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**Probability Theory and Related Fields**



# Limiting Behavior of the Norm of Products of Random Matrices and Two Problems of Geman-Hwang *

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## 1. Introduction

In the theory of large random matrices, how to dominate the norm of a random matrix is a very important problem. This is the reason why many authors are interested in this problem. For interesting works, see Geman (1980), Jonsson (1983), Silverstein (1984) and Yin et al. (1984). In these papers, they consider the norm of a sample covariance matrix, with different moment requirements. The newest result of Yin et al. requires only the existence of 4th moment.

In this paper, we consider a different type of random matrices, namely $W^k$, i.e. a power of a square random matrix with iid entries.

The first result in this paper (Theorem 2.1) is

$$\overline{\lim_{n\to\infty}} \left\| \left(\frac{W}{\sqrt{n}}\right)^k \right\| \leqq (1+k)\,\sigma^k, \qquad \text{a.s. } (n \text{ is the size of } W),$$

here $\sigma^2$ is the variance of the entries of $W$. We assume only the existence of the 4th moment of the entries of $W$. From this result it is easy to show that the spectral radius of $W/\sqrt{n}$ is not greater than $\sigma$ with probability 1.

In proving the above result, a new kind of graphs has to be discussed carefully, (§3), and the truncation method used in Yin et al. (1984) is also important here.

As applications of the above result, we have solved two open problems announced in the paper Geman-Hwang (1982). The solutions are in §5, §6 and §7.

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## 2. Limiting Behavior of Matrix Product Norm

In Sects. 2–4, we will prove the following theorems.

**Theorem 2.1.** *Let* $\{w_{ij}: i=1,2,\ldots,j=1,2,\ldots\}$ *be iid random variables, and* $W_n$ *be the* $n\times n$ *matrix* $(w_{ij})$ $i,j=1,2,\ldots,n$. *Suppose*

$$E w_{11}=0, \quad E w_{11}^2=\sigma^2, \quad E w_{11}^4<\infty. \tag{2.1}$$

*Then, for any positive integer k, we have*

$$\limsup_{n\to\infty}\left\|\left(\frac{W_n}{\sqrt{n}}\right)^k\right\| \leqq (k+1)\,\sigma^k \quad \text{a.s.} \tag{2.2}$$

*Here* $\|A\|$ *denotes the operator norm of the matrix A.*

*Denote by* $\lambda_i(A)$, $i=1,2,\ldots,n$, *the n eigenvalues of the* $n\times n$ *matrix A. We have*

**Theorem 2.2.** *Under the same conditions as in Theorem* 2.1, *we have*

$$\limsup_{n\to\infty}\max_{1\leqq i\leqq n}\left|\lambda_i\left(\frac{W_n}{\sqrt{n}}\right)\right|\leqq\sigma \quad \text{a.s.}$$

This result was earlier proved by Geman (see Geman 1984 or Hwang 1985) under stronger conditions that $Ew_{11}=0$, $Ew_{11}^2=\sigma^2$ and $Ew_{11}^n\leqq n^{\beta n}$ for all $n\geqq 3$ and some $\beta>0$.

Theorem 2.2 can be easily deduced from Theorem 2.1 as follows: For any integer $k\geqq 1$, by Theorem 2.1,

$$\begin{aligned}\limsup_{n\to\infty}\max_{1\leqq i\leqq n}\left|\lambda_i\left(\frac{W_n}{\sqrt{n}}\right)\right| &= \limsup_{n\to\infty}\max_{1\leqq i\leqq n}\left|\lambda_i\left[\left(\frac{W_n}{\sqrt{n}}\right)^k\right]\right|^{1/k}\\ &\leqq \limsup_{n\to\infty}\left\|\left(\frac{W_n}{\sqrt{n}}\right)^k\right\|^{1/k}\leqq(k+1)^{1/k}\,\sigma \quad \text{a.s.}\end{aligned}$$

Letting $k\to\infty$ we get Theorem 2.2.

## 3. Some Lemmas

At first we state the following lemma which can be found in Yin et al. (1984).

**Truncation lemma.** *Let r be a number in the interval* $[\frac{1}{2},2]$, $\{w_{ij}: i,j=1,2,\ldots\}$ *be a set of iid random variables with* $E w_{11}=0$, $E|w_{11}|^{2/r}<\infty$. *For each n, let* $W_n$ *denote the* $p\times n$ *matrix whose* $(i,j)$*-entry is* $w_{ij}$, *here* $p=p(n)$ *satisfies* $p/n\to y\in(0,\infty)$, *as* $n\to\infty$.

*Then there exists a sequence of positive numbers* $\delta=\delta_n$ *such that*

1. $\delta\to 0$, *as* $n\to\infty$,
2. $P(W_n\neq\hat{W}_n, \text{ i.o.})=0$; *here* $\hat{W}_n$ *is the* $p\times n$ *matrix, with the* $(i,j)$ *entry*

$$w_{ijn} = w_{ij} 1_{\{|w_{ij}| < \delta n^r\}},$$

*and the convergence speed of $\delta$ to zero can be slower than any preassigned speed.*

In fact, the truncation lemma can easily follow from the fact that for any fixed $\eta > 0$

$$\begin{aligned}
P\left(\bigcup_{n=2^k}^{\infty}\left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{n}|W_{ij}|\geqq \eta n^r\right)\right) &\leqq \sum_{l=k}^{\infty} P\Big(\max_{2^l \leqq n < 2^{l+1}} \max_{1\leqq i,j\leqq n} |W_{ij}| \geqq \eta 2^{lr}\Big)\\
&\leqq \sum_{l=k}^{\infty} P\Big(\max_{2^l \leqq n \leqq 2^{l+1}} \max_{1\leqq i,j\leqq 2^{l+1}} |W_{ij}| \geqq \eta 2^{lr}\Big)\\
&= \sum_{l=k}^{\infty} P\Big(\max_{1\leqq i,j\leqq 2^{l+1}} |W_{ij}| \geqq \eta 2^{lr}\Big)\\
&\leqq 4 \sum_{l=k}^{\infty} 2^{2l} P(|W_{11}| \geqq \eta 2^{lr})\\
&= 4 \sum_{l=k}^{\infty} 2^{2l} \sum_{m=l}^{\infty} P(\eta 2^{mr} \leqq |W_{11}| < \eta 2^{(m+1)r})\\
&\leqq 8 \sum_{m=k}^{\infty} 2^{2m} P(\eta 2^{mr} \leqq |W_{11}| < \eta 2^{(m+1)r})\\
&\leqq 8 \eta^{-2/r} E|W_{11}|^{2/r} I[|W_{11}| \geqq \eta 2^{kr}] \to 0,
\end{aligned}$$

hence there exists a sequence of positive constant $\delta_n$, $\delta_n \downarrow 0$ such that

$$P\left(\bigcup_{n=2^k}^{\infty}\left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{n}|W_{ij}|\geqq \delta_n n^r\right)\right) \to 0, \qquad \text{as } k \to \infty.$$

