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Probability Theory and Related Fields

On Bilinear Forms in Gaussian Random Variables and Toeplitz Matrices*

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Summary. We improve a result of Szegő on the asymptotic behaviour of the trace of products of Toeplitz matrices.

As an application, we improve also his results on the limiting behaviour of the bilinear forms

$$B_n = \sum_{i,j=1}^{n} a_{i-j} X_i X_j,$$

where X_i is a stationary Gaussian sequence.

1. Statement of Results

A. We study below the asymptotic behaviour of bilinear forms

$$B_n = \sum_{i,j=1}^{n} a_{i-j} X_i X_j \tag{1.1}$$

where X_i is a zero mean stationary Gaussian sequence.

This problem was first studied in the book of Grenander and Szegő [6], as an application of their theory of the asymptotic behaviour of the trace of products of Toeplitz matrices.

Recently, there has been a renewed interest in this problem. See Fox and Taqqu [3, 4] and Taniguchi [11].

In Theorem 1 below we improve the results of Grenander and Szegő on the asymptotics of the trace of products of Toeplitz matrices. As explained in the appendix, this theorem can be viewed also as a generalization of Parseval's

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relation. As a corollary of Theorem 1, we get a result which improves Theorem 11.6 of Genander and Szegő on the bilinear forms B_n (see Theorem 2).

The proof of Theorem 1 is based on a norm inequality. Lemma 1, communicated to us by Larry Brown. The lemma is stated and proved in Sect. 2.

B. Let:

$$r_n = EX_0 X_n \tag{1.2}$$

denote the covariance of the sequence X_n. The key fact about the bilinear form B_n is that its cumulants are:

$$\operatorname{cum}_k(B_n) = 2^{k-1}(k-1)!\ \operatorname{Tr}(A_n R_n)^k. \tag{1.3}$$

where A_n, R_n are the $n \times n$ Toeplitz matrices:

$$A_n(i,j) = a_{i-j}, \qquad R_n(i,j) = r_{i-j}, \qquad \text{for } i,j = 1, \ldots, n$$

(Formula 1.3 is an easy application of the "diagram" formula; see Rosenblatt [9], Theorem 2.2).

The first step in studying B_n should be thus the investigation of the asymptotic behaviour of the trace of products of Toeplitz matrices.

C. Let $\hat{f}_k^{(v)}$, $v = 1, \ldots, s$ be the sequences of the Fourier coefficients of the s complex valued functions $f^{(v)}(x)$, $v = 1, \ldots, s$, i.e.:

$$\hat{f}_k^{(v)} = \int_0^1 e^{2\pi ikx} f^{(v)}(x)\, dx, \tag{1.4}$$

and let $T_n(f^{(v)})$, $v = 1, \ldots, s$ denote the corresponding $n \times n$ Toeplitz matrices, i.e.:

$$T_n(f^{(v)})(i,j) = \hat{f}_{i-j}^{(v)}, \qquad \text{for } i,j = 1, \ldots, n.$$

For convenience of notation, we denote by $\mathscr{L}_p$ the closure of the trigonometric polynomials in the space $L_p[0,1]$, for $1 \leqq p \leqq \infty$. Thus,

$$\mathscr{L}_p = \begin{cases} L_p & \text{if } 1 \leqq p < \infty \\ C & \text{if } p = \infty \end{cases}$$

(see for example Katznelson [79], Theorem 2.11).

Theorem 1. *Suppose that $f^{(v)}(x) \in \mathscr{L}_{p_v}$, with $1 \leqq p_v \leqq \infty$, for $v = 1, \ldots, s$,*

a) if $\sum_{v=1}^{s} (p_v)^{-1} \leqq 1$, then

$$\lim_{n \to \infty} \frac{1}{n} \operatorname{Tr}\left(\prod_{v=1}^{s} T_n(f^{(v)}\right) = \int_0^1 \prod_{v=1}^{s} (f^{(v)}(x))\, dx \tag{1.5}$$

b) if $\alpha > 1$, and $\alpha \geqq \sum_{v=1}^{s} (p_v)^{-1}$, then

$$\lim_{n \to \infty} \frac{1}{n^\alpha} \operatorname{Tr}\left(\prod_{v=1}^{s} T_n(f^{(v)})\right) = 0. \tag{1.6}$$

Note. Formula (1.5) was first obtained by Grenander and Szegő [6], 7.4, under the assumption that $f^{(v)}(x)$ are bounded.

