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hp-version finite elements for the space-time domain

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Abstract. A bilinear formulation of elasto-dynamics is offered which includes, as a special case, "Hamilton's law of varying action". However, the more general bilinear formulation has several advantages over Hamilton's law. First, it admits a larger class of initial-value and boundary-value problems. Second, in its variational form, it offers physical insight into the so-called "trailing terms" of Hamilton's law. Third, numerical applications (i.e., finite elements in time) can be proven to be convergent under correct application of the bilinear formulation, whereas they can be demonstrated to diverge for specific problems under Hamilton's law. Fourth, the bilinear formulation offers automatic convergence of the "natural" velocity end conditions; while these must be constrained in present applications of Hamilton's law. Fifth, the bilinear formulation can be implemented in terms of a Larange multiplier that gives an order of magnitude improvement in the convergence of velocity. This implies that, in this form, the method is a hybrid finite-element approach.

List of symbols

b	arbitrary constant
A_i, A_i'	vector of integrals, i = 0, j
$A(v)$	linear operator on v
$\bar{A}(v)$	Hamilton's form of A
$B(u,v)$	bilinear operator u, v
$\bar{B}(u,v)$	Hamilton's form of B
B_{ij}, $\bar{B}_{ij}$, B'_{ij}	matrix of integrals
C	constant, N/m
c	number of floating-point operations per coef. evaluation
$f, f(x)$	force per unit length, N/m
F, F_0, F_L	forces, N
J	number of functions in series for $\hat{u}$
k	spring rate per unit length, N/m^2
K	spring rate, N/m
K_{max}	maximum value of K
L_a	Lagrangian, non-dimensional
L	length of beam, m
m	mass per unit length, kg/m
M	mass, kg
M_{max}	maximum value of M
n	number of functions in series for $\hat{v}$
N	number of elements in domain
p	momentum density, kg/sec
P, P_0, P_T	momentum, kg-m/sec
q_i	generalized coordinates
r_j	coefficients of ψ_j
t	time, sec
t_0, t_1	limits of action integral, Hamilton's law
T	end of time period, sec
u	solution for displacement, m
$\hat{u}$	approximation to u, m
u_0	initial value for u, m
v	test function, m
$\hat{v}$	limited class of v, m
x	spatial coordinate, m
β	flapping angle, rad
γ	Lock number
Δ	time increment, sec
λ	Lagrange multiplier
μ	longitudinal stiffness EA, N (Eqs. 1–18)
μ	advance ratio of rotor (Eqs. 33–34 and figures)
ϕ_i, ψ_i	polynomial functions
ψ	non-dimensional time, azimuth angle
$\delta( )$	variation of ( )
δW	virtual work
$( )'$	d( )/dx
$\dot{( )}$	d( )/dt
$(^*)$	d/dψ
[ ]	matrix
{ }	column vector
⟨ ⟩	row vector

1 Introduction

1.1 Background

Direct time marching has long been a standard procedure for the solution of initial-value problems in the time domain. In recent years, however, researchers have been studying alternative methods of solution that are more compatible with the finite-difference and finite-element methodologies

used for spatially-dependent problems. There are several potential advantages to such an approach. First, finite elements in time offer a unified solution strategy for the space-time domain which could give synergistic improvements in computational efficiency. Second, finite elements in time could be applied to the energy (or action) rather than to the differential equations, thus saving the cost of derivation of these equations. Third, finite elements in time could be tuned to give minimum error at points for which the greatest accuracy is desired.

Several such methodologies are now in the literature under various titles such as Petroff-Galerkin, weighted residuals, Hamilton's law, time finite elements, etc. Although all of these are based on the same fundamental laws of mechanics (Newton's laws or Hamilton's formulation), they differ greatly in their numerical application, and none offers a proof of convergence. In this paper, we offer one particular method, based on bilinear operation theory, for which convergence is assured. We also show that most of the other methods can be demonstrated to diverge under certain conditions.

1.2 Previous work

The concept of finite elements in time was introduced independently by Fried and Argyris (Fried 1969; Argyris and Sharpf 1969; Oden 1972). Their work is based on direct numerical application of Hamilton's principle

$$\delta \int_{t_0}^{t_1} L_a \mathrm{d}t = 0$$

Argyris correctly notes that, for this equation to hold, $\delta q(t_0)$ and $\delta q(t_1)$ must be constrained to be zero in the numerical approach. The same observation is made by Smith and Smith (1974). Cecil Bailey (Bailey 1975, 1976, 1977) performs a direct numerical solution of the more general Hamilton's law of varying action.

$$\delta \int_{t_0}^{t_1} L_a \mathrm{d}t - \sum_{i=1}^{n} \frac{\partial L_a}{\partial \dot{q}_i} \delta q_i \Bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} \delta W \mathrm{d}t = 0$$

Although the trailing terms are often seen as irrelevant in the derivation of equations, they are important for numerical solutions. Bailey notes that the retention of these terms allows solutions for which δq need not be constrained at t_1. However, he does constrain $\delta q(t_0)$ and $\delta \dot{q}(t_0)$ to be zero due to the fact that, in an initial-value problem, $q(t_0)$ and $\dot{q}(t_0)$ are specified. Virgil Smith (Smith and Smith 1977; Smith 1977) questions Bailey's interpretation of Hamilton's principle; and he proposes a Galerkin procedure for space-time problems that is numerically equivalent to Hamilton's law under certain restrictions on $q_i(t)$ and $\delta q_i(t)$.

It was not long after this that other authors were attacking the direct numerical solution of Hamilton's law. In (Hitzl and Levinston 1980), the method is applied to problems of celestial mechanics. Simkins (1978, 1981) notes that, by breaking the domain into small segments, the numerical applications of Hamilton's law gives rise to *h*-version "finite elements" in the time domain. Smith (1979) replies, that the method of weighted residuals gives the same result but without the semantics problems associated with the word "variational". Further vigorous discussion continues in the literature on this subject (Bailey 1980; Smith 1981); however, most of it centers on the philosophical arguments associated with the different applications and not on the accuracy of numerical results.

Baruch and Riff (1982) offer six possible formulations of finite elements in time, each based on Hamilton's law with various constraints on δq and $\delta \dot{q}$ at $t = t_0$ and $t = t_1$. Each formulation gives a slightly different numerical solution algorithm. The authors note that their fourth method, for which $\delta q(t_1) = 0$, gives by far the best convergence; but their first method, Hamilton's law as used by Bailey, gives worst convergence. (Baruch and Riff 1982) also use very small elements to obtain a marching algorithm, thus closing the gap between numerical integration and timewise finite elements. In (Riff and Baruch 1984), however, the same authors note that (under certain conditions) the finite element formulation can become numerically unstable; and this instability is demonstrated

mathematically. As a solution to the stability problem, the authors replace δq, which appears in Hamilton's law, with $\delta \ddot{q}$ as suggested by Smith (1984). Thus, they make the observation that "δq" in the formulation does not need to be restricted to a literal variation of q. Instead, Hamilton's law must hold true for all functions, δq, regardless of their origin. In essence, Baruch and Riff offer a weighted-residual (rather than variational) form of Hamilton's law. These ideas are further developed and applied in (Riff and Baruch 1984). In (Borri et al. 1985), a comment on (Riff and Baruch 1984), the authors claim that the six variations are artificial and do not adhere to the true meaning of Hamilton's law.