In order to prove Theorem 2.1, we need some combinatorics. Let $i_1, i_2, \ldots, i_{2km}$ be a sequence, we define a multigraph $\Gamma(k, m; i_1, \ldots, i_{2km})$ as follows:

1. The vertices of this graph are $i_1, i_2, \ldots, i_{2km}$. Some of them may be equal.

2. There are $2km$ edges $e_1, e_2, \ldots, e_{2km}$. The ends of $e_a$ are $i_a$ and $i_{a+1}$ ($i_{2km+1} = i_1$). Any two of these edges are different even when they have the same end sets. Sometimes we write $i_a i_{a+1}$ instead of $e_a$.

3. To each edge $e_a$ there corresponds a number $\operatorname{dir}(e_a)$, called the *direction* of $e_a$, such that

$$\operatorname{dir}(e_a) = \begin{cases} +1, & \text{if } [(a-1)/k] \text{ is even} \\ -1, & \text{if } [(a-1)/k] \text{ is odd.} \end{cases}$$

Two different edges $e_a = i_a i_{a+1}$, $e_b = i_b i_{b+1}$ are said to be coincident, if either $i_a = i_b$, $i_{a+1} = i_{b+1}$ and $\operatorname{dir}(e_a) = \operatorname{dir}(e_b)$, or $i_a = i_{b+1}$, $i_{a+1} = i_b$ and $\operatorname{dir}(e_a) = -\operatorname{dir}(e_b)$.

A *chain* is a subgraph with vertex set $\{i_a, i_{a+1}, \ldots, i_b\}$ $(1 \leqq a < b \leqq 2mk+1)$ and edge set $\{e_a, e_{a+1}, \ldots, e_{b-1}\}$. We will denote such a chain by $i_a i_{a+1} \ldots i_b$.

In the graph $\Gamma(k, m; i_1, i_2, \ldots, i_{2km})$, we classify the edges as follows.

1. An edge $i_{a-1} i_a$ is called an innovation if $i_a$ is new, i.e. $i_a \neq i_1, \ldots, i_a \neq i_{a-1}$. The set of all innovations will be denoted by $I$.

2. Let $S$ be the set of all edges $i_{a-1}i_a$ which coincides with an innovation, and for any $b<a$, $i_{b-1}i_b$ does not coincide with that innovation.

3. All other edges consist a set called $T$.

If $i_a i_{a+1}$, $i_b i_{b+1}$ are two edges satisfying the following properties:

(1) $b<a$;

(2) $i_b i_{b+1}$ is single up to $i_a$, i.e. it does not coincide with any edge of the chain $i_1 i_2 \ldots i_a$.

(3) Either $i_b=i_a$ and $\mathrm{dir}(i_b i_{b+1})=\mathrm{dir}(i_a i_{a+1})$, or $i_{b+1}=i_a$ and $\mathrm{dir}(i_b i_{b+1})=-\mathrm{dir}(i_a, i_{a+1})$, then we say that $i_a i_{a+1}$ is coincidable with $i_b i_{b+1}$.

An edge of $S$ is called singular if it is coincidable with just one innovation.

An edge of $S$ is called regular if it is not singular, i.e. it is coincidable with more than one edge.

The proofs of Lemma 3.1, 3.2, 3.3 below are similar to the proofs of Lemma 3.1, 3.2, 3.3 in Yin et al. (1984).

**Lemma 3.1.** *If in the chain $i_a i_{a+1} \ldots i_b$, $i_a i_{a+1}$ is single up to $i_b$ and $i_b$ has been visited by $i_1 i_2 \ldots i_a$ then $i_a i_{a+1} \ldots i_b$ contains an edge of $T$.*

**Lemma 3.2.** *Let $t$ be the number of equivalence classes of $T$ under the equivalence relation "coincidence". Then if $i_a i_{a+1}$ is a regular edge of $S$, the number of edges with which $i_a i_{a+1}$ is coincidable is not greater than $t+1$.*

**Lemma 3.3.** *The number of regular edges of $S$ is not greater than twice the number of edges in $T$.*

*The chain*

$$L_1 = i_1 i_2 \ldots i_k i_{k+1},$$
$$L_2 = i_{k+1} i_{k+2} \ldots i_{2k+1},$$
$$\ldots \ldots,$$
$$L_{2m} = i_{(2m-1)k+1} i_{(2m-1)k+2} \ldots i_{2mk} i_1$$

*are called segments.*

**Lemma 3.4.** *Let $l$ be the number of innovations. Then the number of different ways to appoint the $2km$ edges to be of $I$, or $S$, or $T$, does not exceed $\binom{2km}{2l}(k+1)^{2km-2l+2m}$.*

*Proof.* Since the number of innovations are $l$, the numbers of $S$ and $T$ must be $l$ and $2km-2l$, respectively. So there are $\binom{2km}{2l}$ different ways to select $2km-2l$ edges from the $2km$ edges which are appointed to be of $T$, and the others to be of $I$ or of $S$.