D. **Definition.** The singular values of a matrix A are the eigenvalues of the matrix $(AA^*)^{1/2}$, where A^* denotes the adjoint of A.

As an application of Theorem 1a, consider the case when $f^{(2\nu-1)}=f$, and $f^{(2\nu)}=\bar{f}$ (the complex conjugate), for $\nu=1, \ldots, s$. We get then an improvement of Theorem II of Parter [8] on the distribution of the singular values of $T_n(f)$:

Corollary. *Let $f \in L_\infty(0,1)$, let $M=\sup_x |f(x)|$ and let F be a continuous function, $F\colon [0,M] \to R$. Then*

$$\lim_{n\to n} \frac{1}{n} \sum_{j=1}^{n} F(s_{j,n}) = \int_0^1 F(|f(x)|)\,dx,$$

where $s_{j,n}$ are the singular values of $T_n(f)$.

Proof. Since F can be uniformly approximated on $[0,M]$ by even polynomials, it is enough to establish the case $F(x)=x^{2s}$, for $s=1,2,\ldots$. But this case follows from Theorem 1a (by letting $f^{(2\nu-1)}=f$, $f^{(2\nu)}=\bar{f}$).

E. As another corollary of Theorem 1 we get:

Theorem 2. Let a_k and r_k in (1.1) and (1.2) be the Fourier coefficients of the real, even functions $a(x)$ and $r(x)$, and suppose $a(x) \in \mathscr{L}_{p_1}$, $r(x) \in \mathscr{L}_{p_2}$, $1 \leqq p_1$, $p_2 \leqq \infty$ and

$$(p_1)^{-1} + (p_2)^{-1} \leqq 2^{-1}. \tag{1.7}$$

Then,

$$\frac{B_n - E(B_n)}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2), \tag{1.8}$$

where

$$\sigma^2 = 2 \int_0^1 a^2(x)\, r^2(x)\, dx.$$

Proof. Use the method of cumulants:

$$\operatorname{cum}_k\left(\frac{B_n - EB_n}{\sqrt{n}}\right) = \begin{cases} 0 & \text{for } k=1 \\ 2\operatorname{Tr}(A_n R_n)^2/n & \text{for } k=2 \\ 2^{k-1}(k-1)!\,\dfrac{\operatorname{Tr}(A_n R_n)^k}{n^{k/2}} & \text{for } k \geqq 3 \end{cases}$$

$$\xrightarrow[n\to\infty]{} \begin{cases} 0 & \text{for } k=1 \\ 2\int_0^1 a^2(x)\, r^2(x)\, dx & \text{for } k=2, \text{ by Theorem 1a} \\ 0 & \text{for } k \geqq 3, \text{ by Theorem 1b.} \end{cases}$$

Notes. 1) (1.8) was first established by Grenader and Szego [6] Theorem 11.6, under the assumption that $a(x)$ and $r(x)$ are bounded.

2) Fox and Taqqu [4] extended the result of Grenander and Szegő under a set of assumption different from ours. They show that if $a(x)$ and $r(x)$ are continuous, except maybe at 0, and are regularly varying at 0, then $a(x)\,r(x) \in L_2$ (which is a weaker assumption than (1.7)) is sufficient for (1.8) to hold.

