On the Need to Rewrite the History of Greek Mathematics

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Communicated by W. HARTNER

'History is the most fundamental science, for there is no human knowledge which cannot lose its scientific character when men forget the conditions under which it originated, the questions which it answered, and the function it was created to serve. A great part of the mysticism and superstition of educated men consists of knowledge which has broken loose from its historical moorings.'
BENJAMIN FARRINGTON[1]

'It would not occur to the modern mathematician, who uses algebraic symbols, that one type of geometrical progression [*i.e.*, 1, 2, 4, 8] could be more perfect or better deserving of the name than another. For this reason algebraic symbols should not be employed in interpreting such a passage as ours [*i.e.*, Plato, *Timaeus*, 32A, B].'
FRANCIS M. CORNFORD[2]

'Any historian of mathematics conscious of the perils and pitfalls of Whig history quickly discovers that the translation of past mathematics into modern symbolism and terminology represents the greatest danger of all. The symbols and terms of modern mathematics are the bearers of its concepts and methods. Their application to historical material always involves the risk of imposing on that material, a content it does not in fact possess.'
MICHAEL S. MAHONEY[3]

The previous string of quotations is (most certainly) **not** illustrative of the ways in which the history of mathematics has traditionally been written. The authors of the quotations themselves have not always practiced what they occa-

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[1] *Greek Science Its Meaning For Us* (Harmondsworth: Penguin Books, 1953), 311.

[2] *Plato's Cosmology* (New York: The Liberal Arts Press, 1957), 49.

[3] *The Mathematical Career of Pierre de Fermat (1601–1665)* (Princeton, N.J.: Princeton University Press, 1973), XII–XIII.

sionally preached.[4] Indeed, the discipline is exceedingly rich in works written (as it were) as a living illustration of P. W. BRIDGMAN's exhortation:

> ... the past has meaning only in terms of the present. The impartial recovery of the past, uncontaminated by the influence of the present, is held up as a professional ideal, and a criterion of technical competence is the degree to which this ideal is reached. This ideal is, I believe, impossible of attainment, and cannot even be formulated without involvement with meaningless verbalisms.[5]

The situation is particularly scandalous in the history of ancient and medieval mathematics. It is in truth deplorable and sad when a student of ancient or medieval culture and ideas must familiarize himself first with the notions and operations of *modern* mathematics in order to grasp the meaning and intent of modern commentators dealing with ancient and medieval mathematical texts. With very few and notable exceptions, Whig history *is* **history** in the domain of the history of mathematics; indeed, it is still, largely speaking, the standard, acceptable, respectable, 'normal' kind of history, continuing to appear in professional journals and scholarly monographs. It is *the* way to write the history of mathematics. And since this is the case, one is faced with the awkward predicament of having to learn the language, techniques, and ways of expression of the modern *mathematician* (typically the manufacturer of 'historical' studies) if one is interested in the *historical* exegesis of *pre-modern* mathematics; for it is a fact that the representative audience of the mathematician fathering 'historical' studies consists of historians (or people who identify themselves as historians) rather than mathematicians. The latter look condescendingly upon their (usually older) colleagues in their new and somewhat strange hypostasis which seems to indicate to the working mathematician an implicit, but public, confession of professional (*i.e.*, mathematical) impotence.

As to the goal of these so-called 'historical' studies, it can easily be stated in one sentence: to show how past mathematicians hid their modern ideas and proce-

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[4] Ironically, the very works of FARRINGTON and MAHONEY mentioned above are cases in point for the very popular syndrome referred to by HEINE in the following phrase: 'Sie predigen öffentlich Wasser, Und drinken heimlich Wein'; the difference being, however, that, in this instance, both the 'preaching' and the 'drinking' take place openly, in the public domain. For an analysis of FARRINGTON's work see LUDWIG EDELSTEIN, 'Recent Trends in the Interpretation of Ancient Science', *Roots of Scientific Thought A Cultural Perspective*, P. P. WIENER & A. NOLAND (eds.) (New York: Basic Books, 1957), 90–121; as to MAHONEY's book, I will be dealing with it in a future essay review in *FRANCIA-Forschungen zur westeuropäischen Geschichte*, the journal of the *Institut Historique Allemand* in Paris.

[5] 'Impertinent Reflections on the History of Science', *Philosophy of Science*, **17** (1950), 63–73, at 64; it is also there that BRIDGMAN says (among other things): 'It seems to me that there is a very real danger in a too assiduous devotion to the historical point of view...' (*ibid.*, 72). Without denying the pregnant philosophical problems stemming from the reconstruction of the past, and accepting the obvious conclusion that 'the impartial recovery of the past', *etc.* is indeed an impossible ideal, it does not follow that abandoning irrevocably this unattainable ideal is tantamount to an abandonment of the historical method. Indeed, to repeat a truism, the fact the historian knows that it is in principle impossible to relive the past and that his reconstruction is inherently deficient and inadequate represents for him the utmost challenge *to try* and look at the past through sympathetic and understanding eyes and to achieve a reconstruction which does no patent violence to that which is to be reconstructed. That there is something to be reconstructed and understood is taken for granted by any mentally healthy historian worth his salt.

dures under the ungainly, *gauche*, and embarrassing cloak of antiquated and out-of-fashion ways of expression; in other words, the purpose of the historian of mathematics is to unravel and disentangle past mathematical texts and transcribe them into the modern language of mathematics, making them thus easily available to all those interested.

If the preceding description seems unconvincing and written by a reckless partisan of hyperbole, the balance of this paper should correct this mistaken impression. Indeed it is the purpose of this paper to show what is historically wrong with the traditional way the history of ancient Greek mathematics has been written and to call to the new generation of historians of Greek mathematics to rewrite that history on a new and historically sane basis.

I

One of the central concepts for the understanding of ancient Greek mathematics has customarily been, at least since the time of PAUL TANNERY and HYERONIMUS GEORG ZEUTHEN, the concept of 'geometric algebra'. [6] What it amounts to is the view that Greek mathematics, especially after the discovery of the 'irrational' by the PYTHAGOREAN school, is *algebra* dressed up, primarily for the sake of rigor, in geometrical garb. The reasoning of Greek mathematics, the line of attack of its various problems, the solutions provided to those problems, *etc.* all are essentially *algebraic*, though, to be sure, for reasons that have never been fully elaborated, attired in geometrical accouterments. We are, then, not only authorized to look for the algebraic 'subtext' (so to speak) of any geometrical proof, but it is indeed wise (historically!) always to transcribe the geometrical content of any proposition in the symbolic language of modern algebra, especially when the former is particularly cumbersome and awkward, while the recourse to the latter always makes the logical structure of the proof clear and convincing, without thereby losing

[6] *Cf.*, for instance PAUL TANNERY, *Mémoires Scientifiques*, **1**, *Sciences Exactes Dans L'Antiquité*, J.-L. HEIBERG & H.G. ZEUTHEN (eds.) (Toulouse: Edouard Privat and Paris: Gauthier-Villars, 1912), 254–280. Characteristically, the title of this study is "De la solution géométrique des problèmes du second degré avant Euclide"! See also, *Mémoires Scientifiques*, **3** (1915), 158–187 and 244–250. For ZEUTHEN'S views see his *Die Lehre von den Kegelschnitten im Altertum* (Hildesheim: Georg Olms, 1966, being a photographic reproduction of the Copenhagen, 1886, edition), 1–38 and *Geschichte der Mathematik im Altertum und Mittelalter* (Copenhagen: Andr. Fred. Höst & Sön, 1896), 32–64.

P. TANNERY & H.G. ZEUTHEN were not the originators of the concept of 'geometric algebra'. PIERRE DE LA RAMÉE seems to deserve the doubtful credit for this invention. It was he, apparently, who 'discerned' that the algebraic art must underlie some parts of EUCLID'S *Elements* (Books II and VI) and, perhaps, also Greek analysis. (*Cf.* MICHAEL S. MAHONEY, 'Die Anfänge der algebraischen Denkweise im 17. Jahrhundert', *Rete*, **1** (1971), 15–31, especially p. 25.)

It is quite interesting (and, as will become clear later, strongly supportive of one argument of this paper) that practically all the founders of modern mathematics (VIÈTE, DESCARTES, and FERMAT) followed RAMUS in his belief that algebra lies at the root of Greek analysis! Remarkably enough, WILLIAM OUGHTRED in the seventeenth century, in his most famous mathematical textbook, *Clavis mathematicae*, is also of the opinion that algebra can serve as a means of understanding difficult problems in EUCLID, ARCHIMEDES, APOLLONIOS, and DIOPHANTOS (*cf. ibid.*, n. 49, 28). Our nineteenth and twentieth century historians of mathematics can indeed be proud of their lengthy and aristocratic mathematical lineage; in truth they have made OUGHTRED'S 'insight' the keystone of their methodological and interpretive approach!

anything not only in generality but also in any possible *sui generis* features of the ancient way of doing things.[7]

[7] Thus spake TANNERY: 'Je veux parler de tout le livre *X* d'Euclide et de la théorie des irrationnelles qui s'y trouve renfermée... Ce n'est, rien moins que le détail complet de la solution géométrique de l'équation bicarrée et le commencement de celle de l'équation tricarée, avec l'invention d'une nomenclature destinée à suppléer au défaut de notations' (*op. cit.*, **1** 263). In a historical appendix written for his brother's (JULES) *Notions de Mathematiques* (Paris: Ch. Delagrave, 1903), PAUL emphatically states: 'Quoique leurs [*i.e.*, the ancients] procédés d'exposition aient toujours présenté, par rapport aux nôtres, des différences essentielles, leur méthode zététique était au fond beaucoup plus voisine de la nôtre qu'on n'est porté à le croire au premier abord. C'est que, tandis que leur symbolisme algébrique [*sic* !] se développait péniblement, ils en avaient, dès le quatrième siècle avant notre ère, constitué un pour la géométrie, ... Ce langage présentait en même temps tous les avantages de l'emploi des lettres dans l'analyse de Viète [!] au moins pour les puissances 2 et 3. *Ils avaient dès lors pu constituer, probablement dès le temps des premiers pythagoriciens, une véritable algèbre géométrique pour les premiers degrés, avec la conscience très nette qu'elle correspondait exactement à des opérations numeriques*' (*op. cit.*, **3**, 167, my italics). Then he goes on to say: 'Quoiqu'ils [*i.e.*, the Greeks] ne se soient pas élevés... au concept général des coordonnées, leur facon de considérer les coniques est tout à fait analogue à celle de notre géométrie analytique [!]... *L'équation qu'ils établissent* [!] revient à la forme générale moderne: $y^2 = px \pm \frac{p}{a} x^2$...

Les procédés de transformation des coordonnées chez les anciens sont imparfaits, par suite du défaut de conception générale du problème... Mais ces procédés n'en existent pas moins' (*ibid.*, 168, my italics). Such examples could be multiplied *ad nauseam*.

H. G. ZEUTHEN has stated his views repeatedly and in various places. The most cogent and complete statement, however, appears in the two works quoted in the previous note. Thus, in *Die Lehre von den Kegelschnitten im Altertum*, ZEUTHEN entitles his first chapter 'Voraussetzungen und Hülfsmittel; Proportionen und geometrische Algebra' (*op. cit.*, 1). It is there that, in a paragraph remarkable for its *non sequiturs*, ZEUTHEN says that though the Greeks did not possess the concept of a system of coordinates, they nevertheless used 'rechtwinklige und schiefwinklige Koordinaten', and though Algebra was unknown to them, the historian must establish what they used in its stead (*op. cit.*, 2)! He continues by saying that the Greek theory of proportions '...enthielt Sätze, welche es ermöglichen die wichtigsten algebraischen Operationen... auszuführen (*ibid.*, 4) and 'Auf diese Weise hat man einen Apparat, mit Hülfe dessen man die Zusammensetzung algebraischen Grössen ausdrücken kann (*ibid.*). Furthermore, after the discovery of incommensurability by PYTHAGORAS or one of his disciples, '...wurde der unmittelbaren Anwendung von Zahlen und daran geknüpften Proportionen in der Geometrie, welche Anspruch auf Stringenz sollte erheben dürfen, ein Halt geboten ... Indessen konnte es nicht fehlen, dass man praktisch Zahlen und Proportionen auch auf die Geometrie anwandte, wenn auch mit dem Bewusstsein, dass man, um die gewonnenen Resultate anerkannt zu sehen, dieselben hinterher [!] auf einem anderen Wege beweisen müsse' (*ibid.*, 5).

But, the modern manipulation of proportions is a direct outgrowth of the existence of a symbolic mathematical language in which the symbols themselves are manipulated and operated on. On the other hand, 'Das Altertum hatte allerdings keine Zeichensprache, aber ein Hülfsmittel *zur Veranschaulichung dieser sowie anderer Operationen besass man in der geometrischen Darstellung und Behandlung allgemeiner Grössen und der mit ihnen vorzunehmenden Operationen*' (*ibid.*, 6). And now comes the pregnant statement:

> In dieser Weise entwickelte sich eine geometrische Algebra, wie man sie nennen kann, da dieselbe als Algebra teils allgemeine Grössen, irrationale sowohl wie rationale, behandelt, teils andere Mittel als die gewöhnliche Sprache benutzt um ihr Verfahren anschaulich zu machen und dem Gedächtnisse einzuprägen. Diese geometrische Algebra hatte zu Euklids Zeiten eine solche Entwicklung erreicht, dass sie dieselben Aufgaben bewältigen konnte *wie unsere Algebra solange diese nicht über die Behandlung von Ausdrücken zweiten Grades hinausgeht*, ein Gebiet, welches sie auch, ... in ihrer Anwendung auf die Lehre von den Kegelschnitten ausgefüllt hat. Eine solche Anwendung entspricht der Anwendung unserer Algebra in der analytischen Geometrie (*ibid.*, 7).

Having dealt with the ancient theory of proportions, ZEUTHEN passes on to 'geometric algebra' proper and establishes that the Greeks had the means to represent the equation $\alpha x + \beta y + \gamma z + \cdots = d$

In other words, there is nothing unique and (ontologically) idiosyncratic concerning the way in which ancient Greek mathematicians went about their proofs, which might be lost in the process of translation from the geometrical to the algebraic language; the main reason for this being that the ancient mathematical reasonings and structures *are* indeed substantially algebraic. As B. L. VAN DER WAERDEN put it:

> Theaetetus and Apollonius were at bottom algebraists, they thought algebraically even though they put their reasoning in a geometric dress.
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> Greek algebra was a geometric algebra, a theory of line segments and of areas, not of numbers. And this was unavoidable as long as the requirements of strict logic were maintained. For "numbers" were integral or, at most, fractional, but at any rate rational numbers [?], while the ratio of two incommensurable line segments cannot be represented by rational numbers. It does honor to Greek mathematics that it adhered inexorably to such logical consistency.[8]

Nevertheless, adopting such a procedure necessarily implied imposing very stringent limitations upon the kind of problems one could solve and, therefore, upon the results one could achieve. VAN DER WAERDEN, following in the footsteps of his illustrious predecessors but adding pinch, sharpness, and pungency to their sometimes (by comparison) mild, moderate, and gentle statements, goes on to ascribe to the ancient Greek mathematicians (and he is **not** referring to DIOPHANTOS here) the solution of '*equations*' in their geometrical propositions:

> Equations of the first and second degree can be expressed clearly in the language of geometric algebra, and, if necessary, also those of the third degree. But to get beyond this point, one has to have recourse to the bothersome [?] tool of proportions.

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as follows:

> ... auf einer Geraden neben einander Stücke abgetragen würden, die in den Verhältnissen α, β, γ ... zu x, y, z ... stehen. Der Abstand zwischen dem Anfangspunkt und dem Punkt, den man durch successives Abtragen der Stücke erreicht, wird dann d sein. Auf ähnliche Weise kann man verfahren, wenn andere Vorzeichen in der Gleichung [!] vorkommen. Ebenso wie wir bei der jetzt gebräuchlichen Darstellung im Gedächtnis behalten müssen, was jeder einzelne von unseren Buchstaben bedeutet, ebenso mussten die Alten behalten, was das für Stücke waren, die man abgetragen hatte; *dann aber hatten die Alten ebenso wie wir eine Darstellung der Gleichung* ...
>
> Mit Hülfe einer solchen Darstellung werden Gleichungen ersten Grades auf Wegen gelöst, welche viel mit unserer algebraischen Behandlung gemeinsam haben (*ibid.*, 10, my italics).

This should suffice here. More about TANNERY's and ZEUTHEN's views on 'geometric algebra' will come to the fore in the balance of this study.

[8] B. L. VAN DER WAERDEN, *Science Awakening* (New York: John Wiley & Sons, Inc., 1963), 265–66. In this instance too, as in so many others (*cf.* footnote 15, *passim*), VAN DER WAERDEN mirrors OTTO NEUGEBAUER's views on the fundamentally algebraic character of APOLLONIUS' *Conics*. Thus, NEUGEBAUER thinks that '... auch in der scheinbar rein geometrischen Theorie der Kegelschnitte vieles steckt, das uns Aufschlüsse geben kann, über die sozusagen latente algebraische Komponente in der klassischen griechischen Mathematik' ('Apollonius-Studien', 216; full reference in footnote 15). And, speaking of the structure of the *Conics*, NEUGEBAUER says: 'Die Behandlung des Evolutenproblems ohne jede Benutzung von Infinitesimal methoden *aus rein algebraischen Betrachtungen* ist überhaupt ein besonderes Glanzstück des ganzen Werkes. *Ebenso ist das ganze Arsenal von Identitäten und zugehörigen Ungleichungen aus Buch VII ... rein algebraischer Natur*' (*ibid.*, 218, n. 4, my emphasis).

Hippocrates, for instance, reduced the cubic equation [!] $x^3 = V$ to the proportion

$$a:x = x:y = y:b,$$

and Archimedes wrote the cubic [?]

$$x^2(a-x) = bc^2$$

in the form

$$(a-x):b = c^2:x^2$$

In this manner one can get [*Who* can get?] to equations of the fourth degree; ... But one can not get any farther; besides, one has to be a mathematician of genius, thoroughly versed in transforming proportions with the aid of geometric figures, to obtain results by this extremely cumbersome method [?]. Any one can use our algebraic notation, but only a gifted mathematician can deal with the Greek theory of proportions and with geometric algebra.[9]

[9] *Op. cit.*, 266. It is clear that *not* 'any one can use our algebraic notation'. For somebody to use it, he must *have* such a notation at his disposal in the first place and he must know to use it, *i.e.*, he must be aware of and conversant with the algebraic way of thinking! The position exemplified by the above quotation is also (though, perhaps, not always presented with the same bluntness) that of PAUL TANNERY and ZEUTHEN. Thus, TANNERY begins his study on the geometrical solution of second degree problems before EUCLID with the following statement: 'Si nous nous proposons de parler de la solution géométrique des problèmes du second degré avant Euclide, il est clair cependant que ce n'est que dans l'oeuvre de ce dernier que nous pouvons trouver l'exposition de cette solution' (*Mem. Scient.*, **1**, 254) and ZEUTHEN, who, according to his own confession, adopts the point of view of TANNERY (*Die Lehre*, note 1, 5), says: 'Um ... zu erfahren, wie weit die Bekanntschaft der Alten mit gemischten quadratischen Gleichungen und deren Lösung oder Reduktion auf rein quadratische Gleichungen sich erstreckte, wird es zweckmässig sein zu prüfen, welche Gestalt die quadratische Gleichung in der Sprache der geometrischen Algebra annehmen musste ...' (*ibid.*, 15). ZEUTHEN also categorically proclaims: 'Wir sehen also, dass die Alten alle Formen der Gleichung zweiten Grades behandelt haben ...' (*Gesch. der Math. im Alt. und Mittel.*, 50). This is also the position of NEUGEBAUER in his 'Apollonius Studien' when playing havoc among APOLLONIUS' geometrical propositions, by transcribing the latter's rhetorical descriptions into the language of algebraic, manipulative symbolism. There is very little of APOLLONIUS in NEUGEBAUER'S transcriptions as even a glance at HEIBERG'S edition (or VER EECKE'S translation) will show. 'Bei den vorangehend geschilderten Überlegungen', says NEUGEBAUER, 'bin ich nirgends anders von den Apolloniusschen Text abgegangen als durch die äussere Form' (*op. cit.*, 250). As if this is not precisely the supreme sin a historian of mathematics may perform! (More on the relation between form and content in mathematics, below.) Furthermore, this statement is not even true, since NEUGEBAUER has not respected (among other things) APOLLONIUS' division into propositions. NEUGEBAUER goes on: 'Es wäre selbstverständlich auch bei der griechischen Ausdrucksweise der Beweise ohne weiteres möglich gewesen [?] bei analogen Beweisen gleiche Bezeichnungen einzuführen. [This is retrospective, hindsightful history! It is obvious for us that identical notation in analogous proofs is preferable to arbitrary notation, only because for us matters of notation are more than mere name-calling, baptizing! We *operate on* our notations; for the Greeks this was utterly inconceivable.] Dass die antike Mathematik so gänzlich unempfindlich gegen diese uns so sehr lästige Unsystematik gewesen ist, zeigt, dass man sehr vorsichtig damit sein muss, wenn man behauptet, die Unübersichtlichkeit der Beweise habe ihre Weiterentwicklung schliesslich verhindert. Offenbar [?] überblickte man das Buchstabengewirr einer Konstruktion mit derselben Selbstverständlichkeit vie wir heute komplizierte Formeln' (*ibid.*, 250, n. 28). *This is incredible!* It presupposes that formulae exist somehow independently of their actual, *i.e.*, *written* presence, that they are 'hidden' within the Greek notational chaos, to be merely disentangled by the penetrating eye of the modern historian of mathematics!