Recently, the method of finite elements in time has also been applied to systems of equations with periodic coefficients, such as are present in the modeling of helicopter dynamics (Borri et al. 1985; Borri 1986; Izadpanah 1985, 1986). Borri (Borri et al. 1985) applies a time-marching version of Hamilton's law (analogous to Euler integration) to helicopter problems. However, in contrast to (Baruch and Riff 1982) and (Riff and Baruch 1984), he notes that the trailing terms in Hamilton's law should be written in terms of unknown momenta (P_{1i} and P_{2i}) rather than explicitly in terms of $\partial L_a / \partial \dot{q}_i$.

$$\delta \int_{t_0}^{t_1} L_a \mathrm{d}t + \sum_{i=1}^{n} [-P_{2i} \delta q_i(t_1) + P_{1i} \delta q_i(t_0)] + \int_{t_0}^{t_1} \delta W \mathrm{d}t = 0$$

This allows a natural convergence to $\dot{q}_i(t_0)$ rather than a constraint of $\dot{q}_i(t_0)$. This concept is further developed in (Borri 1986). Izadpanah (1985, 1986) offers a fully bilinear formulation of finite elements in time, which is the basis of this work.

1.3 Scope of work

In this paper, we offer a more general formulation of elasto-dynamics than that presented in (Izadpanah 1985). In particular, we present a bilinear formulation that is applicable to boundary-value, initial-value, and periodic problems of elasto-dynamics. This more general formulation is stated in a completely generic way, but specific examples are given for beams and spring-mass systems to illustrate the implementation. The development here leads to several important advantages of the present formulation over previous numerical work based on Hamilton's law, Hamilton's principle, or Galerkin methods. Of primary importance is the establishment of a convergence proof for the new formulation as well as the demonstration that numerical applications of Hamilton's law can (and often do) fail to converge. Applications of the new and old methods to problems of helicopter stability are also presented for the case of simple helicopter blade flapping, and these yield insight into the numerical effectiveness of the method.

2 Formulation

2.1 Special case of spring-mass system

Perhaps the best way to introduce the bilinear formulation of dynamic systems is to compare it with the standard, bilinear formulation of elasticity. We begin with the governing equation for a uniform beam-segment of length L on elastic foundation (Fig. 1).

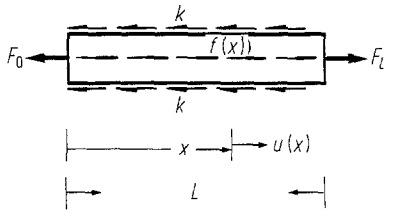
$$ku - \mu u'' = f \tag{1a}$$

$$F_L = \mu u'(L), \quad F_0 = \mu u'(0) \tag{1b,c}$$

$$u(0) = u_0, \quad u(L) = u_L \tag{1d,e}$$

We consider the segment to be an isolated free-body element so that F_0, F_L, u_0, and u_L may or may not be known, depending on the problem.

Next, we compare Eq. (1) with the equation of motion for a simple spring-mass oscillator over a given length of time $0 < t < T$ (Fig. 2),



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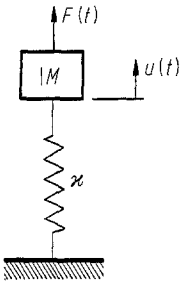

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Figs. 1 and 2. 1 Schematic of beam; 2 schematic of spring-mass-system

$$Ku + M\ddot{u} = F \tag{2a}$$

$$P_T = M\dot{u}(T), \quad P_0 = M\dot{u}(0) \tag{2b,c}$$

$$u(T) = u_T, \quad u(0) = u_0 \tag{2d,e}$$

where P_0, P_T, u_0 and u_T may or may not be given, depending on the problem. Except for the sign of the second-derivative term, Eqs. (1) and (2) are exact mathematical analogies of each other.

Now, the bilinear formulation of the spatial problem, Eq. (1), is well known, (Babuska and Szabo, to be publ.; Rektorys 1980) and can be written in operator notation as

$$B(u,v) = A(v) \quad \text{for all} \quad v \tag{3}$$

where

$$B(u,v) = \int_0^L (kuv + \mu u'v')\,dx, \qquad A(v) = \int_0^L fv\,dx + F_L v(L) - F_0 v(0) \tag{4a,b}$$

That Eqs. (3) and (4) are equivalent to Eq. (1) is easily seen from integration by parts

$$B(u,v) - A(v) = \int_0^L (ku - \mu u'' - f)\,v\,dx - [F_L - \mu u'(L)]\,v(L) + [F_0 - \mu u'(0)]\,v(0) = 0 \tag{5}$$

For Eq. (5) to equal zero for all $v(x)$, clearly each of Eqs. (1a)–(1c) must hold.

By analogy, it would seem that a bilinear formulation of dynamics could be set in the same way as Eq. (3) with

$$B(u,v) = \int_0^T (Kuv - M\dot{u}\dot{v})\,dt, \qquad A(v) = \int_0^T Fv\,dt - P_T v(T) + P_0 v(0) \tag{6a,b}$$

Again, integration by parts shows that Eqs. (3) and (6) are equivalent to Eq. (2).

$$B(u,v) - A(v) = \int_0^T (Ku + M\ddot{u} - F)\,v\,dt + [P_T - M\dot{u}(T)]\,v(T) - [P_0 - M\dot{u}(0)]\,v(0) = 0 \tag{7}$$

For Eq. (7) to be valid for all $v(t)$, clearly each of Eqs. (2a)–(2c) must hold. Therefore, Eqs. (3) and (6) comprise a bilinear formulation of the dynamics of a spring-mass system.

2.2 Variational form

In order to obtain greater insight into the nature of this formulation, it is instructive to consider a special case which has importance in the spatial problem. In particular, we refer to the case $v = \delta u$, in which v is taken as the variation of the displacement. In that case, the spatial problem, Eqs. (3) and (4), reduces to

$$\delta \int_0^L \left[\frac{1}{2} k u^2 + \frac{1}{2} \mu u'^2\right] \mathrm{d}x = \int_0^L f \delta u \, \mathrm{d}x + F_L \delta u(L) - F_0 \delta u(0) \tag{8}$$

Equation (8) has important physical significance in that it equates the variation of the potential energy to the virtual work done on the system by $f(x), F_L$, and F_0. This, then, is a variational (or energy) formulation of the spatial problem.

By analogy, we can write a variational formulation of the temporal problem, Eqs. (3) and (6), when $v = \delta u$.

$$\delta \int_0^T \left[\frac{1}{2} K u^2 - \frac{1}{2} M \dot{u}^2\right] \mathrm{d}t = \int_0^T F \delta u \, \mathrm{d}t - P_T \delta u(T) + P_0 \delta u(0) \tag{9}$$

Equation (9) also has a direct physical interpretation. The left-hand term is the negative variation of action. The integral term on the right-hand side can be thought of as the "virtual action" applied to the system over the time interval $0 < t < T$. Here, "virtual action" has the precise definition of the time-integral of virtual work. This definition of virtual action leads to a physical interpretation of the last two terms in Eq. (9). Consider the following manipulation.

$$\text{Virtual action} = \int_0^T \delta u(t) F \, \mathrm{d}t = \int_0^T \delta u(t) \frac{\mathrm{d}P}{\mathrm{d}t} \, \mathrm{d}t \tag{10a}$$

$$\text{Virtual action} = \int_{P_0}^{P_T} \delta u \, \mathrm{d}P \tag{10b}$$

where P is the momentum.