3) When $a(x)$, $r(x)$ are continuous, the Legendre transform of the large deviations rate of B_n/n can also be obtained explicitly. Indeed, by a standard large deviations result (see for example Lemma 1 of Cox and Griffeath [2], it equals the limit

$$\lim_{n\to\infty} \frac{\log E\, e^{sB}n}{n}. \tag{1.9}$$

But this cumulant generating function of B_n can be computed explicitly:

$$\log E\, e^{sB}n = -(1/2) \sum_{i=1}^{n} \log(1 - 2s\lambda_{i,n}),$$

where $\lambda_{i,n}$ are the eigenvalues of $A_n R_n$, and $s \leqq [\max 2\lambda_{i,n}]^{-1}$; then, the classical result of Szegő [6], 5.2 yields the limit in (1.9) explicitly.

2. Proof of Theorem 1

Definition. Let s_j denote the singular values of a matrix A. For $1 \leqq p \leqq \infty$, the p-Schatten norm of A is:

$$\|A\|_p = \begin{cases} [\sum_j (s_j)^p]^{1/p} & \text{for } 1 \leqq p < \infty \\ \max_j s_j & \text{for } p = \infty. \end{cases}$$

This definition yields indeed norms (see Simon [10], Theorem 2.7a). We collect now several properties of the Schatten norms needed below:

$$\|AB\|_1 \leqq A\|_p \|B\|_q, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1 \tag{2.1}$$

(see Simon [10], Theorem 2.8)

$$|\mathrm{Tr}\, A| \leqq \|A\|_1 \tag{2.2}$$

(see Simon [10], Theorem 3.1)

$$\|A\|_2^2 = [\mathrm{Tr}\,(AA^*)] = \sum_{i,j} |A_{i,j}|^2 \tag{2.3}$$

$$\|A\|_\infty = \max_{\|v\|_2=1} \|Av\|_2, \tag{2.4}$$

where $\|v\|_2$ is the l_2 norm of v (see Simon [10], Theorem 1.5). Thus, if we denote by $\|\,\|A\|\,\|$ the norm of A as an operator of l_2, we have $\|\,\|A\|\,\| = \|A\|_\infty$.

The proof of Theorem 1 is based on the following inequality for the Schatten norms of Toeplitz matrices:

Lemma 1. *For* $1 \leqq p \leqq \infty$.

$$\|T_n(f)\|_p \leqq n^{1/p} \|f\|_p. \tag{2.5}$$

Proof. One can easily check that the Riesz-Thorin interpolation (see Bergh [1], p. 3) usually used in L_p spaces works just as well over the spaces of matrices endowed with the $\|\cdot\|_p$ Schatten norms. Thus, it is enough to establish (2.5) for $p=1$ and $p=\infty$.

Let $T(f)$ denote the double infinite Toeplitz matrix with elements $T(f)(i,j) = \hat{f}_{i-j}$, and let P_n denote the operator of projection of doubly infinite sequences on the subspace of sequences which may have non zero elements only on the components $1, \ldots, n$. We will use the fact that $T_n(f)$ is the restriction of $P_n T(f) P_n$ to $\mathbb{C}^n$.

Case $p=\infty$. We note that:

$$\begin{aligned} \|T_n(f)\|_\infty &= \|\,\|P_n T(f) P_n\|\,\| \quad \text{(by 2.4))} \\ &\leqq \|\,\|T(f)\|\,\| = \|f\|_\infty. \end{aligned} \tag{2.6}$$

To see that the last equality holds, view $T(f)$ as an operator over $L_2[0,1]$, via the Fourier isometry, that is consider the operator $\bar{T}(f)$: $L_2 \to L_2$ defined by:

$$\bar{T}(f)(g) := [T(f)(\hat{g})]^v, \quad \text{where} \quad [c_k]^v := \sum_{r=\infty}^{\infty} c_r\, e^{-2\pi i k x}.$$

It is easy to check that $\bar{T}(f)$ is just multiplication by f, i.e.:

$$\bar{T}(f)(g) = fg, \quad \text{and thus} \quad \|\,\|T(f)\|\,\| = \|\,\|\bar{T}(f)\|\,\| = \|f\|_\infty,$$

yielding (2.6).