What are the grounds for such a view and what are its underlying assumptions? Let me state from the outset that I cannot find any *historically* gratifying basis for this generally accepted view, which, I think, owes its origin, in part, to the fact that those who have been writing the history of mathematics in general, and that of ancient mathematics (including Greek) in particular, have typically been mathematicians, abreast of the modern developments of their discipline, who have been largely unable to relinquish and discard their laboriously acquired mathematical competence when dealing with periods in history during which such competence is historically irrelevant and (I dare say) outright anachronistic. Such an approach, furthermore, stems from the unstated assumption that mathematics is a *scientia universalis*, an algebra of thought containing universal ways of inference, everlasting structures, and timeless, ideal patterns of investigation which can be identified throughout the history of civilized man and which are *completely independent of the form in which they happen to appear at a particular juncture in time.* In other words, such an interpretation takes it for granted that *form* and *content* do not constitute an integral whole in mathematics, that, as a matter of fact, **content** is independent of **form**, and that one can, therefore, transcribe with impunity ancient mathematical texts by means of modern symbolic algebraic notation in order to gain an 'insight' into their otherwise 'cumbersome' content.

Furthermore, exactly because this content (like the inert gases) is essentially unaffected by its formal surroundings, the ability of the modern mathematician to uncover it and give it a 'palatable' (*i.e.*, modern) form constitutes not only the best *modern* reading of ancient 'burdensome' and 'oppressive' mathematical texts but also the only *correct* reading and, at the same time, *the* proof that this is what the ancient mathematician had in mind when he put down (in an awkward fashion, to be sure) for posterity his mathematical thoughts. Thus, if *we* see in a number of EUCLIDEAN propositions in the *Elements* quadratic equations, then this is what EUCLID had in mind when he enunciated and proved those propositions geometrically; if *we* can identify equations of the fourth degree in APOLLONIUS,[10] this is what APOLLONIUS had in mind, though this identification of the algebraic kernel of APOLLONIAN thought is not always easy and requires, obviously, modern mathematical training:

> Reading a proof in Apollonius requires extended and concentrated study. Instead of a concise algebraic formula, one finds a long sentence, in which each line segment is indicated by two letters which have to be located in the figure. *To understand the line of thought, one is compelled to transcribe these sentences in modern concise formulas.*[11]

Besides, and as a natural corollary of such a view, when the mathematician succeeds in showing that two apparently unrelated mathematical texts, belonging to two alien cultures and to two different time periods have the same algebraic[12] content, in spite of their totally different formal outlook and context – like, say, a Babylonian tablet involving lists and manipulations of numbers and the Greek

[10] VAN DER WAERDEN, *ibid.*

[11] *Ibid.*, italics provided.

[12] This qualifier is actually superfluous since this is the only possible content according to the view expounded here.

geometrical propositions dealing with application of areas –, it becomes legitimate to inquire into possible influences, questions of priority, ways of transmission, *etc.* This is precisely the way followed by OTTO NEUGEBAUER in his *Vorgriechische Mathematik*[13], which has recently been reprinted, and (in greater detail and cogency) by VAN DER WAERDEN in *Science Awakening*,[14] where it is argued, exclusively on the basis of this type of reasoning, that Greek 'geometric algebra' is nothing but 'Babylonian algebra' in geometrical attire![15]

[13] *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften*, Band I: *Vorgriechische Mathematik* (Berlin-Heidelberg-New York: Springer-Verlag, 1969); this is an unrevised reprint of a book first published in 1934.

[14] *Cf. op. cit.*, 82–147.

[15] *Ibid.*, 118–124. VAN DER WAERDEN has essentially adopted *in toto* NEUGEBAUER'S approach and findings in the latter's three "Studien zur Geschichte der antiken Algebra'; the first study (I) appeared in *Quellen und Studien zur Geschichte der Mathematik Astronomie und Physik*, **2** (1932), Abteilung B: *Studien*, 1–27; the second (II), carrying the additional title 'Apollonius-Studien', came out in the same volume, same section (*Studien*) of the same journal, pp. 215–254; finally, the third (III), entitled 'Zur geometrischen Algebra', saw the light of the day in volume **3** (1936) of the same journal, same section, pp. 245–259. NEUGEBAUER summarized his well-known views on Greek 'geometric algebra' in his *The Exact Sciences in Antiquity* (Princeton, N.J.: Princeton University Press, 1952); I have used the second edition of this work (New York: Dover Publications, Inc., 1969). In *The Exact Sciences*, NEUGEBAUER confesses that there is no documentary evidence for what he calls 'Oriental influence' on theoretical Greek mathematics (p. 147). His 'working hypothesis', however, is: 'the theory of irrational quantities and the related theory of integration [?] are of purely Greek origin, but the contents of the "geometrical algebra" utilize results known in Mesopotamia' (*ibid.*). The only evidence for this mathematically beautiful 'working hypothesis' that NEUGEBAUER is able to produce is the fact that both the Babylonian numerical-arithmetical material and some Greek geometrical propositions lend themselves rather easily to an algebraic rendering which, when performed, shows them to be identical. There is no question, indeed, about their identity for NEUGEBAUER (and any modern mathematician) who has at his disposal the algebraic language and the rules of translations *into* it. The real question is: Did the ancients (Babylonians and Greeks) know the algebraic language and the rules of translating *it* into either number manipulations or geometric propositions? Pointing out that the problem of application of areas, which he calls 'the central problem of the geometrical algebra' (p. 149) is 'rather difficult to motivate' (*ibid.*) in any other way than by translating it into the language of equations (the same procedure as that followed in the transcription and solution of so-called Babylonian 'problems of second degree'), NEUGEBAUER is really telling us something about his own motivations, idiosyncrasy, and background rather than anything significant about the ancients. Why is the problem of the application of areas a 'strange geometrical problem' (*ibid.*, 150)? What is *strange* about it? Was it strange for the Greeks? Why does it need any motivation? Why is the Babylonian method of solution by reduction to the 'normal form' not in need of any motivation? NEUGEBAUER acknowledges that attempts to explain the problem of application of areas independently of algebraic translations have been made, but he claims that the algebraic explanation is 'by far the most simple and direct explanation' (*ibid.*). Fully aware that simplicity does not amount to historical proof, NEUGEBAUER rests his case on the plausibility of his algebraic interpretation and on the historical likelihood of contacts between the Babylonian and the Greek civilizations in Hellenistic times (*ibid.*, 150–151). NEUGEBAUER has dealt at great length with the historical problem of the alleged relations between Babylonian and Greek mathematics in the 'Schlussbemerkungen' to his third study on ancient algebra (*q.v.*). It is there that after stating that in the realm of elementary geometry as well as in the realms of the theory of proportions and the theory of equations (!), Babylonian mathematics contains the entire substantive material on which Greek mathematics continued to erect its structures, NEUGEBAUER points out that, in spite of the total lack of explicit citations of sources, he is convinced of the indubitable influence of Babylonian on Greek mathematics. His conviction is based on the following three factors: 1) The specific evidence of the relation between the two (by which he means their identity when submitted to the same algebraic treatment); 2) The historical fact of a widely spread Hellenistic culture reaching the 'Orient'; and, finally, 3) The numerous Greek citations referring to Greeks having studied in the 'Orient' (*loc. cit.*, 258). According to NEUGEBAUER, the period during which contacts between Greek and Babylonian mathematics took place should be taken as the period from PLATO to HIP-

Is this an acceptable position? As a historian, I must answer this question with an emphatic '***no***'! *This position,* I happen to believe, *is historically unacceptable.* Among other obvious drawbacks, it fails to answer the most stringent and manifest question, *viz.*, why did Greek mathematics stick throughout its development to the 'cumbersome', 'awkward', 'highly difficult' method of 'geometric algebra' with its application of areas, transformation of proportions by means of geometrical figures, *etc.*? This question gains even more in acuity when one keeps in mind that the perpetrators of the view embodied in the concept of 'geometric algebra' presume without any qualms (and rest assured) that there has been an underlying algebraic edifice to Greek geometry throughout its development. Why, then, did this algebraic framework remain all the time in the background, hidden, camouflaged, concealed?

Answering by pointing out that the Greek system of numeration employed the letters of the Greek alphabet as number symbols and thus made those letters unavailable to the mathematician to serve him as algebraic symbols, leaving at his disposal 'only' the geometrical representation, is missing the point entirely by begging the question.[16] If a necessary ingredient of the algebraic way of thinking is the existence of an operational symbolism, and if the Greeks were thinking algebraically, then, they possessed such an operational symbolism. The graphical shape of the symbols is immaterial; if the letters of the alphabet could not be used (and this is far from clear), then some *other* symbols *had to be* used for an algebraic mode of reasoning to become reality. The geometrical diagrams which we encounter in Greek mathematical texts most certainly are ***not*** algebraic symbols in the proper sense of the word; besides, they are not brought into play *operationally*.

So the question remains unanswered: If thinking algebraically simplifies things, as everybody would agree, and if the great Greek mathematical geniuses were algebraists at heart, then why did they put their relatively simple algebraic reasonings in the clumsy and unwieldy molds of geometrical form? Furthermore, if they thought algebraically, and if the most fundamental difference between the

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PARCHUS. A notable result of these contacts is the Greek geometric algebra, which was later applied to conic sections, achieving there its most remarkable results (*ibid.*) A few questions naturally arise. If the Greeks were so smart to take over 'Babylonian algebra' and geometrize it, why did they adopt the Babylonian dainties rather selectively? Specifically, why did they not adopt a positional number system from the Babylonians rather than clinging to a dreadful one? Why did they fail to see the great 'advantages' of the Babylonian approach to astronomy, sticking exclusively to geometrical models rather than to arithmetical sequences? Why did they not deal with the 'irrational' like the Babylonians? (There would not have been then any 'crisis of the irrational'!) The above is by no means an exhaustive list of troublesome questions stemming from NEUGEBAUER's hypothesis.

[16] This is precisely ABEL REY's point of view in *La Science dans l'Antiquité*, **3** (*La Maturité de la Pensée Scientifique en Grèce*), (Paris: Albin Michel, 1939), note 1, 391. In that note REY seems to imply that there was in existence an 'algebra numerosa' and that the ancient Greeks had, therefore, a real choice between this algebra and the geometrical symbolism of 'geometrical algebra' and that, furthermore, they preferred (wisely) the latter! But if this is the case, where are the traces of this 'algebra numerosa' during the first six centuries B.C.? There are no such traces, and this is for a very simple reason, mentioned by A. REY on a previous page of his book: 'Le mathématicien grec est un geomètre. Il n'arrivera à l'algèbre numérique, et bien imparfaitement encore, qu'à l'extrême fin de la période gréco-romaine, au IV^{e} siècle après J.C.' (*ibid.*, 349). Moreover, this point of view of A. REY clearly contradicts what its author said elsewhere in this work (and in other works) about the inherent fundamental differences between geometry and algebra, the latter requiring a new way of thinking, *etc.* (More about this, below.)

algebraic and the geometric mode of reasoning lies, as I think it does, in the distinction between *symbolic* and *extensive* (*i.e.*, *spatial*) magnitude, then why did they systematically fail to use any algebraic symbolism whatever in their writings? How can one reasonably explain such a failure? Is the unwarranted assumption of such mathematical schizophrenia accountable in any convincing historico-rational manner?

II

In the previous paragraph, I touched on the characteristic features of geometry and algebra. Let us pursue this matter a little further. What are the most fundamental traits of geometrical thinking? Geometry is thinking about space and its properties; furthermore, it is thinking embodied in, fused with graphic, diagrammatic representation. If the diagrams of geometry are its 'symbols', then these 'geometrical symbols' display a feature which is totally absent from a true (algebraic) symbol: they are inherently extended because space, which they represent, is extended; they appeal to the eye of the geometer and to his spatial intuition; they are indeed, in a very real sense, the hypostatization of the geometer's spatial intuition. 'True' geometry (not analytical geometry) is inconceivable without diagrams and geometrical constructions. These diagrams are the characters in which the geometrical language is written: no diagrams, no geometrical way of thinking. Though it is true that these diagrams are only poor and imperfect copies of the real geometrical objects and relations, it is only through them that the geometer can pursue those lengthy and involved chains of reasoning which constitute the beauty and the glory of geometry.

Though diagrams constitute an integral and inseparable part of geometrical thinking, they are not its only ingredient. They must usually be accompanied by a rhetorical component, the proof, the most important function of which is to introduce the time parameter necessary in obtaining the finished, polished, wholesome diagrams through all the required intermediary, manipulative steps leading to the desired conclusion. In other words, if there is an operational, manipulative aspect in geometrical thinking, and I think there is, it takes place not at the level of the 'geometric symbol', the diagram (at least not in the written tradition, which is our only concern here), but at the rhetorical, descriptive, hortative level of the actual proof. Simply put, if one wishes to ascribe status of symbol to geometric diagrams (and it is far from clear that this is an entirely legitimate ascription), he will necessarily realize that the 'symbolism' thus constituted is, most certainly, not *operational* symbolism, when compared, say, with modern algebraic symbolism which is truly operational.[17]

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[17] *Cf.* what A. REY has to say concerning the characteristics of the geometrical method: 'La construction géométrique... nécessite... une intuition plus singulière que les formules de l'algèbre... D'abord elle a besoin d'une intuition concrète. L'esprit comme dira Descartes, y est asservi aux lignes, aux angles, et aux figures, aux agencements complexes de leurs traces et, comme les Grecs le professaient, à la règle et au compas. Il y a là effort pénible prétendra encore Descartes, nous ajouterons *limitatif*, d'imagination: limitatif parce que l'image est quelque chose de limité et de singulier en face de l'acte conceptuel, de la relation saisie toute nue.

Ensuite, ... [l]a construction géométrique est une synthèse ou chaque pas prépare le suivant, This brings us to the characteristics of the algebraic mode of thinking as they constituted themselves in the course of the historical development of algebra. According to a recent study,[18] the main features of the algebraic way of thinking are: 1. Operational symbolism; 2. The preoccupation with mathematical relations rather than with mathematical objects, which relations determine the structures constituting the subject-matter of modern algebra. The algebraic mode of thinking is based, then, on relational rather than on predicate logic; 3. Freedom from any ontological questions and commitments and, connected with this, abstractness rather than intuitiveness.[19] It seems, therefore, that the algebraic way of reasoning is *different* from the geometric one. It is completely abstract, free from dependency on perceptional, spatial considerations, it is manipulative, the entities it manipulates are themselves completely abstract, mere signs, it is analytical, functional, it possesses a universality of application missing in geometrical reasoning, and it is, at least to a certain extent, mechanical in the rules of manipulation of its symbols.[20]

III

Let us return now to the concept of 'geometric algebra'. It would seem, from what was said above alone, that it is a monstrous, hybrid creature, a contradiction in terms, a logical impossibility. Indeed it is. And, as we shall see, it is also an historical impossibility. The argument, to be sure, is very simple and straightforward. To

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ou les inventions se lient et se commandent. Mais dans l'invention elle-même, chaque construction nécessite encore un tour de main, un biais, une intuition, une finesse particulière...

Le symbolisme géométrique reste toujours en deça du symbolisme algébrique. Il faut, pour atteindre les articulations de pensée dans l'algèbre s'affranchir de la nécessité de construire, de même que la « construction » permettait de s'affranchir de la necéssité de compter et de calculer, et du spécifisme qui es [*sic*] affectait' (ABEL REY, *Les Mathematiques en Grèce au Milieu du V^e Siècle* (Paris: Hermann & C^ie, 1935), 55–56).

[18] MICHAEL S. MAHONEY, 'Die Anfänge der algebraischen Denkweise im 17. Jahrhundert', *Rete*, **1** (1971), 15–31.

[19] *Op. cit.*, 16–17. There is a slight variation of this characterization, appearing in an enlightening essay review of the reprint of NEUGEBAUER'S *Vorgriechische Mathematik*: MICHAEL S. MAHONEY, 'Babylonian Algebra: Form Vs. Content', *Studies in History and Philosophy of Science*, **1** (1971), 369–380; see particularly p. 372.

[20] ABEL REY, in his highly illuminating analysis of the features of algebraic thinking (which makes some of his other hackneyed and erroneous conclusions stand out like the proverbial sore thumb), is in substantial agreement with the shorter characterization of MAHONEY. To illustrate: 'La condition *sine qua non* d'une algèbre sera... un système de symboles et de règles mécaniques pour agencer ces symboles. C'est une *spécieuse* universelle, et c'est par ce mot que Viète l'a distinguée du calcul numérique. Son signe éminent c'est l'évasion hors de tout concret dans le pur abstrait. Il faut donc y faire abstraction des nombres et du calcul numérique, opérer sur des termes qui en soient des substituts universels, à l'aide d'un symbolisme opératoire. Les inconnues ont par là-même la même nature, et jouent le même rôle dans l'opératoire que les quantités connues' (*Les Math. en Grèce*, 38). '... les signes opératoires... se substituent en algèbre aux articulations du raisonnement' (*ibid.*). [En algèbre] On n'opère plus sur des nombres, sur des quantités, des valeurs des termes. On opère sur des *relations*. Les termes ici sont déjà des relations, car ils sont imbriqués les uns avec les autres, et pour employer le mot dans un sens très général mais qui prélude à son sens technique moderne, ils sont *fonction* les uns des autres' (*ibid.*, 40). 'Le besoin du symbole et sa création montrent que la penseé ne peut plus, pour l'objectif qu'elle vise et qu'elle trouve, utiliser une représentation concrète et particulière. Le saut, le voilà...' (*ibid.*, 45). 'L'algèbre seule peut permettre de transcender l'espace de la perception' (*ibid.*, 56).

have an Y-like X presupposes the prior or concomitant existence of some X, with respect to which alone departures from X-ness make sense and could be assessed. If there is not now and there never has been in the past an X, Y-like X's are impossible creatures both logically and actually. In the same fashion, to speak of 'geometric algebra' in Greek antiquity makes good sense only if contemporaneously or formerly there existed an algebra from which the Greeks departed in certain ways. The fact is that (in spite of many historically unsubstantiated claims to the contrary on behalf of an alleged Egyptian, Babylonian, or even PYTHAGOREAN algebra) there has never been an algebra in the pre-Christian era.[21] Consequently, there could not have been any 'geometric algebra' either.