Now consider a segment $L_c$. Note that every edge in the same segment has the same direction. Suppose that $L_c$ contains $\mu_c$ edges of $T$. Then $L_c$ is split by these $\mu_c$ $T$-edges into at most $\mu_c+1$ subchains consisting of consecutive edges of $I \cup S$. Let the lengths of these subchains be $v_1, v_2, \ldots, v_{\mu_c+1}$, respectively (if there are less than $\mu_c+1$ such chains, then some $v_i$ at the rear part of this list are zero). Consider the $i$th subchain with $v_i$ edges. It is evident that if some edge in this chain is of $I$, then the next one (if any) must be of $I$ because of the same direction of them. So there are only $v_1+1$ possible appointments for this chain, namely, $III \ldots I$, $SII \ldots I$, $SSI \ldots I$, $SSS \ldots SI$, $SSS \ldots S$. So for the whole segment $L_c$, there are at most $\prod_{i=1}^{\mu_c+1}(v_i+1)\leqq(k+1)^{\mu_c+1}$ ways to appoint the $k-\mu_c$ non-$T$ edges to be of $I$ or of $S$. Thus, for the whole graph, there are at most $\prod_{c=1}^{2m}(k+1)^{\mu_c+1}=(k+1)^{\sum_{c=1}^{2m}\mu_c+2m}=(k+1)^{2km-2l+2m}$ ways to appoint the $2l$ non-$T$ edges to be of $I$ or of $S$.

## 4. Proof of Theorem 2.1

Now we apply the truncation lemma for $r=\frac{1}{2}$ and $p(n)=n$. We need only to prove

$$\limsup_{n\to\infty}\left\|\left(\frac{\hat{W}_n}{\sqrt{n}}\right)^k\right\|\leqq(k+1)\,\sigma^k \quad \text{a.s.} \tag{4.1}$$

Define $\tilde{w}_{ijn}=w_{ijn}-E\,w_{ijn}$ and define $\tilde{W}_n=(\tilde{w}_{ijn})$, $i,j=1,2,\ldots,n$. We shall prove that for any $k\geqq 1$

$$\limsup_{n\to\infty}\left\|\left(\frac{\tilde{W}_n}{\sqrt{n}}\right)^k\right\|\leqq(k+1)\,\sigma^k \quad \text{a.s.} \tag{4.2}$$

If (4.2) holds for any $k\geqq 1$, since

$$\begin{aligned}\left|\left\|\left(\frac{\hat{W}_n}{\sqrt{n}}\right)^k\right\|-\left\|\left(\frac{\tilde{W}_n}{\sqrt{n}}\right)^k\right\|\right| &\leqq\left\|\left(\frac{\hat{W}_n}{\sqrt{n}}\right)^k-\left(\frac{\tilde{W}_n}{\sqrt{n}}\right)^k\right\|\\ &\leqq\sum_{l=0}^{k-1}\left\|\left(\frac{\hat{W}_n}{\sqrt{n}}\right)^l\right\|\left\|\frac{\hat{W}_n}{\sqrt{n}}-\frac{\tilde{W}_n}{\sqrt{n}}\right\|\left\|\left(\frac{\tilde{W}_n}{\sqrt{n}}\right)^{k-l-1}\right\|\end{aligned}$$

and

$$\left\|\frac{\hat{W}_n}{\sqrt{n}}-\frac{\tilde{W}_n}{\sqrt{n}}\right\|=\frac{|E\,w_{11n}|}{\sqrt{n}}\left\|\underbrace{\begin{pmatrix}1,1,\ldots,1\\1,1,\ldots,1\\ \\1,1,\ldots,1\end{pmatrix}}_{n}\right\|=\sqrt{n}\,|E\,w_{11n}|\to 0,$$

by (4.2) we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\left|\left\|\left(\frac{\hat{W}_n}{\sqrt{n}}\right)^k\right\|-\left\|\left(\frac{\tilde{W}_n}{\sqrt{n}}\right)^k\right\|\right|\\ &\leqq\limsup_{n\to\infty}\sum_{l=0}^{k-1}\left\|\left(\frac{\hat{W}_n}{\sqrt{n}}\right)^l\right\|\sqrt{n}|Ew_{11n}|(k-l)\sigma^{k-l-1}\end{aligned} \tag{4.3}$$

from which and by induction we can deduce (4.1). Hence to prove Theorem 2.1, we need only to prove (4.2).

For saving notations, we can assume that $W_n$ is an $n\times n$ matrix with iid random entries $w_{ij}$, such that

$$Ew_{11}=0,\quad |w_{11}|\leqq\delta\sqrt{n},\quad Ew_{11}^2\leqq 1 \text{ and } Ew_{11}^4\leqq d. \tag{4.4}$$

Here, without any loss, we suppose $\sigma=1$, and instead of $2\delta$ we write $\delta$.

Under the condition (4.4), it is easy to see that

$$E|w_{11}^l|\leqq\begin{cases}(\delta\sqrt{n})^{l-2}, & \text{for } l\geqq 2,\\ d(\delta\sqrt{n})^{l-3}, & \text{for } l\geqq 3.\end{cases} \tag{4.5}$$

It is enough to show that for any number $z>(1+k)$

$$\sum_{n=1}^{\infty} P\left(\left\|\left(\frac{W_n}{\sqrt{n}}\right)^k\right\|\geqq z\right)<\infty. \tag{4.6}$$

But since

$$\begin{aligned}\left\|\left(\frac{W_n}{\sqrt{n}}\right)^k\right\|^{2m} &\leqq \left(\lambda_{\max}\left(\left[\left(\frac{W_n}{\sqrt{n}}\right)^k\right]^T\left(\frac{W_n}{\sqrt{n}}\right)^k\right)\right)^m\\ &\leqq \operatorname{tr}\left\{\left(\frac{W_n}{\sqrt{n}}\right)^k\left[\left(\frac{W_n}{\sqrt{n}}\right)^k\right]^T\right\}^m.\end{aligned}$$

For any sequence $m=m(n)$ of positive integers,

$$\begin{aligned}\sum_{n=1}^{\infty} P(\|(W_n/\sqrt{n})^k\|\geqq z) &\leqq \sum_{n=1}^{\infty} P(\operatorname{tr}(W_n^k(W_n^k)^T)^m\geqq z^{2m}n^{mk})\\ &\leqq \sum_{n=1}^{\infty} z^{-2m}n^{-mk}E\operatorname{tr}(W_n^k(W_n^k)^T)^m.\end{aligned}$$