Case $p=1$. Here, we decompose $f\in L_1$ as $f=gh$, in such a way that $\|f\|_1=\|g\|_2\,\|h\|_2$. Note that $T(f)=T(g)\cdot T(h)$. We get thus

$$\begin{aligned}\|T_n(f)\|_1&=\|P_n T(f)\,P_n\|_1=\|P_n T(g)\,T(h)\,P_n\|_1\leqq\|P_n T(g)\|_2\,\|T(h)\,P_n\|_2 \quad \text{(by (2.1))}\\&=\left[\sum_{i=1}^{n}\sum_{j=-\infty}^{\infty}|\hat{g}_{i-j}|^2\right]^{1/2}\left[\sum_{i=-\infty}^{\infty}\sum_{j=1}^{n}|\hat{h}_{i-j}|^2\right]^{1/2} \quad \text{(by (2.3))}\\&=n^{1/2}\,\|g\|_2\cdot n^{1/2}\,\|h\|_2=n\,\|f\|_1.\quad\square\end{aligned}$$

Proof of Theorem 1. a) Let m be the number of $f^{(\nu)}$ in (1.5) which are non-polynomials (have infinitely many non zero Fourier coefficients). We will use induction on m. For $m=0$ (i.e. all $f^{(\nu)}(x)$ are polynomials), it is easy to check that (1.5) holds (by multilinearity, it is enough to check the case $f^{(\nu)}=e^{2\pi i k_\nu x}$). Suppose now (1.5) holds whenever we have at most m non-polynomials.

Consider then any set of $f^{(\nu)}(x)$ which has at most $m+1$ non-polynomials, and suppose w.l.o.g. that $f^{(1)}(x)$ is a non-polynomial. Let then $f_k^{(1)}(x)$ denote the k'th Fejer sum of $f^{(1)}(x)$ and $f^{(1),k}(x)=f^{(1)}(x)-f_k^{(1)}(x)$ be the k'th remainder. Then

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{Tr}\left[T_n(f_k^{(1)})\prod_{\nu=2}^{s}T_n(f^{(\nu)})\right]=\int_0^1 f_k^{(1)}(x)\prod_{\nu=2}^{s}f^{(\nu)}(x)\,dx \tag{2.7}$$

by the induction hypothesis, and the r.h.s. of (2.7) converges as $k\to\infty$ to $\int_{-\pi}^{\pi}\prod_{\nu=1}^{s}(f^{(\nu)}(x)\,dx$, since $f^{(1)}\in\mathscr{L}_{p_1}$ implies that $\|f_k^{(1)}-f^{(1)}\|_{p_1}\xrightarrow[k\to\infty]{}0$, for any $1\leqq p_1\leqq\infty$ (see Katzelnelson [7], Theorem 2.11) and $\prod_{\nu=2}^{s}f^{(\nu)}(x)\in L_{q_1}$, where $(p_1)^{-1}+(q_1)^{-1}\leqq 1$. To show then that (1.5) holds with up to $m+1$ nonpolynomials it remains only to note that:

$$\begin{aligned}&\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\frac{1}{n}\left|\operatorname{Tr}\left[T_n(f^{(1),k})\prod_{\nu=2}^{s}T_n(f^{(\nu)})\right]\right|\\&\leqq\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\frac{1}{n}\left\|T_n(f^{(1),k})\prod_{\nu=2}^{s}T_n(f^{(1)})\right\|_1 \quad \text{(by (2.2))}\\&\leqq\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\frac{1}{n}\|T_n(f^{(1),k})\|_{p_1}\prod_{\nu=2}^{s}\|T_n(f^{(\nu)})\|_{p_\nu} \quad \text{(by (2.1))}\\&\leqq\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\frac{n^{(\sum_\nu(p_\nu)^{-1})}}{n}\|f^{(1),k}\|_{p_1}\prod_{\nu=2}^{s}\|f^{(\nu)}\|_{p_\nu} \quad \text{(by Lemma 1)}\\&=0.\end{aligned}$$

b) Assume first w.l.o.g. $\sum_{\nu=1}^{s}(p_\nu)^{-1}>1$. (Otherwise the result follows from a).)