If in spite of the preceding, however, various authors speak of Greek 'geometric algebra', this is due exclusively to the fact that these authors happen to live in a period after the invention of algebra and its application to geometry (analytical geometry) and assume, therefore, unwarrantedly and ahistorically that the symmetric case, *i.e.*, the application of *geometry* to *algebra*, has also taken place. This conclusion, however, is historically inadmissible. There is (broadly speaking) in the historical development of mathematics an *arithmetical* stage (Egyptian and Babylonian mathematics) in which the reasoning is largely that of elementary arithmetic or based on empirically paradigmatic rules derived from successful trials taken as a prototype,[22] a *geometrical* stage, exemplified by and culminating in classical Greek mathematics, characterized by rigorous deductive reasoning presented in the form of the postulatory-deductive method, and an *algebraic* stage, the first traces of which could be found in DIOPHANTOS' *Arithmetic* and in AL-KHWARIZMI's *Hisab al-jabr w'al muqàbalah*, but which did not reach the beginning of its full potentiality of development before the sixteenth century in Western Europe;[23] it is characterized, as we saw, by its supreme degree of abstract-

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[21] See M. MAHONEY, 'Babylonian Algebra: Form Vs. Content' and 'Die Anfänge der algebraischen Denkweise im 17. Jahrhundert'; ABEL REY, *Les Math. en Grèce*, 30, 32, 34, 36–37, 41, 44, *passim*; also LÉON RODET, *Sur les Notations Numériques et Algébriques antérieurement au XVIe Siècle* (Paris: Ernest Leroux, 1881), *passim*, especially 69–70; JACOB KLEIN, *Greek Mathematical Thought and the Origin of Algebra* (Cambridge, Mass.: M.I.T. Press, 1968), *passim*; Á. SZABÓ, *Anfänge der griechischen Mathematik* (München-Wien: R. Oldenburg, 1969) *passim*, but especially 28, 34, 35–36, and primarily the 'Anhang' appearing on 455–488; PAUL-HENRI MICHEL, *De Pythagore à Euclide: Contributions à l'histoire des mathématiques Préeuclidiens* (Paris: Les Belles Lettres, 1950), 639–646; G.A. MILLER, 'Weak Points in Greek Mathematics', *Scientia*, **39** (1926), 317–322.

Some of the works cited here establish the point solidly and unambiguously (SZABÓ and to a lesser extent KLEIN); some of them are, at best, ambiguous (MAHONEY and REY) succeeding in determining at one and the same time the ontological incommensurability of the geometric and the algebraic way of thinking and, yet, accepting (openly or implicitly), without realizing the contradiction involved, the historical legitimacy of the concept 'geometric algebra'; finally, some, though presenting a less clearcut point of view (RODET), or an unacceptable, ahistorical point of view (MICHEL and, especially, MILLER), enable the astute eye of the historically minded reader to reach easily a conclusion opposite to that presented by the author.

[22] *Cf.* A. REY, *Les Math. en Grèce*, 34, 41.

[23] *Ibid.*, 43, 45, 91–92. Against this view, for NEUGEBAUER, it seems, mathematics has always historically been algebra in various disguises and shapes. Thus, the first stage in the development of mathematics (algebra) was represented by the Babylonian sexagesimal place-value system and the operations with numbers made possible by the existence of such a system ('Zur geometrischen Algebra', 247). The second stage was represented by 'Babylonian algebra' proper, in which problems are reduced to quadratics of the normal form; the third stage is illustrated by the translation of algebraic techniques

ness, by its operational symbolism of universal applicability, and by its preoccupation with relations and structures.

It was only after the instauration of the algebraic stage that 'algebraic geometry' (*i.e.*, analytical geometry) could and did occur. The symmetric counterpart of this 'algebraic geometry', *i.e.*, 'geometric algebra' is not a historical entity, but only the fruit of the mathematico-historical lucubrations of mathematicians born during the algebraic stage in the development of mathematics. It is a figment of their mathematical imaginations, rather than anything real. It is an invention of the modern mathematician reading ancient texts through modern glasses, *i.e.*, an immediate and net outcome of the modern mathematician's ability to read geometry algebraically, to transcribe geometrical propositions into the language of algebraic equations, and to assume 'therefore' that this is what geometry is all about always, everywhere.

Additionally, if it is possible to supplement the above argument by showing that the assumption (for the sake of the argument) of a 'geometric algebra' leads to absurdities in the *specific* analysis of ancient mathematical texts, this should finally dispose of this monstrous concept and lead urgently to its demise. This is exactly what we plan to do by using the great classic of Greek mathematics, EUCLID's *Elements*, as our source of illustrations; let us hope, therefore, that the hours of 'geometric algebra', this arbitrary and aberrant concept, are indeed numbered!

IV

It is at least in principle possible that a partisan of the view embodied in the concept of 'geometric algebra' might counter the argument expounded above in the following fashion: 'I will grant you (though reluctantly!) that since there are no *explicit* instances of algebraic texts in Egyptian and Babylonian mathematical

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into the language of geometry – Greek mathematics or geometrical algebra (*ibid.*), and, finally, the fourth stage in the development of mathematics (algebra) is '... [die] Periode der neuerlichen Rückübersetzung der geometrischen Algebra in eine "algebraische" Algebra' (*ibid.*, 249). Besides, for NEUGEBAUER, geometry has always had secondary, derivative character: 'Die grossen Fortschritte der Geometrie sind in allen Phasen [!] immer unlösbar mit der Entwicklung anderer Disziplinen verknüpft (analytische Geometrie und elementare Algebra, Differentialgeometrie und Analysis, Topologie und Riemannsche Flächen + abstrakte Algebra), so dass das Geometrische an sich immer erst nachträglich [!] wieder aus dieser Verknüpfung gelöst werden musste [?]. Für die Frühgeschichte der Mathematik ist eine „reine" („synthetische") Geometrie viel zu schwierig [Why should this be so? Is there any substantiation for this unqualified claim, except algebraic hindsight?]. Das primäre Hilfsmittel ist hier die Verknüpfung mit dem Bereich der (rationalen) Zahlen und ein wesentlicher Fortschritt der Geometrie ist immer erst möglich, wenn die ungeometrischen Hilfsmittel weit genug entwickelt sind' (*op. cit.*, 246). It is true that NEUGEBAUER knows very well the importance of symbolism for the development of mathematics (*cf. op. cit.*, 246–247); (after all this should not come as a surprise from the part of somebody who, for all practical purposes, identifies mathematics with algebra!) But he draws from this awareness what seems to me to be unwarranted conclusions. Even if he is right about the precedence of computational techniques in pre-Greek civilizations over geometrical considerations, it does *not* follow that algebra preceded geometry in the Hellenic civilization. Logistic is most certainly not algebra, and quoting ARCHYTAS (p. 245) to the effect that logistic takes precedence over the arts, including geometry (DIELS-KRANZ, *Die Fragmente der Vorsokratiker*, 5th ed., 47B4) does not prove that algebra preceded geometry in Greek mathematics!

sources, there was no pre-Hellenic algebra. The situation is, however, totally different with Greek mathematics. It is clear to the shrewd and trained eye that Greek geometry is nothing but geometrically clad algebra. So the Greeks must be taken as the inventors of algebra.[24] However, for reasons that are immaterial to our issue, they decided not to use the standard type of algebraic symbolism, but to dress their algebraic formulas in geometrical outfits. So the very existence of Greek geometry is the best proof for the existence of an ancient algebra, Greek algebra. As to the alleged irreconcilableness and incommensurability of the two ways of thinking (geometrical and algebraic), I do not buy this, modern mathematics doesn't buy it, and, obviously, the ancient Greeks didn't buy it either!'

What does one answer to such an interlocutor? I happen to believe that his 'argument' is really no argument and that it has been taken care of already in what was said above. However, in a more substantive fashion and in order to adapt the general analysis presented above to the *ad hoc*-ness of the interlocutor's alleged counter-argument, the following reply is in place: *Language* is the immediate reality of *Thought*. The differences between the two ways of thinking are real differences, which could not be dismissed off hand, rooted as they are in the features of perceptible space on the one hand (geometry) and the universal denotativeness of the supremely abstract manipulable symbol (algebra) on the other hand. Different ways of thinking imply different ways of expression. It is, therefore, impossible for a system of mathematical thought (like Greek mathematics) to display such a discrepancy between its alleged underlying *algebraic* character and its *purely geometric* mode of expression. Furthermore, why did the ancient Greeks hide their ways of reasoning? What was there to hide?

That this last question is pertinent indeed – and not at all gratuitous as it may, *prima facie*, seem – is shown by the following quotation from VAN DER WAERDEN in which the learned author discusses Book X of EUCLID's *Elements*:

> Up to X 28 it goes fairly well, but when the existence proofs start with X 29 … one does not see very well what purpose all of this is to serve. *The author succeeded admirably in hiding his line of thought* by starting with his constructions, even before having introduced the concept of binomial which does throw some light on the purpose of these constructions, and by placing at a still later point the division into 6 types of binomials.[25]

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[24] Indeed the nineteenth century originators of the concept of 'geometric algebra', ZEUTHEN and TANNERY, wrote before NEUGEBAUER (the 'discoverer' of Babylonian algebra) and VAN DER WAERDEN (the articulate spokesman for the view that 'Babylonian algebra' became Greek 'geometric algebra') and yet, did not hesitate to speak freely of Greek 'geometrical algebra' when they encountered in the *Elements* propositions which seemed to them out of place, unwieldy, incongruous!

[25] *Sci. Awak.*, 172, my italics. Once more, VAN DER WAERDEN follows in the footsteps of O. NEUGEBAUER. It was NEUGEBAUER who, in his 'Apollonius Studien (Studien zur Geschichte der antiken Algebra II.)', already stated that, though there are recognizable structures and algorithms all over the *Conics*, which the trained eye of the mathematician can disentangle, these structures, algorithms, and methods of proof have been subsequently completely hidden, camouflaged ('… nachträglich völlig erdeckt' …; see *loc. cit.* (footnote 15), 253). Furthermore, speaking of his own analytical transcriptions and manipulations of APOLLONIUS' geometrical rhetoric, NEUGEBAUER says: '*Diese höchst einfache Schlussweise gibt den Schlüssel zu sämtlichen hier zusammengestellten Konstruktionen.* Bei Apollonius ist nur alles mit grosser sorgfalt auf den Kopf gestellt und verschleiert' (*ibid.*, 251).

Lest the readers believe that TANNERY, ZEUTHEN, NEUGEBAUER, and VAN DER WAERDEN are the only 'villains', I would like to state that practically 'anybody who counts' in the writing of the history of Greek mathematics in modern times has adopted the stand that 'geometric algebra' provides one with an illuminating insight into the inner workings of Greek mathematics.[26] This certainly is true of contemporary writers of textbooks like CARL B. BOYER and HOWARD EVES;[27] it is true of J.F. SCOTT, DIRK STRUIK, FLORIAN CAJORI, DAVID EUGENE SMITH, EDNA E. KRAMER, and so forth.[28] And, of course, it is true of the greatest writer

[26] NEUGEBAUER's unmitigated enthusiasm for geometric algebra (for which he erroneously takes ZEUTHEN as the originator) is typical: 'Zeuthen verdankt man die für das Verständnis der ganzen griechischen Mathematik grundlegende Einsicht, dass es sich insbes. in den Büchern II und VI von Euklids Elementen um eine geometrische Ausdrucksweise eigentlich algebraischer Probleme handelt. Insbesondere hat er an vielen Stellen darauf hingewiesen, dass in den „Flächenanlegungs"-Aufgaben von Buch VI und zugehöriger Sätze der Data die vollständige Diskussion der Gleichungen zweiten Grades steckt. Er hat dann weiter gezeigt, wie diese „geometrische Algebra" die Basis für die „analytische Geometrie" der Kegelschnitte des Apollonius bildet, deren Bezeichnungen „Ellipse", „Hyperbel", „Parabel" noch heute auf die Fundamentalfälle der „Flächenanlegung" zurückwiesen' ('Zur geometrischen Algebra', *Q.U.S.*, **3** B (1936), 249). There are hardly any unambiguous, clear-cut exceptions to the rule. Even those who, for one reason or another, began doubting the inherited interpretation (and this 'doubting' got under way only in recent times) did not, as a rule, abandon the concept of 'geometric algebra'. A case in point is represented by MICHAEL MAHONEY. (*Cf.*, for instance, his otherwise enlightening article, 'Another Look at Greek Geometrical Analysis', *Archive for History of Exact Sciences*, **5** (1968), 318–48, where he says: 'For example, Proposition VI, 28 [of the *Elements*], which is part of the "geometric algebra" of the Greeks...' (328) or 'The earliest techniques of analysis evolved from the researches of the... Pythagoreans, and are brought together in the major contribution of this mathematico-philosophical school to Greek mathematics: geometrical algebra. Geometrical algebra was one of the basic tools of the mathematical analyst. In the *Data*... Euclid gave prominent place to the doctrine of the application of areas, which is the essence of Greek geometrical algebra' (*ibid.*, 330–31); even in his cogent and powerful criticism of NEUGEBAUER's ahistorical procedures ('Babylonian Algebra: Form *Vs.* Content'), MAHONEY somehow considers the ahistorical concept of 'geometrical algebra' as legitimate, since he says that 'Greek geometrical algebra' could construct '... a quadratic system of equations in two unknowns from the values of those unknowns...' (376), but could not construct '... a single quadratic equation from its two roots...' (*ibid.*).). Another, earlier instance of the same syndrome is illustrated by ABEL REY's writings quoted above. (Incidentally, 'Chapitre IX' of 'Livre III' of *La Maturité de la Penseé Scientifique en Grèce* reproduces verbatim, *in toto*, 'Chapitre IV' ('Arithmetique et Système Métrique Algèbre, Géométrie et Algèbre Géométrique') of *Les Mathématiques en Grèce au Milieu du Ve Siècle*, without any hint whatever to the reader!) To my knowledge, it is only ÁRPÁD SZABÓ, who, in the introduction and (primarily) in an appendix appearing in *op. cit.*, 455–88 (about which more will be said below), unequivocally and forcefully calls attention to what is wrong with the concept of 'geometric algebra' and asks for its abandonment. I had arrived at my ideas concerning the historical unsoundness of the notion of 'geometrical algebra' independently, while, as a graduate student, I immersed myself in reading Greek mathematical texts and the modern commentaries on them. I reached my final conviction about the necessity to discard and repudiate 'geometric algebra' as an explanatory device in the study of the history of Greek mathematics and about the need, growing out of this rejection, to rewrite that history on a sound basis, while teaching a graduate seminar on EUCLID's *Elements* at the University of Oklahoma in the fall of 1972. I gave a talk on this topic at the Hebrew University of Jerusalem in the late fall of the same year, which got (so far as I can judge) a mixed reception: historians and the (very few) historically-minded mathematicians present seemed to like its conclusions, while the mathematicians (to put it mildly) remained unconvinced.

[27] See CARL B. BOYER, *A History of Mathematics* (New York-London-Sydney: John Wiley & Sons, Inc., 1968), 85–87, 114–15, 121–131, *passim* and HOWARD EVES, *An Introduction to the History of Mathematics* (New York: Holt, Rinehart and Winston, 1964, rev. ed.), 64–69, *passim.*

[28] See J.F. SCOTT, *A History of Mathematics* (London: Taylor & Francis Ltd., 1960), 23, *passim*;

on the history of Greek mathematics in the English language in modern times, Sir THOMAS LITTLE HEATH.

HEATH interests us here since he is the author of (among other things) the English editions of the writings of the great classics of Greek mathematics, EUCLID, ARCHIMEDES, and APOLLONIUS.[29] His translations (when he is satisfied to limit himself to the role of translator![30]) are considered reliable and insightful. Since we shall largely confine our discussion in what follows to EUCLID's *Elements*, let us see what HEATH's views on 'geometric algebra' are, as they pertain to the *Elements*. HEATH thinks that after the discovery of the irrational, '... it was possible to advance from a geometrical arithmetic to a geometrical *algebra*,[31] which indeed by EUCLID's time (and probably long before) had reached such a stage of development that it could solve the same problems as our algebra so far as they do not involve the manipulation of expressions of a degree higher than the second.'[32] HEATH goes on to say that '... Book II gives the geometrical proofs of a number of algebraical formulae [!]'[33] and then, without apparently grasping the inconsistency involved, continues:

> It is important however to bear in mind that the whole procedure of Book II is *geometrical;* rectangles and squares are shown in the figures, and the equality of certain combinations to other combinations is proved by those figures. We gather that this was the classical or standard method or proving such propositions, and that the algebraical method of proving them, with no figure except a line with points marked thereon,[34] was a later introduction.[35]

Finally, HEATH finishes his introductory remarks to Book II of the *Elements* with a description of what he calls '... the geometrical equivalent of the algebraical operations'[36] allegedly undertaken by Greek geometers in their *geometrical* treatises, noting, among other things, that 'The division of a product of two quantities by a third is represented in the geometrical algebra by the finding of a rectangle with one side of a given length and equal to a given rectangle or square. This is the problem of *application of areas* ...'[37]

DIRK J. STRUIK, *A Concise History of Mathematics* (New York: Dover Publications, Inc., 1948, 2nd rev. ed.), 58–60, *passim*; FLORIAN CAJORI, *A History of Mathematics* (New York: The MacMillan Co., 1919), 32–33, 39; DAVID EUGENE SMITH, *History of Mathematics*, 2 vols. (New York: Dover Publications, Inc., 1958), **1**, 106 and **2**, 290; EDNA E. KRAMER, *The Nature and Growth of Modern Mathematics*, 2 vols. (Greenwich Connecticut: Fawcett Publ. Inc., 1974), **1**, 108, 137–40, 146.

[29] *The Works of Archimedes* (Cambridge: At the University Press, 1897), *Apollonius of Perga Treatise On Conic Sections* (Cambridge: W Heffer & Sons Ltd., 1961), *The Thirteen Books of Euclid's Elements*, 3 vols. (Cambridge: At the University Press, 1908); HEATH's edition of EUCLID will be referred to in the future as EUCLID, *Elements*.

[30] *Cf.*, in this context, SZABÓ's remark: 'Es wurde also eben betont, dass man auf die Übersetzungen der Quellen – vom Gesichtspunkt der Mathematikgeschichte aus – sich häufig nicht verlassen kann, auch dann nicht, wenn die fraglichen Übersetzungen manchmal philologisch so gut wie *tadellos* sind' (*op. cit.*, 16).

[31] This is also ZEUTHEN's view. *Cf.*, for instance, *Gesch. der Math. im Alt. und Mittel.*, 42.

[32] EUCLID, *Elements*, **1**, 372.

[33] *Ibid. Cf.* also ZEUTHEN, *Die Lehre von den Kegel.*, 12.

[34] Why is such a procedure '*algebraical*'?

[35] *Elements*, **1**, 373.

[36] *Ibid.*, 374.

[37] *Ibid. Cf.* also ZEUTHEN, *Die Lehre*, 14 and TANNERY, *Mem. Scient.* **1**, 256–57.