And we need only to show that for some positive integers $m=m(n)$,

$$\sum_{n=1}^{\infty} z^{-2m}n^{-mk}E\operatorname{tr}(W_n^k(W_n^k)^T)^m<\infty. \tag{4.7}$$

We have

$$\begin{aligned}E_n=E\operatorname{tr}(W_n^k(W_n^k)^T)^m=\sum E(&w_{i_1i_2}w_{i_2i_3}\ldots w_{i_ki_{k+1}})\\ &\cdot(w_{i_{k+2}i_{k+1}}w_{i_{k+3}i_{k+2}}\ldots w_{i_{2k+1}i_{2k}})\ldots\\ &\cdot(w_{i_{(2m-1)k+2}i_{(2m-1)k+1}}\ldots w_{i_{2mk+1}i_{2mk}}).\end{aligned}$$

Here, $i_1, i_2, \ldots, i_{2mk}$ run over $\{1, 2, \ldots, n\}$ and $i_{2mk+1}=i_1$. For each $i_1, i_2, \ldots, i_{2mk}$ we can define a graph $\Gamma(k, m)$ as in Sect. 3.

By Lemma 3.4, there are at most $\binom{2km}{2l}(k+1)^{2km-2l+2m}$ different ways to appoint the $2km$ edges to be of $I$ or of $S$ or of $T$.

Let $t$ denote the number of noncoincident $T$-edges. Because our graphs do not have single throughout edges, we have $l\leqq mk$ and $1\leqq t\leqq 2km-2l$ if $l\leqq mk-1$

Next we bound the number of different ways to appoint each edge in a canonical graph with given positions of the $l$ innovations, $l$ $S$-edges and $2km-2l$ $T$-edges and with $t$ different $T$-edges. Since each edge is an element of the left-upper $2km \times 2km$ submatrix of $W_n$ so there are at most $\binom{(2km)^2}{t} t^{2km-2l}$ different ways to appoint the $t$ different $T$-edges into their $2km-2l$ different positions.

Each innovation in a canonical graph is uniquely determined by the edges before it, and so is each singular $S$ edge. By Lemma 3.2 and 3.3, there are at most $(t+1)^{4km-4l}$ different ways to appoint the regular edges of $S$ to their positions. Here we should note that whether an $S$-edge is singular or regular is determined by all the edges before it.

From the above arguments and (4.8), we get

$$\begin{aligned}|E_n| &\leqq \sum_{l=1}^{mk} \binom{2km}{2l} (k+1)^{2km-2l+2m} n^{l+1} \sum_{t=1}^{2km-2l} \binom{(2km)^2}{t} t^{2km-2l} \\ &\quad \times (t+1)^{4km-4l} m^t (\delta \sqrt{n})^{2km-2l-t} \\ &\leqq n^{km+1} \sum_{l=1}^{mk} \binom{2km}{2l} (k+1)^{2km-2l+2m} \sum_{t=1}^{2km-2l} (2km)^{3t} \\ &\quad \cdot (t+1)^{6km-6l} \delta^{2km-2l} (\delta \sqrt{n})^{-t}.\end{aligned}$$

Here $\sum_{t=1}^{0} A_t = 1$, convenienced for saving notations.

By the elementary inequality

$$a^t (t+1)^b \leqq a^{-1} \left( -\frac{b}{\log a} \right)^b \quad \text{for } (0<a<1, b>0)$$

we get

$$|E_n| \leqq n^{km+\frac{3}{2}} \sum_{l=1}^{mk} \binom{2km}{2l} (k+1)^{2km-2l+2m} (2km) \left( \frac{6km\,\delta^{1/6}}{\log \frac{\delta \sqrt{n}}{(2km)^3}} \right)^{6km-6l} \delta^{km-l}.$$

If we select $m = m(n) = A(n) \log n$ such that

1. $A(n) \to \infty$.
2. $A(n)\, \delta^{1/6} \to 0$ then

$$\frac{6km\,\delta^{1/6}}{\log \frac{\delta \sqrt{n}}{(2km)^3}} \to 0, \quad (n \to \infty).$$

Thus we obtain for large $n$

$$\begin{aligned}|E_n| &\leqq n^{km+2} \sum_{l=1}^{mk} \binom{2km}{2l} ((k+1)^2 \delta)^{km-l} (k+1)^{2m} \\ &\leqq n^{km+2} (1 + (k+1)\, \delta^{1/2})^{2km} (k+1)^{2m}.\end{aligned}$$

Since $z > (1+k)$ and $\delta \to 0$, we have

$$\sum_{n=1}^{\infty} z^{-2m} n^{-km} |E_n| \leqq C \sum_{n=1}^{\infty} (n^{2/m}(1+(k+1)\delta^{1/2})^{2k}(k+1)/z)^m$$
$$\leqq C \sum_{n=1}^{\infty} \eta^m < \infty$$

where $0<\eta<1$ is a constant. Here the last series converges because $m/\log n \to \infty$. The proof is finished.

*Remark.* In the proof of Geman (1984), he used the fact that the spectral radius of a matrix does not exceed its Euclidean norm. The crutial step in his proof, equivalent to the inequality below (4.6), is to estimate

$$\left\| \left( \frac{W_n}{\sqrt{n}} \right) \right\|_E^{2m} = \left( \operatorname{tr} \frac{1}{n} W_n W_n' \right)^m.$$

In the computation, there is a little difference between the method given by Geman and that in this paper.

## 5. Two Problems of Geman-Hwang

In Geman-Hwang (1982), they suggested the following system of linear equations with unknown $n \times 1$ vector $X_n$

$$X_n = 1_n + \frac{1}{\sqrt{n}} W_n X_n \tag{5.1}$$

where $W_n$ is an $n \times n$ matrix whose $(i, j)$-entry is $w_{ij}$ and $W = \{w_{ij}\colon i, j = 1, 2, \ldots\}$ is an infinite matrix of iid random variables, and $1_n$ is the $n \times 1$ vector $(1, 1, \ldots, 1)^T$.

