didn't consider adjoining ideal elements to render some of his sets compact.

The relationship between FRÉCHET and SCHOENFLIES seems to have been cordial, with no lasting effect of the ruffling of FRÉCHET that I mentioned as occurring early in their correspondence. At the request of SCHOENFLIES FRÉCHET read drafts and proof sheets of parts of the SCHOENFLIES report, and SCHOENFLIES expressed gratitude for this. The correspondence was continued, intermittently, at least until December of 1912. It shows that SCHOENFLIES continued to discuss some of his work with FRÉCHET and to seek help from him with certain questions.

In a paper published in 1910 [FRÉCHET, 39] FRÉCHET devoted a few pages (pp. 23–25) to discussion of his own point of view versus that of SCHOENFLIES on certain topics dealt with in the latter's report. FRÉCHET strongly maintains and gives reasons for his preference for not trying to introduce ideal elements (he calls them improper elements) in order to render sets compact. He adds that the difference between him and SCHOENFLIES on this matter is not one of deep principle, but, rather, a difference in bent or preference. On the issue of his manner of dealing with curves, FRÉCHET undertakes to show that he can express everything geometrically (rather than analytically), provided only that he can retain the notion of order on a curve. And he insists that, in his handling in his thesis of the conditions for compactness of a family of curves, his conditions are genuinely geometric while those of ARZELÀ are not entirely so, because they involve the intervention of parametric representations in an essential way, while his own do not.

## 8. The Early Work of F. Riesz on Abstract Point Set Topology

The Hungarian mathematician FRÉDERIC RIESZ published some important work on abstract point set topology in the years 1906–08. It was rudimentary, and remained undeveloped by RIESZ, however. In a paper of 1905 [F. RIESZ, 1] there is a brief reference to the idea of neighborhoods, and this may give a hint of the beginning of RIESZ's thinking about such things. These early publications of RIESZ show that he was acquainted with FRÉCHET's work on the same general subject in the years 1904–1906. For a proper perspective on the work of FRÉCHET, it is important to see what RIESZ did, for it had an important influence on FRÉCHET some years later (but not at the time he was working on his thesis).

F. RIESZ was slightly younger than FRÉCHET; he was born on January 2, 1880. He received his doctorate in Budapest in 1902, four years before FRÉCHET attained his doctorate.

RIESZ's paper of 1905 is on multiple order-types (mehrfache Ordnungstypen), an extension of the theory of simple order-types that is bound up with CANTOR's theory of ordinal numbers. CANTOR's theory of sets, including the concept of a limit point, is employed in the theory of order-types. RIESZ explains his desire to examine the unexplored field of multiple order-types as a natural result of the then recent upsurge of attention to point set theory in the plane and higher dimensions. But, he says, it is necessary to dispense with such concepts as distance and JORDAN'S écart, which do not fall within the group of basic notions that occur in the theory of order-types. One must, RIESZ says, carry over into this theory the concept of neighborhood in the general setting that was then current in the investigations of point set theory.[18] There is no reference to FRÉCHET or his work in this paper; nor is this paper a contribution to abstract point set topology. The reference to the role of neighborhoods and the desire to avoid the use of distance do nevertheless represent an early indication of the way RIESZ's ideas were taking shape.

RIESZ's long paper of 1907 in German on the origins of the concept of space [F. RIESZ, 4] is a translation of a paper [F. RIESZ, 3] originally published in Hungarian in two installments in 1906 and 1907. The original paper was presented to the Hungarian Academy of Sciences on January 22, 1906. This paper is not, in the main, about general point set topology, either concrete or abstract. It is a

[18] I quote here RIESZ's words from page 406 in [F. RIESZ, 1]. Dabei können von den Methoden der Mengenlehre nur jene angewandt werden, welche ausschliesslich mit dem Begriffe des Ordnungstypus operieren. Der Begriff der Distanz oder des Jordanschen "écart" wird zu entbehren sein, und es muss also der Begriff der Umgebung in jener allgemeineren Fassung übertragen werden, die auch schon sonstigen mengentheoretischen Untersuchungen geläufig ist.

quasi-philosophical paper in which RIESZ attempts to construct a mathematical model for the geometry of space as needed or used in physics. In so doing he formulates a notion of what he called a mathematical continuum. This notion is, in fact, that of an abstract space with a rudimentary topology defined axiomatically. (However, RIESZ does not himself use the word topology in this connection.) It is a bit strange that RIESZ does not include the idea of connectedness or unicoherence as a part of the idea of a continuum, but his mathematical continuum is not required to be connected. Thus his usage is contrary to customary usage of the term "continuum" at that time.

Some writers, discussing the role of RIESZ in the beginnings of abstract point set topology, appear either to have overlooked this paper or to have failed to appreciate its significance. The paper of RIESZ that is usually cited is the text of his talk at the Rome International Congress of Mathematicians in 1908 [F. RIESZ, 9]. In the published form of this talk are cited the papers [F. RIESZ, 3, 4]. However, the form of the material on abstract point set topology in the paper for the Congress at Rome is less specific and contains much less detail than the earlier papers. Both the German paper [RIESZ, 4] and its Hungarian original are now available in RIESZ's collected works, which were not published until 1960.

RIESZ's mathematical continuum is a class (he refers to it as a Mannigfaltigkeit) of elements in which there is a rule, subject to four axioms, that specifies, for each element $A$ and each set $t$, one and only one of the relationships: either $A$ is isolated from $t$ or is a limit element (Verdichtungsstelle) of $t$. It is clear from the paper that "Verdichtungsstelle" should *not* be translated as "condensation point" in the special sense given to that term by LINDELÖF. The set of all limit elements (if any) of $t$ is denoted by $t'$ and is called the derived set (Ableitung) of $t$. To say that $A$ is isolated from $t$ means merely that $A$ is not an element of $t'$.

In stating the four axioms given by RIESZ I shall use symbolism more than he did. The axioms are:

(1) If $t$ is a finite set, every element is isolated from $t$. (This is the same as saying that the derived set of a finite set is empty.)
(2) If $t$ is a subset of $u$, $t'$ is a subset of $u'$.
(3) If $t$ is the union of $u$ and $v$, $t'$ is contained in the union of $u'$ and $v'$. (This, together with (2), implies that $t'$ is the union of $u'$ and $v'$. RIESZ did not use the set-theoretical term "union.")
(4) If $A$ is in $t'$ and $B \neq A$, there exists a subset $u$ of $t$ such that $A$ is in $u'$ but $B$ is not.

RIESZ then introduces concepts as follows: $A$ set $u$ is called a neighborhood (Umgebung) of $A$ if $A$ is in $u$ but is isolated from the complement of $u$. RIESZ points out that the elements common to a finite number of neighborhoods of $A$ (that is, their intersection) is a neighborhood of $A$. An element is called an interior element of a set $t$ if $t$ is a neighborhood of $A$. $A$ set is called open if all of its elements are interior elements. (RIESZ says nothing about the status of an empty set or of the entire class, whose complement is empty; indeed, as was characteristic of much of the writing on set theory at that time, there is no specific mention of the concept of the empty set.) The boundary (Grenze) of a set $t$ is the set of elements of $t$ that are limit elements of the complement of $t$, plus the set of elements of the complement that are limit elements of $t$.

RIESZ does not, in this context, define the concept of a closed set. He does, however, introduce a symbol for the union of $t$ and $t'$, namely $\{t, t'\}$. RIESZ points out in a footnote that $(t')'$ is not necessarily contained in $t'$. Then, forgetting that he has not formally defined the property of being closed, he says that the fact that in ordinary set theory the derived set is closed must be done without (literally, missed or regretted) in the general theory of continua.[19]

RIESZ proves that if $A$ is in $t'$, every neighborhood of $A$ contains infinitely many elements of $t$. He asserts (but does not prove) that the inverse is true, meaning that if $A$ and $t$ are such that every neighborhood of $A$ contains infinitely many elements of $t$, then $A$ is in $t'$. In fact, it is a straightforward matter to prove that if every neighborhood of $A$ contains an element of $t$ which is not $A$, then $A$ is in $t'$. It is also true (although RIESZ does not point this out) that if $A$ is in $t'$ for some $t$, then $A$ is a limit element of every one of its neighborhoods. There may of course exist elements that are not limit elements of any set. For such an element any set that contains it is one of its neighborhoods.