Let us try, then, to sum up the views of those who see in Greek geometry (at least in some crucial parts of it) a 'Geometric Algebra' by referring (copiously) to VAN DER WAERDEN, not because I particularly pick on him as my 'bouc émissaire', but for the simple reason that his assertions are among the most shocking in their bluntness and outspokenness, and because his book is one of the most recent pronouncements on the issue, and (in addition) is easily available to the interested (but needy student) in paperback.[38]

'When one opens Book II of the Elements', says VAN DER WAERDEN, 'one finds a sequence of propositions, *which are nothing but geometric formulations of algebraic rules .... We have here, so to speak, the start of an algebra textbook, dressed up in geometrical form.*'[39]

And,

> Quite properly, Zeuthen speaks in this connection of a "geometric algebra." Throughout Greek mathematics, one finds numerous applications of this "algebra." *The line of thought is always algebraic, the formulation geometric.* The greater part of the theory of polygons and polyhedra is based on this method; the entire theory of conic sections depends on it. Theaetetus in the 4th century, Archimedes and Apollonius in the 3rd are perfect virtuosos [*sic*!] on this instrument.[40]

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[38] *Science Awakening* (which was originally published in Dutch as *Ontwakende Wetenschap* (Groningen, 1950) appeared first in English translation at Groningen in 1954; since then the scholarly world was supplied with a paperback edition (New York: John Wiley & Sons, 1963), used in this study, in which the beautiful illustrations of the hard cover edition are marred by imperfect typographical reproductive processes, and, very recently (what a dream for a publishing house!), with a new hardcover 'Third Edition' in English (Groningen: Noordhoff, n.d.).

[39] *Sci. Awaken.*, 118, my italics. *Cf.* also, ZEUTHEN, *Die Lehre*, 12–13. Interestingly, G.H.F. NESSELMANN in his *Die Algebra der Griechen* (Berlin, 1842)—a photographic reprint (Frankfurt: Minerva, 1969) is available—considers Book II as *arithmetical* (*not* algebraic) in character: "Jedenfalls ... müssen wir ... das zweite Buch ... zu den arithmetischen zählen, da von seinen vierzehn Sätzen die ersten zehn gleichfalls nur geometrisch ausgesprochene und bewiesene, aber ihrem Wesen nach [?] lauter arithmetische Wahrheiten enthalten' (*op. cit.*, 154). NESSELMANN (about whose book L. RODET remarked that a better title would be '*le calcul chez les Grecs*' (*op. cit.*, 57)) then goes on to transcribe the first ten propositions in algebraic symbolism! Incidentally, we do possess an *arithmetical* translation of these ten propositions dating from the 14th century by a Byzantine monk, BARLAAM, entitled *ἀριθμητικη ἀπόδειξις των γραμμιχως ἐν τω δευτέρω των στοιχειων ἀποδειχθέντων* and another *arithmetical* translation by CONRAD DASYPODIUS published with the original Book II of EUCLID in 1564. The proofs in these arithmetical translations are patterned after those appearing in the so-called 'arithmetical Books' of the *Elements* (VII–IX). For an example of BARLAAM'S translation and proofs, see NESSELMANN, *op. cit.*, 155, where the proposition dealt with is II.4.

[40] *Op. Cit.*, 119, my italics. Again, NEUGEBAUER espoused similar views long before VAN DER WAERDEN. Thus, describing the contents of the *Conics*, NEUGEBAUER said: 'Im ersten Buch werden die *Grundgleichungen* [!] der Kurven und ihrer Tangenten entwickelt ...' ('Apollonius Studien', 218, italics added). Referring in a more detailed fashion to the contents of Book I, NEUGEBAUER again spoke of, '*Gewinnung der Grundgleichung* a) *zunächst in unmittelbar geometrischen Form*, b) Umformung in eine solche Gestalt, wie sie für die Anwendung bequemer ist ... Schliesslich wird gezeigt: ist ein Kegelschnitt durch seine *Gleichung* gegeben, so gibt es auch ... einen Kegel ... auf dem er liegt. *Zusammen mit der ursprünglichen Gewinnung der Gleichung aus dem räumlichen Schnitt ist damit die volle Äquivalenz von räumlicher und analytischer Darstellung bewiesen*' (*ibid.*, 219, my emphasis).

For VAN DER WAERDEN, Greek '... geometric algebra is the continuation of Babylonian algebra.'[41] However, the Greeks, unlike their Mesopotamian forerunners, translated *everything* into geometric terminology. '*But since*

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[41] *Ibid.* Needless to say, the originator of this view is OTTO NEUGEBAUER. Thus, in his 'Zur geometrischen Algebra', speaking of the title he chose for this study, NEUGEBAUER confesses that, although it may be too narrow for his purposes, it was selected, '... um anzudeuten welchen Punkt ich für den eigentlichen Schlussstein für das Verständnis des Verhältnisses der griechischen Mathematik zur babylonischen halte' (*op. cit.*, 246). Having accepted ZEUTHEN's views on the nature of 'geometric algebra' *in toto*, NEUGEBAUER goes on to ask what was the historical origin of the problem of application of areas, a question left unanswered by ZEUTHEN. NEUGEBAUER's answer runs as follows: 'Die Antwort ... liegt einerseits in der aus der Entdeckung der irrationalen Grössen folgenden Forderung der Griechen der Mathematik ihre Allgemeingültigkeit zu sichern durch Übergang vom Bereich der rationalen Zahlen zum Bereich der allgemeinen Grössenverhältnisse, andererseits in der daraus resultierenden Notwendigkeit [Is this *logical* necessity or *historical* necessity? Clearly, the former! And so, NEUGEBAUER has sinned once more against history, by substituting logical for historical criteria in his analysis.], *auch die Ergebnisse der vorgriechischen "algebraischen" Algebra in eine "geometrischie" Algebra zu übersetzen*' (*ibid.*, 250). Is there any historical proof for the above italicized statement? As NEUGEBAUER would ask: 'Ist diese mächtige Behauptung textlich belegt?' **No!** What, then, is NEUGEBAUER's basis for making such a statement? He tells us in what immediately follows the above passage: 'Hat man das Problem einmal in dieser Weise formuliert, so ist alles Weitere vollständig trivial und liefert *den glatten Anschluss der babylonischen Algebra an die Formulierungen bei Euklid* (*ibid.*) In other words, NEUGEBAUER begins with what one would normally expect the historian to conclude (namely, that the Greeks knew the Babylonian stuff and 'translated' it into geometrical language), and from this historically totally unfounded assumption, by transcribing both the Greek geometrical propositions and the Babylonian numerical manipulations into algebraic symbolism, 'manages to show' that they are both the same and 'therefore' the Greeks copied the Babylonians. The vicious circle of his reasoning is obvious! Having thus shown the complete mathematical equivalence between the Babylonian 'normal form' and the simplest case of application of areas, NEUGEBAUER then exclaims in pleasant amazement: 'Das ist aber genau die einfachste Formulierung der Flächenanlegungsaufgabe des „elliptischen" Falles, wie sie bei Euklid VI, 28 steht ... Euklid VI, 29 steht dann die Übersetzung der Normalform (2), d.h. der „hyperbolische" Fall' (*ibid.*). What does this prove? To my mind, nothing else than the fact that if one performs the historically impermissible translation of the Babylonian and Greek mathematical stuff into algebraic symbolism, one can see that they are the same. *It certainly does not prove that the Greeks knew the Babylonian stuff!* But all this is not enough, since NEUGEBAUER goes on: 'Damit ist gezeigt, dass die ganze Flächenanlegung nichts anderes ist, als die mathematisch evidente geometrische Formulierung der babylonischen Normalformen quadratischer Aufgaben' (*ibid.*, 251). Aber ist was ist mathematisch evident auch historisch evident? Das scheint mir nicht! Neugebauer continues in the same vein by showing '... dass auch die griechische Lösungsmethode nicht anderes ist als die wörtliche Übersetzung der babylonischen Formel ... $\left.\begin{matrix}x\\y\end{matrix}\right\}=\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c^2}$...' (*ibid.*,). Having done this, he remarks: 'Die einzig neue Überlegung ist hier die Bemerkung über die Grösse der Gnomon figur, also etwas, was so nahe liegt, dass es gewiss keiner besonderen Motivierung bedarf, *wenn man den Ausgangspunkt so wählt, wie es hier geschehen ist, nämlich in der Aufgabe, die algebraische Formel* (2) $\left[\text{i.e., } \left.\begin{matrix}x\\y\end{matrix}\right\}=\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c^2}\right]$ *ins Geometrische zu übersetzen.* Hat man aber (2) nicht zur Verfügung, d.h. müsste man auf jede algebraische Formulierung verzichten, so ist gar nicht einzusehen, wie man auf eine derartige Konstruktion verfallen kann' (*ibid.*, my emphasis). In other words, it is impossible to understand Greek geometry without seeing it as derivative, secondary, illustrative of a *truly* algebraic background and motivation! Indeed NEUGEBAUER procceds to show by the same historically indefensible method (*i.e.*, by transcription of both Babylonian number computations and Greek geometrical propositions into algebraic manipulative symbolism) that: ... *die ganze Flächenanlegungsaufgabe wird sowohl hinsichtlich Fragestellung wie hinsichtlich Lösungsmethode unmittelbar verständlich wenn man sie nur als die sinngemässe Übersetzung der babylonischen Methode in die Sprache der geometrischen Algebra auffasst*' (*ibid.*, 252).

it is indeed a translation which occurs here and the line of thought is algebraic, there is no danger of misrepresentation, if we reconvert the derivations into algebraic language and use modern notations.' [42]

VAN DER WAERDEN goes on to say that we can use, without conscience qualms, modern algebraic symbolism, '... provided we take good care, *not to use algebraic transformations, which can not immediately be reformulated in the Greek terminology.*' [43] After doing exactly that, VAN DER WAERDEN obtains (no surprise, since he is the prestigious author of *Moderne Algebra*) for some of the propositions in Book II what he calls '... normalized forms of systems of equations ...,' [44] identical to those equations he has previously obtained by a similar procedure from Babylonian cuneiform tablets, and concludes that:

> *Apparently the Pythagoreans formulated and proved geometrically the Babylonian rules for the solution of these systems.* [45]
>
> Further:
>
> Thus we conclude, that *all the Babylonian normalized equations have, without exception, left their trace* [46] *in the arithmetic and the geometry of the Pythagoreans.* It is out of the question to attribute this to mere chance. [47] What could only be surmised before, has now become certainty [!], namely that the Babylonian tradition supplied the material which the Greeks, the Pythagoreans in particular, used in constructing their mathematics. [48]

I think this should be more than enough!

As I already intimated above, I believe such a view is offensive, naive, and historically untenable. It is certainly indefensible on the basis of the historical

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[42] *Ibid.*, my italics.

[43] *Ibid.*

[44] *Ibid.*, 124.

[45] *Ibid.*

[46] The reader may like being reminded that, strictly speaking, *there are no Babylonian equations*, normalized or 'abnormalized'. There are only tablets containing numbers and operations executed on these (specific) numbers, which the modern mathematician can translate, if he so wishes, into equations. On this whole issue, I urge the reader to peruse MAHONEY'S 'Babylonian Algebra: Form *Vs.* Content'. It is there that MAHONEY says: 'All that Babylonian texts contain is series of arithmetical operations that lead to (usually) correct results. The rest is interpretation by the historian. The Babylonians state the problem and compute the solution; the derivation of that solution is the work of the historian, and one may question whether the derivation tells us more about the historian's mathematics than about Babylonian mathematics' (*op. cit.*, 375). Indeed if NEUGEBAUER was able at all to speak of a 'Babylonian Algebra' and to tie it in with Greek geometry, this was due to his rather radical methodological innovation. As he tells us in his 'Studien zur Geschichte der antiken Algebra I' (full reference in footnote 15): 'Dabei verstehe ich unter „antiker Algebra" einen wesentlich weiteren Problemkreis als dies üblicherweise der Fall ist. Einerseits fasse ich das Wort „Algebra" sachlich möglichst weit, d.h. *ich ziehe auch stark geometrisch betonte Probleme mit in Betracht, wenn sie mir nur auf dem Wege zu einen letztlich „algebraisch" zu nennenden formalen Operieren mit Grössen zu liegen scheinen.* Andererseits gehe ich zeitlich weit über das übliche Mass hinaus, ...' (*op. cit.*, 1, italics added).

[47] Indeed it is far from chancy! It is due to the conscious *premeditated* talents of the modern mathematician, turned historian, who has managed to translate ('traduttore traditore') both Babylonian *numerical* manipulations and Greek *geometrical* propositions into algebraic language.

[48] *Ibid.*

record, *i.e.*, on the basis of a study of the documents of Greek mathematics, undertaken *not* from the point of view of the achievements, results, and methods of modern mathematics, which, it should be unequivocally understood, are completely irrelevant in attempting to understand Greek mathematics for its own sake, but from the standpoint adopted by the ancient Greek mathematicians themselves, inasmuch as this standpoint could be grasped by a modern mind. To read ancient mathematical texts with modern mathematics in mind is the safest method for misunderstanding the character of ancient mathematics, in which philosophical presuppositions and metaphysical commitments played a much more fundamental and decisive role than they play in modern mathematics. To assume that one can apply automatically and indiscriminately to any mathematical content the modern manipulative techniques of algebraic symbols is the surest way to fail to understand the inherent differences built into the mathematics of different eras.

'Mathematics is a reflection of culture ...'[49] It is, clearly, not immune to the intellectual and cultural environment in which it grows. Nothing is. This is the most fundamental reason why we have an *Egyptian*, a *Babylonian*, and a *Greek* mathematics (to limit ourselves to Antiquity only), and not just *Ancient* mathematics. It is indeed a truism that in some very substantive and irreducible aspects, Egyptian mathematics is *not* Babylonian mathematics, and Babylonian mathematics is *not* Greek mathematics. They become practically indistinguishable only if one commits the deadliest of sins a historian may be tempted to commit, namely that of inflicting upon them the ultimate historiographical insult of considering them mere adumbrations of modern mathematics and, therefore, proceed to translate them into modern algebraic symbolism.[50]

Whig history, a dead horse nowadays – one would like to believe – in most branches of history, is alive and thriving in the history of mathematics, where its dangers are no less real than in the more traditional types of intellectual history. It seems perfectly obvious to me that the ultimate implication of the historiographical view which allows one to read ancient mathematical texts through modern glasses must be that in mathematics, unlike any other domain of intellectual endeavour, the 'real stuff', the 'hard-core' mathematical content, the very essence of the discipline, its true fabric is immune to historical development and change, representing, in good Platonic fashion, a given, permanent, universal, stable structure, which man somehow grasped from the very beginning of his preoccupation with mathematical topics and which can easily be identified,

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[49] M. MAHONEY, 'Babyl. Alg.', 370.

[50] Recently, more attention is being paid to mathematics as a reflection of culture. *Cf.* in this context DAVID BLOOR, 'Wittgenstein and Mannheim on the Sociology of Mathematics', *Studies in History and Philosophy of Science*, **4** (1973), 173–91. It is there that one finds the following interesting remark: 'As evidence for the idea that mathematical notions are cultural products, consider the historical case of the concept zero. Our present concept is not the one that all cultures have used. The Babylonians, for example, used a place-value notation but had a different, though related, concept. Their nearest equivalent to zero operated in the way that ours does when we use it to distinguish, say, 204 from 24. They had nothing correspond – [*sic*!] to our use when we distinguish, say 240 from 24. As Neugerbauer [*sic*!] puts it, 'context alone decides the absolute value in Babylonian mathematics'... *If the Babylonians used a zero which left some aspects of a calculation context dependent, then, thus far, their concept of zero differs from ours*' (*ibid.*, 186, italics provided).

recognized, and labeled by the perceptive and skilled sage, *i.e.*, by the individual trained in (modern) mathematics, who knows with certitude what *Mathematics* is all about.

In a very real sense such a view must lead one to look at the mathematics of bygone eras as *preparatory* for modern mathematics, in the sense that the essential structure is already there, the only real difference being that this structure is expressed in 'cumbersome', 'awkward', 'unnecessarily difficult' form or language. In other words, the development of mathematics becomes the (almost) exclusive development of mathematical form, the groping for the 'right' kind of language to express the universal 'truths' which were there and were apprehended all the time. This approach, I submit, is not just naive and offensive historiographically, but it undermines the very fiber of the history of mathematics as a historical discipline. In short, it is, I believe, unacceptable to us as historians and should, therefore, be relinquished.

I am not about to enter into an exhaustive analysis of the sociological roots of such a scandalous situation. Let me only suggest again, however, that the fact that the history of mathematics has been typically written by mathematicians might have something to do with it; and in many instances it was not just 'broad-minded' mathematicians who engaged in such ventures;[51] on the contrary, these were mathematicians who have either reached retirement age and ceased

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[51] As an illustration of the kind of mathematician I have in mind, I shall refer the reader to G. A. MILLER (see reference in note 21, above). His article on 'Weak Points in Greek Mathematics' is a genuine *chef-d'oeuvre* and should be read in its entirety. Space limitations, however, permit me to quote here only sparingly. Thus, after deploring the '... undue emphasis on the geometric view ...' (317) of the ancient Greeks, MILLER declares emphatically that '*the lack of emphasis on the formal algebraic side of mathematics doubtless constituted the greatest inherent weakness of Greek mathematics*' (*op. cit.*, 317–18, my italics). He illustrates this 'drawback' with 'the Greek attitude towards the solution of the quadratic equations. Not only did they solve certain quadratic equations geometrically, but ... it appears clear that they had three general formulas [! ?] for the algebraic solutions of such equations ... they failed to see the general significance of the formulas and hence ... did not succeed in obtaining a general solution of the quadratic equation in the modern sense. It seems therefore unfortunate that many writers claim that they solved the quadratic equation' (*ibid.*, 318). 'By overlooking the fact that the algebraic equation frequently gives us much more than what we explicitely put into it, *the Greeks made a blunder and failed to put into their work one of the most fruitful ideas of later mathematics*' (*ibid.*, my italics).

'The awe inspired by the immortal *Elements* ... is partly offset by the short-sightedness exhibited by the Greeks when they failed to extend the number concept so as to include the negative and the ordinary complex numbers [!]. In fact, the earlier Greek writers did not include the irrational numbers in their concept of numbers [Imagine, such nasty behaviour!]' (*ibid.*, 318–19).

'It was well that the Greeks developed the theory of conic sections *without awaiting the discovery of the usefulness of this theory in the study of our solar system* ... (*ibid.*, 320, my italics).

Finally: 'The painstaking care which the modern scientist employs in making accurate measurements was foreign to the Greek mind. They devoted their attention to the shorter and easier routes leading to scientific truths' (*ibid.*, 320–21).

I have burdened the reader with this string of quotations for two main reasons: 1. Most of the views expressed by MILLER relate to issues discussed in this study and 2. These views, though representing a much lower degree of sophistication than those embodied in the term 'geometric algebra', stem ultimately from the same condemnable approach to the history of mathematics. If they, rightly, seem offensive and simple-minded, let the reader keep in mind that they, at least, condemn Greek mathematics for not being algebraic rather than (which I think is potentially much more dangerous) *making* it algebraic and then discussing it as such.

to be productive in their own specialties or became otherwise professionally sterile. However, both of these categories had something in common: in order to serve humanity and expend untapped remnants of scholarly energy, they decided to employ their creativity in a field, *history* of *mathematics*, 'half' of which – the *history* – was too alien and exotic to them while the other 'half' – the *mathematics* – was, alas, too familiar to them; the underlying assumption being that history does not really require any training, its narrative, reportorial methods and techniques being common-sensical and self-evident; and since they were highly proficient in mathematics they had *all* which was required to become successful historians of mathematics! ...

If the above suggestion is correct, then, the reader may judge for himself how wise a decision it is for a professional to start writing the history of his discipline, when his only calling lies in professional senility which bars him from encroaching on more friendly, familiar, and hospitable territory!

V

In this section I shall select a few examples from EUCLID'S *Elements* and analyse them in detail, in order to show, I think peremptorily, the inherent deficiencies of the time-honoured and venerable viewpoint that Greek geometry (at least some very important parts of it) is algebra in disguise.