If $X_n = (X_{n1}, \ldots, X_{nn})^T$, then for any integer $m \geqq 1$, Geman and Hwang proved that as $n \to \infty$,

$$(X_{n1}, X_{n2}, \ldots, X_{nm})^T \to N\left(1_m, \frac{\sigma^2}{1-\sigma^2} I_m\right) \quad \text{weakly,} \tag{5.2}$$

under the conditions

1. $E w_{11} = 0$, $0 < E w_{11}^2 = \sigma^2 < \frac{1}{4}$;
2. $E|W_{11}^n| \leqq n^{\alpha n}$ for any integer $n \geqq 1$; $\alpha$ is a positive constant.

Geman and Hwang pointed out that the computer simulations support (5.2) even in the case of uniform distribution on $[-1, 1]$, where $\sigma^2 = \frac{1}{3}$.

We will prove that (5.2) is true even when $\sigma^2 < 1$ and $E|w_{11}^4| < \infty$.

**Theorem 5.1.** *Let $X_n$ be the solution of* (5.1) *whenever $\left(I - \frac{1}{\sqrt{n}} W_n\right)$ is non-singular, otherwise define $X_n = 0$. Then* (5.2) *holds when $E w_{11} = 0$, $E w_{11}^2 = \sigma^2 < 1$ and $E|w_{11}^4| < \infty$.*

Geman and Hwang (1982) suggested a system of differential equations

$$\dot{X}_n(t)=\alpha X_n(t)+\frac{1}{\sqrt{n}} W_n X_n(t), \qquad X_n(0)=1_n. \tag{5.3}$$

They proved that for any integer $m\geqq 1$, real $T>0$, $X_{n1}(\cdot), \ldots, X_{nm}(\cdot)$ (the first $m$ components of the vector $X_n(\cdot)$, the solution of (5.3)) tend to $m$ iid Gaussian processes weakly, as $n\to\infty$, on $[0, T]$. Each of these $m$ processes has mean $\mu(t) = e^{\alpha t}$ and covariance function

$$C(t, s)=e^{\alpha(t+s)}\sum_{k=1}^{\infty}\frac{(ts)^k}{(k!)^2}.$$

They supposed among others the following moment requirement

$$E|w_{11}|^n\leqq n^{\beta n} \qquad \text{for all } n\geqq 2, \text{ and some } \beta>0.$$

In the same paper, they conjectured that the analogous theorem should hold for the equation

$$\dot{X}_n(t)=\alpha X_n(t)+\frac{W_n}{\sqrt{n}} X_n(t)+1_n, \qquad X_n(0)=1_n. \tag{5.4}$$

We will prove

**Theorem 5.2.** *Suppose $Ew_{11}=0$, $Ew_{11}^2=1$, and $Ew_{11}^4<\infty$. Let $X_n(t)$ be the solution of*

$$\dot{X}_n(t)=\alpha X_n(t)+\frac{1}{\sqrt{n}} W_n X_n(t)+\beta 1_n, \qquad X_n(0)=1_n. \tag{5.5}$$

*Then for any integer $m\geqq 1$, real $T>0$, $X_{n1}(t), \ldots, X_{nm}(t)$ tend to $m$ iid Gaussian processes weakly on $[0, T]$ as $n\to\infty$. The mean of these processes is*

$$\mu(t)=e^{\alpha t}+\beta\int_0^t e^{\alpha s}\,ds=e^{\alpha t}+\frac{\beta}{\alpha}(e^{\alpha t}-1), \tag{5.6}$$

*the covariance function is*

$$C(t, s)=\sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(t^k e^{\alpha t}+\beta\int_0^t u^k e^{\alpha u}\,du\right)\left(s^k e^{\alpha s}+\beta\int_0^s u^k e^{\alpha u}\,du\right). \tag{5.7}$$

*Remark.* When $\beta=0$, Theorem 5.2 reduces to an extension of Geman-Hwang theorem. When $\beta=1$, Theorem 5.2 includes a proof of Geman-Hwang's conjecture.

## 6. Proof of Theorem 5.1

By the Truncation lemma; we can assume that the entries of $W_n$ are bounded by $\sqrt{n\delta}$, here $\delta=\delta_n\to 0$ arbitrarily slow. We suppose $\delta$ is defined as in the proof of Theorem 2.1.

Write $Y=X_n-1_n$, $A=W_n/\sqrt{n}$. (5.1) is equivalent to

$$(I_n-A)\,Y=A\,1_n.$$

Multiply both sides by $\sum_{i=0}^{k-1} A^i$, we get

$$Z_n \stackrel{\text{def}}{=} (I_n-A^k)\,Y=\sum_{i=1}^{k} A^i\,1_n. \tag{6.1}$$

We need the following lemma.

**Lemma 6.1.** *Suppose*

1. $\{w_{ij};\, i,j=1,2,\ldots\}$ *are iid random variables; and* $W_n$ *is the matrix* $(w_{ij};\, 1\leqq i,j\leqq n)$;

2. $Ew_{11}=0$, $Ew_{11}^2=\sigma^2$, $Ew_{11}^4<\infty$. *Then if* $\alpha(i,k,n)$ *denotes the i-th component of the vector* $\left(\frac{W_n}{\sqrt{n}}\right)^k 1_n$, *for any distinct ordered pairs* $(i_1,k_1),\ldots,(i_m,k_m)$, *as* $n\to\infty$,

$$(\alpha(i_1,k_1,n),\ldots,\alpha(i_m,k_m,n))^T \xrightarrow{w} N_m(0,\Lambda_m),$$

*where* $\Lambda_m=\operatorname{diag}(\sigma^{2k_1},\ldots,\sigma^{2k_m})$.