RIESZ discusses the concept of connectedness. He calls a mathematical continuum *connected* (zusammenhängend) when it is impossible to express it as the union of two complementary sets, each of them open. (He neglects to specify that the sets should also be nonempty.) He calls a set connected if it cannot be broken into two sets $t_1$, $t_2$ such that $\{t_1, t_1'\}$ and $\{t_2, t_2'\}$ are disjoint. He calls a set *absolutely connected* if whenever $t$ is expressed as the union of two disjoint subsets $t_1$, $t_2$, there exists an element of one subset that is a limit element of the other. He then proves the theorem: A mathematical continuum is connected if and only if to each pair of its elements corresponds an absolutely connected set that contains the pair. It should be noted that what RIESZ defines as an absolutely connected set is what is today called merely a connected set and what he calls a connected set is not necessarily connected in today's sense of the term.

There are other interesting things touched upon very briefly by RIESZ. He mentions the possibility that in some particular instances of a mathematical continuum it is sufficient to deal with a special collection of neighborhoods, less than the totality of *all* neighborhoods of all elements. A sufficient collection of special neighborhoods is a collection such that, for each element $A$ that is not isolated from all sets, any given neighborhood of $A$ shall contain a special neighborhood of $A$. RIESZ also introduces a notion of *types* of mathematical continua. If $M_1$ and $M_2$ are two mathematical continua and if there is a one-to-one correspondence between the elements of $M_1$ and those of $M_2$ such that, given any element $A_1$ and any set $t_1$ in $M_1$, with correspondents $A_2$ and $t_2$ in $M_2$, the relationship of $A_1$ to $t_1$ is the same as that of $A_2$ to $t_2$ (that is, $A_1$ is in $t_1'$ if and only if $A_2$ is in $t_2'$), then $M_1$ and $M_2$ are called similar (RIESZ calls them *ähnlich verdichtet*). The class of all mathematical continua similar to a given $M$ is called the type of $M$ (RIESZ calls it the *Verdichtungstypus* of $M$). RIESZ stresses that the

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[19] RIESZ's words, on page 320 of the cited paper: „Die Abgeschlossenheit der Ableitung, die doch für die Theorie der Punktmengen eine so fruchtbare Prämisse ist, must somit in einer allgemeinen Theorie der mathematischen Kontinua vermisst werden".

believes consideration of the notion of classifying mathematical continua into types is important as a systematizing principle. He also states (page 322) that he believes his presentation of ideas is new and original. It may be noted that there is no use of RIESZ's fourth axiom in the development I have described. He does use the fourth axiom when he applies his general ideas to the construction of a mathematical model for the geometry of physical space.

In this paper [F. RIESZ, 4] RIESZ refers to three of FRÉCHET's notes in the Comptes Rendus [FRÉCHET, 9, 16, and 18], but not to FRÉCHET's thesis. He calls the results in the third note particularly interesting.

RIESZ's paper for the Rome Congress in 1908 presents, but in less extended form, the same notions of RIESZ I have described in the foregoing paragraphs. He does not include the fourth axiom, but states instead the assumption that each element $A$ of the derived set $t'$ of a given set $t$ is uniquely determined by the totality of those subsets of $t$ of which $A$ is a limit element. RIESZ expressed the view that to go further with a theory one would need additional axioms. Of the first three axioms alone he wrote (on page 19 of [F. RIESZ, 9]) that they are so broadly conceived that one cannot build much more on them alone.[20] RIESZ did not carry on with his early ideas about abstract general topology. RIESZ's ideas did eventually bear fruit at the hands of FRÉCHET, however. In work published in 1917 [FRÉCHET, 63], 1918 [FRÉCHET, 66], and 1921 [FRÉCHET, 75], FRÉCHET used the idea of an axiomatic development of general topology from axioms about derived sets. He used the first three axioms from RIESZ's work plus the axiom that derived sets are closed to define what he called $H$-classes. The $H$ is for HEDRICK. FRÉCHET also gave an alternative set of axioms for $H$-classes; this set of axioms differs from HAUSDORFF's four axioms for neighborhoods in just one respect. FRÉCHET's separation axiom is weaker. If $A$ and $B$ are distinct elements, FRÉCHET assumes the existence of a neighborhood of each one of the elements that does not contain the other, whereas HAUSDORFF assumes the existence of two disjoint neighborhoods, one of $A$ and the other of $B$.

## 9. Fréchet and Linear Functionals

In 1904, at roughly the same time that he was at work on the ideas that led to his early notes in the Comptes Rendus and ultimately to his thesis, FRÉCHET was writing two papers on linear functionals [FRÉCHET, 8 and 15]. Both were sent to his American friend E. B. WILSON in 1904 (see § 3) and the papers were published in the Transactions of the American Mathematical Society in 1904 and 1905 respectively. There was a third paper in this series; it was sent off to America early in 1907 and published in October of that year. This third paper is the most interesting of the three and the only one which contains results of lasting importance. The principal result in it, the representation theorem for continuous linear functionals on $L^2$ (to describe it in modern terms), was also announced by FRÉCHET to the Paris Academy of Sciences in June of 1907. The same result, discovered

[20] RIESZ's words: „Die drei Vorderungen ... sind so weit gefasst dass man auf ihnen allein nur sehr wenig weiter bauen kann."

independently by F. RIESZ, was announced by RIESZ in the same issue of the Comptes Rendus that carried FRÉCHET's announcement [FRÉCHET, 26) and [F. RIESZ, 1]. I shall now discuss FRÉCHET's work on linear functionals, concentrating most of the attention on the paper of October, 1907. I shall also discuss the work of RIESZ, showing how he was led to the representation theorem and describing the correspondence between RIESZ and FRÉCHET that took place in connection with their theorem.

The first paper [FRÉCHET, 8] starts from the base provided by a short note published by HADAMARD in 1903 [HADAMARD, 2]. It must be understood that at that time there did not exist a clearly formulated general (abstract) notion of a linear space with a topological structure in terms of which one could speak about continuous linear operations on a "space" of functions. The notion of linearity for a class of functions was based on the naturally occurring idea of linear combinations of functions. When notions of continuity of linear operations were involved they were based on notions of convergence that were naturally present in the context of the situation under discussion. HADAMARD, considering real continuous functions of the real variable $x$ on a finite closed interval $(a, b)$, showed that if $U$ is an opération fonctionnelle linéaire[21] on this class of functions, it can be represented in the form

$$U[f] = \lim_{n\to\infty} \int_a^b f(t)\, \Phi(t, n)\, dt, \tag{1}$$

where $\Phi(t, n)$ is, for each $n$, a special kind of continuous function of $t$. For example, it may be the result of applying $U$ to the function of $x$ defined by the expression

$$\frac{n}{\sqrt{\pi}} e^{-n^2(x-t)^2}.$$

The key to the situation is the known formula

$$f(x) = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} \int_b^a f(t)\, e^{-n^2(x-t)^2}\, dt. \tag{2}$$

HADAMARD's understanding about $U$ was that it acts on linear combinations of functions by the rule

$$U[c_1 f_1 + c_2 f_2] = c_1 U[f_1] + c_2 U[f_2]$$

and that

$$\lim_{n\to\infty} U[f_n] = U[f]$$

when $f_n(x)$ converges uniformly to $f(x)$.

FRÉCHET used, instead of (2), the then new theorem of FEJÉR about the uniform summability by arithmetic means of the FOURIER series of a continuous pe-

[21] This was prior to HADAMARD's decision to use the word *fonctionnel* as a noun. See § 5.

riodic function.[22] For the case in which $(a, b)$ is $(0, \pi)$ FRÉCHET obtained the result

$$U[f] = \lim_{n\to\infty} \int_0^\pi f(t)\, K_n(t)\, dt,$$

where $K_n$ is the continuous function of $t$ that results from applying $U$ to the function of $x$ defined by the expression

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{n+1-k}{n+1} \cos kx \cos kt.$$

FRÉCHET really did not go significantly further than HADAMARD in the quest for a general representation theorem for continuous linear functionals on the class of continuous functions. He speculated about conditions under which a representation of the form

$$U[f] = \int_a^b f(t)\, K(t)\, dt$$

might be valid, where the integral is a LEBESGUE integral and $K$ is not continuous, but merely LEBESGUE integrable; but he obtained no general result. A general representation theorem solving this problem was first obtained by F. RIESZ in 1909 [F. RIESZ, 10].

The second of this series of papers [FRÉCHET, 15] followed soon after the first. It embroidered on what he had done in the first paper, but contained no striking results. FRÉCHET explored the possibility of extending HADAMARD'S method to a more extensive class of functions—for example, to bounded measurable functions or to LEBESGUE summable functions.