My examples will be taken from Books II and VI (the books, *par excellence*, containing the so-called 'geometric algebra of the Greeks'), which will enable me to say something about the characteristic features of Greek geometry, from one of the so-called 'arithmetical books' (Book IX) and from Book X, dealing with incommensurable lines and their classification. Let me state that these are just a few examples out of a luxuriant plethora of similar illustrations which 'beautify' the thirteen books of the *Elements*. I have selected them because they represent striking illustrations of my point, namely:

Greek geometry is not algebra (geometric or otherwise) but simply geometry. Clearly, since there is (and this is obvious for us) a complete isomorphism between geometry and algebra – what else if not this is the message of analytical geometry? –, one can practically always use algebraic techniques for transferring the geometrical form and structure to their algebraic, analytical counterparts. There is no quarrel about this. However this is ***not*** the crucial historiographical point! The crucial historiographical point is that in this transfer-process one does irreparable violence and inflicts unrectifiable damage to the unique, peculiar, *sui generis* traits of Greek geometry which are not, let me state this emphatically, reducible to something 'simpler', less 'clumsy', *etc.* There is nothing 'clumsy', 'awkward', 'cumbersome', and so forth about Greek mathematics when it is not taken out of its own context. It certainly was not cumbrous, unwieldy, and oppressive for EUCLID, ARCHIMEDES, and APOLLONIUS, and this is what is historically important and, from our point of view, constitutes the most decisive clincher.

This being my point of view, I shall display, in what I consider to be an irrefragable fashion, the absurd consequences of the traditional interpretation when this interpretation is submitted to the most important test, *i.e.*, *the test of Greek* *mathematics itself.* Let me state from the outset that, to my mind, the traditional interpretation does not withstand such a test, – indeed it collapses noisily under its own unwarranted assumptions.

I have included Book X among the sources of my examples because the book is considered by many as the crowning achievement of the *Elements*, the most 'powerful' of all the thirteen books, and because historians of mathematics have traditionally analyzed its contents in purely algebraic terms.[52] Thus, VAN DER WAERDEN thinks that, 'In X 33–35, the solution of ... equations is indicated, for various cases, by the use of geometric algebra.'[53] And, in perusing HEATH's edition of the *Elements* one is immediately struck by the whole tenor of HEATH's commentaries on Book X, in which, from the very beginning, he speaks freely (and abundantly) of 'quadratic equations',[54] 'roots of equations of the second degree as are incommensurable with the given magnitudes', 'a classification of ... irrational magnitudes ... arrived at by successive solution of equations of the second degree',[55] and, finally (to cut the quotations short), of the fact that '... the Greeks would write the equation leading to negative roots in another form so as to make them positive, *i.e. they would change the sign of x in the equation.*'[56] HEATH also says, to mention one last shocking example, that the *binomial* and the *apotome* (which he writes as '$\rho \pm \sqrt{k}\,\rho$') '... are the positive roots of the biquadratic (reducible to a quadratic) $x^4 - 2(1+k)\,\rho^2 \cdot x^2 + (1-k)^2\,\rho^4 = 0$.'[57]

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Let us now proceed to the promised examples.

Proposition II.5 states: 'If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.'[58]

EUCLID's proof advances through the following stages: Let AB be given and bisected at C; let it also be divided into two unequal segments at D. Construct the square on CB and draw BE. Let $DG \parallel CE$. Through H, the point of intersection of DG and BE, let KM be drawn $\parallel AB$, and through A let AK be drawn $\parallel BM$. By I.43, the complement $CH =$ the complement HF. Consequently, $CM = DF$. But $CM = AL$ (because $AC = CB$ by hypothesis); therefore $AL = DF$. By adding CH to each of the preceding rectangles, it follows that $AH =$ gnomon NOP. But AH

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[52] *Cf.* TANNERY, *Mém. Scient.*, **1**, 264–67; ZEUTHEN, *Gesch. d. Math. im. Alt. u. Mittel.*, **56**, 158–161; ZEUTHEN, *Die Lehre*, 24–26; HEATH's edition of EUCLID, *Elements*, **3**, *passim;* VAN DER WAERDEN, *Sc. Awaken.*, 168–172; BOYER, *A Hist. of Math.*, 128–29; *etc., etc.*

[53] *Op. cit.*, 170.

[54] *Elements*, **3**, 5.

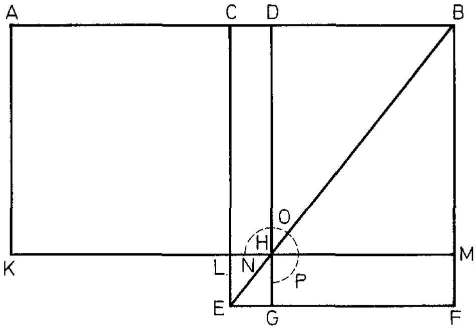
[55] *Ibid.*, 4–5; these expressions are taken over approvingly by HEATH from ZEUTHEN's *Gesch. d. Math. im. Alt. u. Mittel.*, 56.

[56] *Ibid.*, 5, my italics.

[57] *Ibid.*, 7. By the way, Sir THOMAS avows somewhat belatedly, in a confessional slip, while discussing the character of the first five propositions of Book XIII of the *Elements*, that, '... the method of [proof of] the propositions is that of Book II., being strictly geometrical and not algebraical ...' (*ibid.*, 441).

[58] EUCLID, *Elements*, **1**, 382.

is the rectangle AD, DB (because $AK=DH=DB$), therefore the gnomon $NOP=$ rectangle AD, DB. Let LG (the square on CD) be added to each member of the previous equality. ∴ Square on $CB=$ rectangle AD, $DB+$ square on CD, q.e.d.[59]



These are EUCLID's enunciation and proof. There is no trace of equations here as there is no trace of equations anywhere in Greek classical mathematics, *i.e.*, in Greek geometry. The proof is purely geometrical, constructive, intuitive (or visual), in the sense of its appeal to the eye, and it consists of a logical concatenation of statements about geometrical objects (in this case, rectangles, squares, and gnomons). There are no symbols and, consequently, there are no operations performed on symbols; the proof appeals to spatial perception rather than being abstract and it is essentially rooted in what has become known as Aristotelian predicate logic. All these are the very characteristics of Greek geometry.[60]

And yet what do we find in HEATH's commentary on II.5? 'Perhaps the most important fact about II.5,6 is however their bearing on the *Geometrical Solution of a quadratic equation*.'[61] How does HEATH discern such a bearing? This is very simple:

Suppose, in the figure of II.5, that $AB=a$, $DB=x$; then

$$ax-x^2=\text{the rectangle } AH$$
$$=\text{the gnomon } NOP.$$

Thus, if the area of the gnomon is given ($=b^2$, say), and if a is given ($=AB$), the problem of solving the equation

$$ax-x^2=b^2$$

is, in the language of geometry, *To a given straight line (a) to apply a rectangle which shall be equal to a given square (b^2) and shall fall short by a square figure*, i.e. to construct the rectangle AH or the gnomon NOP.[62]

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[59] *Ibid.*, 382–83.

[60] *Cf.* 'Babyl. Algebra', 372; see also 'Die Anfänge der algebr. Denkweise', 17–18, *passim*.

[61] *Elements*, 383.

[62] *Ibid.*; *cf.* also ZEUTHEN, *Die Lehre*, 19 and TANNERY, *Mem. Scient.*, **1**, 257–59. Also, ZEUTHEN, *Geschichte d. Math. im. Alt. u. Mittel.*, 47–48 and 52.

What does this prove? That the Greeks solved quadratic equations? *Not at all!* The only thing it proves is that HEATH (and ZEUTHEN, and TANNERY, and VAN DER WAERDEN, and P.H. MICHEL, *etc., etc.*) can transcribe EUCLID's geometry into algebraic symbolism and obtain (in this case) a quadratic equation. Does this tell us anything about the Greeks in general, and about II.5 in particular? *Nothing. There is not the slightest shred of genuine historical evidence that* EUCLID *(or the other great Hellenistic mathematicians, let alone the* PYTHAGOREANS*) ever used equations in their geometrical works. The sources do not contain equations.* This, however, does not prevent historians of mathematics from applying foreign (algebraic) techniques to Greek geometry and obtaining thus algebraic counterparts to Greek geometrical propositions, which they, then, illegitimately consider as being the genuine Greek stuff.

PAUL-HENRI MICHEL is a case in point. It seems to me highly interesting and significant that he begins his discussion of 'geometrical algebra' with the following statement: 'Pour faire comprendre comment la géométrie *pouvait* "jouer le rôle d'algèbre", nous prendrons un cas très simple.'[63] He then takes the equation $bx=c$, shows how it is mathematically equivalent to simple Greek geometrical techniques of application of areas, and soon claims, serenely and coolly, that the Greeks *solved the equation* by means of the application of areas! Then he says: 'Telle fut longtemps l'algèbre des Grecs.'[64] From here to the next claim there is just one step: 'Les solutions géométriques d'équations du deuxième degré abondent chez Euclide.'[65] As an example, MICHEL uses EUCLID II.5, the same proposition we discussed above. Let us see what it becomes in his skillful hands:

> Si une droite [b] est coupeé en deux parties égales [$b/2$] et en deux parties inégales [x et y], le rectangle [xy ou c] formé par les deux segments inégaux de la droite entière, plus le carré du segment placé entre les sections [$(b/2-y)^2$] est égal au carré de la moitié de la droite entière [$(b/2)^2$].[66]

The preceding is a beautiful illustration, I think, of the despicable methods of historians of mathematics, which enable them so easily to 'discover' equations in ancient Greek mathematics. Their procedures are clearly unveiled by MICHEL's square brackets used to transcribe EUCLID's geometrical proposition into algebraic symbolism, symbolism which does *not* appear in the EUCLIDEAN text at all. Indeed, MICHEL goes on to say, 'Pour démontrer ce théorème,[67] Euclide fait usage du gnomon des Pythagoriciens'[68], and then he summarizes EUCLID'S geometrical

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[63] *De Pythagore à Euclide*, 639, my italics.

[64] *Op. cit.*, 640. It is clear that MICHEL has a weird (if unoriginal) view of both algebra and the historical method. Thus he says: 'Nous ne considérons pas l'algèbre comme nécessairement liée à un certain système de symboles, mais, à la suite de M. Thureau-Dangin, comme "une application de la méthode analytique [des Grecs] à la résolution des problèmes numériques" ... Pour qu'il y ait algèbre (non pas algèbre "lettrée" mais algèbre "parlée") il faut mais il suffit qu'une quantité inconnue soit posée d'emblée comme connue. Dès que le mathématicien adopte cette méthode, *son discours est susceptible d'être traduit en équations, ce que nous faisons couramment pour la commodité du lecteur'* (*ibid.*, 641–42, my italics).

[65] *Ibid.*, 643.

[66] *Ibid.*, 643–44.

[67] This is what II.5 is: a theorem, a *geometrical* theorem and not a quadratic equation, or a problem leading to a quadratic equation!

[68] *Op. cit.*, 644.

proof, but not without adding (again in square brackets) the corresponding algebraic expressions (missing in EUCLID), as if EUCLID's procedure and the algebraic manipulations are exactly one and the same thing! Furthermore, he continues his anachronistic analysis of II.5 by saying the following about EUCLID's diagram:

> On peut d'ailleurs constater immédiatement sur la figure... l'égalite du gnomon... et du rectangle xy, ou c; et en conséquence *deduire*:
>
> $$x = b/2 + \sqrt{(b/2)^2 - c} = \frac{b + \sqrt{b^2 - 4c}}{2}$$
>
> et
>
> $$y = b/2 - \sqrt{(b/2)^2 - c} = \frac{b - \sqrt{b^2 - 4c}}{2}.$$
>
> Nous sommes ainsi ramenés aux formules par lesquelles se traduisent les opérations et les résultats de l'algèbre numérique babylonienne.[69]

And so, having translated in good traditional fashion, on the one hand, *both* the Babylonian specific numbers and the Babylonian cookbook-recipe type of solution procedure into algebraic symbols and algebraic operations and, on the other hand, the EUCLIDEAN purely geometric procedure into the same symbolic and operational language, MICHEL may now (like others before him) marvel at their 'identity' and, consequently, establish the necessary historical connections and influences between the two mathematical cultures! History? Perhaps, but certainly not sane, acceptable history. As if to crown his entire discussion, P.H. MICHEL continues by making this profound historical statement:

> Une même équation est donc susceptible d'être résolu par deux méthodes bien différentes, sur la valeur desquelles nous n'avons pas à nous prononcer.[70]

Beautiful! History? Perhaps, but certainly not sound, acceptable history. It is rather 'logical history', *i.e.*, in more cases than not, *non-history. It is history as it should be* rather than an honest attempt to establish it as it was; it is, in other words, a logical rather than a historical reconstruction.

Noting that, historically, the geometrical treatment (coming after the 'arithmetical atomism' of the PYTHAGOREANS) represented an advancement in Greek mathematics, P.H. MICHEL asserts: 'La géométrie (*tenant lieu d'algèbre*) permettait en effet la généralisation des calculs arithmetiques et l'inclusion des quantités irrationnelles dans ces calculs généralisés.'[71] Why did geometry take the place of algebra? Where is the algebra which it allegedly replaced? No trace of it is found in the known sources of Greek mathematics and this is for a very good reason: There was no algebra preceding geometry. The *arithmetic* of the PYTHAGOREANS, which (even according to the standard treatment) was replaced by the geometrical approach, was **not** algebra and cannot therefore be identified

[69] *Op. cit.*, 645, my italics.
[70] *Ibid.*
[71] *Ibid.*, 646, my italics.

historically with it.[72] If words have (within given historical periods) more or less settled meanings, then, most certainly, PYTHAGOREAN arithmetic (with its treatment of discrete entities), which was replaced by the geometrical approach (with its treatment of continuous magnitudes), was not algebra.

Historians of mathematics cannot have it both ways! It is logically impossible to claim at one and the same time that, on the one hand, one of the main reasons for the general decline of mathematics in the post-Hellenistic era was due to the Greek emphasis on Geometry,[73] and that, on the other hand, Greek geometry was nothing but algebra in disguise; had the latter been the case, it would have been very easy to abandon the disguise, to drop the mask, and pursue undisguised algebra, while, as is known, in reality one must wait until the sixteenth century for this to start happening.[74]

It seems to me that it is a considerably more appealing (and certainly historically more defensible) thesis that Greek mathematics, as found in the *Elements*, is an outgrowth of PYTHAGOREAN mathematics, the arithmetical discreteness of the latter (with all its accompanying inherent weaknesses) having been replaced in the former by the continuity of geometrical magnitude; thus, in EUCLID numbers are not collections of points anymore, but segments of straight lines, *etc.* This replacement enabled Greek geometry to deal 'honourably' (and vigorously) with the alleged 'scandal' generated by the discovery of the irrational.[75]

It also seems true that the 'figurative', numerical approach of the PYTHAGOREANS contained somehow in germ another possibility of generalization (and, potentially, of removal of contradictions) than that actually taken by classical Greek mathematics (*i.e.*, the purely geometric approach), and this is the possibility of distinguishing visually relations between numbers of the same kind, by means of the gnomonic differences in their punctiform representation, which relations could, perhaps, be seen retrospectively as a step in the direction of algebra proper. (But algebra did *not* develop in the sixteenth century out of this consideration!)

The only contention one can make in this context with a reasonable degree of accuracy (and it does not amount to an earth-shaking position) is that for the Greek mathematician living before the discovery of the irrational and working within the tradition of arithmetical geometry, the very way of representing numbers geometrically by points and punctiform figures contained intrinsic possibilities of grasping visually numerical relations; in other words, the PYTHAGOREAN way of representing numbers gave the PYTHAGOREAN mathematician an intuitive, visual means of generalization which, undoubtedly, contributed to the progress of mathematics.

Before leaving proposition II.5, I would like to call the attention of the reader

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[72] *Cf.* the following: '... chaque type de problème arithmétique nécessite une invention de l'esprit particulière à ce problème, adaptée à sa solution, et que ne peut pas servir à d'autres types dérectement [*sic*]; car, indirectement, toute opération contribue bien à former l'esprit arithmétique et a faciliter les inventions nouvelles' (*Les Math. en Grèce*, 55–56).

[73] See MICHEL, *op. cit.*, 646; A. REY, *La Science dans l'Antiquité*, **3**, 388–91; G. A. MILLER, *op. cit.*, *passim.*

[74] See A. REY, *Les Math. en Grèce*, 32, 45 (note 1), *passim;* MAHONEY, 'Die Anfänge der algebr. Denkweise', 18, 23, *passim.*

[75] That there was such a 'scandal' in the *mathematical* world is, at best, doubtful. *Cf.*, in this respect, Á. SZABÓ, *op. cit.*, 115.

to VAN DER WAERDEN's discussion of II.5 and II.6 and to SZABÓ's criticism thereof. For VAN DER WAERDEN both II.5 and II.6 are nothing but the geometrical expression of one and the same algebraic formula: $a^2-b^2=(a-b)(a+b)$.[76] 'But', as VAN DER WAERDEN says, 'it can not have been the sole purpose of the two propositions to give [the above] formula... a geometric dress and to prove it in that way; for, why should two propositions be given for *one* formula?[77] Indeed, why? VAN DER WAERDEN's answer is that the two propositions are really not propositions but '... *solutions of problems*; II 5 calls for the construction of two segments x and y of which the sum and product are given, while in II 6 the difference and the product are given.'[78]

ÁRPÁD SZABÓ, in one of the most effective criticisms ever leveled against the term of 'geometric algebra', thoroughly takes apart VAN DER WAERDEN's interpretation, based as it is on unbridled manipulations of algebraic symbols, leading to equations, *etc.* According to SZABÓ, II.5 is a purely geometrical proposition, more exactly a lemma, necessary in the proof of the very important purely geometrical proposition II.14. This follows not only from modern editions of EUCLID, in which during the proof of II.14 one is referred back to II.5, but also from the essentially identical wording of large parts of both propositions in the original, ancient Greek text. Indeed, the very 'clumsiness' and sluggishness of the language in which II.5 is enunciated seems to indicate that it was meant to represent a 'pre-fabricated' constitutive part, to be used ready-made in the proof of II.14.

Furthermore, this is not the only instance of such a procedure in the *Elements*. Other purely geometrical propositions, which were previously taken to illustrate Greek 'geometrical algebra', display a similar relation, in that one of a couple of propositions represents a modular unit necessary in the proof of the other member of the couple. A case in point is represented by II.6 and II.11, where II.6 constitutes such an integral 'pre-fabricated' part of II.11, both being, again, purely geometrical theorems. The reason that II.6 looks as a special case of II.5, says SZABÓ, is simply that II.11 (in whose proof II.6 is used as a module) is indeed a special case of II.14 (in whose proof II.5 is used as a module).[79] Another case

[76] *Sci. Awaken.*, 120. ZEUTHEN, closer to EUCLID, transcribes II.5 as $(a-b)\,b+(\frac{1}{2}a-b)^2=(\frac{1}{2}a)^2$ or as $(a-b)\,b+\left(b-\frac{1}{2a}\right)^2=(\frac{1}{2}a)^2$ and II.6 as $(a+b)\,b+(\frac{1}{2}a)^2=(\frac{1}{2}a+b)^2$ or $b(b-a)+(\frac{1}{2}a)^2=(b-\frac{1}{2}a)^2$ (*Die Lehre*, 12). NESSELMANN, for whom, as we saw, these are *arithmetical* propositions, chooses another, equivalent algebraic form for II.5, $ab+\left(\frac{a-b}{2}\right)^2=\left(\frac{a+b}{2}\right)^2$, and one of the two variants of ZEUTHEN as II.6 (*Die Algebra der Griechen*, 154). This variety and richness in transcription is, in itself, a clear-cut indication that the venerable authors are performing geometricide!