The proof of Lemma 6.1 is almost the same as the proof in the Appendix of Geman-Hwang (1982). In fact, if we truncate all the entries of $W_n$ according to the truncation lemma and then centralize them, without loss of generality we can assume that

$$Ew_{11}=0, \quad Ew_{11}^2 \xrightarrow{\leqq} \sigma^2, \quad |w_{11}|<\delta\sqrt{n} \quad \text{and} \quad Ew_{11}^4\leqq d<\infty.$$

Checking the proof of the Appendix, we find that in the expansion of $E\prod_{j=1}^{m}\alpha^{s_j}(i_j,k_j,n)$ the main terms remain the same except the factor $\sigma^{s_1k_1+\ldots+s_mk_m}$ is exchanged by $(Ew_{11}^2)^{(s_1k_1+\ldots+s_mk_m)/2}$ which tends to $\sigma^{s_1k_1+\ldots+s_mk_m}$. On the other hand, if $v_1,\ldots,v_t$ are given integers satisfying $v_1+\ldots+v_t=s_1k_1+\ldots+s_mk_m$, $v_1\geqq 2,\ldots,v_t\geqq 2$ and at least one of them is strict, then

$$\left|\prod_{j=1}^{t} E\left(\frac{w_{11}}{\sqrt{n}}\right)^{v_j}\right|\leqq \delta\sigma^{2t}n^{-t}$$

and the total number of those terms with the factor $\prod_{j=1}^{t}E\left(\frac{w_{11}}{\sqrt{n}}\right)^{v_i}$ is $O(n^t)$. Hence the sum of all those terms tends to zero by the fact that $\delta\to 0$. Therefore, we get the same limits of $E\prod_{j=1}^{m}\alpha^{s_j}(i_j,k_m,n)$ as that gotten in the Appendix of Geman and Hwang. This implies Lemma 6.1.

By truncation lemma and Lemma 6.1, it is not difficult to see that

$$(I_m 0)\,Z_n \xrightarrow{w} N_m\left(0, \sum_{i=1}^{k}\sigma^{2i} I_m\right), \quad \text{as} \quad n\to\infty. \tag{6.2}$$

Here $I_m$ is the $m\times m$ identify matrix and $(I_m 0)$ is of order $m\times n$. Also, if $(Z_n)_i$ is the $i$th component of $Z_n$, $E(Z_n)_1^2 \to \sum_{i=1}^{k}\sigma^{2i}$ as $n\to\infty$. Here the reader has to note that we have truncated the entries of $W_n$ at $\sqrt{n}\,\delta$.

In order to prove Theorem 5.1, we notice that

$$X_n = 1_n + Y = 1_n + Z_n + A^k Y.$$

Then, if $t=(t_1, \ldots, t_m)^T$, $i=\sqrt{-1}$,

$$\begin{aligned}\left|E e^{it'(I_m 0)(X_n-1_n)} - \exp\left\{-\frac{1}{2}t't\sum_{j=1}^{\infty}\sigma^{2j}\right\}\right| &\leqq |E e^{it'(I_m 0)(X_n-1_n)} - E e^{it'(I_m 0)Z_n}| \\ &+ \left|E e^{it'(I_m 0)Z_n} - \exp\left\{-\tfrac{1}{2}t't\sum_{j=1}^{k}\sigma^{2j}\right\}\right| \\ &+ \left|\exp\left\{-\tfrac{1}{2}t't\sum_{j=1}^{k}\sigma^{2j}\right\} - \exp\left\{-\tfrac{1}{2}t't\sum_{j=1}^{\infty}\sigma^{2j}\right\}\right| \\ &= a_1 + a_2 + a_3.\end{aligned}$$

As $n\to\infty$, $a_2\to 0$, by (6.2).

Now we estimate $a_1$. We have for any $\varepsilon>0$

$$a_1 \leqq E|e^{it'(I_m 0)A^k Y} - 1| \leqq 2P(\|(I_m 0)A^k Y\| \geqq \varepsilon) + \phi(\varepsilon).$$

Here $\phi(\varepsilon) = \sup_{\|x\|\leqq\varepsilon} |e^{t'x}-1| \to 0$ as $\varepsilon\to 0$.

We consider only those $k$, for which $(1+k)^{1/k}\sigma<1$.

Let $\Delta=\Delta_{n,k}=\{\omega\in\Omega\colon \|A^k\|<\eta^k\}$, where $(1+k)^{1/k}\sigma<\eta<1$, $\eta$ is fixed. Evidently $P(\Delta)\to 1$ as $n\to\infty$ by Theorem 2.1. Thus

$$\begin{aligned}P(\|(I_m 0)A^k Y\| \geqq \varepsilon) &\leqq P(\|(I_m 0)A^k Y\| \geqq \varepsilon,\ \|A^k\|<\eta^k) + P(\|A^k\|\geqq\eta^k) \\ &\leqq \frac{1}{\varepsilon^2} E\|(I_m 0)A^k Y\|^2 1_\Delta + P(\|A^k\|\geqq\eta^k) \\ &\leqq \frac{m}{\varepsilon^2 n} E\|A^k Y\|^2 1_\Delta + 1 - P(\Delta),\end{aligned} \tag{6.3}$$

since the components of $A^k Y 1_\Delta$ have the same distribution.

We have

$$A^k Y = A^k(I-A^k)Y + A^k A^k Y = A^k Z_n + A^k(A^k Y),$$

so

$$\|A^k Y\| \leqq \|A^k\|\,\|Z_n\| + \|A^k\|\,\|A^k Y\|,$$

and

$$\|A^k Y\|\,1_\Delta \leqq \frac{\|A^k\|}{1-\|A^k\|}\,\|Z_n\|\,1_\Delta \leqq \frac{\eta^k}{1-\eta^k}\,\|Z_n\|\,1_\Delta. \tag{6.4}$$

By (6.3) and (6.4),

$$P(\|(I_m 0)A^k Y\| \geqq \varepsilon) \leqq \frac{m}{\varepsilon^2 n}\left(\frac{\eta^k}{1-\eta^k}\right)^2 E\|Z_n\|^2 + 1 - P(\Delta).$$

Let $n\to\infty$, we get

$$\overline{\lim_{n\to\infty}}\, P(\|(I_m 0)\, A^k Y\| \geqq \varepsilon) \leqq \frac{m}{\varepsilon^2}\left(\frac{\eta^k}{1-\eta^k}\right)^2 \sum_{j=1}^{k} \sigma^{2j}.$$

So

$$\begin{aligned}\overline{\lim_{n\to\infty}} & \left| E e^{it'(I_m 0)(X_n - 1_n)} - \exp\left\{-\tfrac{1}{2} t' t \sum_{j=1}^{\infty} \sigma^{2j}\right\}\right| \\ & \leqq \overline{\lim_{n\to\infty}}\, a_1 + a_3 \leqq \frac{m}{\varepsilon^2}\left(\frac{\eta^k}{1-\eta^k}\right)^2 \sum_{j=1}^{k} \sigma^{2j} + \phi(\varepsilon) \\ & \quad + \left|\exp\left\{-\tfrac{1}{2} t' t \sum_{j=1}^{k} \sigma^{2j}\right\} - \exp\left\{-\tfrac{1}{2} t' t \sum_{j=1}^{\infty} \sigma^{2j}\right\}\right|.\end{aligned}$$

Letting $k\to\infty$, and then $\varepsilon\to 0$, we see that the left hand side tends to zero.