A comparison of Eq. (10b) with Eq. (9) identifies the last two terms in Eq. (9) as the virtual action entering $(P_0 \delta u_0)$ and leaving $(P_T \delta u_T)$ the system at the boundaries of the time interval. Therefore, just as the right-hand side of Eq. (8) contains both the virtual work done on the spatial domain $0 < x < L$ and the virtual work done across the boundaries, the right-hand side of Eq. (9) represents both the virtual action done during the time domain and the virtual action that crosses the boundaries. Thus, we interpret Eq. (9) as a variational statement of dynamics. *Namely, the variation of the action plus the virtual action over any time interval* $0 < t < T$ *must sum to zero.* This is easily generalized to a complete theory of elastodynamics. In particular, the variation of the action plus the virtual action done on the domain must sum to zero (Fig. 3). On the spatial boundaries $[(x = 0, L), (0 < t < T)]$, the "virtual action" is the temporal integral of the virtual work. On the time boundaries, $[(t = 0, T), (0 < x < L)]$, the "virtual-action" is taken to mean the spatial integral of virtual-action density which is defined as $p \delta u$, momentum per unit length times virtual displacement. On boundaries that cut across space time (such as a moving consraint), the virtual action takes the form of a convective momentum/force balance.

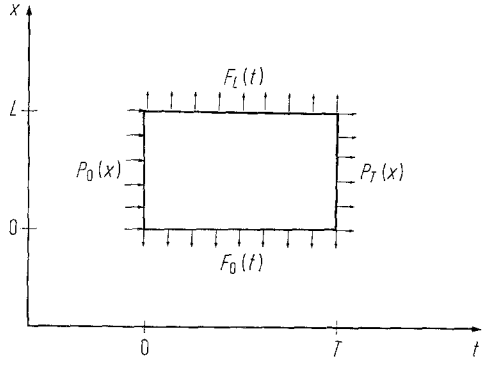


Fig. 3. Space-time domain and boundaries

2.3 Comparison with Hamilton's law

At first glance, the manipulation in Eqs. (1)–(9) may seem almost trivial because the various forms of the equations are all equivalent when $u(t)$ is the exact solution (provided no impulsive forces exist). However, these various forms can give quite different results when applied to approximations for $u(t)$, $\hat{u}(t) \simeq u(t)$, and when enforced over only a limited class of $v(t)$, $\hat{v}(t)$. The true numerical convergence depends upon the nature of the operators in Eq. (3), $B(\hat{u},\hat{v})$ and $A(\hat{v})$.

For the bilinear formulation presented here and in (Borri et al. 1985; Borri 1986; Izadpanah 1985, 1986), these operators are defined by Eqs. (6a) and (6b). We note that $B(\hat{u},\hat{v})$ is a symmetric, bilinear operator; and $A(\hat{v})$ is a linear operator that contains possibly unknown momenta, P_0 and P_T. The numerical method of *every other reference* involves replacement of these unknown momenta by their $M\dot{u}$ counterparts. This changes the basis of numerical convergence because it places the trailing terms into the bilinear operator (both u and v appear in $M\dot{u}v$). Thus, we have for the methods (Fried 1969; Argyris and Scharpf 1969; Oden 1972; Smith and Smith 1974, 1977; Bailey 1975, 1976, 1977, 1980; Smith 1977, 1979, 1981; Hitzl and Levinston 1980; Simkins 1978, 1981; Baruch and Riff 1982; Riff and Baruch 1984a, b).

$$\bar{B}(\hat{u},\hat{v}) = \bar{A}(\hat{v}), \qquad \bar{B}(\hat{u},\hat{v}) = \int_0^T (K\hat{u}\hat{v} - M\dot{\hat{u}}\hat{v})\,\mathrm{d}t + M\dot{\hat{u}}(T)\,\hat{v}(T) - M\dot{\hat{u}}(0)\,\hat{v}(0) \tag{11a,b}$$

$$\bar{A}(\hat{v}) = \int_0^T F\hat{v}\,\mathrm{d}t \tag{11c}$$

with various constraints on $\hat{v}(0)$ and $\hat{v}(T)$. The Galerkin formulation

$$\bar{B}(\hat{u},\hat{v}) = \int_0^T (K\hat{u} + M\ddot{\hat{u}})\hat{v}\,\mathrm{d}t \tag{12}$$

is also suggested by Smith (1977). It is equivalent to Eq. (11b) provided $\hat{u}$ has C^1 continuity.

Several observations are important in the comparison of the present formulation, Eq. (6), with the majority of previous work, Eqs. (11) and (12). First, we note that the trailing terms in Eq. (11b) destroy the symmetry of the operator. (In the following section we will also show that they destroy convergence.) The only means to eliminate them in Hamilton's law is to constrain $v(0) = v(T) = 0$, which results in Hamilton's principle. This is done in (Argyris and Scharpf 1969) and (Smith and Smith 1974), and in method 2 of (Baruch and Riff 1982). However, with or without this constraint, the equivalence of Eqs. (11b) and (12) implies that the natural boundary conditions ($P_0 = M\dot{u}_0$ and $P_T = M\dot{u}_T$) will not be enforced by $\bar{B}(\hat{u},\hat{v}) = A(\bar{v})$. Thus, these methods require constraints on the velocity end conditions. Therefore, the trial functions $\hat{u}$ must be chosen from C^1 (velocity and displacement continuity) rather than from the more general C^0; and this over-constraint of the problem impacts convergence. Equation (12) suffers from the same drawback, since the $\ddot{\hat{u}}$ terms demand a piecewise continuous $\ddot{u}$ which simplies $\hat{u}$ must be from C^1, not C^0.

In contrast, Eqs. (6) show that the present formulation preserves a symmetric operator; and Eq. (7) shows that $M\dot{\hat{u}}$ will converge naturally to the desired momentum as $B(\hat{u},\hat{v}) - A(\hat{v})$ approaches zero. In fact, Eq. (7) implies a quite different constraint on v than has been previously prescribed for initial-value problems. In particular, at an end for which P is known, v must *not* be zero so that $M\dot{\hat{u}}$ will converge to P. On the other hand, at an end for which P is not known, v must be set to zero in order to eliminate the unknown. (An alternate strategy, suggested in (Borri et al. 1985; Borri 1986) is to leave P_0 or P_T as unknown variables. This however, is not allowed in the convergence proof.)

For boundary-value problems, the variational from $v = \delta u$ exactly fulfills this requirement since, if u is prescribed at some point (P unknown), then $v = \delta u = 0$. For initial-valsue problems [$u(0)$ and $\dot{u}(0)$ prescribed], however, $v = \delta u$ gives exactly the wrong v constraints ($v(0) = 0$, $v(T) \neq 0$, rather than ($v(0) \neq 0$, $v(T) = 0$).

3 Convergence

3.1 Approximate solutions

In the numerical formulation of the problem, we assume a solution for u from some limited class of functions ϕ_j, $j = 1, J$.

$$\hat{u}(t) = \sum_{j=1}^{J} \phi_j(t) q_j \tag{13}$$

Now, $\hat{u}$ is only an approximation to u (except in the limit as $J \to \infty$) and can exactly satisfy neither the differential equations and boundary conditions of Eq. (1) nor those of Eq. (2). Similarly, $\hat{u}$ cannot satisfy the bilinear formulation, $B(\hat{u}, v) = A(v)$, for all possible v. A numerical solution *can* be obtained, however, if one restricts the class of v to some subspace, $\hat{v}$, such that $B(\hat{u}, \hat{v}) = A(\hat{v})$ for all $\hat{v}$ in the subspace. For example, we can write

$$\hat{v}(t) = \sum_{i=1}^{n} \psi_i(t) r_i \tag{14}$$

Any mathematical proof for the numerical solution for $u(t)$ must show that, as $\hat{v}$ is expanded to cover more and more of the space of admissable functions, then $\hat{u}$ will converge to u.