[77] *Ibid.*

[78] *Ibid.*, 121.

[79] I wonder if this really solves the 'problem'! Let me state emphatically that there is a problem *only* if one transcribes II.5 and II.6 into modern symbolism. It is only due to this totally unacceptable procedure that VAN DER WAERDEN could make his initial claim that II.5 and II.6 are nothing but the same algebraic formula! If one stays within the EUCLIDEAN realm (and this is the only admissible procedure), *i.e.*, if one does not transcend the boundaries of geometry, then, most clearly, II.5 and II.6 are **not** the same proposition. Specifically, in the language of application of areas, II.5 asks to apply a rectangle to a given line such that it will be equal to a given square and *fall short* by a square figure, while II.6 asks for the application of a rectangle to a given line such that it will be equal to a given square and *exceed* by a square figure! (*Cf.* EUCLID, *Elements*, **1**, 385–86). These can be shown to be the same only by somebody who has the benefit of formulaic expression.

in point (on the authority of PROCLUS – in his commentary on PLATO'S *Republic*) is provided by II.10 which represented a module for a proposition not included in the *Elements*, but which was reconstructed on the very basis of PROCLUS' remarks.[80]

One does not necessarily have to accept SZABÓ'S preceding interpretation for the 'similarities' between II.5 and II.6, in order to agree with the main thrust of his argument against 'geometric algebra'. To begin with, SZABÓ shows that even if such a creature as 'Babylonian Algebra' ever existed (and this is rather doubtful), '... *auch dann hat man bisher noch mit gar keiner konkreten Angabe* wahrscheinlich machen können, dass die Griechen in voreuklidischer Zeit eine solche Algebra wirklich gekannt hätten, geschweige denn, dass sie dieselbe übernommen und geometrisiert hätten. (Die Griechen haben nicht einmal die positionelle Bezeichnungsart der Zahlen von den Babyloniern übernommen!)'[81]

Furthermore, those so-called 'geometrically clad algebraic propositions' in EUCLID are 'algebraic' only in the sense that **we** can rather easily make them algebraic. 'Aber es kann gar keine Rede davon sein dass diese Theoreme ursprünglich "*algebraische* Sätze" oder Lösungen für "*algebraische* Aufgaben" gewesen wären. Nein, diese sind alle sowohl die Sätze wie auch die Aufgaben – *rein geometrischen Ursprungs*. Auch II.5 ist ein rein geometrischer Satz. Wohl kann man diesen Satz in der modernen Interpretation mit einer "algebraischen Aufgabenlösung" *vergleichen*. Aber man nehme sich in acht, damit ein solcher Vergleich den ursprünglichen und echt geometrischen Sinn des Satzes nicht verdunkle!'[82]

SZABÓ points out that the problem of incommensurability itself was originally a *geometrical* problem,[83] and he chooses to talk of 'Pythagorean geometry of surfaces',[84] rather than the hackneyed and wrong term of 'geometric algebra'. Summing up his criticism, SZABÓ says:

> Es wäre irreführend, diesen Satz [i.e., II.5] als "Lösung einer algebraischen Gleichung" aufzufassen. Die algebraische Auslegung – auch wenn sie dem Satz EUKLIDS äquivalent ist – verdunkelt den wahren geometrischen Sinn dieses Satzes, und historisch erweckt sie den falschen Schein, als hätten die Griechen in voreuklidischer Zeit in der Tat mit "algebraischen Gleichungen" operiert.[85]

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[80] Á. SZABÓ, *Anfänge der griechischen Mathematik*, 458–59.

[81] *Ibid.*, 457.

[82] *Ibid.*, 458. SZABÓ aims a scathing criticism at the TANNERY-ZEUTHEN thesis: 'H. G. ZEUTHEN, dessen "Verdienste" um die Entdeckung der sog. "geometrischen Algebra der Griechen" durch O. NEUGEBAUER ... so übertrieben hervorgehoben wurden, hat in Wirklichkeit in seinen beiden Werken (Die Lehre von den Kegelschnitten im Altertum, ... und Geschichte der Mathematik im Altertum und Mittelalter, ...) – was die "geometrische Algebra" betrifft – nur den irreführenden *Vergleich* von P. TANNERY weitergebaut. (Man hätte sich nämlich erst einmal fragen müssen, inwiefern überhaupt *erlaubt* ist, im Zusammenhang mit EUKLIDS *geometrischen Konstruktionen* über Lösungen von *algebraischen Gleichungen* zu reden!)' (*ibid.*, note 6, 457). For additional elements of this criticism, *cf. op. cit.*, 35–36, 474, 488.

[83] *Ibid.*, 36.

[84] *Ibid.*, 465, *passim.*

[85] *Ibid.*, 487.

And finally:

> … auch die übrigen Sätze der sog. "geometrischen Algebra der Pythagoreer" sich als rein geometrische Sätze erklären lassen. Dagegen hat man Spuren von echt algebraischen Gedankengängen aus der voreuklidischen und Euklidischen Überlieferung bisher *nicht* nachweisen können.[86]

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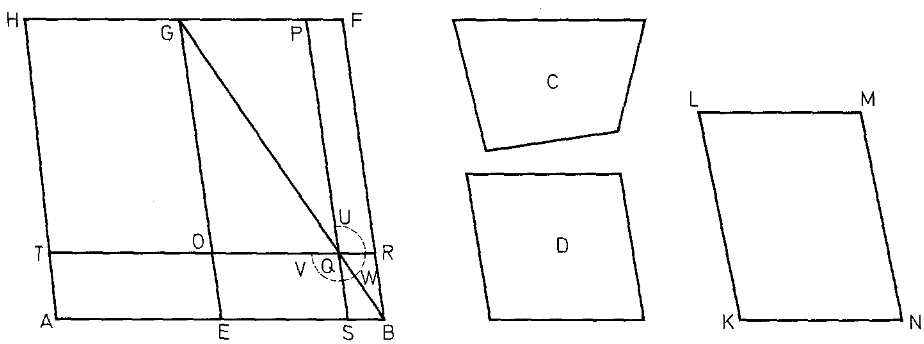
Our next example, from a Book considered a mainstay of 'geometric algebra' is VI.28. EUCLID's enunciation is:

> To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogramic figure similar to a given one: thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.[87]

EUCLID's proof proceeds as follows: Let AB be the given line, C the given rectilineal figure (not greater than the parallelogram described on half of AB and similar to the defect), and D the parallelogram to which the defect is to be similar.



It is required to apply to AB a parallelogram equal to C and deficient by a parallelogramic figure similar to D. Bisect AB at E. On EB construct the parallelogram $EBFG$ similar and similarly situated to D. (This is done by VI.18.) Complete the parallelogram AG.

Now, if ▱$AG = C$, then the requirement is obviously fulfilled. If however, ▱$AG \neq C$, then the only remaining possibility (due to the διόρισμόσ included in the enunciation) is that ▱$AG > C$. If this is the case, then ▱$GB > C$ since ▱$AG =$ ▱GB, by construction. Let, now, ▱$KLMN$ be constructed such that it is at one and the same time equal to the excess of ▱GB over C and similar (and similarly situated) to D. (This can be done by VI.25.)

Since ▱$GB \sim$ ▱D, ∴▱$KM \sim$ ▱D (by VI.21). Assume that in the two similar parallelograms, GB and KM, the corresponding sides are respectively GE and KL and GF and LM.

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[86] *Ibid.*

[87] EUCLID, *Elements*, **2**, 260.

Since $▱GB = C + ▱KM$ (by construction), $\therefore ▱GB > ▱KM$. EUCLID now concludes (and this is a tacit assumption) that

$$GE > KL \quad \text{and} \quad GF > LM$$

Let, then, $GO = KL$ and $GP = LM$ (by construction), and let the $▱GOPQ$ be constructed. Obviously, $▱GQ \cong ▱KM$. $\therefore ▱GQ \sim ▱GB$ (by VI.21), and, by VI.26, $\therefore ▱GQ$ is about the same diameter with $▱GB$. Let the common diameter GQB be described. Since $▱BG = C + ▱KM$ and $▱GQ = ▱KM$, $\therefore$ gnomon $UWV = C$. Furthermore, since $▱PR = ▱OS$ (by I.43), $\therefore ▱PB = ▱OB$. But $▱OB = ▱TE$ (by I.36), since $AE = EB$. $\therefore ▱TE = ▱PB$, $\therefore ▱TS =$ gnomon UWV. But gnomon $UWV = C$, $\therefore ▱TS = C$, q.e.d.[88]

Again, no algebraic symbols, no equations; a typical, purely geometrical, proposition belonging to the PYTHAGOREAN geometry of surfaces. Only somebody already steeped in the modern algebraic wisdom can 'discern' the 'algebraic line of thought' behind the traditional geometrical reasoning, transcribe the proposition into modern symbolism, and, then (if he does not discern the historical blunder involved and the *non sequitur* on which his conclusion is based) claim that this proposition is nothing but geometrically clad algebra. This is exactly what VAN DER WAERDEN[89] and T.L. HEATH are doing.

For HEATH this proposition '... is the geometrical equivalent of the solution of the quadratic equation $ax - \frac{b}{c}x^2 = S$, subject to the condition necessary to admit of a real solution, namely that $S \ngtr c/b \cdot a^2/4$.'[90] Who can see that? If you face a smart student of Greek synthetic geometry, whose mind was never exposed to the algebraic way of thinking, with HEATH's statement quoted above, there is not the slightest doubt whatever that he would fail to understand it. Indeed, I think that EUCLID *himself would have failed to understand* HEATH'S *statement*; not because EUCLID was less smart than HEATH, but because, living when he did, he did not have at his disposal what HEATH had in the nineteenth and twentieth centuries (primarily algebra and analytical geometry) and because (and this is another way of saying the same thing) his pattern of mathematical thought was different than HEATH'S. HEATH approached mathematics *algebraically*, EUCLID (like all the Greeks) approached it *geometrically*, and never did the twain seriously meet before the sixteenth century.[91]

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[88] *Ibid.*, 260–62.

[89] *Op. cit.*, 121–22; *cf.* also ZEUTHEN, *Die Lehre*, 19–20, 29–31 and *Gesch. d. Math. im Alt. u. Mittel.* 47–48.

[90] *Elements*, **2**, 263.

[91] HEATH goes on, in his commentary, 'To exhibit the exact correspondence between Euclid's geometrical and the ordinary algebraical method of solving the equation ...' (*ibid.*, 263). He transforms the equation in various ways and, finally reaches the following expression for the root:

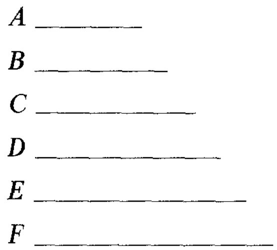
$$x = \frac{c}{b} \cdot \frac{a}{2} \pm \sqrt{\frac{c}{b}\left(\frac{c}{b} \cdot \frac{a^2}{4} - S\right)}.$$

Then he manages to show how he can find for every step in EUCLID'S proof a corresponding algebraic expression, until he reaches an expression for QS (see figure above) identical to the expression he found

We are now going to discuss two propositions belonging to Book IX. In proposition IX.8, EUCLID says:

> If as many numbers as we please beginning from an unit be in continued proportion, the third from the unit will be square, as will also those which successively leave out one; the fourth will be cube, as will also all those which leave out two; and the seventh will be at once cube and square, as will also those which leave out five.[92]

The EUCLIDEAN proof proceeds as follows (and my paraphrase *is faithful* to EUCLID's way of reasoning):



Let there be given as many numbers as one pleases, starting from the unit, like A, B, C, D, E, and F. By the definition of 'continued proportion', it follows that as the unit is to A so is A to B. But the unit measures A; so, as many times as the unit measures A, A measures B (by Definition VII.20). Now, since the unit measures A according to the units in A, A measures B also according to the units in A. Therefore A by multiplying itself makes B, and so B is a square. Furthermore, since B, C, D are in continued proportion and B is square it follows (by VIII.22) that D is also square. For the same reason, F is square and so are all those which leave out one.

Now since as the unit is to A, so is B to C, the unit measures A the same number of times that B measures C. But, again, the unit measures A according to the units in the latter; thus B measures C according to the units in A, *i.e.*, A by multiplying B makes C.

Now, since A by multiplying itself makes B and by multiplying B makes C, it follows that C is cube. Furthermore, since C, D, E, and F are in continued proportion and C is cube, it follows (by VIII.23) that F is also cube. But F was

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previously for x (with *minus* before the radical). No quarrel with HEATH's procedure as long as he does not ascribe it to EUCLID. However, that is exactly what HEATH does when he says: '... Euclid *really* finds GO from the equation $GO^2 \cdot \frac{b}{c} = \frac{c}{b} \cdot \frac{a^2}{4} - S$' (*ibid.*, 264, my italics). This is inexcusable! There are other unacceptable historical blunders in HEATH's commentary. For instance, realizing that EUCLID's solution 'corresponds' to only one root of HEATH's equation, the latter makes the following remark: 'He [*i.e.*, EUCLID] cannot have failed to see [?] how to *add* GO to GE would give another solution' (*ibid.*); then HEATH shows how '... the other solution can be arrived at ... [!]' (*ibid.*). The root of the last blunder can be found in ZEUTHEN, *Die Lehre*, 19–21.

[92] *Elements*, 2, 390.

also proved to be square so that the seventh from the unit is both cube and square. In a similar fashion one can prove that all other numbers which leave out five are both cube and square, q.e.d.[93]

So much for EUCLID. This proposition can serve, I think, as a beautifully striking example of the inherent limitations and the built-in chronic inadequacies of the beloved method (practiced by mathematicians posing as historians) of automatically and ahistorically transcribing EUCLID's language into the modern symbolism of algebra. A proposition for the proof of which EUCLID has to toil energetically (perhaps, it would be no great exaggeration to say, with might and main) and in the course of whose proof he had to rely on many previous propositions and definitions (*e.g.*, VIII.22, VIII.23, def. VII.20), becomes a trivial commonplace, which is an immediate outgrowth, a trite after-effect of our symbolic notation:

$$1, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, \ldots$$

As a matter of fact, if we use modern algebraic symbolism, this *ceases altogether to be a proposition* and its truthfulness is an immediate and trivial application of the definition of a geometric progression in the particular case when the first member equals one and the ratio, q, is a positive integer (for EUCLID)!

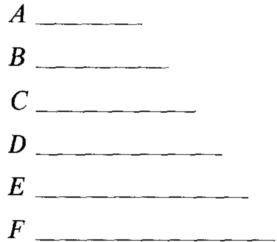
Second Example

In proposition IX.9, EUCLID states that:

> If as many numbers as we please beginning from a unit be in continued proportion, and the number after the unit be square, all the rest will also be square. And, if the number after the unit be cube, all the rest will also be cube.[94]

The Euclidean Proof

Let there be given as many numbers as we please starting from an unit and in continued proportion, namely A, B, C, D, E, and F, where A, the number after the unit, is square. By the previously proved proposition (IX.8), B and all those which leave out one are square. A, B, C being in continued proportion and A being square, it follows (by VIII.22) that C is also square. In the same manner, one can prove that all the rest are also square.



[93] *Ibid.*, 390–91.
[94] *Ibid.*, 392.

Now, let A be cube. By IX.8, C is cube and so are all those which leave out two. Since as the unit is to A so is A to B, the unit measures A as many times as A measures B. But the unit measures A according to the units in it; so A measures B according to the units in A. Consequently, A by multiplying itself makes B. But A is cube, and a cube by multiplying itself makes a cube (by IX.3). So B is cube. Now, A, B, C, D are four numbers in continued proportion, the first of which is cube; consequently, by VIII.23, D is also cube. And, for exactly the same reason, E is also cube, and so are all the rest, q.e.d.[95] So much for EUCLID.

What happens to this proposition if we sin once more and employ the modern notation? Clearly, its 'propositional' character – if I may be allowed to use this term in such a context – vanishes and *the proposition* becomes once more a trivial consequence of the general definition of a geometric progression:

$$1, a^2, a^4, a^6, a^8, \ldots, a^{2n}, \ldots$$

$$1, a^3, a^6, a^9, a^{12}, \ldots, a^{3n}, \ldots$$

There is nothing to be proved, there is no proposition any more. Again, in the *definition* of a geometric progression, some particular values have been substituted for the first term and the ratio, and the whole thing is nothing but this particularized form of the definition!

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Let me now switch Books and pick up some EUCLIDEAN propositions from the longest and one of the most remarkable books of the *Elements*, *viz.*, Book X, most of the results of which are, apparently, due to THEAETETUS.[96] EUCLID, however, as usual, systematized, made precise definitions and distinctions, and clarified.

Proposition X.9

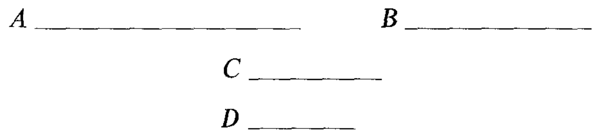
> The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number; and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.[97]

Now EUCLID'S proof proceeds along the following lines: First, if A, B are commensurable in length, then A has to B the ratio which a number has to a number (by X.5). Let this ratio be equal to the ratio of C to D, i.e., A is to B as C is to D.

[95] *Ibid.*, 392–93.

[96] This follows from a scholium to X.9 and from PAPPUS' commentary on Book X, preserved in Arabic; *cf.*, however, concerning the veracity of these sources, Á. SZABÓ, *op. cit.*, 100–111.

[97] *Elements*, 3, 28.



But the square on A has to the square on B a ratio which is the duplicate of the ratio which A has to B, since similar figures are in the duplicate ratio of their corresponding sides (by VI.20 Porism); and the ratio of the square on C to the square on D is the duplicate of the ratio of C to D, for between two square numbers there is one mean proportional number, and, by VIII.11, the square number has to the square number a ratio duplicate of that which the side has to the side. Consequently, as the square on A is to the square on B, so is the square on C to the square on D.

Next, let us assume that as the square on A is to the square on B so is the square on C to the square on D. We must prove that A is commensurable in length with B.

From the hypothesis it follows, says EUCLID, that as A is to B so is C to D.[98] Consequently, A has to B the ratio which a number (C) has to a number (D), *i.e.*, A is commensurable in length with B (by X.6).

Next, assume that A is incommensurable in length with B. (The proof proceeds by *reductio ad absurdum.*) If the square on A has to the square on B the ratio which a square number has to a square number, then, by the immediately preceding, it would follow that A is *commensurable* in length with B, which it is not; therefore the square on A cannot have to the square on B the ratio which a square number has to a square number.

Again, assume now that the square on A has *not* to the square on B the ratio which a square number has to a square number. If A were commensurable in length with B, then, by the preceding, the square on A would have to the square on B the ratio which a square number has to a square number, which is not the case; consequently, A is not commensurable in length with B, q.e.d.[99]

What does this theorem become when one throws away the geometrical néglígé which barely covers its algebraic nudity ..., in order to uncover the hidden charms of the latter?

If A, B be straight lines and C, D be numbers, then, if $A/B = C/D$, $A^2/B^2 = C^2/D^2$ and conversely. That is it! Is *this* what EUCLID says? Is this what EUCLID hid from our view in his geometrical dishabille? Were one to believe the partisans of 'geometric algebra', the answer to these questions should be an unequivocal **yes.** And yet the very serious deficiencies of such an interpretation are, I submit, self-evident in this case. To be sure, even HEATH himself, in his commentary on this proposition, says:

> This inference, which looks so easy when ... symbolically expressed, was by no means so easy for Euclid owing to the fact that $\underline{a}$, $\underline{b}$ are straight lines,

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[98] Incidentally, EUCLID takes this for granted, *i.e.*, without further ado, he assumes that ratios the duplicates of which are equal are themselves also equal; the converse of this assumption was employed in the preceding stage of the proof.