The proof is now similar with that of part a). If all $f^{(\nu)}(x)$ are polynomials, the limit is 0 since $\alpha > 1$. Otherwise an induction on the number of non polynomials works: assume w.l.o.g. that $f^{(1)}(x)$ is a nonpolynomial, replace $f^{(1)}$ by $f^{(1),k} + f_k^{(1)}$, and split (1.6) in two parts.

The double limit in k and n of the second part is 0 by the induction hypothesis; for the first part, let $\theta = \sum_{\nu=1}^{s} (p_\nu)^{-1}$, and note that

$$\begin{aligned}
\frac{1}{n^\alpha}\left|\operatorname{Tr}\left[T_n(f^{(1),k})\prod_{\nu=2}^{s} T_n(f^{(\nu)})\right]\right| &\leqq \frac{1}{n^\alpha}\|T_n(f^{(1),k})\|_{\theta p_1}\prod_{\nu=2}^{s}\|T_n(f^{(\nu)})\|_{\theta p_\nu} \quad \text{(by (2.1))}\\
&\leqq \frac{1}{n^\alpha}\|T_n(f^{(1),k})\|_{p_1}\prod_{\nu=2}^{s}\|T_n(f^{(\nu)})\|_{p_\nu} \quad (\text{since } \theta > 1)\\
&\leqq \frac{n^{(\sum_\nu (p_\nu)^{-1})}}{n^\alpha}\|f^{(1),k}\|_{p_1}\prod_{\nu=2}^{s}\|f^{(\nu)}\|_{p_\nu} \quad \text{(by Lemma 1)}\\
&\leqq \|f^{(1),k}\|_{p_1}\prod_{\nu=2}^{s}\|f^{(\nu)}\|_{p_\nu} \xrightarrow[k\to\infty]{} 0. \quad \square
\end{aligned}$$

Appendix

A. The relationship with Convolution Sums and Parseval's Relation

It is interesting to compare Theorem 1a with the

Convolution Theorem. *Suppose that $f^{(\nu)} \in L_{p_\nu}$, with $1 < p_\nu < \infty$, for $\nu = 1, \ldots, s$, and that $\sum_{\nu=1}^{s} (p_\nu)^{-1} \leqq 1$. Then*

$$\lim_{n\to\infty} \sum_{\substack{j\in\{-n,\ldots,n\}^s \\ j_1+\ldots+j_s=0}} \hat{f}_{j_1}^{(1)} \ldots \hat{f}_{j_s}^{(s)} = \int_0^1 \prod_{\nu=1}^{s} f^{(\nu)}(x)\,dx. \tag{A.1}$$

This theorem holds since if we let $S_n(f)$ denote the n'th Fourier sum of f, then the l.h.s. of (A.1) is precisely $\int_0^1 \prod_{\nu=1}^{s} S_n(f^{(\nu)})\,dx$. Since for $1 < p < \infty$, $\|S_n(f) - f\|_p \to 0$ (see Katznelson [7], II.1.5), and since the condition $\sum_{\nu=1}^{s} (p_\nu)^{-1} \leqq 1$ implies the continuity of the integral as a functional on $L_{p_1} x \ldots x L_{p_s}$, the result follows.

The sums appearing in the l.h.s. of (1.5) in our Theorem 1 are also related to some convolution sums. One can obtain, after some algebra, the following identity:

$$\frac{1}{n} T_r\left[\prod_{v=1}^{s} T_n(f^{(v)})\right] = \frac{A_0 + \dots + A_{n-1}}{n}, \tag{A.2}$$

where A_n are the "skew" convolution sums:

$$A_n = \sum_{\substack{j_1 + \dots + j_s = 0 \\ \max_{1 \leq v \leq s} \sum_{i=1}^{v} j_i - \min_{1 \leq v \leq s} \sum_{i=1}^{v} j_i \leq n}} \hat{f}_{j_1}^{(1)} \dots \hat{f}_{j_s}^{(s)}. \tag{A.3}$$

Thus, Theorem 1 is a result about the Cesaro convergence of "skew" convolution sums. We have been however unable to use in the proof of Theorem 1 its simpler counterpart, the Convolution theorem, except when $s=2$ or 3; in these cases, the sums in (A.1) and (A.3) coincide in fact, and Theorem 1 reduces to the classical Parseval relation (see Katznelson [7], p. 35).