Clearly, the choice of ϕ_i and ψ_i is related to the convergence in a very direct way. In the bilinear formulation, ψ_i and ϕ_i are completely independent. In the variational case, however, $\psi_i = \phi_i$, and $r_i = \delta q_i$. Clearly, then, the convergence will be affected by the choice of bilinear formulation. In this section, we wish to address this convergence.

3.2 Sufficient proof of convergence

In this section, we deal the coup de grace to Hamilton's law (as *a* computational tool) by showing that: (1) the nonsymmetric terms in $\bar{B}(\hat{u}, \delta\hat{u})$ preclude proof of convergence, and (2) specific cases of divergence can be demonstrated for well-formulated problems. On the other hand, the bilinear formulation is *proven* to converge; and specific numerical examples are given for which the bilinear formulation eliminates the divergence found with Hamilton's law.

For spatial boundary-value problems, convergence can be proven based on the following properties of $B(u, v)$ and $F(v)$, (Babuska and Szabo, to be publ.; Rektorys 1980).

(1) $F(v)$ linear: $F(bv) = bF(v)$
(2) $B(u, v)$ bilinear: $B(bu, v) = B(u, bv) = bB(u, v)$
(3) $B(u, v)$ symmetric: $B(u, v) = B(v, u)$
(4) $B(u, v)$ positive definite, i.e.

$$B(u, u) > 0 \quad \text{if} \quad \int_0^T u^2 \, dt > 0$$

For the temporal problem, the positive-definite property is lost; but convergence can still be proven for both initial-value and boundary-value problems (even for a nonsymmetric B) provided that: (1) $\hat{u}$ and $\hat{v}$ are chosen from C^0, and (2) and alternative property holds in lieu of numbers (3) and (4) above. This property is the Lax-Milgram Lemma (Babuska and Szabo, to be publ.; Rektorys 1980; Aubin 1979). It is a sufficient condition for convergence and is given by

$$|B(u, v)| \leqq C \left[\int_0^T (u^2 + \dot{u}^2) \, dt \right]^{1/2} \times \left[\int_0^T (v^2 + \dot{v}^2) \, dt \right]^{1/2} = C \, \|u, \dot{u}\| \times \|v, \dot{v}\| \tag{15}$$

Where C is *a* constant independent of u and v. The above condition ensures that small perturbations in problem parameters will result in small perturbations to the generalized solution.

The verification of Eq. (15) for the bilinear formulation given in this paper, Eq. (6), follows from the Schwarz inequality:

$$
\begin{aligned}
|B(u,v)| &\leqq \int_0^T K|u| \times |v| \, dt + \int_0^T M|\dot{u}| \times |\dot{v}| \, dt \\
&\leqq K_{max} \int_0^T |u| \times |v| \, dt + M_{max} \int_0^T |\dot{u}| \times |\dot{v}| \, dt \\
&\leqq K_{max} \left[\int_0^T u^2 dt \right]^{1/2} \times \left[\int_0^T v^2 dt \right]^{1/2} \\
&+ M_{max} \left[\int_0^T \dot{u}^2 dt \right]^{1/2} \times \left[\int_0^T \dot{v}^2 dt \right]^{1/2} \\
&\leqq \sqrt{K_{max}^2 + M_{max}^2} \left[\int_0^T (u^2 + \dot{u}^2) dt \right]^{1/2} \times \left[\int_0^T (v^2 + \dot{v}^2) dt \right]^{1/2}
\end{aligned} \tag{16}
$$

Thus, the property is demonstrated to hold if we take

$$C = \sqrt{K_{max}^2 + M_{max}^2}$$

3.3 Failure of Hamilton's law

In contrast, in the numerical formulation of Hamilton's law, $\bar{B}(u,v)$ has two extra bilinear terms that prevent the establishment of the above property

$$|\bar{B}(u,v)| \leqq |B(u,v)| + \underline{M|\dot{u}(0)||v(0)|} + \underline{M|\dot{u}(T)||v(T)|} \tag{17}$$

The two underlined terms in $|\bar{B}(u,v)|$ cannot be limited to less than the norm in the above equation. For example, consider

$$v(t) = 1, \quad \dot{v}(t) = 0 \quad \text{for all } t \tag{18}$$

$$u(t) = \begin{cases} 0, & 0 \leqq t \leqq T - \Delta \\ 1 - \dfrac{T-1}{\Delta}, & T - \Delta < t \leqq T \end{cases} \tag{19}$$

$$\dot{u}(t) = \begin{cases} 0, & 0 \leqq t \leqq T - \Delta \end{cases} \tag{20}$$

$$\dot{u}(t) = \begin{cases} \dfrac{1}{\Delta}, & T - \Delta < t \leqq T \end{cases} \tag{21}$$

It follows that

$$|\dot{u}(T)| \times |v(T)| = \frac{1}{\Delta} \tag{22}$$

$$\left[\int_0^T (v^2 + \dot{v}^2) dt \right]^{1/2} = T^{1/2} \tag{23}$$

$$\left[\int_0^T (u^2 + \dot{u}^2) dt \right]^{1/2} = \left[\frac{1}{\Delta} + \frac{\Delta}{3} \right]^{1/2} \tag{24}$$

Thus, no matter how large one maskes C, there is always a Δ small enough such that

$$M|\dot{u}(T)||v(T)| = \frac{M}{\Delta} > \|u, \dot{u}\| \times \|v, \dot{v}\| = C \left[\frac{T}{\Delta} \right]^{1/2} \tag{25a,b}$$

This does *not* imply that numerical application of Hamilton's law will *never* converge, as convergence has been demonstrated in a great number of cases. However, it *does* imply that one can find individual examples for which convergence will not occur.

4 Numerical applications

4.1 Matrix formulation

Numerical solutions to dynamics problems by use of the above, bilinear formulation can be couched in a matrix framework. We consider an approximate solution, $\hat{u}$, as in Eq. (13) with a restricted class of test functions, $\hat{v}$, as in Eq. (14). Substitution into Eqs. (3) and (6) gives a matrix formulation of an approximate solution for a temporal problem.

$$\langle r_i \rangle [B_{ij}] \{q_j\} = \langle r_i \rangle \{A_i\} \tag{26}$$

where

$$B_{ij} = \int_0^T (K\psi_i \phi_j - M\dot{\psi}_i \dot{\phi}_j)\,\mathrm{d}t, \qquad A_i = \int_0^T F\psi_i \mathrm{d}t - P_T \psi_i(T) + P_0 \psi_i(0) \tag{27a,b}$$

Since Eq. (26) must be valid for all r_i, we can eliminate r_i from the equation to obtain n equations in J unknowns.