In order to formulate and prove the representation theorem for continuous linear functionals on $L^2$ it is, of course, necessary to have in mind the nature of $L^2$ as a linear function class and an understanding of what it means for a linear functional on it to be continuous. FRÉCHET had not mentioned this class among his examples of $E$-classes in his thesis. RIESZ was clearly ahead of FRÉCHET in recognizing the interest in using the expression

$$\left\{\int_a^b |f(x) - g(x)|^2\, dx\right\}^{\frac{1}{2}} \tag{3}$$

as a metric (écart). His first exhibition of interest in this matter occurs in a note in the Comptes Rendus of November 12, 1906 [F. RIESZ, 2]. There RIESZ is interested in a theorem recently announced in Göttingen by E. SCHMIDT, to the effect that a family of continuous functions on a finite closed interval, forming a complete orthogonal system, is denumerable. He shows how to prove this theorem by relying on FRÉCHET'S demonstration of the fact that the class of continuous functions is a separable $E$-class (with écart as defined in Example A in my description of the second half of FRÉCHET'S thesis in § 5). The key to RIESZ'S argument

[22] This result appeared in FEJÉR'S paper which was published in the first issue of volume **58** (1904) of Mathematische Annalen. This issue appeared in December of 1903. See the Bibliography.

is the fact that each member of a set of functions that are mutually orthogonal is isolated from the other members of the set.

Seeking to generalize this argument to include a wider class of functions and use the LEBESGUE integral, RIESZ defines the distance (FRÉCHET's écart) of two functions $f$ and $g$ by the expression in (3). He observes at once that it will be necessary to regard $f$ and $g$ as identical if the expression in (3) is equal to zero. For convenience he considers merely bounded measurable functions that are LEBESGUE integrable on the interval $(0, 2\pi)$, saying that there is no difficulty in generalizing the reasoning. Then he observes that the resulting function class is separable (a denumerable and everywhere dense set being furnished by finite trigonometric sums with rational coefficients). In this way he extends SCHMIDT's theorem. Here we see FRÉCHET's thesis serving in an essential way in a thought process by which RIESZ formulated the concept of the function class and the metric for it that constitute what we now call the metric space $L^2(0, 2\pi)$, namely, the class of functions $f$ that are LEBESGUE measurable on $(a, b)$ and such that $f$ and $f^2$ are LEBESGUE integrable, two functions being regarded as the same if and only if they are equal except on a set of measure zero.

RIESZ's next step, accomplished in two more notes, in the Comptes Rendus of March 18, 1907 [F. RIESZ, 5] and April 8, 1907 [F. RIESZ, 6], was to show that there is a one-to-one correspondence between the elements of the real function class $L^2(a, b)$ and the class of real number sequences $\{\xi_n\}$ such that the series

$$\sum_{n=1}^{\infty} \xi_n^2 \tag{4}$$

is convergent. Such a correspondence can be exhibited by starting with a complete orthonormal system (that is, a maximal system of orthogonal functions each having the integral of its square equal to unity) of functions $\{\phi_n\}$ in the function class. Then, given a sequence $\{\xi_n\}$ such that the series (4) is convergent, there is a function $f$ of our class such that

$$\int_a^b f(x)\, \phi_n(x)\, dx = \xi_n, \qquad n = 1, 2, 3, \ldots . \tag{5}$$

Conversely, given a function $f$ of the class in question, the sequence $\{\xi_n\}$ defined by (5) is such that the series (4) is convergent. Finally, if $f$ and $g$ correspond to $\{\xi_n\}$ and $\{\eta_n\}$ respectively, then

$$\int_a^b f(x)\, g(x)\, dx = \sum_{n=1}^{\infty} \xi_n \eta_n . \tag{6}$$

The class of sequences $\{\xi_n\}$ such that the series in (4) is convergent is now designated by $l^2$. It is an $E$-class (metric space) with distance between $\{\xi_n\}$ and $\{\eta_n\}$ given by

$$\left\{\sum_{n=1}^{\infty} |\xi_n - \eta_n|^2\right\}^{\frac{1}{2}} . \tag{7}$$

RIESZ's work established not only that the classes $L^2(a, b)$ and $l^2$ are in one-to-one correspondence, but (by virtue of (6)) that the correspondence preserves distances between pairs of elements.

This work of RIESZ arose out of his interest in HILBERT'S published work of 1906 [HILBERT] on integral equations of FREDHOLM type, in which the integral equation problem is changed into a problem of a system of infinitely many linear equations in an infinite number of unknowns. HILBERT, however, was not using LEBESGUE integrals and all his functions were continuous.

Thus far in RIESZ'S work there is nothing about a representation theorem for continuous linear functional operations. Let us now turn to see what FRÉCHET was doing in 1907. In February of that year FRÉCHET, then teaching in a lyceé in Besançon, finished and sent off to America the third of his papers on linear operations [FRÉCHET, 24]. As published, it carries a statement that it was received by the editors on March 8. A letter from E. B. WILSON to FRÉCHET, dated May 24, informed FRÉCHET that VAN VLECK, one of the editors of the Transactions of the American Mathematical Society, wanted FRÉCHET to know that the paper would be published in either July or October. It appeared in the October issue. The published paper contains the representation theorem with proof in § 2, pages 439–441. However, it is evident from internal evidence in the paper (which I shall explain later) that some changes were made in the text of the paper subsequent to its original submission.

Further information about the course of events is provided by three letters from RIESZ to FRÉCHET, of dates May 21, June 19, and July 7, 1907. I know of no other correspondence bearing on this matter. In the correspondence RIESZ comments on his difficulties with the French language and thanks FRÉCHET for his "conseils grammatique dont je veux profiter". RIESZ was in Paris when he wrote on May 21, but he said that he would soon be leaving for Germany. He said he would like to meet FRÉCHET but couldn't go to see him in Besançon. He thanked FRÉCHET for a letter. Whether this exchange was the opening of the correspondence is not certain.

In the letter of May 21 RIESZ expressed interest in generalizing and completing the work of HADAMARD and FRÉCHET on linear functional operations. By that date, we know, RIESZ was in possession of knowledge of a method of studying the "space" of functions of class $L^2$ by means of its isometric correspondence with the "sequence space" $l^2$. RIESZ tells FRÉCHET that he has found the theorem that states (here I use modern terminology) that $L^2$ is complete. I quote from the letter:

"En ce qui concerne les applications de mes résultats à la théorie des opérations fonctionnelles, non seulement des opérations ordonnant à chaque fonction un nombre, mais aussi aux transformations linéaires (ou alors distributives) de l'espace des fonctions, moi je pensais bien que vos resultats et ceux de M. Hadamard peuvent être complètés et généralisés en même temps. M. Hadamard m'avait communiqué le même espoir. J'aimerais bien de pouvoir écrire une publication sur ce sujet. Naturellement, pour pouvoir appliquer la méthode, il faudra augmenter le domaine des fonctions et même temps élargir le sens de la notion de "opération continue." Alors, les problèmes reviendront à la résolution des systèmes d'équations linéaires d'une infinité d'inconnus.

"Je veux vous communiquer un théoreme qui suit immediatement de mes recherches mais que je n'avais (sic) pas encore communiqué. Pour la classe des fonctions sommables, de carrés sommables, et pour la notion de fonction limite que j'ai introduit dans ma publication antérieure sur les ensembles de fonctions (à l'aide de la notion de distance), pour qu'une série de fonctions $f_n$ converge (au sens donné) vers une fonction limite, il faut et il suffit que la distance de deux fonctions de rang assez élevé devienne telle petite que l'on veut. Donc l'ensemble consideré rentrera dans votre classification normale."[23]

In RIESZ's letter of June 19 (sent from Göttingen) he indicated that he had received two cards from FRÉCHET (via his brother M. RIESZ) but had not yet received copies of FRÉCHET's publications. FRÉCHET had evidently told RIESZ about a condition he had found as a criterion for compactness of a set of functions of integrable square. In RIESZ's letter he expresses pleasure that his own results have been useful to FRÉCHET and says "Quant à la condition de compacteté que vous avez trouvez, n'est-ce pas, au fond, c'est qu'il existe une fonction majorante à tous les fonctions de l'ensemble, relative à leurs séries de Fourier." Then he adds "Quant à votre envie de pouvoir mentionner mon résultat que la classe de *fs* (RIESZ thus abbreviates fonctions) de carré *s.* (he means sommables) est normale, vous avez peut-être déjà reçu la note (C. R. 13 mai) de M. Fischer qui retrouve le même résultats."

FRÉCHET's result about compactness for a set of functions in $L^2$ is given in his note of June 24 in the Comptes Rendus [FRÉCHET, 26]. He is dealing with the FOURIER coefficients $\{a_n\}$ and $\{b_n\}$ of functions $f$ of class $L^2$. The necessary and sufficient conditions for compactness of a set of functions are that (1) there exist a positive constant $M$ such that

$$\frac{a_0^2}{2} + \sum_{i=1}^{n} (a_i^2 + b_i^2) < M$$

for all $n$'s and all $f$'s in the set, and that (2) to each positive $\varepsilon$ there corresponds an $n$ such that

$$\sum_{i=n+1}^{\infty} (a_i^2 + b_i^2) < \varepsilon$$

for all $f$'s in the set.