[99] *Ibid.*, 28–30. There is a porism (and a lemma) after this proposition; they do not interest us here.

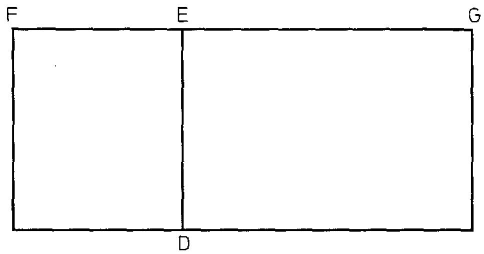
and $\underline{m}$, $\underline{n}$ numbers.[100] He has to pass from $\underline{a}:\underline{b}$ to $a^2:b^2$ by means of VI.20, Por. through the duplicate ratio; the square on $\underline{a}$ is to the square on $\underline{b}$ in the duplicate ratio of the corresponding sides $\underline{a}$, $\underline{b}$. On the other hand, $\underline{m}$, $\underline{n}$ being *numbers*, it is VIII.11 which has to be used to show that $m^2:n^2$ is the ratio duplicate of $\underline{m}:\underline{n}$.[101]

What HEATH says is, in effect, an unwitting confession of the ahistoricity lying at the very root of the concept of 'geometric algebra' (which, incidentally, does not prevent Sir THOMAS from using indiscriminately modern symbolism in the very same commentary, a few paragraphs below the above quotation)![102]

I shall not belabour this point anymore. Let us now go over to some other examples culled from Book X. The lemma before proposition X.22 states that:

> If there be two straight lines, then, as the first is to the second, so is the square on the first to the rectangle contained by the two straight lines.[103]

EUCLID proves this in the following manner:



FE and *EG* being two straight lines, as *FE* is to *EG* so is the square on *FE* to the rectangle *FE*, *EG*. Let us describe on *FE* the square *FD*, and let *GD* be completed. By VI.1, it follows that as *FE* is to *EG* so is *FD* to *GD*. But *FD* is the square on *FE* and *GD* is the rectangle *GE*, *DE*, *i.e.*, the rectangle *GE*, *FE*. Hence, as *FE* is to *EG* so is the square on *FE* to the rectangle *FE*, *GE*. Similarly, EUCLID goes on, as the rectangle *GE*, *FE* is to the square on *FE*, that is, as *GD* is to *FD*, so is *GE* to *EF*, q.e.d.[104]

Compare the above proof with the algebraic content of the lemma, which says that $a/b = a^2/ab$. In its algebraic form, the triviality of the entire enterprise becomes striking. The lemma becomes nothing but an inane, vapid, banal illustration of the simplification of fractions!

A similar instance is provided by the lemma after proposition X.53. In it EUCLID shows that if *AB*, *BC* are two squares, placed as they are in the accompanying diagram, and if the parallelogram *AC* be completed, then *AC* is a square,

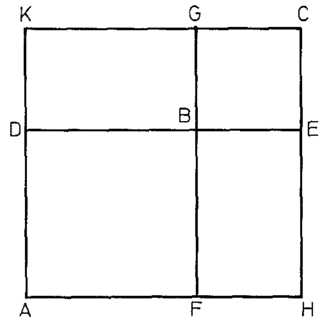
[100] This is how HEATH transcribed EUCLID's enunciation: 'If *a*, *b* be straight lines, and $a:b=m:n$, where *m*, *n* are numbers, then $a^2:b^2=m^2:n^2$ and conversely' (*ibid.*, 30).

[101] *Ibid.*, 31.

[102] *Ibid.*

[103] *Ibid.*, 50.

[104] *Ibid.*, 50–51.



DG is a mean proportional between AB and BC, and, finally, DC is a mean proportional between AC and CB.[105]

EUCLID's proof contains the following steps:

$$DB = BF$$
$$BE = BG$$
$$\therefore DE = FG.$$

But $DE = AH = KC$, and $FG = AK = HC$, by I.34.

$\therefore AH = KC = AK = HC$, hence the ▱$AC$ is equilateral. It is also rectangular; therefore it is a square.

Now since FB is to BG as DB is to BE and

$$\left.\begin{array}{l}\text{as } FB \text{ is to } BG, \text{ so is } AB \text{ to } DG, \text{ and}\\ \text{as } DB \text{ is to } BE, \text{ so is } DG \text{ to } BC,\end{array}\right\} \text{(by VI.1)},$$

then it follows that as AB is to DG, so is DG to BC (by V.11). Consequently, DG is a mean proportional between AB and BC.

Next, since as AD is to DK, so is KG to GC (for they are respectively equal) and, *componendo*, as AK is to KD, so is KC to GC (by V.18), while,

$$\text{(by VI.1)} \left\{\begin{array}{l}\text{as } AK \text{ is to } KD, \text{ so is } AC \text{ to } DC, \text{ and}\\ \text{as } KC \text{ is to } GC, \text{ so is } DC \text{ to } BC, \text{ it follows}\end{array}\right.$$

that as AC is to DC, so is DC to BC (by V.11), *i.e.*, that DC is a mean proportional between AC and BC, q.e.d.[106]

In algebraic notation, this lemma asserts that

$$x^2/xy = xy/y^2 \quad \text{and} \quad (x+y)^2/(x+y)\,y = (x+y)\,y/y^2.$$

Not only is this a beautiful example of the inadequacies inherent in transcribing geometrical propositions into algebraic notation, which transform once more a

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[105] *Ibid.*, 115.
[106] *Ibid.*, 115–16.

rather involved – though straightforward – proposition into a trivial matter of simplifying fractions, but, what is even more significant, the first half of the proposition becomes *obviously* (in algebraic form) a repetition of something already proved by EUCLID during his proof of X.25![107] Uneasy about this fact, HEATH suggests that the lemma may not be genuine! ...[108] There are some real dangers lurking behind the back of 'geometric algebra' ...

Passing now to more complicated propositions, let me mention – without reproducing the proof, which is long and rather difficult – proposition X.92. EUCLID's enunciation reads:

> If an area be contained by a rational straight line and a second apotome, the "side" of the area is a first apotome of a medial straight line.[109]

EUCLID's proof is, as usual, *completely* geometrical in character (relying on *two* diagrams and on *many* previous propositions), and in its course EUCLID uses the method of application of areas and the EUDOXEAN theory of proportions developed in Book V.[110] Clearly, there is not, and there could not be, any talk of 'equations', 'square roots', *etc.*, and no algebraic symbolism whatever is used. Yet HEATH, in his lengthy commentary,[111] starts by saying:

> This proposition amounts to finding and classifying
>
> $$\sqrt{\rho\left(\frac{k\rho}{\sqrt{1-\lambda^2}}-k\rho\right)}.$$
>
> The method is that of the last proposition. Euclid solves, first, the equations
>
> $$\begin{aligned} u+v &= \frac{k\rho}{\sqrt{1-\lambda^2}} \\ uv &= \tfrac{1}{4}k^2\rho^2. \end{aligned} \qquad (1)$$

[107] *Cf. ibid.*, 56–57, especially the beginning of 57.

[108] *Ibid.*, 116.

[109] *Ibid.*, 194. To enable the reader to grasp the meaning of the proposition and, at the same time, to give him an inkling of its complexity, I shall define the crucial concepts appearing in it: An *apotome* is an irrational straight line obtained by subtracting from a rational straight line another rational straight line, the two rational straight lines being commensurable in square only.

The *Annex* is the straight line which, when added to a compound irrational straight line obtained by subtraction (like an *apotome*) makes up the greater term, *i.e.* the *annex* is the negative term in an apotome.

A *second apotome* is an apotome having the following characteristics: Given a rational straight line and an apotome, if the square on the whole be greater than the square on the annex by the square on a straight line commensurable in length with the whole, and the annex be commensurable in length with the rational straight line set out, the apotome is called a *second apotome*.

A *medial* straight line is a mean proportional between two rational straight lines commensurable in square only.

A *first apotome of a medial* straight line is an irrational line obtained by subtracting from a medial straight line another medial straight line commensurable with the former in square only and which contains with it a rational rectangle.

[110] *Ibid.*, 194–97.

[111] *Ibid.*, 197–98.

Then, using the values of u, v so found, he puts

$$\begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \tag{2}$$

and $(x-y)$ is the square root required.[112]

A greater discrepancy than that between what EUCLID is doing and HEATH's 'translation' of it is indeed hard to come by, although, from the point of view of modern mathematics, what HEATH is doing is correct, and the two ways of approaching the proposition are mathematically equivalent. Historically, however, there is an unbridgeable chasm between EUCLID's *way* and HEATH's *way!* Sir THOMAS' procedure is, I think, a vivid exemplification of what the Italians must have meant when they came up with the phrase 'traduttore traditore'!

For whatever it is worth, let me note that in a graduate seminar on EUCLID's *Elements*, the students and I have found that one of the greatest difficulties in the study and understanding of Book X – typically considered among the most difficult, if not **the** most difficult in the *Elements* by historians of mathematics – consists in HEATH's modern interpretation of it and in the dangerous exercises in intellectual equilibrium required by the continuous adjustment to the two incommensurable ways of thinking when switching from EUCLID to HEATH and *vice versa*. To read Book X **not** through modern eyes, it would appear, removes the brunt from such an undertaking. A good edition for such a purpose is the copy of the *Elements* in Great Books of the Western World, which contains HEATH's translation without his commentaries! ...

I shall not burden the patience of the reader with many more examples of the type I have been lastly discussing. Let me only note that beautiful instances of the historical incompatibility between EUCLID's *geometry* and HEATH's *algebra* are offered by propositions X.100, X.101, X.102, and X.103.[113]

What I would like to do now is to call the reader's attention to a string of three consecutive propositions in Book X,[114] namely, 112, 113, and 114, which, I believe, show beyond any reasonable doubt that what EUCLID is doing is **not** algebra, but geometry. These are all long and (relatively) involved propositions, which I do not intend to prove here (for obvious reasons). I must, however, reproduce the enunciations so that the reader can get the flavour of EUCLID's ideas and, then, see my point more easily.

Proposition X.112

The square on a rational straight line applied to the binomial straight line[115] produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and moreover in the same ratio; and further

[112] *Ibid.*

[113] *Ibid.*, 221–231.

[114] *Ibid.*, 243–53.

[115] A *binomial* straight line is an irrational line obtained from the addition of two rational straight lines commensurable in square only.

the apotome so arising will have the same order as the binomial straight line.[116]

Proposition X.113

The square on a rational straight line, if applied to an apotome, produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome.[117]

Proposition X.114

If an area be contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, the "side" of the area is rational.[118]

It is easy for us to discern the complete symmetry of these three enunciations. There is no symmetry, however (and this is of utmost significance), in the three proofs. About X.112 HEATH says that '... it is the equivalent of rationalising the denominators of the fractions $\frac{c^2}{\sqrt{A}+\sqrt{B}}, \frac{c^2}{a+\sqrt{B}}$, by multiplying numerator and denominator by $\sqrt{A}-\sqrt{B}$ and $a-\sqrt{B}$ respectively [!].'[119] HEATH goes on to say that 'Euclid proves that $\frac{\sigma^2}{\rho+\sqrt{k\rho}}=\lambda\rho-\sqrt{k}\cdot\lambda\rho(k<1)$, and his method enables us to see that $\lambda=\sigma^2/(\rho^2-k\rho^2)$.'[120] In the continuation of his commentary, HEATH considers it a certainty that '... the Greeks must have had some analytical method which suggested the steps of such proofs',[121] the entire burden of the statement seeming to be that 'analytical' is used here as synonymous to 'algebraic'.[122]

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[116] *Ibid.*, 243.

[117] *Ibid.*, 248.

[118] *Ibid.*, 252.

[119] *Ibid.*, 246. Isn't this a 'faithful' translation? ...

[120] *Ibid.*

[121] *Ibid.*

[122] On 'genuine' Greek analysis see R. ROBINSON, 'Analysis in Greek Geometry', *Mind*, N.S., **45** (1936), 464–73 and M. MAHONEY, 'Another Look at Greek Geometrical Analysis' (full reference in note 26, above). MAHONEY's article is, on the whole (as are his other studies), perceptive, penetrating and insightful. Concerning his attitude toward 'geometrical algebra', however, these qualities are lacking. Thus, he says: 'As they are given in the *Data*, however, the theorems pertaining to geometrical algebra are *cumbersome* [?], involving as they do the intricate construction of plane figures. Working mathematicians used a simpler form of geometrical algebra, an algebra of line lengths ... Although it is an example of theoretical, rather than problematical, analysis, the analysis of Euclid XIII, 1 ... illustrates the use of the simplified algebra of line lengths ...' (*op. cit.*, 331, my italics).

Where is there any concrete, specific proof for the use of 'geometric algebra' in pre-EUCLIDEAN, or even EUCLIDEAN, times? *There is none!* The reference to the scholiast's interpolation to XIII.1 misses the point, I think, since the exact date of the interpolation is unknown; furthermore, one of the few positive statements one can make about this interpolation (and others) is that it is *spurious*, or, as HEATH put it, '... altogether alien from the plan and manner of the *Elements*' (*Elements*, **3**, 442). It was interpolated perhaps as late as 500 years after the writing of the *Elements* (HEATH says that all the interpolations to XIII.1-5 '... took place before Theon's time ...' (*ibid.*) – *i.e.*, fourth century A.D.), and the method of proof it displays is totally foreign to classical Greek mathematics. There is not one shred of reliable historical evidence to support the speculations of BREITSCHNEIDER, HEIBERG, *etc.* that this method represents '... a relic of analytical investigations by Theaetetus or Eudoxus ...' (*ibid.*); indeed the whole history of Greek mathematics seems to exclude such an inference. But even if one believes HEIBERG's later dating (in 'Paralipomena zu Euklid', *Hermes*, **38** (1903), 46–74, 161–201, 321–356), namely that the author of these interpolations is HERON OF ALEXANDRIA, this would still make these additions some 400 years younger than EUCLID and would place them comfortably (to the exclusion of PAPPUS and DIOPHANTUS) *after* the decline of classical Greek mathematics.

MAHONEY also speaks of the '... increased use of an informal, but subtle and penetrating, *algebra of line lengths*'... (*op. cit.*, 337, my italics) in the works of the post; EUCLIDEAN mathematicians of the 3rd century, APOLLONIUS and ARCHIMEDES. Then he goes on saying, 'ARCHIMEDES provides an example of these analyses' (*ibid.*). His reference to proposition II.1 of *On the Sphere and the Cylinder*, however, does not warrant any allusion to ARCHIMEDES' proof as an 'algebra of line lengths.' (*Cf.* J. L. HEIBERG, ed., *Archimedes Opera Omnia*, 2nd ed. (Leipzig: Teubner, 1910) **1**, 170–74.) The proof is still essentially geometrical in the best tradition of EUCLID's *Elements!* MAHONEY, then, proceeds to give a detailed example of '... Greek geometrical analysis in action, *one which proceeds by an algebra-like manipulation of line lengths* ...' (*ibid.*, my italics). His chosen example is proposition II.4 of ARCHIMEDES' *On the Sphere and the Cylinder.* I must say, however, that I am unconvinced. Again, what ARCHIMEDES is doing in II.4 of *On the Sphere and the Cylinder* is very much like what EUCLID is doing in the *Elements* (though there are obviously differences, some of which do point toward a freer manipulation of lines; interstingly, however, in both examples given by MAHONEY the line lengths *are closely associated* with two or three-dimensional figures!); I cannot see how somebody whose mind was not 'corrupted' by algebraic reasoning and manipulations can describe ARCHIMEDES' proof as 'algebra-like manipulation of line lengths', though, to be sure, this name is less offensive than 'geometrical algebra'.

One more remark. Speaking of EUCLID's *Porisms*, MAHONEY says that it represents '... the best example of the sort of treatise included in the Treasury of Analysis. It also illustrates well TANNERY's remark that the Greeks lacked not so much the methods as the language to express them' (*ibid.*, 343–44). There is another laudatory reference to TANNERY's saying at the end of the article. According to MAHONEY, the *Porisms* indicates '... why the lack of a suitable mode of exposition – such as symbolic algebra – prevented the Greeks from pursuing geometrical analysis further and from being able to express clearly what they had accomplished [!]. In the realm of geometrical analysis in particular, TANNERY's remark holds true; the Greeks did not so much lack methods of mathematics as means to express them' (*ibid.*, 348). Finally the motto itself of MAHONEY's article is, once more, TANNERY's original saying: 'Ce qui manque aux mathématiciens grecques [*sic*] ce sont moins les méthodes... que des formules propres à l'exposition des méthodes' (*ibid.*, 318). MAHONEY is not alone in praising this famous 'fliegende Wort' of TANNERY; so do ZEUTHEN, HEATH, *etc.* And yet, is not this famous saying an unwitting confession that 'geometric algebra' is a pernicious and historically stillborn concept to use? Furthermore, is it not absurd to talk of *the methods* when *the means to express them*, i.e., *to use them*, are not available? How could one use a method which is *de facto* inexpressible, *i.e.*, unthinkable? Within the given limits of coherence of a mathematical culture, the *methods* available to that culture are exactly those by means of which the culture reached and *expressed* its mathematical achievements. The methods are contained in the tangible products of that mathematical culture. In the absence of treatises on the methodology of mathematics, the methods are those embodied in and displayed by the actual mathematical works available to the historian. The question is really very simple: To what extent does one possess the method if he lacks the means to put it to use? And the answer seems to me obvious. 'Wovon man nicht sprechen kann darüber muss man schweigen' has not only hortatory and prescriptive consequences; it is also, *mutatis mutandis*, a correct description of the *historical* state of affairs in intellectual history: 'Wovon man nicht sprechen kann darüber schweigt man'. If a culture (any culture!) cannot speak it does **not** speak. It remains silent. It certainly does not hide its impotence. Part of being ignorant of something is being ignorant of your ignorance. If you know that you are ignorant, your ignorance *stricto sensu* has disappeared. And the Greeks, clearly, did not know that they did not know algebra. So they did not hide their ignorance behind a geometrical screen. There is nothing lurking in hiding behind Greek geometry!

Now I happen to think that this is not at all certain. It is certain only (and I am not referring specifically to HEATH, whose serious scholarly contributions to the history of mathematics are firmly extablished) for somebody who has grown arrogant of the past and cannot consequently think anymore, when complicated geometrical questions are involved, but in analytical terms. In other words, it is certain for somebody who knows how to get out of geometrical difficulties by translating them into analytical terms. What I am saying, I guess, is that if **we** do not see any other way, it does **not** mean that the *Greeks*, who obviously did not have our algebra, did not see any either! So the Greeks did **not** use '... geometry as the equivalent of our algebra' – this is infatuation of the twentieth century with its own great achievements and it certainly is anachronistic – *they used geometry.*' It is **we** who are using algebra, with remarkable dexterity, I must confess, as the equivalent of *their geometry!*

Furthermore, in a more substantive fashion (and in a less polemical vein), if EUCLID's *lines* were *general algebraic symbols* (which they are not), which could be manipulated like such symbols, then the essence of X.112 could be expressed as follows: If $R^2 = B \cdot A$, where R is a rational line and B is a binomial, then A is a corresponding apotome. Under such circumstances, X.113 would follow immediately and trivially from X.112, as a consequence of the unicity of algebraic operations and the commutativity of multiplication, since X.113 states only that

If $R^2 = A \cdot B$, where R is rational and A an apotome, then B is a corresponding binomial.

In such a setting, all of EUCLID's efforts to prove X.113[123] would have been in vain, and therefore incomprehensible. Indeed, under such circumstances, no proof at all of X.113 would have been necessary and X.113 would have become, *at best*, a Porism[124] and not an independent proposition. But this is certainly not the case in the *Elements*, and this is, I believe, a beautiful substantiation and corroboration of my view: *Greek geometry is geometry!* It is not algebra (without qualification), *i.e.*, it is not even 'geometrical algebra' if the term is understood as it has been traditionally understood since TANNERY and ZEUTHEN.