The constraints on $\hat{u}$ and $\hat{v}$ must now be included, and this can be handled in a variety of ways. For example, for an initial-value problem, we have $(\hat{u}(0) = u_0,\ M\dot{u}(0) = P_0)$ as given, with $\hat{u}(T)$ and P_T unknown. Thus, we hve two constraints.

$$\hat{u}(0) = \sum_{j=1}^{J} \phi_j(0) q_j = u_0, \qquad \hat{v}(T) = \sum_{i=1}^{n} \psi_i(T) \mathrm{r}_i = 0 \tag{28a,b}$$

Equation (28a) can be included as an augmented equation in Eq. (26). Equation (28b) can be included through multiplication by an aribitrary Lagrange multiplier, λ, and then by addition of that term to the left-hand side of Eq. (26). This gives, as a constrained equation,

$$\begin{bmatrix} [B_{ij}] & \{\psi_i(T)\} \\ \langle \phi_j(0) \rangle & 0 \end{bmatrix} \begin{Bmatrix} \{q_j\} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} A_i' \\ u_0 \end{Bmatrix}, \qquad A_i' = \int_0^T F\psi_i \mathrm{d}t + P_0 \psi_i(0) \tag{29a,b}$$

where $\psi_i(T) P_T$ has been eliminated from A' due to the constraint $v(T) = 0$. Equation (29) is taken with $J = n$ (the same number of functions in $\hat{u}$ and $\hat{v}$) to yield $n + 1$ equations in $n + 1$ unknowns to be solved for q_j and λ.

Of course, there are other ways to do this besides use of the Lagrange multiplier. For example, if we take

$$\phi_1(0) = 1, \qquad \phi_j(0) = 0 \qquad j \neq 1 \tag{30a}$$

$$\psi_1(T) = 1, \qquad \psi_i(T) = 0 \qquad i \neq 1 \tag{30b}$$

Then the constraints become trivial

$$q_1 = u_0, \qquad r_1 = 0 \tag{30c}$$

The Lagrange multiplier is, then, effectively eliminated from Eq. (29); and we have in $n - 1$ equations in $n - 1$ unknowns

$$[B_{ij}'] \begin{Bmatrix} q_2 \\ \vdots \\ q_n \end{Bmatrix} = \begin{Bmatrix} A_2' \\ \vdots \\ A_n' \end{Bmatrix} - \begin{Bmatrix} B_{21} \\ \vdots \\ B_{n1} \end{Bmatrix} u_0 \tag{31a}$$

where B_{ij}' is B_{ij} with the first row and column removed, and where

$$\lambda = A_1' - B_{11} u_0 - \langle B_{12} \ldots B_{1n} \rangle \begin{Bmatrix} q_2 \\ \vdots \\ q_n \end{Bmatrix} \tag{31b}$$

In contrast, the same development for Hamilton's law differs slightly (but significantly) in its numerical formulation. Equations (27a) and (27b) become:

$$\bar{B}_{ij} = \int_0^T (K\phi_i\phi_j - M\dot{\phi}_i\dot{\phi}_j)\,\mathrm{d}t + M\phi_i(T)\dot{\phi}_j(T) - M\phi_i(0)\dot{\phi}_j(0) \qquad \bar{A}_i = \int_0^T F\phi_i\,\mathrm{d}t \tag{32a,b}$$

Furthermore, an initial-velocity constraint must be added to Hamilton's law which implies a second auxillary equation. Also, the relation $v = \delta u$ implies constraints on both $v(0)$ and $\dot{v}(0)$ which implies two Lagrange multipliers. The results is

$$\begin{bmatrix} [\bar{B}_{ij}] & \{\phi_i(0)\} & \{\dot{\phi}_i(0)\} \\ \langle\phi_j(0)\rangle & 0 & 0 \\ \langle\dot{\phi}_j(0)\rangle & 0 & 0 \end{bmatrix} \begin{Bmatrix} \{q_j\} \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} \{\bar{A}_i\} \\ u_0 \\ P_0/M \end{Bmatrix} \tag{33}$$

with $\bar{A}_i$ from Eq. (32b). A comparison of Eqs. (29) and (33) reveals the similarities and differences in the two formulations.

Again, λ_1 and λ_2 can be eliminated, if desired, by judicious choice of ϕ, q_1, and q_2. Thus, the Hamilton's formulation is similar to the bilinear formulation but with differences in numerical details. However, it is exactly these details that cause Eq. (29) to converge unconditionally; whereas Eq. (33) can results in divergence.

A similar development occurs for other choices of end conditions or for other constraints on $\hat{u}$. For multiple elements (i.e., many finite-elements in time over a domain), each solution is applied sequentially, exactly as above, over the time domain of interest, with the end values of one segment [$u(T)$ and $\dot{u}(T)$] used as the initial conditions for the next.

4.2 Significance of Lagrange multipliers

Lagrange multipliers, such as those introduced in Eqs. (29)–(33), often have important physical meaning; and this is the case in the present formulation. A comparison of Eqs. (26) and (27) with (29) reveals that λ is numerically identical to P_T, the final momentum. This fact provides an extraction technique to obtain an improved estimate of $\dot{u}(T)$.

Equation (7) shows that P_T must approach $M\dot{u}(T)$ as the number of functions, n, is increased. However, one would expect P_T/M to converge much more rapidly to $\dot{u}(T)$ than does the actual time derivative of $\hat{u}$,

$$\dot{\hat{u}}(T) = \sum_{j=1}^{J} \dot{\phi}_j(T) q_j \tag{34a}$$

The reason for this expectation is that Eq. (34a) involves derivatives of the trial functions, ϕ_j, which can be more sensitive in convergence than ϕ_j itself. On the other hand, the formulation

$$\dot{u}(T) \simeq \lambda/M \tag{34b}$$

is not subject to these sensitivities and represents more of a least-squares estimate of the final velocity. A similar effect is present in the space domain for which it is well known the summation of forces and moments on a beam is a much more accurate measure of stresses at an end than are the second and third derivatives of deflection at that end. Equation (31b), the special case of $v(T)$ constrained by choice of ψ_i, shows this clearly. The extraction equation for λ (in this case separate from the solution for q) is expressed as a summation of external forces and internal momenta.

Therefore, whether or not one explicitly invokes a Lagrange multiplier to enforce $v(T) = 0$, one should calculate $\lambda = P_T$ in order to obtain the most accurate estimate of $\dot{u}(T)$. In the case of multiple elements, this is extremely important. Thus, $u(T)$ and P_T for a particular segment should be used directly as $u(0)$ and P_0 for the next segment. Such a formulation is analogous to the mixed or hybrid finite-element method in space for which deflections and stresses are the state variables. It should be noted here that, in the marching algorithm of (Borri 1986), P_T is moved to the left-hand side of the equation (along with $u(T)$) as an unknown. Thus, although $v(T)$ is not formally set to zero, the numerical result is the same as the present method with Lagrange multiplier. However, we recommend the formulation *without* an explicit Lagrange multiplier, Eq. (31a).

Once we have made the above observations, it is quite natural to extend this concept in order to obtain a better approximation for the momentum $P(t)$ (i.e., for the velocity) at any point in the domain. Having recognized that the Lagrange multiplier represents a momentum balance, we can write the momentum at time t as

$$P(t) = P_0 + \int_0^t (f - K\hat{u})\,\mathrm{d}t \quad \text{or} \quad P(t) = A_0 - \langle \ldots B_{0j} \ldots \rangle \quad \{q_j\} \tag{35a,b}$$

where

$$A_0 = P_0 + \int_0^t f\,\mathrm{d}t \quad \text{and} \quad B_{0j} = \int_0^t K\phi_j\,\mathrm{d}t \tag{35c,d}$$

This is again analogous to the spatial problem, in which force equilibrium gives

$$F(x) = F_0 + \int_0^x (K\hat{u} - f)\,\mathrm{d}x \tag{35e}$$

One can easily prove that $P(T)$ in equation (35b) is identically equal to $P_T = \lambda$ of the bilinear formulation, provided that $v(t) = 1$ can be exactly represented by a linear combination of the ψ_i's retained in the numerical results. This is most certainly the case for any practical set of ψ_i's.