There is a passage in RIESZ's letter of June 19 of unclear significance; it evidently responds to something that FRÉCHET wrote to RIESZ, but we don't know what that was. RIESZ writes: "Malheureusement, il me faut insister sur la notion de distance fondamentale, non seulement pour la notion de limite mais aussi pour une géométrie métrique des fonctions *s.* (sommable)."

Finally, in this letter of June 19, RIESZ tells FRÉCHET about his discovery of the representation theorem. It would seem that FRÉCHET had not yet told RIESZ of his own discovery of that theorem, for if he had, RIESZ would surely have made reference to the fact in the report of his own result to FRÉCHET. Here is what he wrote:

"Encore un résultat qui vous intéressera. En étendant le champ des opérations continues à l'ensemble des *fs* (functions) de carré *s.* (sommables) et en adop-

[23] Here RIESZ is referring to the fact that, in FRÉCHET's terminology, to be normal, an *E*-class must be complete (as well as perfect and separable).

tant une définition convenant des opérations continues, je parvenais au résultat que chaque opération linéaire peut être exprimé par

$$A[f(x)] = \int_a^b K(x) f(x)\, dx,$$

$K(x)$ étant une fonction bien définie sommable de c.s." (He means carré sommable.)

I shall postpone discussion of RIESZ's letter of July 7 to FRÉCHET until after I discuss the notes published by FRÉCHET and RIESZ in the Comptes Rendus of June 24.

In his note FRÉCHET denotes by (R) the function class $L^2(0, 2\pi)$ (of course he does not use the $L^2$ symbolism). He mentions RIESZ's note in the Comptes Rendus of November 12, 1906 and refers indirectly to RIESZ's note of March 18 by citing the result there mentioned (as a special case of his main theorem) that a trigonometric series, with coefficients $\{a_n\}$, $\{b_n\}$ such that the series

$$\sum_n (a_n^2 + b_n^2)$$

is convergent, is the FOURIER series of a function of the class $(R)$. He then states the representation theorem, giving the integral formula for $U[f]$, where $U$ is the linear operation. After that he states his theorem about conditions for compactness of a set in $(R)$. He does not mention RIESZ's note in the Comptes Rendus of April 8. Nor does he mention FISCHER.

Both RIESZ's note of June 24 and that of FRÉCHET were presented to the Académie by ÉMILE PICARD. RIESZ starts off by remarking that, in a lecture at Göttingen on February 26, 1907, he announced the results of his investigations of systems of summable functions, after which he communicated the principal results in the Comptes Rendus in March and April, and also in the Göttingen Nachrichten [F. RIESZ, 8]. He says that he had intended to wait until he could publish details and applications in a longer paper, but that the two notes by E. FISCHER in the Comptes Rendus (May 13, 27) forced him to alter his plans. RIESZ then writes discursively about his aims and his point of view, and, near the end of his note, states the representation theorem. Here is what he says: "Pour l'ensemble des fonctions sommables, de carré sommable, j'appelle *opération continue* chaque opération faisant correspondre à toute fonction $f$ de l'ensemble un nombre $U(f)$ et telle que, quand $f_n$ converge en moyenne vers $f$, $U(f_n)$ converge vers $U(f)$. L'opération est dite linéaire si $U(f_1 + f_2) = U(f_1) + U(f_2)$ et $U(cf) = cU(f)$. Alors pour chaque opération linéaire continue il existe une fonction $k$ telle que la valeur de l'opération pour une fonction quelconque $f$ est donnée par l'intégrale du produit des fonctions $f$ et $k$." It may be noted that, although RIESZ is not specific here about the class to which $k$ belongs, he was quite specific about this in his letter to FRÉCHET of date June 19. RIESZ had mentioned the term 'convergence en moyenne' earlier in the note of June 24, remarking that this notion amounted to the same as his notion of a limit introduced via a distance, given in his Comptes Rendus note of November 12, 1906.

I turn now to RIESZ' letter of July 7, 1907 to FRÉCHET. It was written from Göttingen. It is clear from the contents that he has at some time since June 19 received a letter from FRÉCHET in which the latter gave some indication of his method of proof of the representation theorem. He had also received (perhaps with the same letter) proof sheets of a paper by FRÉCHET on linear operations which (as I shall explain presently) must have been the paper that was published later that year in America [FRÉCHET, 24]. There is no way of telling whether FRÉCHET sent these things before of after the simultaneous appearance of the notes of FRÉCHET and RIESZ in the Comptes Rendus of June 24. RIESZ begins by begging FRÉCHET's pardon for not having hastened to reply. He says that when he last wrote he had already sent his note to PICARD for the Comptes Rendus. Then: "C'est comme ça que nous avons publié tous deux le même théorème dans le même numéro. Mais ça ne fait rien. Ce qui est très intéressant, c'est votre condition de compacteté.

"La démonstration de notre théorème commun que je possède et[24] la même que celle que vous m'avez indiqué. C'est un fait très naturel, connaissant le théorème que si $\Sigma a_i x_i$ converge pour chaque $\Sigma x_i^2$ convergent, $\Sigma a_i^2$ converge, le théorème devient conséquence immédiate de mon théorème fondamental.[25] J'aimerais apprendre, Monsieur, si vous avez trouvé vous même le lemme cité ou l'avez d'autre part; je connais ce lemme simple d'une note des M. M. Hellinger et Toeplitz parue il y a quelques mois dans les Gött. Nachr."[26]

Next in the letter RIESZ thanks FRÉCHET for having sent proof sheets of his interesting paper on linear operations and offers to send back the proofs if FRÉCHET so requests. He makes a suggestion about the first section of the paper, proposing that FRÉCHET mention the name of FEJÉR in connection with his use of LEBESGUE's result that the arithmetic means of the partial sums of the FOURIER series of a bounded measurable function $f$ are convergent to $f(x)$ except on a set of measure zero. RIESZ's suggestion is that FRÉCHET write that "M. Lebesgue a montré que la méthode de sommation de M. Fejér est applicable aux fonctions bornées et mesurables excepté pour un ensemble de mesure 0".[27]

It is time now to turn to consideration of the paper of FRÉCHET here in question. Section 1 of the paper is devoted to a study of continuous linear functionals on the class (which FRÉCHET denotes by $M$) of real functions defined on $(0, 2\pi)$ that are bounded and measurable. This is made into an $L$-class by defining $\lim f_n = f$ to mean that the $f_n$'s are uniformly bounded and that $f_n(x) \to f(x)$ except on a set of measure zero. FRÉCHET obtains a representation theorem for

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[24] RIESZ surely wrote *et* when he meant *est*.

[25] Here RIESZ is referring to the one-to-one correspondence between $L^2(0, 2\pi)$ and $l^2$.

[26] The work of HELLINGER & TOEPLITZ to which RIESZ is here referring is that cited in the bibliography.

[27] FEJÉR had shown in 1904 that the arithmetic means $\sigma_n(x)$ of the partial sums of the FOURIER series of $f$ converge to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ at each point where the one-sided limits $f(x+0)$ and $f(x-0)$ exist. He was using RIEMANN integrals.

a linear functional $U$ in the form

$$U(f) = \lim_{n\to\infty} \int_0^{2\pi} \int_0^{2\pi} f(x)\, u(y)\, \theta_n(y-x)\, dx\, dy,$$

where $\theta_n(t)$ is a simple linear combination of trigonometric functions, independent of $U$ and $f$, and $u(y)$ is a continuous function uniquely determined by $U$. In the argument FRÉCHET follows the method of FEJÉR in making use of the arithmetic means $\sigma_n(x)$ derived from the FOURIER series of $f(x)$. It seems clear from the wording that FRÉCHET followed the suggestions made by RIESZ in his letter of July 7, and thus we have one of the pieces of evidence that indicates that FRÉCHET made changes in the original text of the manuscript.

In Section 2 of the paper FRÉCHET states and proves the representation theorem. There is evidence here, also, of some modification of the original manuscript, which was mailed to America in February and received by the editors, early in March. FRÉCHET has put in citations to three of RIESZ's notes in the Comptes Rendus: those of November 12, 1906, March 18, 1907, and April 8, 1907. There is no citation of the note of June 24, however. FRÉCHET also cites the two notes in the Comptes Rendus in May by FISCHER, and uses the same symbol as FISCHER, namely $(\Omega)$, for the class $L^2(0, 2\pi)$. He had denoted it by $(R)$ in his note of June 24. He also uses FISCHER's term "convergence in mean". There is another thing that may be worthy of notice. FRÉCHET writes the representation theorem formula as

$$U_f = \int_0^{2\pi} f(x)\, k(x)\, dx,$$

$k$ being a fixed member of $(\Omega)$ determined by $U$. The somewhat curious thing is that FRÉCHET had used the letter $u$, not $k$, in his note in the Comptes Rendus of June 24, but RIESZ had used $k$ in *his* note of the same date, as well as in his letter of June 19 to FRÉCHET.