Moreover, proposition X.114 is another case in point. In the same notation I used above, it merely states that

If $A \cdot B = R^2$, where A is an apotome and B is a corresponding binomial, then R is rational,

which, once more, *algebraically* is nothing but X.112, or for that matter, X.113 read in the opposite direction so to speak. Had algebra been in the background of EUCLID's mind, he would **not** have spent great intellectual energies to prove *thrice exactly the same thing*. The conclusion is clear: For EUCLID, who did not think algebraically, the triplet of propositions I discussed did not represent one and the same proposition; and, indeed, *geometrically*, they are different. It would be enough to go laboriously through the proofs to convince oneself of the truth of this, my last assertion.

[123] *Elements, ibid.*, 248–50.
[124] In its sense of 'corollary'.

VI

The last topic I want to deal with in this paper is the question of its originality. How original is it? In its form, the thrust of its argument, most of the examples cited to substantiate different points, its 'radicalism', and in its main conceptual emphasis the paper is original. Still I know very well that 'there is nothing new under the sun'. This is why I was not completely surprised when, in the course of my research, I encountered isolated and sporadic ideas which I had, naively, considered my exclusive intellectual property. Thus, in the 1930's, two lengthy articles appeared under the title 'Die griechische Logistik und die Entstehung der Algebra'.[125] Their author was JACOB KLEIN, who, recently, had the articles published in book form in an English translation by EVA BRANN.[126] This is a book which I personally consider to be one of the most substantial contributions to the literature on the history of mathematics; by the same token, it seems to be one of the least influential. In this book, KLEIN deals primarily with the differences between the Greek and the modern concept of number, and his conclusions completely match mine. I shall return to KLEIN's ideas below.

Two other authors whose works contain interpretations similar to mine are ABEL REY and MICHAEL S. MAHONEY. Strangely, however, each of them adheres to a view which accepts the legitimacy of the term 'geometrical algebra'. Nevertheless this does not make those of their ideas which are strongly supportive of my interpretation less interesting. Thus, ABEL REY questioned the propriety of ZEUTHEN's interpretation of PYTHAGOREAN and EUCLIDEAN mathematics.[127] Both he and MAHONEY pointed out forecefully and, I think, convincingly (and in this they also agree with KLEIN), that algebra is a product of modern times, starting its full flowering with the sixteenth and seventeenth centuries.[128] Both, as we saw, emphasized the differences between the geometrical and the algebraic way of thinking,[129] and both attacked the view which identified pre-Hellenic mathematics as algebraical.[130]

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[125] *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* (Abteilung B: Studien), **3** (fasc. 1, 1934), 18–105 and **3** (fasc. 2, 1936), 122–235. Ironically, KLEIN's insightful articles were followed in each case by NEUGEBAUER's unbridled transcriptions of ancient mathematical texts into algebraic language; thus, in fasc. 1, NEUGEBAUER published 'Serientexte in der babylonischen Mathematik' (*ibid.*, 106–114), while in fasc. 2 KLEIN's article was succeeded by 'Zur geometrischen Algebra'! To make things even more piquant, the article immediately following the last part of KLEIN's study was 'Eudoxos Studien III. Spuren eines Stetigkeitaxioms in der Art des Dedekind'schen zur Zeit des Eudoxos' (*op. cit.*, 236–244), by OSKAR BECKER!

[126] *Greek Mathematical Thought and the Origin of Algebra* (Cambridge, Mass.: 1968, M.I.T. Press). The book includes an 'Appendix', containing VIETA's *In artem analyticem* [*sic*] *isagoge* (Tours, 1591), translated into English by the Reverend J. WINFREE SMITH. (Parenthetically, let me urge those readers who have a choice and wish to read KLEIN's highly interesting study to refer back to the original German articles: somehow the pomposity, stuffiness, and turgidity of the author's style are better accomodated by the Teutonic cadences than by the more friendly sounds of the perfidious Albion...)

[127] *Les Math. en Grèce*, 30.

[128] REY, *op. cit.*, 32, 44, 45, 48, 91; MAHONEY, 'Die Anfänge der algebr. Denkweise', *passim.*

[129] This makes their acceptance of the legitimacy of the term 'geometric algebra' the more difficult to understand. Instances of this acceptance in REY can be found in *op. cit.*, 33, 46, 49–51, 52 ('Le théorème de Pythagore lui-même est une résolution intuitive de l'equation $x^2+y^2=z^2$'), 56–57; also, *La Science dans l'Antiquité*, 352; for examples of MAHONEY's acceptance see note 26 above.

[130] MAHONEY, 'Babyl. Algebra', *passim*; Rey, *op. cit.*, 34, 36–37, 41, 91–92. The last reference

The only scholar (so far as I know) who, in a book remarkable for its solid, penetrating, and far-sighted analysis, rejected peremptorily the historical value of the concept of 'geometric algebra' is the Hungarian philologist ÁRPAD SZABÓ; the book, to which we already referred before, is *Anfänge der griechischen Mathematik.* Though his remarks about 'geometric algebra' are a mere aside to the main endeavour of his analysis,[131] they fit *one* of the main messages of his book, *viz.*, that essentially, fundamentally, Greek mathematics became *very* early in its historical development Greek *geometry*, and that it grew and matured in very close nearness to PYTHAGOREAN musical theory. The main issues of Greek mathematics were *geometrical* issues, not the least of which is the issue of incommensurability.[132] These issues were *Greek* issues, not borrowed and disguised ones. Indeed,

> Jene Vermutungen die die "geometrische Algebra der Pythagoreer" als Übernahme bzw. griechische Weiterentwicklung von ursprünglich babylonischen Gedankengängen auffassen wollten, waren voreilig. Der Zusammenhang dieser Art Kentnisse mit der "babylonischen Wissenschaft" ist in Wirklichkeit *nirgends erwiesen.* Im Gegenteil! Man hat eher den Eindruck, dass die hier behandelte 'Flächengeometrie der Pythagoreer' eine *rein griechische Errungenschaft* war.[133]

Algebra is not geometry and, therefore, algebraic transcriptions of non-algebraic mathematical texts are historically inadmissible.[134] Besides, there are no traces in the Greek mathematical tradition (of both the pre-EUCLIDEAN and the EUCLIDEAN period) of any genuine algebraic ways of thinking.[135] This is why I think ABEL REY was right to state:

> Elle [la mathématique greque] sera géométrique. Lorsqu'elle ne le sera plus – lorsqu'elle tendra à devenir calculante, sous l'influence orientale sans doute, donc par l'affaiblissement même de ses forces créatrices, voire chez le plus puissant de ses derniers représentants, Diophante, à la fin de la civilization antique – elle sera tout près aussi de ne plus être.[136]

It is now time to return to JACOB KLEIN'S book. *Number* for the Greeks meant *positive integer. Numbers* are represented by EUCLID as *line segments.* After the discovery of the irrational, it became obvious that it is not the case that 'all *line segments* could be associated with numbers'; however, it does not follow from here that the converse statement is also false, and, indeed, EUCLID associates *numbers*

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in REY is a scathing attack against NEUGEBAUER'S interpretation of Babylonian mathematics. It is there that REY says: 'Du reste la preuve convaincante c'est que si, bien avant l'ère chrétienne et surtout avant Diophante, on avait eu l'idée algébrique des équations et, peu on prou, la pensée algébrique, toute la face de la mathématique en eût été changée' (*ibid.*, 91).

[131] Most of them appear in an appendix to the book.

[132] *Op. cit.*, 28, 36, *passim.*

[133] *Op. cit.*, 488.

[134] *Ibid.*, note 21, 487. It seems to me, therefore, a concession to the prevailing mode of writing the history of mathematics, which SZABÓ so eloquently denounced, when he himself starts, somewhat indiscriminately, using algebraic notation in his geometrical discussions (*cf. op. cit.*, 483).

[135] *Op. cit.*, 472–73.

[136] *La science dans l'antiquité*, 390.

with *line segments* throughout his so-called 'arithmetical books', *i.e.*, Books VII, VIII, and IX of the *Elements*.

The Greek concept of **arithmos** (number), *i.e.*, a 'number of things' (what KLEIN calls *Anzahl*), was replaced in the sixteenth century by a *new* concept of number as an abstract symbol. Instrumental in this change was FRANCOIS VIÈTE (VIETA), 1540–1603, who transformed the concept of *arithmos* into the modern concept. This transformation marks the beginnings of *modern* mathematics. Greek **arithmos** and modern **number** do *not* mean the same thing. As KLEIN has it, the two concepts differ in 'Begrifflichkeit', *i.e.*, *conceptualization* and *intentionality*. (By the latter KLEIN understands '... the mode in which our thought, and also our words, signify or intend their objects.'[137]) For the Greeks, *arithmos* always meant a *number of things* (Anzahl), although 'things' did not have to be mentioned explicitly; for modern mathematics after VIÈTE, on the other hand, *Number is a concept*; it is *the concept of quantity*! As *numbers* come to be regarded as *abstract* and *symbolic* entities, a 'new' mathematics (and by the same token, a 'new' science on the long run) came into being, the mathematics in which the *symbolic form* of a statement is inseparable from its content; indeed, if I may put it this way, **the form is the content**! *Mutatis mutandis*, this separation is also, to a very great extent, impossible in modern (physical) science, where *mathematical form* and *physical content* are irreducibly intertwined and hopelessly enmeshed.

With VIÈTE and his successors (STEVIN, DESCARTES, WALLIS, *etc.*), then, a radical conceptual change has occurred. It is, therefore, historically unwarranted to apply mechanically to Greek mathematics the manipulations and jugglings of modern mathematical symbolism. Historians of mathematics, however, have been doing exactly this. Being themselves immersed in modern ways of thought, they have been reading Greek mathematical texts through modern glasses and, to nobody's surprise, were rather successful in identifying in these texts an *inexistent* Greek algebra.[138] They could achieve such a fantastic result only by

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[137] *Op. cit.*, 118.

[138] Modern algebra (the only true algebra) is a creation of the sixteenth and seventeenth centuries. Its great protagonists are VIÈTE, DESCARTES, and FERMAT. It marks the passage from an old way of thinking in mathematics (the geometrical way, the *mos geometricus*) to a new way (the symbolic way, the *mos per symbola*). Its historical development is rightly connected with the reintroduction into the West of the great works of classical Greek mathematics which, however, contained the old way of thinking, to be discarded by modern mathematics. With VIÈTE algebra becomes the very language of mathematics; in DIOPHANTUS' *Arithmetica*, on the other hand, we possess merely a refined auxiliary tool for the solution of arithmetical problems (*cf.* M. MAHONEY, 'Die Anfänge der algebr. Denkweise', *passim*). In the seventeenth century, algebra was called *ars analytica*, a pregnant name indeed. It shows the difference between the Greek approach and that of the seventeenth century. For the Greeks, mathematics was not an *art*, a manipulative technique (*techne*) but a science (*episteme, scientia*). Furthermore, for the Greeks *analysis* was merely a means of discovery, a heuristic tool. Mathematics, *episteme*, was limited to *synthesis*. In the seventeenth century, on the other hand, one is faced with algebraical analysis without any synthesis. This new approach meant (among other things) a certain loosening of the Greek strictures of rigor and a new mathematical style. MAHONEY identifies the necessarily *external* factors which led to this development as PETRUS RAMUS' pedagogical endeavours and the search for a universal symbolism (*characteristica universalis*) starting with RAMON LULL in the thirteenth century. These two factors were united in RAMUS, who contributed to a separation of the universal symbolism from its ties with magic *via* the *ars memoriae*. According to MAHONEY, RAMUS seems to have been the first to demand more respectability and status for the algebraic *art*, practiced, as a rule, outside the walls of the university establishment (*cf. ibid.*, 25).

betraying Greek mathematics, only by applying to it foreign categories of post-Renaissance mathematical thinking.

One should not apply modern symbolism to Greek mathematics with impunity, as if modern symbolism were nothing but a temporally universal (*i.e.*, historically indifferent) means for organizing and simplifying *any* given conceptual content. The fact that it is *modern* symbolism that one applies is, in itself, the best evidence for the ahistoricity of such a procedure. KLEIN has shown, and I think successfully, that '... symbolic formalism is at the *core* of the modern concept of number, and that to translate Greek mathematics into its terms obscures completely both the meaning of the Greek concept and the genuine Greek achievement in the theory of number.'[139]

In his commentary on proposition VIII.4, HEATH talks '... of the cumbrousness of the Greek method of dealing with *non-determinate numbers*. The proof in fact is not easy to follow', he goes on, 'without the help of modern symbolical notation. If this be used, the reasoning can be made clear enough.'[140] The question, however, is: Did the Greeks in general, and EUCLID in particular, ever use 'non-determinate' numbers?

In his *Die Algebra der Griechen*, G.H.F. NESSELMANN produced a since famous trichotomous classification of the historical development of algebra.[141] The *three* stages distinguished by NESSELMANN are: *Rhetorical*, *Syncopated*, and *Symbolic* Algebra. In NESSELMANN's classification, DIOPHANTUS' *Arithmetica* fell in the second category (syncopated).[142] NESSELMANN's analysis, however, is very approximative and, at best, faulty.[143] LÉON RODET evolved another, dichotomous classification: 1. The 'algebra of abbreviations and *given* numbers' and 2. Symbolic algebra (*i.e.*, the only true algebra, algebra proper).[144]

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[139] From the dust jacket of KLEIN's book.

[140] *Elements*, **2**, 353, my italics.

[141] *Op. cit.*, 301–303.

[142] KLEIN, I think rightly, sees DIOPHANTUS' *Arithmetica* as an exercise in *theoretical logistic* (*cf. op. cit.*, 127–149, *passim*).

[143] LÉON RODET in *op. cit.* (see note 21 above for full reference) demolishes NESSELMANN's taxonomy. It is there that RODET says: '... il faut reconnaître que cette distinction des trois étapes successives du langage algébrique a quelque chose de séduisant. Il n'y a qu'un malheur: c'est qu'elle est bâtie uniquement sur un échafaudage d'inexactitudes...' (*ibid.*, 56). RODET points out that even admitting the truthfulness of NESSELMANN's classification, *it is wrong to call it historical*! The three stages do not correspond to three *historically successive* stages even on NESSELMANN's own account, since the lowest rank of this classification is occupied by the Arabs and by Italian mathematicians writing between the Crusades and the sixteenth century, while DIOPHANTUS (3rd century A.D.) corresponds to the middle stage and the Hindus, reported masters of the Arabs, are occupying the highest rank, *i.e.*, the same spot as modern symbolic algebra! RODET destroys especially this characterization of Hindu mathematics and reveals its absolute historical falsehood due to NESSELMANN's ignorance of '... les notations algébriques des Indiens' (*ibid.*, 57). Speaking of Hindu 'algebraic notation', LÉON RODET says: 'Il lui manque, pour être mise en parallèle avec la nôtre, deux choses essentielles: des signes spéciaux pour les deux opérations directes de l'addition et de la multiplication, et le moyen de représenter autrement que par des nombres particuliers les *paramètres* qui entrent, simultanément aux *variables* proprement dites, dans nos expressions algébriques. Enfin, comme chez Diophante, les symboles qu'elle emploie ne sont que les initiales des noms des quantités qu'elle veut représenter. L'algèbre Indienne mérite tout autant que celle des Grecs et des Européens entre le XII^{e} et le XVII^{e} siècles, le nom d'*Algèbre* syncopée...' (*ibid.*, 60).

[144] *Op. cit.*, 69–70.

According to RODET even DIOPHANTUS' algebra belongs to the first type.[145] Modern algebra '... n'a pris naissance que lorsqu'on eut l'idée

> de représenter les données du problème sous forme générale par un symbole, de symboliser également les opérations chacune par un signe spécial, et d'arriver ainsi non plus à resoudre avec plus on moins de facilité un problème particulier, mais à trouver des formules donnant la solution de tous les problèmes d'une même espèce, et, parce qu'elle servait à caractériser chaque espece de probléme, servant à exprimer les propriétés générales de certaines catégories des nombres, de certaines familles de figures, où a formuler les lois de certaines classes de phénomènes naturels.[146]

Do we find, then, any *algebra* in EUCLID? I doubt it! EUCLID's *numbers* are *given* line-segments, no abstract symbols, and EUCLID's presentation is *not* symbolic. *It always deals with determinate numbers* of units of measurement which are **not** seen as representing specific illustrations, instances of a *concept of general magnitude.*[147] From here on, allow me to quote JACOB KLEIN:

> In *illustrating* each determinate number of units of measurement by measures of distance it [*i.e.*, the EUCLIDEAN presentation] does *not* do two things which constitute the heart of the symbolic procedure: It does *not* identify the object represented with the means of its representation, and it does *not* replace the real determinateness of an object with a *possibility* of making it determinate, such as would be expressed by a sign which, instead of *illustrating* a determinate object, would *signify* possible determinacy... when in the arithmetical books an arithmetical, or more exactly a logistical proposition is demonstrated *generally* with the aid of lines, this does not in the least mean that there exists either a general number or the concept of a "general," i.e., indeterminate, number corresponding to this general proof... the *general* "linear approach"... intends only *determinate* numbers... Since... in Euclid... the single illustrative lines are additionally identified by a letter, the possibility arises of representing the numbers intended by those letters. This does not, however, in the least amount to the introduction of symbolic designations. Letters for indicating magnitudes and numbers seem to have been used already by Archytas... [As TANNERY put it, however,] *the letter does not symbolize the value of a number, and does not lend itself to being operated on.* Aristotle, too made use of such mathematical letters, e.g. in the *Physics* and in *On the Heavens*; and he even introduced them into his "logical" and ethical investigations. But such a letter is never a "symbol" in the sense that that which is signified by the symbol is in itself a "general" object.[148]

It simply cannot be said any better!

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[145] *Cf.*, in this connection, MICHEL's statement: 'D'une façon générale, le vocabulaire de Diophante reste imprégné de géométrie, comme en témoignent ces énoncés de problèmes' (*op. cit.*, 641); also, NESSELMANN: 'So finden wir wirklich selbst bei Diophant Beispiele von gänzlicher Vernachlässigung des Gebrauches der Abbreviaturen..., die also ganz der rhetorischen Stufe angehören' (*op. cit.*, note 15, 304).

[146] *Ibid.*

[147] KLEIN, *op. cit.*, 123.

[148] *Op. cit.*, 123–24.

Now, *Symbolic Algebra* (*i.e.*, algebra proper) was not born, as RODET has shown,[149] 'before someone had the idea of representing what is given in a problem in a general form by means of a symbol, and of similarly symbolizing each of the operations by a special sign.'

Such an idea, so far as I am aware, certainly does **not** appear in the *Elements*, in which EUCLID, according to PROCLUS, collected '... many of the theorems of Eudoxus, perfecting many others by Theaetetus, and bringing to irrefragable demonstration the things which had only been somewhat loosely proved by his predecessors.'[150]

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Editorial Note: A defense of his views will be published by Professor van der Waerden in a succeeding issue.

[149] See text to note 145 above; *cf.* also, KLEIN, *op. cit.*, 146–47.

[150] EUCLID, *Elements*, *1*, 37. I have striven in this paper to demolish the validity of the concept of 'geometric algebra' as a useful historiographic term. In this, if POPPER is right, I must have achieved the highest level of understanding of the true underpinnings of that concept... According to Sir KARL, there are three levels of understanding: 1. *The lowest* represented by the pleasant feeling of having grasped the argument. 2. *The medium level*, represented by the ability to repeat the argument. 3. *The highest level*, represented by the ability to refute the argument. (*Cf.* IMRE LAKATOS, 'Proofs and Refutations (II)', *British Journal for the Philosophy of Science*, **14** (1963–64), 120–139, at 131.)