5 Numerical results

In this section, we apply three finite-element formulations to the flapping dynamics of a helicopter rotor blade. The system is a second-order differential equation with periodic coefficients.

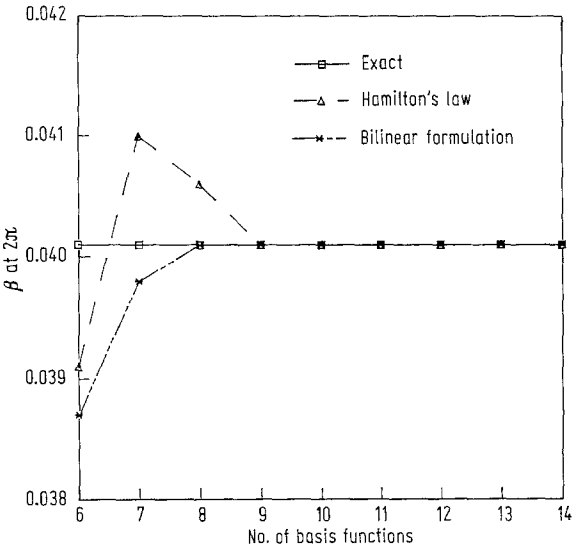
$$\overset{*}{\beta} + \frac{\gamma}{8}\left[1 + \frac{4}{3}\mu\sin\psi\right]\overset{*}{\beta} + \left[1 + \frac{\gamma}{8}\left(\frac{4}{3}\mu\cos\psi + \mu^2\sin 2\psi\right)\right]\beta = 0 \tag{36}$$

The solution of this equation must be found over one period ($0 < \psi < 2\pi$) in order to find stability information from the Floquet transition matrix, [24].

We will now compare numerical results from application of Hamilton's law of varying action (Method H) with results from the new, bilinear formulation, (Methods B1 and B2). In B1, $\dot{u}(T)$ is obtained from the series, Eq. (34a). In B2, $\dot{u}(T)$ is obtained from the Lagrange multiplier, Eq. (34b). The basis functions for both ϕ_i and ψ_i are taken as Legendre polynomials over the range $[-1, +1]$, unless otherwise noted in the results. We have obtained results with other polynomials; and there is little effect of polynomial choice, as long as the polynomials are reasonably orthogonal. All results are for one element ($N = 1$) unless stated otherwise.

5.1 Hover

For comparison purposes in the following results, we will consider $\beta(2\pi)$ and $\overset{*}{\beta}(2\pi)$ in response to the initial conditions $\overset{*}{\beta}(0) = 0$, $\beta(0) = 1$. These values are two of the four elements of the "Floquet transition matrix"; and relative comparisons with these elements are representative of those for the other two elements and for the Floquet eigenvalues themselves. Figure 4 presents the results of Methods H and B as compared to an exact solution for $\mu = 0$. The response, $\beta(2\pi)$, is plotted versus n, the number of polynomials used in the series. At 6 polynomials, the error with Hamilton's law is about 2%; and little improvement is obtained when the 7th polynamial is added. However, by 9 polynomial terms, the errors has rapidly converged to less than 0.1%. With the bilinear formulation, the convergence is even better; and only 8 polynomials are needed to reach 0.1%. accuracy. At 6 polynomials, however, the accuracy of the bilinear formudlation is slightly inferior to that of Hamilton's law. The reason for this cross-over is straightforward. In Hamilton's law, one enforces $\dot{u}(0)$; and this is more accurate when fewer functions are used, provided it converges. However, as more terms are added, this advantage disappears. This is analogous to the advantage of Galerkin over Ritz methods when only a few comparison functions are used. However, once enough terms are used so as to converge on $\dot{u}(0)$, this difference is lost.



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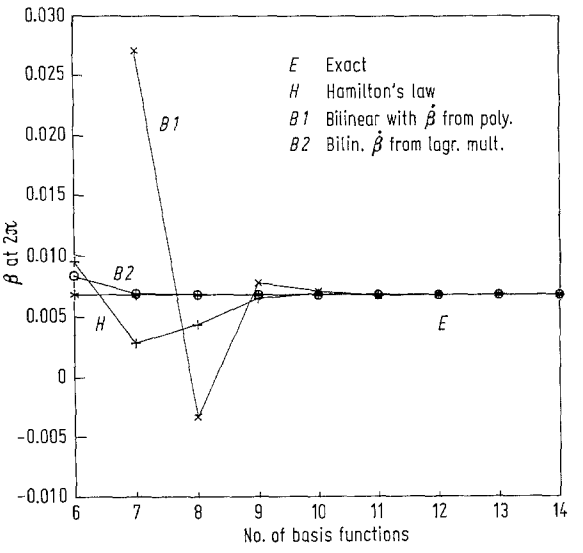

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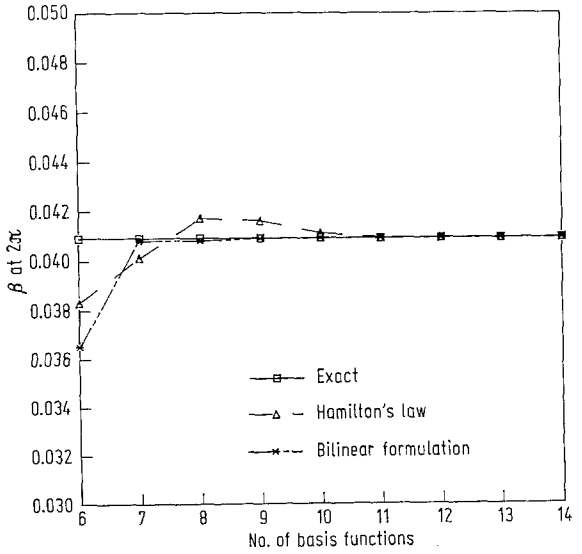
Figs. 4 and 5. Flapping at end of period, $\mu = 0.0$; Flapping velocity at end of period, $\mu = 0.0$

In Fig. 5, we present the same data for $\overset{*}{\beta}(2\pi)$. In this figure, two curves are provided from the bilinear formulation. B1 is $\overset{*}{\beta}$ from the polynomial derivatives and B2 is $\overset{*}{\beta}$ from the Lagrange multiplier. Comparison of the H and B1 curves provides the same relative conclusions as in Fig. 4. However, with B1, the convergence for $\overset{*}{\beta}$ is seen to be much slower than that for β. With B2, on the other hand, there is a dramatic improvement in $\overset{*}{\beta}$ convergence; and the accuracy of $\overset{*}{\beta}(2\pi)$ from λ rivals that of $\beta(2\pi)$ for accuracy. Thus, our original speculation on the convergence of the Lagrange multiplier is supported.