Because of these circumstances one naturally wonders about the extent to which FRÉCHET was in debt to RIESZ for the representation theorem. What could have been in the original version of Section 2 of FRÉCHET's paper? One must reject any suspicion that FRÉCHET simply stole the representation theorem that RIESZ communicated to him in the letter of June 19. There would not have been time for him to compose his own note for the Comptes Rendus of June 24 *after* receiving RIESZ's letter, and get his note accepted by PICARD and printed. Nor is it credible, I think, that he would have done this.

I can see no way of telling whether FRÉCHET sent the page proofs of his paper to RIESZ before or after he received RIESZ's letter of June 19. It is my judgement, on the basis of RIESZ's behavior subsequent to June 24, that when he received the page proofs from FRÉCHET he found in them FRÉCHET's fully detailed proof of the representation theorem, including the use of the lemma concerning which RIESZ questioned FRÉCHET. My conjecture is that FRÉCHET had an imperfect representation theorem in the original version of his paper—the manuscript he sent off to America in February—and that he was able to perfect the theorem some time after he saw RIESZ's note of March 18 in the Comptes Rendus. He could then have sent off a revised version of Section 2 in time to get back the proof sheets before the end of June. When he sent the corrected proof sheets back to America he would have been able to include the change suggested by RIESZ and also to include references to the work of FISCHER (as well as a reference to RIESZ's paper of April 8). This does not explain, however, why there is no reference to the June 24 issue of the Comptes Rendus, containing the notes by himself and RIESZ.

I shall explain how I support my conjecture and what I think FRÉCHET may have had in his original manuscript as I discuss his derivation of the representation theorem in the October paper in the Transactions of the American Mathematical Society.

To begin with I must refer to results contained in a famous paper by FATOU (see the Bibliography) in which it is proved that if $f$ and $k$ are two functions of the class $L^2(0, 2\pi)$, $f$ having FOURIER coefficients $\{a_n\}$ and $\{b_n\}$, $k$ having FOURIER coefficients $\{\alpha_n\}$ and $\{\beta_n\}$, then

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\,k(x)\,dx = \frac{a_0\alpha_0}{2} + \sum_{p=1}^{\infty}(a_p\alpha_p + b_p\beta_p). \tag{8}$$

If

$$s_n(x) = \frac{a_0}{2} + \sum_{p=1}^{n}(a_p\cos px + b_p\sin px), \tag{9}$$

then it is seen, by an easy calculation, that

$$\frac{1}{\pi}\int_0^{2\pi}|f(x) - s_n(x)|^2\,dx = \frac{1}{\pi}\int_0^{2\pi}|f(x)|^2\,dx - \frac{a_0^2}{2} - \sum_{p=1}^{n}(a_p^2 + b_p^2). \tag{10}$$

From (10) and (8), with $k = f$, it is clear that the expression on the left in (10) approaches 0 as $n \to \infty$.

In his paper, starting to look for a representation of the continuous linear functional $U$, using the metric for the class $L^2(0, 2\pi)$, FRÉCHET observes that as a consequence of the foregoing,

$$\lim_{n\to\infty} U(s_n) = U(f). \tag{11}$$

He introduces numbers $\alpha_n$ and $\beta_n$, not initially as FOURIER coefficients of any function but by the formulas

$$\pi\alpha_n = U(u_n), \qquad n = 0, 1, 2, \ldots,$$

$$\pi\beta_n = U(v_n), \qquad n = 1, 2, \ldots,$$

where the functions $u_n$ and $v_n$ are defined by the formulas

$$u_n = \cos nx, \qquad v_n = \sin nx.$$

Thus from (9)

$$\frac{1}{\pi}U(s_n) = \frac{a_0\alpha_0}{2} + \sum_{p=1}^{n}(a_p\alpha_p + b_p\beta_p). \tag{12}$$

From (11) and (12) it follows that

$$\frac{1}{\pi} U(f) = \frac{a_0 \alpha_0}{2} + \sum_{p=1}^{\infty} (a_p \alpha_p + b_p \beta_p), \tag{13}$$

the series on the right necessarily being convergent.

I pause to remark that FRÉCHET could have come this far in his reasoning without any knowledge of RIESZ's papers of March and April of 1907. All that is needed is the use of the metric for the class $L^2(0, 2\pi)$ and the result of FATOU in (8) for the special case in which $k = f$. FRÉCHET could have been motivated to use the metric for $L^2(0, 2\pi)$ by RIESZ's note of November 12, 1906, and he would most probably have been familiar with the paper of FATOU, with whom he was surely acquainted. In FRÉCHET's previous work on representation theorems he had shown a predilection for resorting to use of the FOURIER series of functions. Therefore I think it highly likely that FRÉCHET got as far as (13) in his original manuscript of the paper. The next, and crucial step, is to recognize that $\{\alpha_n\}$ and $\{\beta_n\}$ are the FOURIER coefficients of a function $k$ of class $L^2(0, 2\pi)$. As soon as this step is accomplished, formula (8) in the general form obtained by FATOU delivers the formula

$$U(f) = \int_0^{2\pi} f(x)\, k(x)\, dx.$$

My conjecture is that FRÉCHET didn't recognize that he might accomplish this step until after he saw RIESZ's note of March 18.

But once he saw RIESZ's note he realized that he should try to prove that the numbers $\alpha_n$, $\beta_n$ in (13) were such that the series

$$\sum_{p=1}^{\infty} (\alpha_p^2 + \beta_p^2) \tag{14}$$

is convergent. Then everything else would follow. Now, the numbers $\alpha_n$, $\beta_n$ are independent of $f$, and are hence independent of $a_n$ and $b_n$. The result in (13), together with RIESZ's note of March 18, shows that the series

$$\sum_{p=1}^{\infty} (a_p \alpha_p + b_p \beta_p)$$

is convergent whenever the series

$$\sum_{p=1}^{\infty} (a_p^2 + b_p^2)$$

is convergent. From this FRÉCHET is able to reason that the series in (14) is convergent. From RIESZ's letter of July 7 we can infer that he followed the same line of reasoning as FRÉCHET in arriving at the representation theorem, with *one* difference: RIESZ used a result of HELLINGER & TOEPLITZ for the last step in the argument, whereas FRÉCHET had to argue in a different way. In his paper he gives an argument based on the use of a theorem by DINI that was called to his attention by PRINGSHEIM, a German in Munich. PRINGSHEIM had written an article containing DINI's result for the Encyklopädie der Mathematischen Wissenschaften. This article was translated into French by J. MOLK, an acquaintance of FRÉCHET, for the French edition of the encyclopedia. Because FRÉCHET mentions a direct communication from PRINGSHEIM, he presumably didn't have to wait to get the information from the translated version of PRINGSHEIM'S article, which is in a part of the French edition dated May 30, 1907.

I believe that the foregoing discussion adequately supports my belief that FRÉCHET had attained a proof of the representation theorem before he learned from RIESZ that the latter had also found the theorem, but that FRÉCHET relied upon RIESZ's note in the Comptes Rendus of March 18 for the clue that enabled him to overcome a difficulty at a crucial stage of the work.

The remainder of this paper by FRÉCHET contains no definitive results of lasting interest.

In subsequent years a good deal of attention was devoted, by RIESZ and others, to the finding of representation theorems for continuous linear functionals on various function spaces. FRÉCHET'S work took him in other directions. RIESZ found an entirely different proof of the representation theorem for the case of the class $L^2(a, b)$. It is given, starting on page 180 of a paper [F. RIESZ, 11] written in Hungarian. He uses the $L^2$ notation and mentions both his own and FRÉCHET'S note about the theorem in 1907. In an important long paper of the same year, written in German [RIESZ, 12], he introduces the spaces $L^p\,(1 < p < \infty)$ and obtains the representation theorem for continuous linear functionals on them, In his paper (on page 476), referring to his announcement of the theorem in 1907 for the case $p = 2$, he writes "Zu gleichen Zeit hat den Satz auch Fréchet entwickelt," thus acknowledging the simultaneity of FRÉCHET'S presentation of the theorem.