B. *Proof of Formula (A.2)*

Let $E(j_1, \dots, j_s)$ denote the range of the sums $\sum_{i=1}^{v} j_i$, $v = 1, \dots, s$, i.e.

$$E(j_1, \dots, j_s) = \operatorname*{Max}_{1 \leq v \leq s} \sum_{i=1}^{v} j_i - \operatorname*{Min}_{1 \leq v \leq s} \sum_{i=1}^{v} j_i,$$

and let

$$D_n = \left\{(j_1, \dots, j_s) : \sum_{v=1}^{s} j_v = 0, E(j_1, \dots, j_s) \leq n\right\}.$$

Thus, the "skew" convolution sums A_n are given by

$$A_n = \sum_{\underline{j} \in D_n} \hat{f}_{j_1}^{(1)} \dots \hat{f}_{j_s}^{(s)}.$$

Then,

$$\frac{1}{n} \operatorname{Tr}\left(\prod_{v=1}^{s} T_n(f^{(v)})\right) = \frac{1}{n} \sum_{\underline{i} \in \{1, \dots, n\}^s} \hat{f}_{i_1 - i_2}^{(1)} \dots \hat{f}_{i_s - i_1}^{(s)} = \frac{1}{n} \sum_{\underline{j} \in D_{n-1}} \hat{f}_{j_1}^{(1)} \dots \hat{f}_{j_s}^{(s)} \sum_{\substack{i_1 - i_2 = j_1, \dots \\ i_s - i_1 = j_s}} 1$$

$$= \frac{1}{n} \sum_{\underline{j} \in D_{n-1}} \hat{f}_{j_1}^{(1)} \dots \hat{f}_{j_s}^{(s)} (n - E(j_1, \dots, j_s)). \tag{A.4}$$

The last equality holds since the set of all $\underline{i}$'s with given $\underline{j}$ differences can be obtained from any of its elements $\underline{i}^{(0)}$, by adding or subtracting $(1, \dots, 1)$

as long as all components are in the range $\{1, \ldots, n\}$; as such, it has $(n - \max_\nu i_\nu^{(0)}) + \min_\nu i_\nu^{(0)}$ elements. Furthermore,

$$\max_\nu i_\nu^{(0)} - \min_\nu i_\nu^{(0)} = \max_\nu(-i_\nu^{(0)}) - \min_\nu(-i_\nu^{(0)}) = \max_\nu(i_1^{(0)} - i_\nu^{(0)}) - \min_\nu(i_1^{(0)} - i_\nu^{(0)})$$

$$= \max_\nu\left(\sum_{k=1}^{\nu} j_k\right) = \min_\nu\left(\sum_{k=1}^{\nu} j_k\right) = E(j_1, \ldots, j_s).$$

Finally, note that

$$\frac{1}{n}\sum_{k=1}^{n-1} A_k = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{\underline{j}\in D_k} \hat{f}_{j_1}^{(1)} \ldots \hat{f}_{j_s}^{(s)}$$

$$= \frac{1}{n}\sum_{j\in D_{n-1}} \hat{f}_{j_s}^{(1)} \ldots \hat{f}_{j_s}^{(s)}\left(\sum_{E(j)\leqq k\leqq n-1} 1\right)\frac{1}{n}\sum_{j\in D_{n-1}} \hat{f}_{j_1}^{(1)} \ldots \hat{f}_{j_s}^{(s)}(n - E(j)).$$

and thus (A.4) yields (A.2).

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