5.2 *Forward flight*

Next, we move on to higher advance ratios, which introduces periodic coefficients into the equations. Figures 6–9 show the evolution of $\beta(2\pi)$ convergence as advance ratio is increased. No exact solutions are available, but high-precision time-marching results are taken as essentially exact. In



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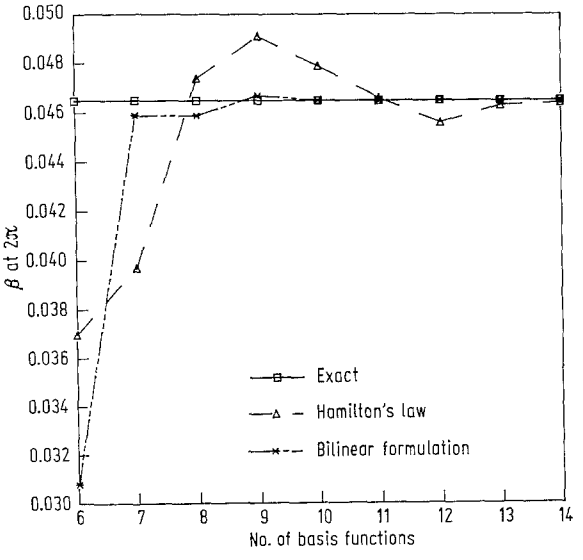

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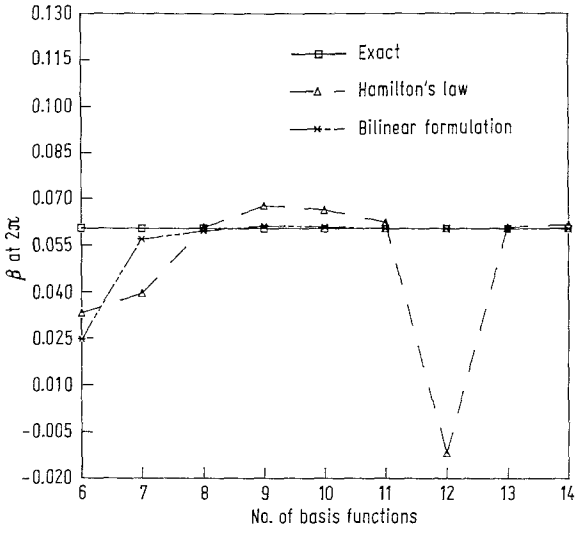
Figs. 6 and 7. Flapping at end of period, **6** $\mu = 0.1$; **7** $\mu = 0.3$



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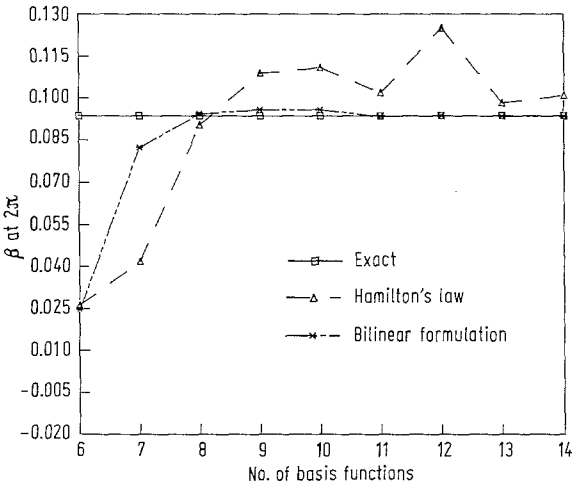

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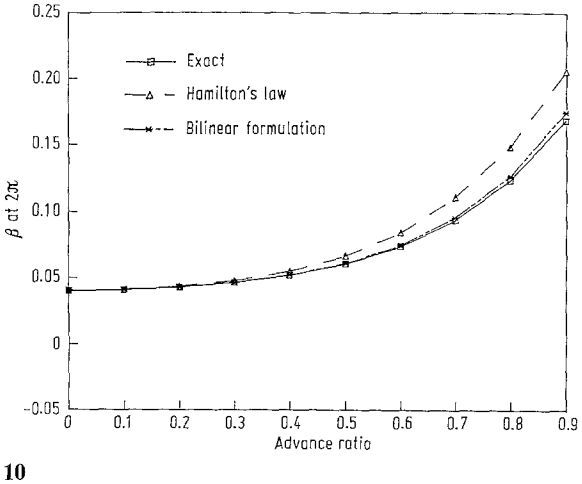
Figs. 8 and 9. Flapping end of period, **8** $\mu = 0.5$; **9** $\mu = 0.7$

Fig. 6, at $\mu = 0.1$, we see a definite retardation in the convergence of Hamilton's law, with 1% error still present at $n = 10$. (Note that Fig. 6 has a compressed scale as compared to Fig. 4.) The bilinear formulation, however, converges rapidly. At $\mu = 0.3$, Fig. 7, a further degradation in Hamilton's law is seen, and the results oscillate about the true solution as n is increased. The error is now 4% for $n = 10$. The bilinear formulation, on the other hand, still converges quickly with less than 0.1% error at $n = 10$. At $\mu = 0.5$, Fig. 8, greatly expanded scale is required to capture the large errors present in results with Hamilton's law. At $n = 12$, the error is over 100%. In contrast, bilinear results are essentially converged at $n = 11$. Last, at $\mu = 0.7$ (Fig. 9), the same scale shows a better, but still poor, results for Hamilton's law; while convergence is achieved at $n = 11$ for the bilinear formulation.

Additional insight into the convergence problems of Hamilton's law can be obtained by a cross-plot of this same data versus advance ratio for specified valves of n. Figure 10, for 10 basis functions, shows that the accuracy decreases with advance ratio and that the bilinear formulation has only 20% as much error as does Hamilton's law. At 11 basis functions, Fig. 11, we see that Hamilton's law and the bilinear formulation each show improved convergence; but the bilinear form is essentially exact whereas Hamilton's law has 10% error at $\mu = 0.9$. As we add one more basis function, $n = 12$, Fig. 12, we see clearly the numerical difficulties encountered by Hamilton's law. Although the error is maximum near $\mu = 0.5$, large errors still persist at all advance ratios greater than 0.5.



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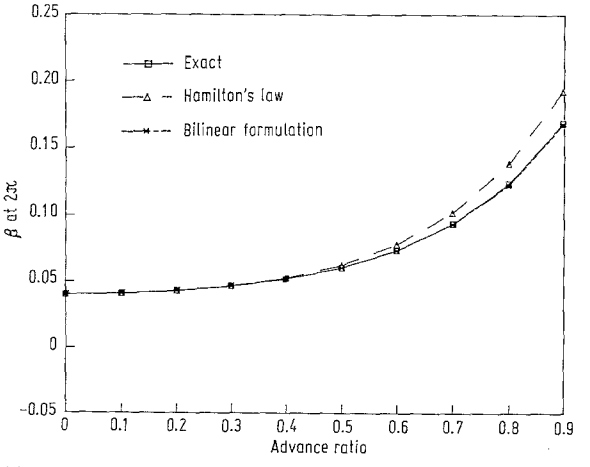

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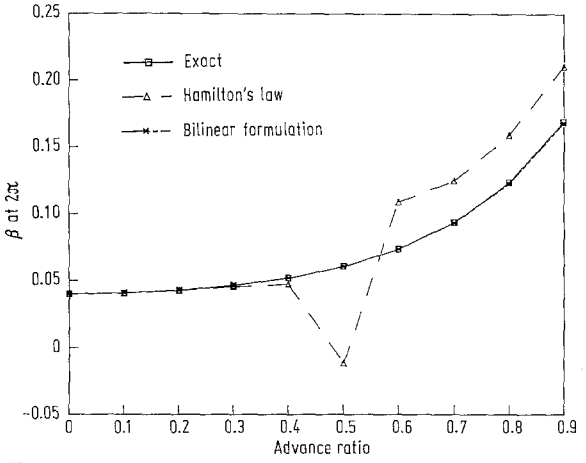
Figs. 10 and 11. Flapping versus advance ratio, **10** $n = 10$; **11** $n = 11$