## 10. Hilbert Space

The first use of the nomenclature "HILBERT space," so far as I know, was that by SCHOENFLIES on page 266 of his second report on point set theory (see § 7). In addition to using this designation for the sequence space now known as $l^2$, SCHOENFLIES mentions the work of RIESZ and FISCHER in 1907 that establishes the isometric isomorphism (not *their* terminology, of course!) between the class of functions of LEBESGUE integrable square and the sequence space $l^2$. The spread of the use of the name HILBERT space for the sequence space seems to have been gradual. Neither RIESZ nor FISCHER used it in 1907. Nor did FRÉCHET use it when he published a paper on the sequence space in 1908 [FRÉCHET, 28]. ERHARDT SCHMIDT, a student of HILBERT whose dissertation was published in 1905 wrote a paper [E. SCHMIDT] in which he discusses what he called, explicitly, geometry in a function space. SCHMIDT does not use the name HILBERT space. He regards a sequence $(x_1, x_2, \ldots)$ in the space as a function defined on the integers, and makes an acknowledgement to G. KOWALEWSKI in connection with the geometric point of view.[28] RIESZ called the sequence space HILBERT space on page 78 of his book

[28] SCHMIDT's words: „Die geometrische Deutung der in diesem Kapitel entwickelten Begriffe und Theoreme verdanke ich Kowalewski. Sie tritt noch klarer hervor, wenn $A(x)$ statt als Funktion als Vektor in einem Raum von unendlichvielen Dimensionen wird." Here $A(x)$ is the function defined for $x = 1, 2, \ldots$ with $\sum_{n=1}^{\infty} |A(n)|^2$ convergent.

[F. RIESZ, 13] published in 1913. HAUSDORFF also refers to the sequence space as HILBERT space, on page 287 of the 1914 edition of his book on Mengenlehre.

The function classes $L^2$ do not seem to have been regularly referred to as HILBERT spaces until after the introduction of the idea of an abstract HILBERT space, as first defined axiomatically in 1929 by VON NEUMANN (see the Bibliography). However, there was an exchange between FRÉCHET and LEBESGUE in 1923 in which FRÉCHET proposed to LEBESGUE that an $L^2$ space should be called HILBERT-LEBESGUE-RIESZ space; LEBESGUE rejected the proposal. This exchange is particularly interesting because of LEBESGUE'S reply. The circumstances were as follows. On October 23, 1923 the young Russian mathematicians PAUL ALEXANDROV and PAUL URYSOHN, both then in their twenties, wrote to FRÉCHET in what was, I think, the opening of an extended correspondence. They sent copies of three notes which they had written, hoping to have them published in the Comptes Rendus of the Paris Academy. The originals were sent to LEBESGUE, who was a member of the Academy and could present the notes. One of the notes, by URYSOHN alone, had the words "l'espace Hilbertien" in the title. It was presented by LEBESGUE in December and published in the January 2, 1924 issue of the Comptes Rendus [P. URYSOHN]. In the Archives of the Academy in Paris there is a two-page document bearing a commentary about the name HILBERT space in FRÉCHET'S handwriting. This was evidently sent to LEBESGUE. On the back, written in pencil, is LEBESGUE'S reply. The document carries no dates, but it is evident that it relates to the manuscript of the note by URYSOHN, and so belongs to the period October–December, 1923.

FRÉCHET seems to have had the intention of submitting a note for the Comptes Rendus expressing his comments on URYSOHN'S use of the name HILBERT space. The following sentences by FRÉCHET are crossed out and in the margin is written "Voir au dos" (see the back). FRÉCHET had written:

"L'espace auquel M. Urysohn attache avec raison le nom de M. Hilbert—espace à une infinité de dimensions où la distance est de la forme $\sqrt{\Sigma(x_n - y_n)^2}$ — n'a acquis toute sa importance actuelle que le jour où M. Riesz, complétant un théorème de M. Fatou, a reconnu l'identité (topoloqique et même métrique) de cet espace avec l'espace dont les éléments sont les fonctions de carrés sommables. Et cette identité disparait si l'on délaisse l'intégrale de M. Lebesgue pour garder celle de Riemann. Il serait donc plus approprié d'appeler cet espace l'espace de Hilbert-Lebesgue-Riesz. Je continuerai comme dans ma Thèse[29] à le désigner par la lettre $\Omega$."

Here is the reply of LEBESGUE, written out in his handwriting on the backs of the two pages:

"Je ne puis pas présenter une note ou il m'est attribué une part de priorité

[29] FRÉCHET was in error in this reference to his thesis, which nowhere contains any reference to the sequence space here designated as HILBERT space, or to the class of functions whose squares are LEBESGUE integrable. FRÉCHET *did* denote the $L^2$ class by $\Omega$ in his paper of 1907 on linear operations.

sur la conception d'un espace dont je n'ai jamais dit un seul mot. (The double underlining of "jamais" and "un seul" is by LEBESGUE.)

Qu'indirectement mes travaux montrent l'importance de cet espace c'est possible, mais si indirectement et tellement sans que je m'en sois jamais douté ni aperçu.

"Au fond, je ne suis pas d'accord avec vous en cet historique. Alors que, je vois bien, il n'y avait encore eu que des laïus[30] très généraux, et par suite un peu flous (*i.e.* blurred, not quite distinct), sur les espaces à une infinité de dimensions, HILBERT a fait de la géométrie métrique, précise de l'espace $\Sigma x_n^2$ convergent. Il a ainsi réuni bien des faits de la théorie des équations intégrales, les a éclairé et a preparé et rendu possible de nombreuses recherches sur ce sujet. Parmi celles-ci se trouvait en particulier celle du développement des fonctions réelles qu'il at étudié personnellement, et avec quel fruit! En continuant sa étude (qui, à elle seule, prouvait l'intérêt énorme de son espace, sans pour cela montrer tous les intérêts de cet espace) ses élèves se sont demandés qu'elle était la portée exacte de cet ensemble de développements en séries orthogonales qu'il considérait, c'est-à-dire qu'elle était la famille de fonctions dont l'espace d'Hilbert était la représentation. Et comme le théorème de Fatou-Parseval donnait une condition necéssaire pour qu'une fonction appartienne à la famille cherchée, la réciproque du théorème de Fatou êtait a étudier. C'est ce que firent simultanément F. Riesz et E. Fischer.

"Et notez ceci: Fatou s'était douté de la réciproque de son théorème. Pourquoi n'en a-t-il pas cherché la démonstration avec assez d'obstination pour la trouver (ce qui ne lui aurait pas été très long)? C'est qu'il n'en a pas compris l'intérêt.

"En réalité l'intérêt des fonctions de carrés sommables vient de ce qu'elles sont celles qui représente l'espace d'Hilbert, et les deux termes de cette observation ne peuvent être transposés. C'est à cause de l'intérêt de l'espace d'Hilbert qu'on a attaché de l'importance à la réciproque de Fatou, qu'on l'a demontré et qu'on a compris l'importance de Fatou lui même et des fonctions à carrés sommables.

"Et moi la dessus, où suis-je? Dans sommable (?), mais j'y suis avec Riemann, Cauchy, Archimède, etc. ... c'est trop." (The questioned word is hardly legible as LEBESGUE wrote it.)

I have encountered a story—I cannot recall where—that in his advanced years HILBERT once asked a younger mathematician, "Now tell me, what really is this Hilbert space, anyway?"

In FRÉCHET's paper of 1908 [FRÉCHET, 28] he makes a study of the geometry of the sequence space of HILBERT, denoting it by $\Omega$. In the first part of the paper he shows how to develop the geometry of lines, spheres, two-dimensional planes, and three-dimensional hyperplanes in $\Omega$, using no basic notion but that of point and no tools but those of distance and simple algebra. He makes only the barest use of addition and multiplication by a scalar, and never defines a norm or inner product. He shows how to set up a system whereby the hyperplane determined by

[30] The word laïus is school slang for a speech or lecture.

four points, not all in a two-dimensional plane, is put into one-to-one correspondence with the Euclidean 3-space of points $(X, Y, Z)$. He investigates the nature of the most general distance-preserving mapping of $\Omega$ onto all of itself. He shows that, if $T$ is such a mapping, and if $U$ is the mapping defined by $U(x) = T(x) - T(0)$, then $U$ is linear. Although not terribly difficult, this is a notable result. The corresponding result for an arbitrary normed linear space was first proved in 1932 by two Polish mathematicians, S. MAZUR and S. ULAM (see the Bibliography). FRÉCHET also investigates the way in which the mapping $U$ is represented by an infinite matrix that determines a linear transformation of the coordinates of an element of $\Omega$. Here he depends on the work of RIESZ; there is a gap in FRÉCHET's argument. Finally, he proves the necessity and sufficiency of the criterion he gave, in his note in the Comptes Rendus of June 24, 1907, for the compactness of a set in $\Omega$, and he gives the general form of a continuous linear functional on $\Omega$. This paper of FRÉCHET probably helped to diffuse awareness of the characteristics of the HILBERT space of sequences.