## 7. Proof of Theorem 5.2

It is easy to verify that

$$X_n(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{W_n}{\sqrt{n}}\right)^k 1_n \left(t^k e^{\alpha t} + \beta \int_0^t s^k e^{\alpha s}\, ds\right) \tag{7.1}$$

is the solution to (5.5).

Theorem 5.2 is a consequence of the following lemma.

**Lemma 7.1.** *Let* $\{w_{ij}: i, j = 1, 2, \ldots\}$ *be a family of iid random variables with* $E w_{11} = 0$, $E w_{11}^2 = 1$ *and* $E w_{11}^4 < \infty$, *and* $W_n = (w_{ij},\ 1 \leqq i \leqq n,\ 1 \leqq i \leqq n)$.

*Let* $\{g_k(\cdot),\ k = 0, 1, \ldots\}$ *be a sequence of continuous functions satisfying*

$$\sum_{k=0}^{\infty} \frac{r^k}{k!} \sup_{0 \leqq t \leqq T} |g_k(t)| < \infty, \tag{7.2}$$

*where* $r > 2$, $T > 0$ *are positive constants.*

*Then for any integer* $m \geqq 1$, *as* $n\to\infty$ *the stochastic process*

$$(I_m 0) \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{W_n}{\sqrt{n}}\right)^k 1_n g_k(t), \qquad t \in [0, T],$$

*tends to an m-dimensional Gaussian process with iid components, each with mean* $g_0(t)$ *and covariance function* $c(t, s) = \sum_{k=1}^{\infty} \left(\frac{1}{k!}\right)^2 g_k(t)\, g_k(s)$.

*Proof.* Let

$$Z_n(t) = (Z_{n1}(t), \ldots, Z_{nn}(t))^T = \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{W_n}{\sqrt{n}}\right)^k 1_n g_k(t).$$

We prove that the sequence $\{(Z_{n1}(\cdot), \ldots, Z_{nm}(\cdot)),\ n = 1, 2, \ldots\}$ of stochastic processes is tight in $C^m[0, T]$. It is easy to see that we need only to show that $\{Z_{ni}(\cdot),\ n = 1, 2, \ldots\}$ is tight in $C[0, T]$, $1 \leqq i \leqq m$.

Let $\Delta_n = \left\{ \omega \in \Omega : \frac{\|W_n\|}{\sqrt{n}}(\omega) \leqq r \right\}$. By Theorem 2.1, $P(\Delta_n) \to 1$. Let

$$\rho_k(\delta) = \sup_{\substack{|t-s|<\delta \\ t,\, s \in [0,\, T]}} |g_k(t) - g_k(s)|,$$

$$\alpha(i, k, n) = \left\{ \left( \frac{W_n}{\sqrt{n}} \right)^k 1_n \right\}_i = \text{the } i\text{th component of } \left( \frac{W_n}{\sqrt{n}} \right)^k 1_n.$$

We have

$$\sup_{\substack{|t-s|<\delta \\ t,\, s \in [0,\, T]}} |Z_{ni}(t) - Z_{ni}(s)| \leqq \sum_{k=1}^{\infty} |\alpha(i, k, n)| \frac{\rho_k(\delta)}{k!},$$

hence

$$\begin{aligned} &\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} P\Big( \sup_{\substack{|t-s|<\delta \\ t,\, s \in T}} |Z_{ni}(t) - Z_{ni}(s)| > \varepsilon \Big) \\ &\leqq \lim_{\delta \to 0} \overline{\lim_{n \to \infty}} P\left( \sum_{k=1}^{\infty} |\alpha(i, k, n)| \frac{\rho_k(\delta)}{k!} > \varepsilon \right) \\ &\leqq \lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \left[ \frac{1}{\varepsilon} E \sum_{k=1}^{\infty} 1_{\Delta_n} |\alpha(i, k, n)| \frac{\rho_k(\delta)}{k!} + (1 - P(\Delta_n)) \right] \\ &= \lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \frac{\rho_k(\delta)}{k!} E 1_{\Delta_n} |\alpha(i, k, n)|. \end{aligned}$$

It is easy to see that $\alpha(i, k, n) 1_{\Delta_n}, \ldots, \alpha(n, k, n) 1_{\Delta_n}$, $i = 1, 2, \ldots, n$, have an identical distribution. Therefore

$$\begin{aligned} E 1_{\Delta_n} |\alpha(i, k, n)| &\leqq E^{1/2} 1_{\Delta_n} |\alpha(i, k, n)|^2 \\ &\leqq \left[ \frac{1}{n} E 1_{\Delta_n} \left\| \left( \frac{W_n}{\sqrt{n}} \right)^k 1_n \right\|^2 \right]^{1/2} \\ &\leqq \left[ E 1_{\Delta_n} \left\| \frac{W_n}{\sqrt{n}} \right\|^{2k} \right]^{1/2} \leqq r^k. \end{aligned}$$

So,

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} P\Big( \sup_{\substack{|t-s|<\delta \\ t,\, s \in T}} |Z_{ni}(t) - Z_{ni}(s)| > \varepsilon \Big) \leqq \lim_{\delta \to 0} \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \frac{\rho_k(\delta)}{k!} r^k = 0.$$

Thus, the tightness of the family $\{Z_{ni}(\cdot);\, n = 1, 2, \ldots\}$ of stochastic processes is established.