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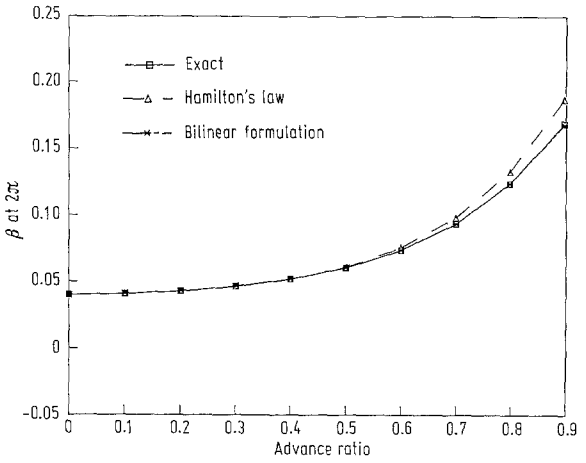

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Figs. 12 and 13. Flapping versus advance ratio, **12** $n = 12$; **13** $n = 13$

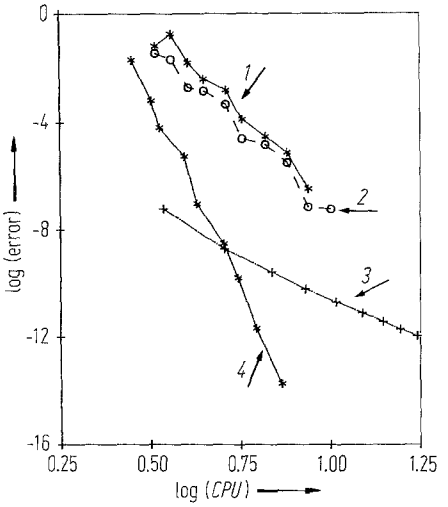
Interestingly, at 13 basis functions, Fig. 13, the convergence anomaly disappears; and we return to a more uniform curve. Still, however, results with the bilinear formulation excel those with Hamilton's law. Thus, we see that Hamilton's law can sometimes yield spurious results despite the fact that it often does converge. Furthermore, for 7 basis functions or more, results with Hamilton's law are always less accurate than those with the bilinear formulation.

5.3 Numerical efficiency

Although a detailed discussion of the numerical efficiency of the bilinear formulation is beyond the scope of this paper, we would like to demonstrate that the method is competitive with conventional time-marching solutions. Figure 14 provides three plots of the error in flap damping in hover (calculated from the Floquet transition matrix) versus the required CPU time on a VAX 750 computer. One plot depicts the performance of "Hamming's modified predictor-corrector" (from the IBM Scientific Subroutine Package). The second plot is the performance of B1, the bilinear formulation with velocity from $\hat{u}$. The third plot is the performance of B2, the bilinear formulation with Lagrange multiplier. The predictor-corrector results, indicated by the squares, are calculated for 100 to 900 time steps in increments of 100. The bilinear results are calculated for 6 to 15 polynomials in increments of 1.

Several important conclusions can be drawn from this figure. First, at low CPU's (i.e., at larger errors), the predicator-corrector is much more efficient than is the bilinear formulation. This is due to the relatively high cost of performing the required integrals. However, as the required error becomes more exacting (and CPU increases), the rate of convergence is quite different among the three methods. For the predictor-corrector, if one neglects the start-up CPU, the error decreases as CPU^{-5} (which is to be expected to a fourth-order method). For B1, however, the error decreases as CPU^{-13}; and, for B2, the error decreases as CPU^{-29}. We conclude, therefore, that the bilinear formulation with Lagrange multiplier is spectacularly better than the same method with velocity from $\hat{u}$. We also conclude that, for small error bounds, the bilinear formulation can actually be more efficient than time marching, even for constant-coefficient cases.

As a final comparison, we look at the numerical efficiency as a function of both the number of elements, N, and the number of polynomials per element, n. Here, we use the integrals of Legendre polynomials as the trial functions. Figure 15 provides this comparison. The vertical axis is the error in $\beta(2\pi)$. The horizontal axis is the total number of operations in the comptutation (normalized CPU). Curves are provided for 2 to 8 polynomials per element. The number of elements is an implicit paramater that varies along the curves. For a given error criteria, there is a specific value of n which provides the lowest CPU. Thus, there exists a locus of optimum points which is formed by these optimum segments. This locus gives the best combination of N and n for an hp finite element in time. The optimum N is usually 2 elements per period. Also shown on the curve are x's



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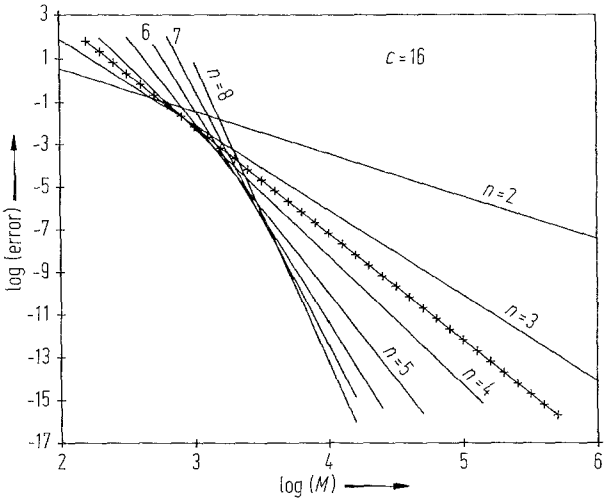

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Figs. 14 and 15. 14 Per cent error vs. CPU, $\mu = 0.0$. *1* Bilinear formulation (Method B1); *2* Hamilton's law (Method H); *3* Hamming's predictor-corrector; *4* bilinear formulation (Method B2); **15** Error as a function of floating-point multiplications, $c = 16$

which represent the efficiency of a fourth-order predictor corrector. For this particular case ($c = 16$ operations per coefficient evaluation), the predictor-corrector results lie tangent to the hp optimum. Thus, there is only a small error region, near 10^{-3}, for which conventional time marching is better than the bilinear formulation. For smaller c, there is a larger range for which this conventional marching technique is superior to finite elements. For larger c, finite elements are always superior to standard marching. More details are provided in (Izadpanah 1986).

6 Summary and conclusions

The conclusions of this work are:

(1) The use of Hamilton's law of varying action, as a basis for numerical solutions of time problems, is not always stable and can result in divergence and incorrect answers even as the number of polynomials is increased.

(2) A bilinear formulation of dynamics is introduced. In one of its special cases, it is a variational statement of dynamics which states that the variation of the "Action" plus the "Virtual action" (taken over a space-time domain and crossing the space-time boundaries) must sum to zero.

(3) The bilinear formulation can be used as a basis for numerical, finite-element solutions of time problems. These can be proven to be convergent, provided that the test functions are constrained in a very precise way depending on the problem. For example, for initial-value problems, one must have

$$\hat{v}(0) \neq 0, \qquad \hat{v}(T) = 0$$

(4) Numerical results with the new formulation (and with the Lagrange multiplier used as an estimate of velocity) eliminate all previous numerical difficulties and display a computational efficiency competitive and often superior to that of time marching. This efficiency is enhanced by the optimum choice of the number of elements, which depends on the desired accuracy for a given problem.

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