## 11. Best Approximation by Trigonometric Sums

FRÉCHET's work during 1903–04 of preparing BOREL's lectures for publication in book form acquainted him with the theory of best approximation to a continuous real function, by polynomials. With this preparation and with his thesis completed, FRÉCHET turned to the problem of best approximation by finite trigonometric sums. Specifically, he asked about best approximation to a continuous function $f$ of period $2\pi$ by functions of the form

$$T_n(x) = u_0 + \sum_{k=1}^{n} (u_k \cos kx + v_k \sin kx)$$

with numerical coefficients $u_k$, $v_k$. FRÉCHET announced some results in a note in the Comptes Rendus [FRÉCHET, 25] of date January 27, 1907. He published his work in full [FRÉCHET, 27] in 1908. In this latter work he shows that for given $f$ and $n$ there is a unique $T_n$ such that the maximum of $|f(x) - T_n(x)|$ is strictly smaller than the maximum $|f(x) - S_n(x)|$ for any *other* trigonometric polynomial $S_n$ of the same order $n$ (where $n \geqq 1$). He shows, moreover, that $T_n$ depends continuously on $f$ in the sense that, if $U_n$ is the trigonometric polynomial of order $n$ that provides the best approximation to another function $g$, then, with $f$ and $n$ fixed, to each positive number $\varepsilon$ there corresponds a positive number $\delta$ such that $\max |T_n(x) - U_n(x)| < \varepsilon$ when $g$ is such that $\max |f(x) - g(x)| < \delta$. Also, $\lim_{n\to\infty} T_n(x) = f(x)$, uniformly in $x$.

FRÉCHET deduced these results as applications of some results on rather general formulations of the problem of best approximation to a function in a certain class by functions from a certain family of functions depending on a finite number of parameters.

When I first read this work of FRÉCHET I was quite surprised, for while these results on best approximation by trigonometric functions are well known in modern literature on the subject, I had never seen any attribution of credit to FRÉ-

CHET for them. As a consquence I made a more thorough examination of the situation in an effort to appraise FRÉCHET's contribution. My conclusion (reasons for which I shall explain a bit later on) is that these results were indeed original with FRÉCHET. It would seem, however, that in the opinion of later experts, FRÉCHET's achievements in this field are not regarded as remarkable. They were not "great finds."

As it happened, FRÉCHET was not alone in this discovery of the results on best approximation by trigonometric polynomials. An American, J. W. YOUNG, published a paper [J. W. YOUNG] in 1907 that contains results overlapping some of those of FRÉCHET. His paper was completed in September of 1906, but was not published until after the appearance of FRÉCHET'S note in the Comptes Rendus early in 1907. In a footnote in his paper (on page 340) YOUNG mentions FRÉCHET's note in the Comptes Rendus and states that FRÉCHET's result is a special case of his own Theorem 5. In his long paper on the subject FRÉCHET refers to the paper by YOUNG. He calls it an interesting paper and says that YOUNG examined the same problem he is considering, and from an analogous point of view. However, says FRÉCHET, YOUNG's reasoning and results are sufficiently different from his own that there is no useless repetition ("double emploi" are the words used by FRÉCHET). FRÉCHET'S comment that his note in the Comptes Rendus appeared before YOUNG'S paper led YOUNG to write to FRÉCHET a few years later (on April 11, 1912). Perhaps it was only then that YOUNG, who was then at Dartmouth College, in New Hampshire, saw FRÉCHET's long paper. The burden of YOUNG's letter was that he thought FRÉCHET appeared to be claiming priority over him because of the times of publication. YOUNG stressed that his results were complete in 1906, were announced to the American Mathematical Society in April and September of that year, and that abstracts of the work were published in 1906 in the Bulletin of the American Mathematical Society. He said he hoped that FRÉCHET would find an opportunity "to correct the misunderstanding." I know of no indication that FRÉCHET ever did anything about this.

The problem of best approximation to a real continuous function of a real variable $x$ by a polynomial in $x$ goes back at least as far as some work of P. L. CHEBYCHEV in 1858. CHEBYCHEV's method was improved by PAUL KIRCHBERGER in a thesis at Göttingen in 1902 and by BOREL in his book of 1905. The works of FRÉCHET and YOUNG use arguments very much like those of BOREL, but they bring in some more general ideas in the formulation of the problem. They consider a class of approximating functions that are characterized by certain properties rather than being defined individually and in explicit form. Moreover, the results on best approximation by trigonometric polynomials are not derived from results on approximation by ordinary (algebraic) polynomials by a mere change of variable. The technique of handling the problem in this way is not without difficulty, although it has been worked out; but the technique was not available at the time we are considering.

DE LA VALLÉE POUSSIN wrote extensively on the theory of approximation by polynomials, but I have not found any recognition of FRÉCHET'S work by DE LA VALLÉE POUSSIN. However, in a review [LEBESGUE], written by LEBESGUE, of a book on this subject by DE LA VALLÉE POUSSIN, LEBESGUE writes to the following effect (I paraphrase): From work by Tchebycheff (LEBESGUE'S spelling of the name), perfected by Kirchberger and Borel, it turns out that, given $f$, to each $n$ corresponds a polynomial $\pi_n$ of degree $n$, unique and of best approximation to $f$, with the properties that there exist $n+2$ values $a_i$ of $x$, $a_1 < a_2 < \ldots < a_{n+2}$, such that $f(x) - \pi_n(x)$ takes on the values $+\varrho_n$ and $-\varrho_n$ (or $-\varrho_n$ and $+\varrho_n$) in alternating succession at $a_1, a_2, \ldots, a_{n+2}$, where $\varrho_n = \max|f(x) - \pi_n(x)|$. Then there is an analogous theory for trigonometric approximation. Tchebycheff himself alluded to the possibility of replacing polynomials by other functions. The case of trigonometric approximations was perfected by Fréchet and Young. (This is the end of my paraphrasing of LEBESGUE.)

Another instance of attribution of credit to FRÉCHET is found in a paper by ALFRED HAAR [A. HAAR]. HAAR cites the papers of FRÉCHET and YOUNG. In a footnote on page 308 HAAR refers to "Fréchet's theorem" as a special case of his own theorem. HAAR considers Euclidean space of $n$ dimensions and examines the problem of finding, for a given closed and bounded set $S$ and a given continuous function $f$, the best approximation to $f$ on $S$ by a certain sort of linear combination of preassigned linearly independent functions. HAAR's method is based on the idea of a unique supporting plane to a convex body.

## 12. Conclusions

My conclusions, based on my study of FRÉCHET's work, on documents which I saw in Paris, and on other reading and conversation, relate only to the accomplishments of FRÉCHET up through 1908, and they are divided into two parts. The first part is about FRÉCHET's role as the first mathematician to make a systematic and rather extensive study of general point set topology using an abstract and axiomatic approach. I think it is unquestionable that he opened the way and that his work had an important impact. I have cited some of the evidence of the impact. More evidence can be cited in a review of the developments after 1906 until 1928, when FRÉCHET's book on abstract spaces was published. I plan to discuss this aspect of the matter in a later essay.

FRÉCHET's inspiration and motivation did not come out of the blue sky, of course. He was clearly influenced by what he knew of CANTOR's work. I think his main inspiration probably came from HADAMARD and BOREL, especially the former, who I am sure was responsible for seeing to it that FRÉCHET learned what he should from the works of VOLTERRA, ASCOLI, and ARZELÀ. However, if HADAMARD was fully frank in what he wrote about FRÉCHET for the Académie des Sciences, FRÉCHET's decision to go at things in a totally abstract way was his own decision. By his own account that decision was influenced by the existence of an abstract theory of groups and by certain things in BOREL's writings (see § 5).

There were, of course, certain trends tending to facilitate FRÉCHET's move into abstraction. The works of RIEMANN, WEIERSTRASS and HILBERT contributed significantly to that trend. However, I have not made a careful historical study of that trend or of the extent to which it might lessen the importance of FRÉCHET's contribution. The fact that F. RIESZ, independently of FRÉCHET, but very slightly later, started a different abstract axiomatic approach to general topology, may be cited as evidence that abstraction was "in the air" at that time. Nevertheless, FRÉCHET's short note of 1904 [FRÉCHET, 9] broke absolutely fresh ground.

Although VITO VOLTERRA's work certainly had some influence on FRÉCHET's work taken as a whole, I think a good deal of it was exerted indirectly, through HADAMARD. I see little or no reason for thinking that VOLTERRA contributed directly to the shaping of FRÉCHET's ideas on *L*-classes, *V*-classes, or *E*-classes.

The extent of the influence of ASCOLI and ARZELÀ is uncertain. In § 4 I made a conjecture about ARZELÀ and FRÉCHET. It is possible that the papers of ASCOLI and ARZELÀ, especially the latter, had more to do with FRÉCHET's formulation of the notion of compactness than I have conjectured.