Finally we show that for any positive integer $l$ and $t_1, \ldots, t_l \in [0, T]$, as $n \to \infty$

$$E \exp\left\{ i \sum_{v=1}^{m} \sum_{j=1}^{l} \lambda_{vj} Z_{nv}(t_j) \right\} \to \exp\left\{ -\frac{1}{2} \sum_{v=1}^{m} \sum_{j=1}^{l} \sum_{q=1}^{l} \lambda_{vj} \lambda_{vq} c(t_j, t_q) \right\}.$$

Here $i = \sqrt{-1}$ and $\{\lambda_{vj}\}$ are real numbers.

Let

$$e_{nv}^{p}(t)=\sum_{k=p+1}^{\infty}\left(\left(\frac{W_n}{\sqrt{n}}\right)^k 1_n\right)_v \frac{g_k(t)}{k!}$$
$$=\sum_{k=p+1}^{\infty}\alpha(v,k,n)\frac{g_k(t)}{k!},\qquad v=1,\dots,n.$$

Let $g_k=\sup_{t\in[0,T]} g_k(t)$. The for any $\varepsilon>0$,

$$\lim_{p\to\infty}\overline{\lim_{n\to\infty}} P(|e_{vn}^{p}(t_j)|\geqq\varepsilon)\leqq\overline{\lim_{p\to\infty}}\,\overline{\lim_{n\to\infty}}\frac{1}{\varepsilon}\sum_{k=p+1}^{\infty}\frac{g_k}{k!}E1_{\Delta_n}|\alpha(v,k,n)|$$
$$\leqq\frac{1}{\varepsilon}\overline{\lim_{p\to\infty}}\sum_{k=p+1}^{\infty}\frac{r^k}{k!}g_k=0. \tag{7.3}$$

On the other hand, by (7.2)

$$\overline{\lim_{p\to\infty}}\left|\sum_{k=p+1}^{\infty}\frac{1}{(k!)^2}g_k(t_j)g_k(t_q)\right|\leqq\overline{\lim_{p\to\infty}}\left(\sum_{k=p+1}^{\infty}\frac{g_k}{k!}\right)^2=0. \tag{7.4}$$

We have

$$\left|E\exp\left\{i\sum_{v=1}^{m}\sum_{j=1}^{l}\lambda_{vj}\sum_{k=1}^{\infty}\alpha(v,k,n)\frac{g_k(t_j)}{k!}\right\}-E\exp\left\{-\frac{1}{2}\sum_{v=1}^{m}\sum_{j=1}^{l}\sum_{q=1}^{l}\lambda_{vj}\lambda_{vq}c(t_j,t_q)\right\}\right|$$
$$\leqq\left|E\exp\left\{i\sum_{v=1}^{m}\sum_{j=1}^{l}\lambda_{vj}\sum_{k=1}^{\infty}\alpha(v,k,n)\frac{g_k(t_j)}{k!}\right\}-E\exp\left\{i\sum_{v=1}^{m}\sum_{j=1}^{l}\lambda_{vj}\sum_{k=1}^{p}\alpha(v,k,n)\frac{g_k(t_j)}{k!}\right\}\right|$$
$$+\left|E\exp\left\{i\sum_{v=1}^{m}\sum_{j=1}^{l}\lambda_{vj}\sum_{k=1}^{p}\alpha(v,k,n)\frac{g_k(t_j)}{k!}\right\}-\exp\left\{-\frac{1}{2}\sum_{v=1}^{m}\sum_{j=1}^{p}\left(\sum_{j=1}^{l}\lambda_{vj}\frac{g_k(t_j)}{k!}\right)^2\right\}\right|$$
$$+\left|\exp\left\{-\frac{1}{2}\sum_{v=1}^{m}\sum_{k=1}^{p}\left(\sum_{j=1}^{l}\lambda_{vj}\frac{g_k(t_j)}{k!}\right)^2\right\}-\exp\left\{-\frac{1}{2}\sum_{v=1}^{m}\sum_{j=1}^{l}\sum_{q=1}^{l}\lambda_{vj}\lambda_{vq}c(t_j,t_q)\right\}\right|$$
$$=a_1+a_2+a_3.$$

By (7.3) $\overline{\lim_{p\to\infty}}\,\overline{\lim_{n\to\infty}}\, a_1=0$. By Lemma 6.1, $\lim_{n\to\infty}a_2=0$. And

$$a_3=\left|\exp\left\{-\frac{1}{2}\sum_{v=1}^{m}\sum_{j=1}^{l}\sum_{q=1}^{l}\lambda_{vj}\lambda_{vq}\sum_{k=1}^{p}\left(\frac{1}{k!}\right)^2 g(t_j)g(t_q)\right\}\right.$$
$$\left.-\exp\left\{-\frac{1}{2}\sum_{v=1}^{m}\sum_{j=1}^{l}\sum_{q=1}^{l}\lambda_{vj}\lambda_{vq}\sum_{k=1}^{\infty}\left(\frac{1}{k!}\right)^2 g(t_j)g(t_q)\right\}\right|$$
$$\leqq\left|1-\exp\left\{-\frac{1}{2}\sum_{v=1}^{m}\sum_{j=1}^{l}\sum_{q=1}^{l}\lambda_{vj}\lambda_{vq}\sum_{k=p+1}^{\infty}\left(\frac{1}{k!}\right)^2 g(t_j)g(t_q)\right\}\right|$$
$$\times\left|\exp\left\{\frac{1}{2}\sum_{v=1}^{m}\sum_{j=1}^{l}\sum_{q=1}^{l}|\lambda_{vj}||\lambda_{vq}|\left(\sum_{k=1}^{\infty}\frac{g_k}{k!}\right)^2\right\}\right|\to 0,\quad\text{as}\quad p\to\infty,$$

by (7.4). We finish the proof.

*Remark.* Throughout this paper, we have assumed $W_n$ comes from a fixed infinite random matrix. If we give up this assumption, and keep the others, then conclusions in Theorem 2.1 and 2.2 are still true in the sense in probability, and those of other theorems remain the same. If we strengthen the condition as to $E|w_{11}|^6 < \infty$, then Theorem 2.1 and 2.2 are also true.

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