FRÉCHET's mathematical development was significantly influenced by his acquaintance with RENÉ BAIRE and HENRI LEBESGUE, both personally and through their work, but there is no reason to think they influenced him toward abstraction. Their influence was in the furtherance and deepening of FRÉCHET's knowledge of real function theory. In the case of LEBESGUE, he played an important role in helping FRÉCHET along with his mastery of LEBESGUE's theory of measure and integration and the application of this theory to FOURIER series. Concrete evidence of LEBESGUE's role is exhibited in his correspondence with FRÉCHET in the years 1904–1906; see [TAYLOR & DUGAC]. It must be remembered that LEBESGUE's great work was quite new at that time. His thesis was published in 1902, his book on integration in 1904, and his book on trigonometric series in 1906.

FRÉCHET's displeasure with BOURBAKI's treatment of him, RIESZ, and HAUSDORFF in the discussion of topology in the Histoire, and the ensuing reverberations of that displeasure lasted through the 1960's and into the 1970's. FRÉCHET's daughter showed me two letters to him that no doubt pleased him and gave him a sense of vindication. One, a reply to a letter from FRÉCHET, was from PAUL ALEXANDROV in Moscow, dated October 21, 1967. As was his wont in their correspondence in the 1920's, he opened his letter with the salutation "Cher Maître et ami." The following paragraph in his letter shows that FRÉCHET had written him about the BOURBAKI Histoire: "Je comprend très bien l'émotion que vous éprouvez en lisant l'histoire des mathématiques écrite par N. Bourbaki. Mais enfin, ce n'est pas cette histoire, mais la vrai Histoire de la Science, telle qu'elle est en Réalité, qui détermine pour chaque savant sa place dans l'Évolution de la Science et son rôle dans cette évolution. Quant à votre place et à votre rôle—c'est la place parmi les plus grandes mathématiciens de notre temps, c'est le rôle d'un vrai MAÎTRE. Je n'ai pas encore atteint l'âge de 89 ans (this was FRÉCHET's age); mais j'ai 71—c'est aussi quelque chose et je crois que j'ai le droit d'avoir mon opinion sur ce que c'est que la Science Mathématique contemporaine et quels sont ses créateurs et ses Maîtres." The other letter was from ARNAUD DENJOY, written July 13, 1971, when DENJOY was eighty seven years old and FRÉCHET was in his ninety third year. DENJOY wrote: "Ta lettre me touche beaucoup et je t'en remercie profondement. Ta appréciation, trop flatteuse de mes travaux mathématiques, m'est très sensible. Tu as joué un rôle immense, tu as ouvert l'avenir aux mathématiques et j'ai l'impression que les Académies sont restées aveugles devant le mouvement universal auquel tu avais donné l'impulsion et n'ont pas su te rendre des hommages dus." DENJOY does not refer to the BOURBAKI Histoire, but he indulges himself by berating "the bourbakists."

FRÉCHET's unhappiness with the BOURBAKI Histoire was not, so far as I know, coupled with a dislike or disdain for the mathematics of BOURBAKI (a dislike and disdain which was held by DENJOY). What bothered FRÉCHET was an injury to his sense of what was his due as the creator of the first systematic point set theory in abstract spaces. Actually, I think, a great deal of his feeling of hurt was due to a fundamental difference between his notion of how the history of topology should be written and the ideas that guided the members of the BOURBAKI group in the composition of their "notes historiques". These notes originally appeared piecemeal at the end of various sections of BOURBAKI'S large treatise on the Elements of Mathematics and were later gathered together in the Histoire. Thus, for example, in the note at the end of the chapter on uniform structures (Chapter II in [BOURBAKI, 1]), FRÉCHET is credited with the creation of metric spaces and recognition of the "principle of CAUCHY" (*i.e.* the criterion for completeness). In the slightly revised version appearing in the Histoire he is more clearly credited, also, with the discovery of a theorem about the notion of total boundedness (page 153 in [BOURBAKI, 3], page 182 in the revised edition of 1974 [BOURBAKI, 4]). There is also a note historique about metric spaces on page 95 in [BOURBAKI, 2], where FRÉCHET is mentioned again. See page 176 in the 1960 edition of the Histoire, page 205 in the 1974 edition. Thus, it is clear, the topical arrangement of BOURBAKI'S Histoire *is based upon the notion of looking back from the present to ascertain where present ideas and results came from and where they can first be identified in a form more or less consonant with current usage.* HAUSDORFF appeared as a more important figure than FRÉCHET or RIESZ in the historical note on topological spaces because it is possible to see in HAUSDORFF'S book of 1914 a more than merely nascent topology that more clearly resembles the topology of 1940 than anything visible in the work of FRÉCHET or RIESZ before 1914. And there is, *in this section of the Histoire*, no reference to FRÉCHET'S important act of creating metric spaces. (A qualifying phrase, to indicate explicitly the non-inclusion of consideration of metric spaces, was added in the 1974 edition of the Histoire, page 179.) Instead, there is a rather denigrating reference which asserts that FRÉCHET, by starting from the notion of the limit of a sequence (notion de limite dénombrable), failed to construct a serviceable or convenient (commode) and fruitful (fécond) set of axioms. This is surely unduly nonappreciative of FRÉCHET, especially since FRÉCHET himself recognized that he would not get far with $L$-classes alone, and promptly moved on to $V$-classes and $E$-classes, devoting most of Part I to these latter classes, and all of Part II to $E$-classes (metric spaces). As I think I can show in a later essay, HAUSDORFF was very likely strongly influenced by the concept of metric spaces before he finally arrived at his four neighborhood axioms for what we now call a HAUSDORFF space.

It is interesting to observe that in a quite recent book by one of the original BOURBAKI group, JEAN DIEUDONNÉ, in which he undertakes a broad and coherent overview of the development of functional analysis [DIEUDONNÉ], the author does not fail to assign to FRÉCHET a very important role. DIEUDONNÉ remarks (on page 1) that functional analysis is a rather complex blend of algebra and topology. He also says "As a matter of fact, it is almost impossible to dissociate the early history of General Topology (and even of the set-theoretic language) from the beginnings of Functional Analysis, since the sets and spaces which (after the subsets of $\mathbb{R}^n$) attracted most attention consisted of *functions*." At the beginning of DIEUDONNÉ'S Chapter 5 (page 97) he writes: "Between 1900 and 1920, there was a sudden crystallization of all the ideas and methods which had been slowly accumulating during the XIXth century and which we have described in the previous chapters. This was essentially due to the publication of *four fundamental papers*:

Fredholm's 1900 paper on integral equations;
Lebesgue's thesis of 1902 on integration;
Hilbert's paper of 1906 on spectral theory;
Fréchet's thesis of 1906 on metric spaces."

Then, later in Chapter 5 (on pages 116–117), DIEUDONNÉ goes into a bit of detail about FRÉCHET'S thesis, and writes: "But the greatest merit of Fréchet lies in the emphasis he puts on three notions which were to play a fundamental part in all later developments of functional analysis: compactness, completeness, and separability. Moreover, he did not limit himself to deriving general theorems in an abstract setting, but more than half of his thesis is devoted to very concrete metric spaces (as they came to be called later) very closely linked to Analysis: ..."

By giving due credit to FRÉCHET there is no belittling of HAUSDORFF, for unquestionably HAUSDORFF had an enormous influence on the development of topology after 1914.

Turning now to FRÉCHET'S early work on linear functionals, I conclude that, although this work was historically significant and important, it does not rank with his thesis. FRÉCHET and RIESZ together (but independently) discovered the first truly satisfactory representation theorem for a continuous linear functional on a function space. FRÉCHET gave the first published proof of the theorem. FRÉCHET'S interest in such representation theorems was sparked by HADAMARD'S effort in his paper of 1903 [HADAMARD, 2]. HADAMARD'S result was not really satisfactory, for it did not establish a unique and tangible class of mathematical objects which was put into one-to-one correspondence with the class of all continuous linear functionals on the specific class of functions he had under consideration. FRÉCHET'S first efforts were not satisfactory either. Only in his third paper on linear functional operations [FRÉCHET, 24] did he achieve a genuinely satisfactory and lasting result. RIESZ'S interest in representation theorems seems to have been stimulated, or at least intensified, by the work of HADAMARD and FRÉCHET. (See the discussion of his letter of May 21, 1907 in § 9.) It is to be noted that FRÉCHET'S success in completing his proof of the representation theorem in his long paper of 1907 was dependent on what he learned about the FOURIER coefficients of functions of class $L^2$ in one of the short notes by RIESZ [RIESZ, 5]. See the discussion in § 9. For mathematical ingenuity in matters of analytical techniques, RIESZ outranks FRÉCHET. One can see this clearly if one reads extensively the research work of each of the two men. FRÉCHET never developed a powerful technique as an analyst.

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